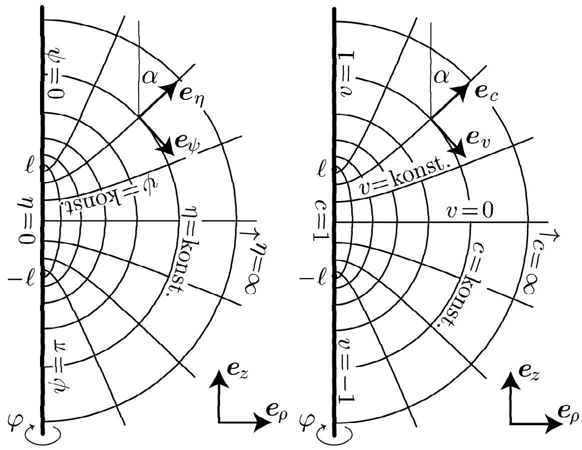
Protáhlé elipsoidální souřadnice



Hodnoty souřadnic:

$$\eta \in \mathbb{R}^+ \quad \psi \in \big(-\pi, \pi\big) \qquad c \in (1, \infty) \quad v \in (-1, 1)$$

Vztah ke kartézským souřadnicím:

$$x = \ell \sinh\eta\, \sin\psi\, \cos\varphi \;= \sqrt{c^2 - \ell^2}\,\sqrt{1 - v^2}\, \cos\varphi$$
$$y = \ell \sinh\eta\, \sin\psi\, \sin\varphi \;= \sqrt{c^2 - \ell^2}\,\sqrt{1 - v^2}\, \sin\varphi$$
$$z = \ell \cosh\eta\, \cos\psi \;= c\, v$$

$$\cosh\eta = \frac{c}{\ell} = \frac{\sqrt{\sigma^2 + r^2 + \ell^2}}{\sqrt{2}\,\ell} = \frac{\sqrt{\rho^2 + (z+\ell)^2} + \sqrt{\rho^2 + (z-\ell)^2}}{2\,\ell} \qquad \sinh\eta = \frac{\sqrt{c^2 - \ell^2}}{\ell} = \frac{\sqrt{\sigma^2 + r^2 - \ell^2}}{\sqrt{2}\,\ell}$$

$$\cos\psi = v = \frac{\sqrt{2}\,z}{\sqrt{\sigma^2 + r^2 + \ell^2}} = \frac{2\,z}{\sqrt{\rho^2 + (z+\ell)^2} + \sqrt{\rho^2 + (z-\ell)^2}} \qquad \sin\psi = \sqrt{1 - v^2} = \frac{\sqrt{2}\,\rho}{\sqrt{\sigma^2 + r^2 - \ell^2}}$$

$$\tan\varphi = \frac{y}{x} \qquad x = \rho\cos\varphi \qquad y = \rho\sin\varphi$$

pomocné funkce:

$$\rho^2 = x^2 + y^2 = \ell^2 \sinh^2\eta\, \sin^2\psi = (c^2 - \ell^2)\,(1 - v^2)$$
$$r^2 = \rho^2 + z^2 = \ell^2\big(\cosh^2\eta - \sin^2\psi\big) = \ell^2\big(\sinh^2\eta + \cos^2\psi\big) = c^2 - \ell^2 + \ell^2 v^2$$
$$\sigma^2 = \sqrt{\rho^2 + (z+\ell)^2}\,\sqrt{\rho^2 + (z-\ell)^2} = \sqrt{(r^2 - \ell^2)^2 + 4\ell^2\rho^2} = \sqrt{(r^2 + \ell^2)^2 - 4\ell^2 z^2}$$
$$= \ell^2\big(\cosh^2\eta - \cos^2\psi\big) = \ell^2\big(\sinh^2\eta + \sin^2\psi\big) = c^2 - \ell^2 v^2$$

Metrika a objemový element:

$$\boldsymbol{q} = \sigma^2\, \mathbf{d}\eta\, \mathbf{d}\eta + \sigma^2\, \mathbf{d}\psi\, \mathbf{d}\psi + \rho^2\, \mathbf{d}\varphi\, \mathbf{d}\varphi$$
$$= \frac{c^2 - \ell^2 v^2}{c^2 - \ell^2}\, \mathbf{d}c\, \mathbf{d}c + \frac{c^2 - \ell^2 v^2}{1 - v^2}\, \mathbf{d}\psi\, \mathbf{d}\psi + (c^2 - \ell^2)\,(1 - v^2)\, \mathbf{d}\varphi\, \mathbf{d}\varphi$$
$$dV = \rho\sigma^2\, d\eta\, d\psi\, d\varphi = (c^2 - \ell^2 v^2)\, dc\, dv\, d\varphi$$

Laméovy koeficienty:

$$h_\eta = \sigma \qquad h_\psi = \sigma \qquad h_\varphi = \rho$$
$$h_c^2 = \frac{c^2 - \ell^2 v^2}{c^2 - \ell^2} \qquad h_v^2 = \frac{c^2 - \ell^2 v^2}{1 - v^2} \qquad h_\varphi^2 = (c^2 - \ell^2)\,(1 - v^2)$$

