Quantum mechanics offers many surprises

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No surprises?

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- Confine a particle in a two-dimensional region Ω. It is well known that among all regions of the same area, the ground-state energy is *minimized* by a disc. However, it Ω is not simply connected, a *'ribbon'* of a fixed width and length, the ground-state is instead *maximized* by a circular annulus.

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- Consider a Dirac particle with a singular *Lorentz scalar interaction* supported by a surface Σ, i.e. $H = -ic\alpha \cdot \nabla + \beta\big(mc^2 + \tau\delta(x - \Sigma)\big)$ is the corresponding Hamiltonian. If $\tau = \pm 2c$, the surfaces becomes *impenetrable*. There is no such effect in nonrelativistic QM,

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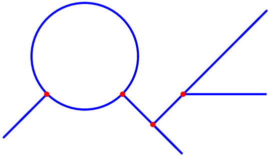
- etc., etc.

Quantum graphs

Since my time is short I will say a few more words on a single class, *quantum graphs*, by which we mean situations when the *configuration space* is a *metric* graph as in this example



Hamiltonian: $-\frac{\mathrm{d}^2}{\mathrm{d}x_j^2} + v(x_j)$
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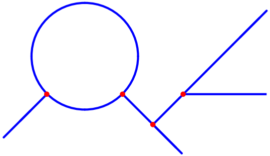

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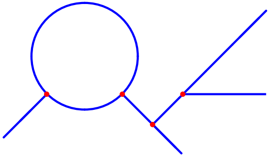

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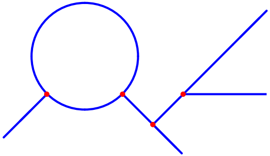

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One has to specify how that wave function at the vertices are coupled, since the Hamiltonian must be a *self-adjoint operator*, which in physicist's language means that the *probability current must be preserved*.

Vertex coupling

In a vertex of *degree n* people usually work with the so-called δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0),\ j,k = 1,\dots,n, \quad \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0)$$

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The simplest example is represented by the following conditions,

$$(\psi_{j+1}(0) - \psi_j(0)) + i\big(\psi_{j+1}'(0) + \psi_j'(0)\big) = 0\,, \quad j \in \mathbb{Z} \pmod n,$$

which are nontrivial for any $n \geq 3$ and manifestly non-invariant w.r.t. complex conjugation.

P.E., M. Tater: Quantum graphs with vertices of a preferred orientation, *Phys. Lett.* **A382** (2018), 283–287.

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The most intriguing property is the dependence on the *vertex parity*. If n is *even*, transport at high energies is possible, while for n *odd* we asymptotically a *full reflection*.

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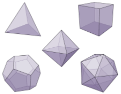
- A lattice with a boundary may be *conducting* there and *insulating* in the *bulk*, or vice versa, depending on the lattice 'texture'.
- Among *Platonic solid graphs*, only the *octahedron* has at high energies eigenfunctions supported on *more than a single edge*.



Source: Wikipedia Commons

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A class of vertex couplings which would be useful to investigate are those invariant with respect to *cyclic permutations*. They may violate the time reversal invariance, but they exhibit *$\mathcal{PT}$-symmetry*, i.e. they are invariant w.r.t. a combination of time reversal and *parity transformation*.

P.E., M. Tater: Quantum graphs: self-adjoint, and yet exhibiting a nontrivial $\mathcal{PT}$-symmetry, *Phys. Lett.* **A416** (2021), 127669

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Thank you for your attention!