

# Fourier Analysis → Ch. 2. z Trefethena

[FT-I]

- pozn: PDE → analýza stability  
 → spektrální metody -- výpočet derivací -- exp. konverg. pro hl. fce  
 → multigrádové metody a iterační metody ... analýza konvergence & dekonvoluce

4 Možnosti:

	Jiného	Prostor. prom.	Frekv. prom.
Aplika ce... filtry, konvoluce, derivace	Fourier. transformace	neomezená, spoj.	spoj. neomezená
přes. reprezent. na interv.	Fourier řady	omezená, spoj.	diskrét. neomez.
interpolace	semidiskrét. F. transf. DFT	neomez. diskrét. omezená, diskrét. 1...∞	spojitá omezená, diskrét. omezená 1...∞

analýza ... bez malených jehou ... ně plati vlnor vlna na  $L^2, L^2$

## 1) Fourierova transformace

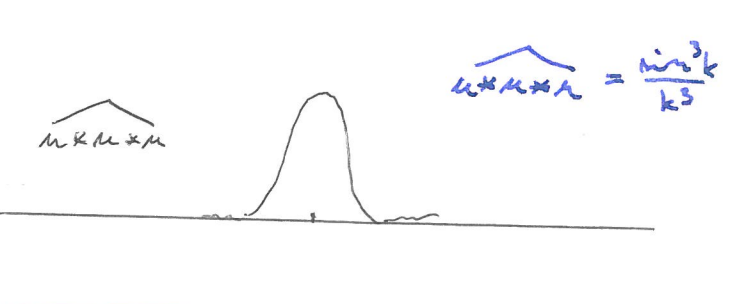
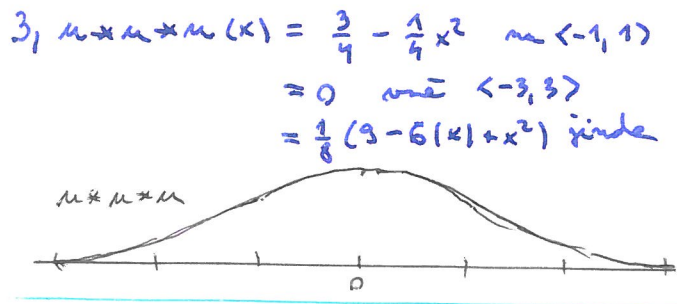
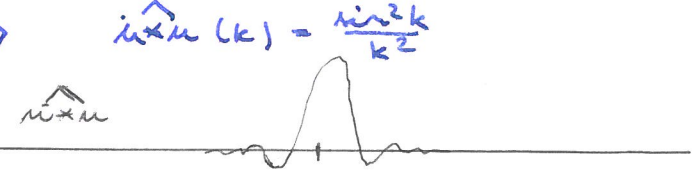
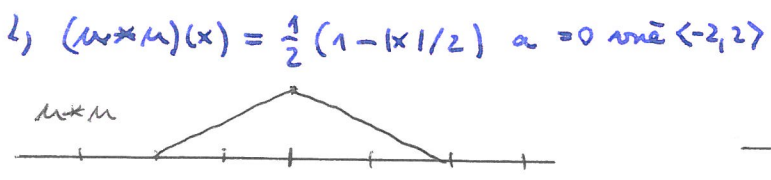
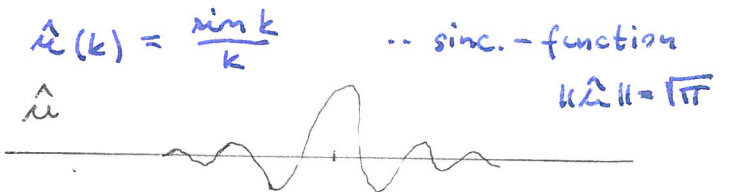
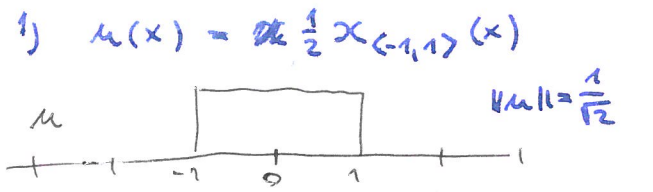
$$\hat{u}(k) = (Fu)(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x) dx \quad k \in \mathbb{R} \quad k \dots \text{vlnová délka}$$

nebo  $x \leftrightarrow t \quad k \leftrightarrow \omega \dots \text{frekvence}$

např. v  $L^2$   $u \in L^2$  pak  $\hat{u}(k) \in L^2$  a plati **VĚTA 9 INVERZE**  
 $u(x) = (F^{-1}\hat{u})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k) dk \dots$  Inverzní F.T.