Vztah eliptických a kartézských bází vektorů:

$$\boldsymbol{e}_x = \cos\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi + \sin\alpha\,\boldsymbol{e}_\eta\big) - \sin\varphi\,\boldsymbol{e}_\varphi \qquad \boldsymbol{e}_\eta = \boldsymbol{e}_c = \cos\alpha\,\boldsymbol{e}_z + \sin\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y\big)$$
$$\boldsymbol{e}_y = \sin\varphi\,\big(\cos\alpha\,\boldsymbol{e}_\psi + \sin\alpha\,\boldsymbol{e}_\eta\big) + \cos\varphi\,\boldsymbol{e}_\varphi \qquad \boldsymbol{e}_\psi = \boldsymbol{e}_v = -\sin\alpha\,\boldsymbol{e}_z + \cos\alpha\,\big(\cos\varphi\,\boldsymbol{e}_x + \sin\varphi\,\boldsymbol{e}_y\big)$$
$$\boldsymbol{e}_z = -\sin\alpha\,\boldsymbol{e}_\psi + \cos\alpha\,\boldsymbol{e}_\eta \qquad \boldsymbol{e}_\varphi = -\sin\alpha\,\boldsymbol{e}_x + \cos\alpha\,\boldsymbol{e}_y$$

kde

$$\sin\alpha = \frac{\ell}{\sigma}\,\cosh\eta\,\sin\psi = \frac{c\sqrt{1 - v^2}}{\sqrt{c^2 - \ell^2 v^2}} = \frac{\rho}{\sigma}\sqrt{\frac{\sigma^2 + r^2 + \ell^2}{\sigma^2 + r^2 - \ell^2}}$$
$$\cos\alpha = \frac{\ell}{\sigma}\,\sinh\eta\,\cos\psi = \frac{v\sqrt{c^2 - \ell^2}}{\sqrt{c^2 - \ell^2 v^2}} = \frac{z}{\sigma}\sqrt{\frac{\sigma^2 + r^2 - \ell^2}{\sigma^2 + r^2 + \ell^2}}$$

verze 1.06 (2006-03-28)

Derivace bázových vektorů a forem:

$$
\begin{aligned}
\nabla \boldsymbol{e}_\eta &= \frac{1}{\sigma}\coth\eta\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\varphi - \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\, \boldsymbol{e}^\eta\, \boldsymbol{e}_\psi + \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\, \boldsymbol{e}^\psi\, \boldsymbol{e}_\psi \\
\nabla \boldsymbol{e}_\psi &= \frac{1}{\sigma}\cot\psi\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\varphi - \frac{\ell^2}{\sigma^3}\sinh\eta\cosh\eta\, \boldsymbol{e}^\psi\, \boldsymbol{e}_\eta + \frac{\ell^2}{\sigma^3}\sin\psi\cos\psi\, \boldsymbol{e}^\psi\, \boldsymbol{e}_\psi \\
\nabla \boldsymbol{e}_\varphi &= -\frac{1}{\sigma}\coth\eta\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\eta - \frac{1}{\sigma}\cot\psi\, \boldsymbol{e}^\varphi\, \boldsymbol{e}_\psi
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\boldsymbol{e}_\eta &= \frac{\ell^2}{\sigma^3}\,\coth\eta\big(2\sinh^2\eta+\sin^2\psi\big) &&= \nabla\cdot\boldsymbol{e}_c = \frac{c\,(2c^2-\ell^2v^2-\ell^2)}{\sqrt{c^2-\ell^2v^2}^{\,3}\sqrt{c^2-\ell^2}} \\
\nabla\cdot\boldsymbol{e}_\psi &= \frac{\ell^2}{\sigma^3}\cot\psi\big(\sinh^2\eta+2\sin^2\psi\big) &&= \nabla\cdot\boldsymbol{e}_v = \frac{v\,(c^2-2\ell^2v^2+\ell^2)}{\sqrt{c^2-\ell^2v^2}^{\,3}\sqrt{1-v^2}} \\
\nabla\cdot\boldsymbol{e}_\varphi &= 0 &&= \nabla\cdot\boldsymbol{e}_\varphi = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\boldsymbol{e}_\eta &= -\frac{\ell^2}{\sigma^3}\,\sin\psi\cos\psi\,\boldsymbol{e}_\varphi &&= \nabla\times\boldsymbol{e}_c = -\frac{\ell^2\,v\,\sqrt{1-v^2}}{\sqrt{c^2-\ell^2v^2}^{\,3}}\,\boldsymbol{e}_\varphi \\
\nabla\times\boldsymbol{e}_\psi &= \frac{\ell^2}{\sigma^3}\,\sinh\eta\cosh\eta\,\boldsymbol{e}_\varphi &&= \nabla\times\boldsymbol{e}_v = \frac{c\,\sqrt{c^2-\ell^2}}{\sqrt{c^2-\ell^2v^2}^{\,3}}\,\boldsymbol{e}_\varphi \\
\nabla\times\boldsymbol{e}_\varphi &= \frac{1}{\sigma}\,\cot\psi\,\boldsymbol{e}_\eta - \frac{1}{\sigma}\,\coth\eta\,\boldsymbol{e}_\psi &&= \nabla\times\boldsymbol{e}_\varphi = \frac{1}{\sqrt{c^2-\ell^2v^2}}\left(\frac{v}{\sqrt{1-v^2}}\,\boldsymbol{e}_\eta - \frac{c}{\sqrt{c^2-\ell^2}}\,\boldsymbol{e}_\psi\right)
\end{aligned}
$$

