Prostor antisymetrických tenzorů

definice:

$V^{[k]} \subset V^k$ je prostor antisymetrických tenzorů k-tého stupně, $k = 0, \ldots, d$; $\dim V^{[k]} = \binom{d}{k}$

$$A \in V^{[k]} \quad \equiv \quad \forall\, \sigma \text{ – permutace } [1, \ldots, k] \ : \quad A^{a_1 \ldots a_k} = \operatorname{sign} \sigma \ A^{a_{\sigma_1} \ldots a_{\sigma_k}}$$

antisymetrizace:

$$A = \mathcal{A} B \qquad A^{a_1 \ldots a_k} = B^{[a_1 \ldots a_k]} = \frac{1}{k!} \sum_\sigma \operatorname{sign} \sigma \ B^{a_{\sigma_1} \ldots a_{\sigma_k}}$$

$$A \in V^{[k]} \quad \Leftrightarrow \quad A^{a_1 \ldots a_k} = A^{[a_1 \ldots a_k]}$$

projektor na $V^{[k]}$:

$${}^{[k]}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} = \delta^{[a_1}_{b_1} \ldots \delta^{a_k]}_{b_k} = \delta^{a_1}_{[b_1} \ldots \delta^{a_k}_{b_k]} \qquad , \qquad {}^{[k]}\delta \in V^{[k]}_{[k]}$$

vlastnosti projektoru:

$${}^{[k]}\delta^{a_1 \ldots a_k}_{r_1 \ldots r_k} \, {}^{[k]}\delta^{r_1 \ldots r_k}_{b_1 \ldots b_k} = {}^{[k]}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} \qquad , \qquad A^{[a_1 \ldots a_k]} = {}^{[k]}\delta^{a_1 \ldots a_k}_{r_1 \ldots r_k} \, A^{r_1 \ldots r_k}$$

$${}^{[k]}\delta^{a_1 \ldots a_{k-l} a_{k-l+1} \ldots a_k}_{b_1 \ldots b_{k-l} r_1 \ \ldots \ r_l} \, {}^{[l]}\delta^{r_1 \ \ldots \ r_l}_{b_{k-l+1} \ldots b_k} = {}^{[k]}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k}$$

$${}^{[k]}\delta^{a_1 \ldots a_l r_1 \ldots r_{k-l}}_{b_1 \ldots b_l r_1 \ldots r_{k-l}} = \frac{(d-l)!\, l!}{(d-k)!\, k!} \, {}^{[l]}\delta^{a_1 \ldots a_l}_{b_1 \ldots b_l} \qquad , \qquad {}^{[k]}\delta^{r_1 \ldots r_k}_{r_1 \ldots r_k} = \dim V^{[k]}$$

$${}^{[k]}\delta^{a_{\sigma_1} \ldots a_{\sigma_k}}_{b_{\sigma_1} \ldots b_{\sigma_k}} = {}^{[k]}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} \qquad \sigma \text{ je permutace } [1, \ldots, k]$$

souřadnice:

$$A = A^{a_1 \ldots a_k} \ \vec{e}_{a_1} \ldots \vec{e}_{a_k} = \sum_{a_1 < \cdots < a_k} A^{a_1 \ldots a_k} \ k! \ \mathcal{A}(\vec{e}_{a_1} \ldots \vec{e}_{a_k})$$

totálně antisymetrické formy a tenzory:

prostory $V_{[d]}$ a $V^{[d]}$, kde d je dimenze prostoru V; $\dim V_{[d]} = \dim V^{[d]} = 1$
souřadnice ($\alpha \in V_{[d]}$):

$$\alpha = \alpha_{a_1 \ldots a_d} \ \underset{\rightarrow}{e}^{a_1} \ldots \underset{\rightarrow}{e}^{a_d} = \alpha_{1 \ldots d} \sum_\sigma \operatorname{sign} \sigma \ \underset{\rightarrow}{e}^{\sigma_1} \ldots \underset{\rightarrow}{e}^{\sigma_d} = \alpha_{1 \ldots d} \ d! \ \mathcal{A}(\underset{\rightarrow}{e}^1 \ldots \underset{\rightarrow}{e}^d)$$

duál:

$$\tilde{} \ : \ V_{[d]} \leftrightarrow V^{[d]} \ , \quad \alpha \to \tilde{\alpha} \ , \quad \tilde{\tilde{\alpha}} = \alpha \ , \quad \alpha_{r_1 \ldots r_d} \, \tilde{\alpha}^{r_1 \ldots r_d} = d!$$

vlastnosti duálu:

