

Úvodní kurz matematických metod fyziky

pro nastupující posluchače 1. ročníku MFF UK

Integrální počet

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Definice integrálu

Integrál reprezentuje “spojitou” sumu – součet velmi mnoha velmi malých hodnot.

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Integrál funkce $f(x)$ na intervalu $x \in \langle a, b \rangle$ je

$$\int_a^b f(x) \, dx = \sum_{i=1}^N f(x_i) \, dx + \text{malá chyba}$$

kde interval $\langle a, b \rangle$ je rozdělen na dostatečně velký počet N malých intervalů délky $dx = \frac{b-a}{N}$

a x_i jsou zvolené hodnoty v těchto intervalech (např. koncové body intervalů $x_i = a + i \, dx$).

Newtonův vzorec

Integrování je “inverzní” operace k derivování

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

často používáme zkrácené označení $[F]_a^b = F(b) - F(a)$.

Primitivní funkce a neurčitý integrál

Primitivní funkce F k funkci f nazýváme “invertovanou” derivaci, tj. funkci splňující

$$F' = f$$

Primitivní funkci označujeme též pomocí neurčitého integrálu (integrálu bez mezí)

$$F = \int f \, dx$$

Newtonův vzorec nám dává vztah integrálu na intervalu a primitivní funkce

$$\int_a^b f \, dx = F(b) - F(a)$$

Primitivní funkce je určena až na konstantu, tj. F a $F + \text{konst.}$ jsou primitivní funkce téže funkce f .

Pravidla pro integrály

$$\text{I} \text{o} \quad \int f' dx = [f] \quad \text{Newtonův vzorec}$$

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$$\mathcal{I}_0 \quad \int f' dx = [f] \quad \text{Newtonův vzorec}$$

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$$\mathcal{I}_0 \quad \int f' dx = [f] \quad \text{Newtonův vzorec} \quad \mathcal{I}_1 \quad \int x^n dx = \frac{1}{n+1} x^{n+1}$$

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$$\mathcal{I}_9 \quad \begin{aligned} \int \operatorname{sh} x dx &= \operatorname{ch} x \\ \int \operatorname{th} x dx &= \log(\operatorname{ch} x) \end{aligned} \quad \begin{aligned} \int \operatorname{ch} x dx &= \operatorname{sh} x \\ \int \operatorname{cth} x dx &= \log(\operatorname{sh} x) \end{aligned}$$

$$\mathcal{I}_{10} \quad \begin{aligned} \int \frac{1}{\sqrt{1+x^2}} dx &= \operatorname{arcsh} x \\ \int \frac{1}{1-x^2} dx &= \operatorname{arcth} x \end{aligned} \quad \begin{aligned} \int \frac{1}{\sqrt{x^2-1}} dx &= \operatorname{arcch} x \\ \int \frac{1}{1-x^2} dx &= \operatorname{arccth} x \end{aligned}$$

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substituce $x = g(\xi)$, meze $a = g(\alpha)$, $b = g(\beta)$

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substituce $x = g(\xi)$, meze $a = g(\alpha)$, $b = g(\beta)$

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(x(\xi)) \frac{dx}{d\xi} d\xi$$

substituce $x = x(\xi)$, meze $a = x(\alpha)$, $b = x(\beta)$

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$$\mathcal{I}_{\text{2}} \quad \begin{aligned} \int \sin x dx &= -\cos x & \int \cos x dx &= \sin x \\ \int \tan x dx &= -\log(\cos x) & \int \cot x dx &= \log(\sin x) \end{aligned}$$

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$$\mathcal{I}_{\text{IVb}} \quad \int_{\alpha}^{\beta} h(g(\xi)) d\xi = \int_a^b h(x) g^{\text{inv}\prime}(x) dx$$

substituce $x = g(\xi)$, meze $a = g(\alpha)$, $b = g(\beta)$

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substituce $x = g(\xi)$, meze $a = g(\alpha)$, $b = g(\beta)$

$$\int_{\alpha}^{\beta} h(x(\xi)) d\xi = \int_a^b h(x) \frac{d\xi}{dx} dx$$

substituce $x = x(\xi)$ s inverzí $\xi = \xi(x)$

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