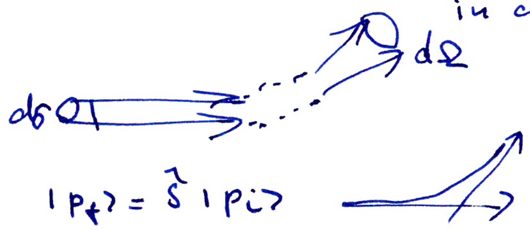
TS1

Toy models in scattering

$V = \lambda\,\delta(x-x_0)$ $V = \lambda\,|g\rangle\langle g|$ point deffect in chain

opakování rozptylu:



$d\sigma$ … $\frac{d\sigma}{d\Omega}$ .. $\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$

inelastic

scattering operator $|p_f\rangle = \hat{S}\,|p_i\rangle$

$$\langle p|S|p'\rangle = \delta(p-p') - 2\pi i\,\delta(E-E')\,t(p,p')$$

Schrödinger eq. approach ..

$$\Big[-\frac{1}{2m}\frac{d^2}{dx^2} + V(x)\Big]\psi(x) = E\psi(x)$$

$\underbrace{-\frac{1}{2m}\frac{d^2}{dx^2} + V(x)}_{H_0+V}$

$H_0 \longrightarrow -\frac{1}{2m}\frac{d^2}{dx^2}\,\phi_{in}(x) = E\phi_{in}(x)$

+ boundary condition

$|p_{in}\rangle$, $R\,|p_{out}\rangle$, $T\,|p\rangle = |p_{in}\rangle + t\,|p_{in}\rangle$

$|p_{in}\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_x x}$ .. $\langle p|p'\rangle = \delta(p-p')$

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scattering theory approach $H = H_0 + V$

$$[E - H_0 - V]|\psi\rangle = 0$$
$$[E - H_0]|\phi_{in}\rangle = 0$$

$\Big\}\ |\psi_{sc}\rangle = |\psi\rangle - |\phi_{in}\rangle$ $[E-H]\,|\psi_{sc}\rangle = -V|\phi_{in}\rangle$

simplier boundary condition (outgoing) $\psi' = ik\psi$

(complex scaling / absorbing potentials) tricks $\leftrightarrow$ Dirichlet Neumann type

solution with G-function method:

$$[E + H]\,G(x,x') = \delta(x-x') \leftarrow \hat{I} \;\rightarrow\; \hat{G}(E) = [E + i\varepsilon - H]^{-1} \leftarrow \text{complicated object}$$

$$\rightarrow |\psi_{sc}\rangle = -GV|\phi_{in}\rangle \;\rightarrow\; |\psi\rangle = |\phi_{in}\rangle - GV|\phi_{in}\rangle = [I - GV]\,|\phi_{in}\rangle\ \big(= \Omega^{(+)}|\phi_{in}\rangle\big)$$

solution with G_0 $[E - H_0]|\psi\rangle = V|\psi\rangle$ … $[E + i\varepsilon - H_0]\,G_0 = I \leftarrow \delta(x-x')$

$$\rightarrow |\psi\rangle = \phi_{homogen} + G\,|rhs\rangle = |\phi_{in}\rangle + G_0 V|\psi\rangle \text{ .. LS eq.}$$

Practically -- finding G_0

(A) directly $G_0(E,x,x') = \int_{-\infty}^{\infty} dp\, \langle x|p\rangle\Big[E - \frac{p^2}{2m} + i\varepsilon\Big]\langle p|x'\rangle = \ldots$ residuum theorem

(B) $\Big[E - \frac{1}{2m}\frac{d^2}{dx^2}\Big]\, G_0(E,x,x') = \delta(x-x')$ .. solution for $x<x'$ | $x>x'$ + conecting boundary cond. not obvious

$\rightarrow$ anyway .. $G_0^{(+)}(E,x,x') = \frac{2m}{2ik}\, e^{+ik|x-x'|}$ $k = \sqrt{2mE}$

$$\boxed{\text{Lippmann - Schwinger equation:}\quad \psi(x) = \phi(x) + \int dx'\, G_0^{(+)}(E,x,x')\,V(x')\,\psi(x')}$$

asymptotic behavior:

$\boxed{x \to \infty}$ $$\psi(x) \to \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{m}{ik} e^{ikx} \int dx' \, e^{-ikx'} V(x') \psi(x')$$

$$= \frac{1}{\sqrt{2\pi}} e^{ikx} \Big(1 - 2\pi i \, \frac{m}{k} \underbrace{\langle +k | V | \psi \rangle}_{T_{++}} \Big)$$

$\boxed{x \to -\infty}$ $$\psi(x) \to \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{m}{ik} e^{-ikx} \int dx' \, e^{ikx'} V(x') \psi(x')$$

$$= \frac{1}{\sqrt{2\pi}} e^{ikx} - \frac{1}{\sqrt{2\pi}} e^{-ikx} \cdot \frac{m}{k} 2\pi i \underbrace{\langle -k | V | \psi \rangle}_{T_{-+}}$$