- Vlastnosti:
- Parsevalova rovnost:  $\| \hat{u} \| = \sqrt{2\pi} \| u \|$  ... podmíněnost F.T.
  - $u \in L^2, v \in L^1$  pak  $u \times v \in L^2$  a  $\widehat{u \times v}(k) = \hat{u}(k) \hat{v}(k)$  (nebo naopak)

## ODBOČKA: B-splines



- $\hat{u}_x = ik \hat{u}(k)$
- linearita  $F(u+v) = \hat{u} + \hat{v}$ ;  $F(cu) = c\hat{u}$
- skalování  $F(u(cx))(k) = \hat{u}(k/c) / |c|$
- kompl. ndrůz.  $F(\bar{u})(k) = \overline{\hat{u}(-k)}$
- Modulace  $F(e^{ik_0x} u(x)) = \hat{u}(k - k_0)$
- Translace  $F(u(x+x_0)) = e^{ikx_0} \hat{u}(k)$
- Inverze  $F^{-1}(u)(k) \equiv \frac{1}{2\pi} \hat{u}(-k)$

Symetrie:

- $u$  sudá (lichá)  $\leftrightarrow \hat{u}$  sudá (lichá)
- $u$  reál. (imag.)  $\leftrightarrow \hat{u}$  hermit (antihermi)
- $u$  reál a sudá  $\leftrightarrow \hat{u}$  reál a sudá
- $u$  reál a lichá  $\leftrightarrow \hat{u}$  imag. a lichá
- $u$  imag a sudá  $\leftrightarrow \hat{u}$  imag a sudá
- $u$  imag a lichá  $\leftrightarrow \hat{u}$  reál a lichá

kde:

Def: sudá  $u(x) = u(-x)$ ; lichá  $u(x) = -u(-x)$ ; imag  $\overline{u(x)} = -u(x)$ , reál  $\overline{u(x)} = u(x)$   
 hermit  $u(x) = \overline{u(-x)}$ ; antihermi  $u(x) = -\overline{u(-x)}$

Důležit. vlast.

$u(x)$	$\hat{u}(k)$
hladká $\leftrightarrow$	rázně klesá pro $ k  \rightarrow \infty$
rychle klesá $ x  \rightarrow \infty \leftrightarrow$	hladká

Přesněji: necht  $u \in \mathcal{L}^2$ :

- (a)  $u$  má  $(p-1)$  spoj. derivací v  $\mathcal{L}^2$  pro  $p \geq 0$  a  $p$ -tá deriv. omez. variací v  $\mathcal{L}^2$   
 pak  $\hat{u}(k) = O(|k|^{-(p+1)})$   $|k| \rightarrow \infty$
- (b)  $u$  má  $u \in C^\infty$  a  $u$  deriv. a  $\mathcal{L}^2$   
 pak  $\hat{u}(k) = O(|k|^{-M})$  pro  $|k| \rightarrow \infty$  a  $\forall M$  a obráceně
- (c)  $u$  lze rozšířit na analyt. fci prom  $z = x + iy$  pro nejakou  $|Im z| < a$   
 pro nějaké  $a > 0$   $\Rightarrow \|u(x + iy)\| \leq C$  stejnoměrně  $\forall y \in (-a, a)$   
 pak  $e^{a|k|} \hat{u}(\xi) \in \mathcal{L}^2$  a obráceně
- (d)  $u$  lze rozšířit na analyt. fci v celém  $z \in \mathbb{C}$  a  $|u(z)| = O(e^{a|z|})$   
 pro  $|z| \rightarrow \infty$  a nějaké  $a > 0$   $\hat{=}$  BAND LIMITED FCION  
 pak  $\hat{u}$  má kompaktní support obměny v  $(-a, a)$  tj  $\hat{u} = 0$  vně  $(-a, a)$   
 a obráceně