Operátory ve sférických souřadnicích:

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$
a_\eta = a_c = \frac{1}{\sigma}\,f_{,\eta} = \sqrt{\frac{c^2-\ell^2}{c^2-\ell^2v^2}}\,f_{,c} \qquad a_\psi = a_v = \frac{1}{\sigma}\,f_{,\psi} = \sqrt{\frac{1-v^2}{c^2-\ell^2v^2}}\,f_{,v} \qquad a_\varphi = \frac{1}{\rho}\,f_{,\varphi} = \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,f_{,\varphi}
$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$
\begin{aligned}
f &= \frac{1}{\sigma}\,a^\eta_{,\eta} + \frac{1}{\sigma}\,a^\psi_{,\psi} + \frac{1}{\rho}\,a^\varphi_{,\varphi} + \frac{\ell^2}{\sigma^3}\coth\eta\big(2\sinh^2\eta+\sin^2\psi\big)\,a^\eta + \frac{\ell^2}{\sigma^3}\cot\psi\big(\sinh^2\eta+2\sin^2\psi\big)\,a^\psi \\
&= \sqrt{\frac{c^2-\ell^2}{c^2-\ell^2v^2}}\,a^c_{,c} + \sqrt{\frac{1-v^2}{c^2-\ell^2v^2}}\,a^v_{,v} + \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a^\varphi_{,\varphi} + \frac{c\,(2c^2-\ell^2v^2-\ell^2)}{\sqrt{c^2-\ell^2v^2}^{\,3}\sqrt{c^2-\ell^2}}\,a^c + \frac{v\,(c^2-2\ell^2v^2+\ell^2)}{\sqrt{c^2-\ell^2v^2}^{\,3}\sqrt{1-v^2}}\,a^v
\end{aligned}
$$

rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$
\begin{aligned}
b_\eta &= \frac{1}{\sigma}\,a_{\varphi,\psi} - \frac{1}{\rho}\,a_{\psi,\varphi} + \frac{\cot\psi}{\sigma}\,a_\varphi = \frac{\sqrt{1-v^2}}{\sqrt{c^2-\ell^2v^2}}\,a_{\varphi,v} - \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a_{v,\varphi} - \frac{v}{\sqrt{c^2-\ell^2v^2}\sqrt{1-v^2}}\,a_\varphi \\
b_\psi &= \frac{1}{\rho}\,a_{\eta,\varphi} - \frac{1}{\sigma}\,a_{\varphi,\eta} - \frac{\coth\eta}{\sigma}\,a_\varphi = \frac{1}{\sqrt{c^2-\ell^2}\sqrt{1-v^2}}\,a_{c,\varphi} - \frac{\sqrt{c^2-\ell^2}}{\sqrt{c^2-\ell^2v^2}}\,a_{\varphi,c} - \frac{c}{\sqrt{c^2-\ell^2v^2}\sqrt{c^2-\ell^2}}\,a_\varphi \\
b_\varphi &= \frac{1}{\sigma}\big(a_{\psi,\eta} - a_{\eta,\psi}\big) + \frac{\ell^2}{\sigma^3}\big(\cosh\eta\sinh\eta\,a_\psi - \cos\psi\sin\psi\,a_\eta\big) \\
&= \frac{\sqrt{c^2-\ell^2}}{\sqrt{c^2-\ell^2v^2}}\,a_{v,c} - \frac{\sqrt{1-v^2}}{\sqrt{c^2-\ell^2v^2}}\,a_{c,v} + \frac{c\sqrt{c^2-\ell^2}}{\sqrt{c^2-\ell^2v^2}^{\,3}}\,a_v + \frac{\ell^2v\sqrt{1-v^2}}{\sqrt{c^2-\ell^2v^2}^{\,3}}\,a_c
\end{aligned}
$$

laplace:

$$
\begin{aligned}
\nabla^2 f &= \frac{1}{\sigma^2}\,f_{,\eta\eta} + \frac{1}{\sigma^2}f_{,\psi\psi} + \frac{1}{\rho^2}f_{,\varphi\varphi} + \frac{1}{\sigma^2}\big(\coth\eta\,f_{,\eta} + \cot\psi f_{,\psi}\big) \\
&= \frac{c^2-\ell^2}{c^2-\ell^2v^2}\,f_{,cc} + \frac{1-v^2}{c^2-\ell^2v^2}\,f_{,vv} + \frac{1}{(c^2-\ell^2)(1-v^2)}f_{,\varphi\varphi} + \frac{2}{c^2-\ell^2v^2}\big(c\,f_{,c} - v\,f_{,v}\big)
\end{aligned}
$$