→ related to s-matrix $\langle p' | S | p \rangle = \delta(p - p') \, S_{n'n}$

$$S_{n'n} = \delta_{n'n} - 2\pi i \, T_{n'n}$$

probabilities $p_R = |S_{-+}|^2 = \left| \frac{2\pi m}{k} T_{-+} \right|^2$ $\quad p_T = |S_{++}|^2$

unitarity $S^\dagger S = S S^\dagger = I \quad \Rightarrow \quad p_T + p_R = 1$

WORFLOW: solve $|\psi\rangle = |\phi_i\rangle + G_0 V |\psi\rangle$

calculate $T_{fi} = \langle \phi_f | V | \psi \rangle$

calculate observables from $|T_{fi}|^2$

$\boxed{\text{TOY model 1}}$ δ-potential $V(x) = \lambda \delta(x - x_0)$ ... $x_0 = 0$ without loss of generality

i.e. $\psi(x) = \phi(x) + G_0(E, x, 0) \lambda \psi(0)$

to find $\psi(0)$ ... $\boxed{x=0}$ in ↑ ; $\psi(0) = \phi(0) + G_0(E, 0, 0) \lambda \psi(0)$

i.e.: $$\psi(0) = \frac{\phi(0)}{1 - \lambda G_0(E,0,0)} = \frac{1/\sqrt{2\pi}}{1 - \frac{m}{ik}\lambda}$$

$$T_{++} = T_{-+} = \langle \pm k | V | \psi \rangle = \frac{1}{\sqrt{2\pi}} \cdot \lambda \cdot \psi(0) = \frac{\lambda / 2\pi}{1 - \frac{m\lambda}{ik}}$$

$$S_{+-} = S_{-+} = -\frac{\lambda i m}{k} \left[1 - \frac{m\lambda}{ik} \right]^{-1} = \frac{\lambda m}{ik} \left[1 - \frac{\lambda m}{ik} \right]^{-1}$$

$$S_{++} = S_{--} = 1 - \frac{\lambda i m}{k} \left[1 - \frac{m\lambda}{ik} \right]^{-1} = \left[1 - \frac{m\lambda}{ik} \right]^{-1}$$

→ matrix $\begin{bmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{bmatrix}$

↓

eigenvalues $e^{2i\delta_\ell}$

ℓ = odd / even

TOY model 2 separable potential $\hat{V} = \lambda |g\rangle\langle g|$ TS 3

$$|\psi\rangle = |\phi\rangle + G_0 \lambda |g\rangle\langle g|\psi\rangle$$ .. already solved!

we just need one number $\langle g|\psi\rangle$: $\langle g|\psi\rangle = \langle g|\phi\rangle + \lambda \langle g|G_0|g\rangle\langle g|\psi\rangle$

$$\Rightarrow \langle g|\psi\rangle = [1 - \lambda \langle g|G_0|g\rangle]^{-1} \langle g|\phi\rangle$$

i.e. T-matrix $T = \langle \phi_f|V|\psi\rangle = \langle \phi_f|V|\phi_i\rangle + \langle \phi_f|V G_0 \lambda |g\rangle\langle g|\psi\rangle$

Born approx.

$$T = \langle \phi_f|g\rangle\langle g|\phi_i\rangle \left\{ \lambda + \lambda^2 \langle g|G_0|g\rangle [1 - \lambda\langle g|G_0|g\rangle]^{-1} \right\}$$

$$\boxed{T = \frac{\langle \phi_f|g\rangle\langle g|\phi_i\rangle \lambda}{1 - \lambda \langle g|G_0|g\rangle}}$$

note: independent of representation, can be in coord $V = \lambda g(x) g(x')$

or in impulse $V = \lambda g(p) g(p')$

simpler express. for G_0

GENERALIZATION: $\hat{V} = \sum_{ij} V_{ij} |g_i\rangle\langle g_j|$

in principle any $\hat{V}$ if $\{|g_i\rangle\}_{i=1}^{N}$ is "sufficiently complete"

then: $|\psi\rangle = |\phi_i\rangle + \sum_{ij} G_0 |g_i\rangle V_{ij} \langle g_j|\psi\rangle$ .. def. vector $\Psi = (\langle g_i|\psi\rangle)$

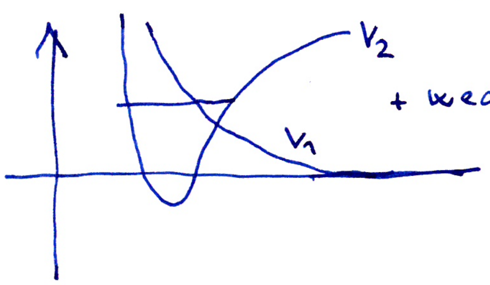
matrix $W \equiv \{V_{ij}\}$ and $\mathbb{G} \equiv \{\langle g_i|G_0|g_j\rangle\}$

then: $\Psi = [I - \mathbb{G}W]^{-1} \Phi_i$

and $T_{fi} = \Phi_f W [W - W\mathbb{G}W]^{-1} W \Phi_I$

gives recipe how to solve any potential numerically

Motivation for model 2: optical potential in Landau-Zener



+ weak coupling C