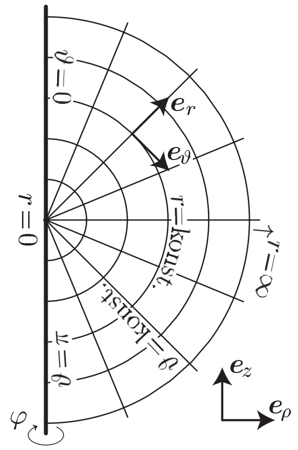
Sférické souřadnice $(r,\ \vartheta,\ \varphi)$



Vztah ke kartézským souřadnicím:

$$x = r\sin\vartheta\,\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\vartheta\,\sin\varphi \qquad \vartheta = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

$$z = r\cos\vartheta \qquad \varphi = \arctan\frac{y}{x}$$

$$\rho = r\sin\vartheta \qquad x=\rho\cos\varphi \qquad y=\rho\sin\varphi$$

Metrika a objemový element:

$$\boldsymbol{q} = \mathbf{d}r\,\mathbf{d}r + r^2\,\mathbf{d}\vartheta\,\mathbf{d}\vartheta + r^2\sin^2\vartheta\;\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad dV = r^2\sin\vartheta\;dr\,d\vartheta\,d\varphi$$

Laméovy koeficienty:

$$h_r = 1 \qquad h_\vartheta = r \qquad h_\varphi = r\sin\vartheta$$

Triáda (normovaná báze):

$$\boldsymbol{e}_r = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} \qquad \boldsymbol{e}_\vartheta = \frac{1}{r}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} \qquad \boldsymbol{e}_\varphi = \frac{1}{r\sin\vartheta}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\boldsymbol{e}^r = \mathbf{d}r \qquad \boldsymbol{e}^\vartheta = r\,\mathbf{d}\vartheta \qquad \boldsymbol{e}^\varphi = r\sin\vartheta\;\mathbf{d}\varphi$$

Vztah sférických a kartézských bází vektorů

$$\boldsymbol{e}_x = \sin\vartheta\cos\varphi\,\boldsymbol{e}_r + \cos\vartheta\cos\varphi\,\boldsymbol{e}_\vartheta - \sin\varphi\,\boldsymbol{e}_\varphi = \frac{x}{r}\,\boldsymbol{e}_r + \frac{z}{r}\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_\vartheta - \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_\varphi$$

$$\boldsymbol{e}_y = \sin\vartheta\sin\varphi\,\boldsymbol{e}_r + \cos\vartheta\sin\varphi\,\boldsymbol{e}_\vartheta + \cos\varphi\,\boldsymbol{e}_\varphi = \frac{y}{r}\,\boldsymbol{e}_r + \frac{z}{r}\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_\vartheta + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_\varphi$$

$$\boldsymbol{e}_z = \cos\vartheta\,\boldsymbol{e}_r - \sin\vartheta\,\boldsymbol{e}_\vartheta = \frac{z}{r}\,\boldsymbol{e}_r - \frac{\sqrt{x^2+y^2}}{r}\,\boldsymbol{e}_\vartheta$$

$$\boldsymbol{e}_r = \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z) = \sin\vartheta\cos\varphi\,\boldsymbol{e}_x + \sin\vartheta\sin\varphi\,\boldsymbol{e}_y + \cos\vartheta\,\boldsymbol{e}_z$$

$$\boldsymbol{e}_\vartheta = \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_x + \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_y - \frac{\sqrt{x^2+y^2}}{z}\,\boldsymbol{e}_z\right)$$

$$= \cos\vartheta\cos\varphi\,\boldsymbol{e}_x + \cos\vartheta\sin\varphi\,\boldsymbol{e}_y - \sin\vartheta\,\boldsymbol{e}_z$$

$$\boldsymbol{e}_\varphi = -\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_x + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_y = -\sin\varphi\,\boldsymbol{e}_x + \cos\varphi\,\boldsymbol{e}_y$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_r = \frac{1}{r}\,\boldsymbol{e}^\vartheta\,\boldsymbol{e}_\vartheta + \frac{1}{r}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi \qquad \nabla\cdot\boldsymbol{e}_r = \frac{2}{r} \qquad \nabla\times\boldsymbol{e}_r = 0$$

$$\nabla \boldsymbol{e}_\vartheta = -\frac{1}{r}\,\boldsymbol{e}^\vartheta\,\boldsymbol{e}_r + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\varphi \qquad \nabla\cdot\boldsymbol{e}_\vartheta = \frac{1}{r}\cot\vartheta \qquad \nabla\times\boldsymbol{e}_\vartheta = \frac{1}{r}\,\boldsymbol{e}_\varphi$$

$$\nabla \boldsymbol{e}_\varphi = -\frac{1}{r}\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_r - \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^\varphi\,\boldsymbol{e}_\vartheta \qquad \nabla\cdot\boldsymbol{e}_\varphi = 0 \qquad \nabla\times\boldsymbol{e}_\varphi = -\frac{1}{r}\,\boldsymbol{e}_\vartheta + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}_r$$

Operátory ve sférických souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_r = f_{,r} \qquad a_\vartheta = \frac{1}{r}\,f_{,\vartheta} \qquad a_\varphi = \frac{1}{r\sin\vartheta}\,f_{,\varphi}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$f = \frac{1}{r^2}(r^2\,a^r)_{,r} + \frac{1}{r\sin\vartheta}\,(\sin\vartheta\,a^\vartheta)_{,\vartheta} + \frac{1}{r\sin\vartheta}\,a^\varphi{}_{,\varphi}$$

rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_r = \frac{1}{r\sin\vartheta}\,(\sin\vartheta\,a_\varphi)_{,\vartheta} - a_{\vartheta,\varphi} \qquad b_\vartheta = \frac{1}{r}\left(\frac{1}{\sin\vartheta}\,a_{r,\varphi} - (r\,a_\varphi)_{,r}\right) \qquad b_\varphi = \frac{1}{r}\,\left((r\,a_\vartheta)_{,r} - a_{r,\vartheta}\right)$$

laplace:

$$\nabla^2 f = \frac{1}{r^2}\,(r^2\,f_{,r})_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\,f_{,\vartheta})_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\,f_{,\varphi\varphi}$$

$$= f_{,rr} + \frac{1}{r^2}f_{,\vartheta\vartheta} + \frac{1}{r^2\sin^2\vartheta}f_{,\varphi\varphi} + \frac{2}{r}f_{,r} + \frac{\cot\vartheta}{r^2}f_{,\vartheta}$$