$$\alpha_{b_1 \ldots b_k r_1 \ldots r_{d-k}} \, \tilde{\alpha}^{a_1 \ldots a_k r_1 \ldots r_{d-k}} = (d-k)!\, k! \ {}^{[k]}\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k}$$

$$\alpha_{b_1 \ldots b_d} \, \tilde{\alpha}^{a_1 \ldots a_d} = d! \ {}^{[d]}\delta^{a_1 \ldots a_d}_{b_1 \ldots b_d} \qquad , \qquad \alpha_{r_1 \ldots r_d} \, \tilde{\alpha}^{r_1 \ldots r_d} = d!$$

$$\tilde{\alpha}^{1 \ldots d} = (\alpha_{1 \ldots d})^{-1}$$

determinant:

$$\det A = {}^{[d]}\delta^{a_1 \ldots a_d}_{b_1 \ldots b_d} \ A^{b_1}_{a_1} \ldots A^{b_d}_{a_d} = \sum_\sigma \operatorname{sign} \sigma \ A^{\sigma_1}_1 \ldots A^{\sigma_d}_d \qquad , \qquad A \in V^1_1$$

Prostor symetrických tenzorů

definice:

$V^{(k)} \subset V^k$ je prostor symetrických tenzorů k-tého stupně, $k \in \mathbb{N}_0$; $\dim V^{(k)} = \binom{k+d-1}{k}$

$$A \in V^{(k)} \quad \equiv \quad \forall\, \sigma - \text{permutace } [1,\dots,k] \ : \quad A^{a_1\dots a_k} = A^{a_{\sigma_1}\dots a_{\sigma_k}}$$

symetrizace:

$$A = \mathcal{S}B \qquad A^{a_1\dots a_k} = B^{(a_1\dots a_k)} = \frac{1}{k!}\sum_\sigma B^{a_{\sigma_1}\dots a_{\sigma_k}}$$

$$A \in V^{(k)} \quad \Leftrightarrow \quad A^{a_1\dots a_k} = A^{(a_1\dots a_k)}$$

projektor na $V^{(k)}$:

$${}^{(k)}\delta^{a_1\dots a_k}_{b_1\dots b_k} = \delta^{(a_1}_{b_1}\dots\delta^{a_k)}_{b_k} = \delta^{\ a_1}_{(b_1}\dots\delta^{a_k}_{b_k)} \qquad , \qquad {}^{(k)}\delta \in V^{(k)}_{(k)}$$

vlastnosti projektoru:

$${}^{(k)}\delta^{a_1\dots a_k}_{r_1\dots r_k}\,{}^{(k)}\delta^{r_1\dots r_k}_{b_1\dots b_k} = {}^{(k)}\delta^{a_1\dots a_k}_{b_1\dots b_k} \qquad , \qquad A^{(a_1\dots a_k)} = {}^{(k)}\delta^{a_1\dots a_k}_{r_1\dots r_k}\,A^{r_1\dots r_k}$$

$${}^{(k)}\delta^{a_1\dots a_{k-l}a_{k-l+1}\dots a_k}_{b_1\dots b_{k-l}r_1\ \ \dots\ \ r_l}\,{}^{(l)}\delta^{r_1\ \ \dots\ \ r_l}_{b_{k-l+1}\dots b_k} = {}^{(k)}\delta^{a_1\dots a_k}_{b_1\dots b_k}$$

$${}^{(k)}\delta^{a_1\dots a_l r_1\dots r_{k-l}}_{b_1\dots b_l r_1\dots r_{k-l}} = \frac{(k+d-1)!\,l!}{(l+d-1)!\,k!}\,{}^{(l)}\delta^{a_1\dots a_l}_{b_1\dots b_l} \qquad , \qquad {}^{(k)}\delta^{r_1\dots r_k}_{r_1\dots r_k} = \dim V^{(k)}$$

$${}^{(k)}\delta^{a_{\sigma_1}\dots a_{\sigma_k}}_{b_{\sigma_1}\dots b_{\sigma_k}} = {}^{(k)}\delta^{a_1\dots a_k}_{b_1\dots b_k} \qquad \sigma \text{ je permutace } [1,\dots,k]$$

souřadnice:

$$A = A^{a_1\dots a_k}\,\vec{e}_{a_1}\dots\vec{e}_{a_k} = \sum_{a_1\le\dots\le a_k} A^{a_1\dots a_k}\ n(a_1,\dots,a_k)\,\mathcal{S}(\vec{e}_{a_1}\dots\vec{e}_{a_k})$$

$n(a_1,\dots,a_k)$ je počet vzájemně odlišných permutací indexů $a_1\dots a_k$