PR: B-splíny

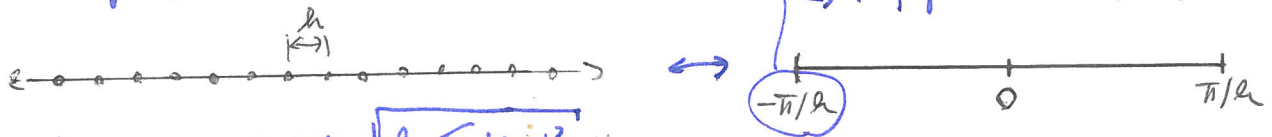
3) Semidiskrétní Fourier. transform.

-- a její inv. obraz Fourier. řady

Pozn: měří se potřebujeme pro fci reprezent. na gride

GRID (mT):  $x_j = jh$   $u(x_j) \equiv \omega_j$

upínání:



def:  $\mathcal{L}^2$  prohor  $\| \omega \| = \sqrt{h \sum_j |\omega_j|^2}$

konvoluce  $(\omega * \omega)_m = h \sum_j \omega_{m-j} \omega_j = h \sum_j \omega_j \omega_{m-j}$

podiskrétní F.T.  $\hat{\omega}(k) = (F_h \omega)(k) \equiv h \sum_j e^{ikx_j} \omega_j$   $k \in (-\frac{\pi}{h}, \frac{\pi}{h})$

Aliasing:  $k$  a  $k + \frac{2\pi}{h} \cdot m$  nerozliš pro libovol  $m$

$\hat{\omega}(k)$  měřena pouze za  $2\pi/h$  period. nebo se omezit na

$$\hat{v} \in \mathcal{L}_h^2 \equiv \mathcal{L}^2 \left( \left\langle -\frac{\pi}{2h}, \frac{\pi}{2h} \right\rangle \right)$$

analog vety a inverzi:

$v \in \mathcal{L}_h^2$  pak  $\hat{v}(k) \equiv (F_h v)(k) = h \sum_{j \in \mathbb{Z}} e^{ikx_j} v_j$   $\{ \in \langle -\pi/2h, \pi/2h \rangle$   
 nativ do  $\mathcal{L}_h^2$  a  $v_j = (F_h^{-1} \hat{v})(x) = \int_{-\pi/2h}^{\pi/2h} e^{ikx_j} \hat{v}(k) \frac{dk}{2\pi}$   $j \in \mathbb{Z}$

- Vlastnosti:
- Parseval:  $\|\hat{v}\| = \sqrt{2\pi} \|v\|$
  - $v \in \mathcal{L}_h^2; \hat{v} \in \mathcal{L}_h^1$  nek  $v * w \in \mathcal{L}_h^2$  a  $\widehat{v * w} = \hat{v} \cdot \hat{w}$  (iebo nerovna)
  - Linearita  $F_h(v+w) = \hat{v} + \hat{w}; F_h(cv) = c\hat{v}$
  - Translace  $F_h(v_{j+j_0}) = e^{ikx_{j_0}} \hat{v}(k)$
  - Modulace  $F_h(e^{ik_0 x_j} v_j) = \hat{v}(k - k_0)$
  - Skalovani  $F_h(v_{mj}) (k) = \hat{v}(k/m) / |m|$
  - kompl. odraz.  $F_h(\bar{v}) = \overline{\hat{v}(-k)}$

Vztahe F.T. a semidiskr. F.T.

ALIASING:  $k$  a  $k + \frac{2\pi}{h} j$  nerozlika pro  $\forall j \in \mathbb{Z}$  na gridu  $h\mathbb{Z}$

Poissonova sumacni formule (Aliasing formula):

$u \in L^2$  dost hladka  $\hat{F}u = \hat{u}$  a  $v \in \mathcal{L}_h^2$  restrikuje na  $h\mathbb{Z}$   
 Pak:  $\hat{v}(k) = \sum_{j=-\infty}^{\infty} \hat{u}(k + \frac{2\pi}{h} j)$   $\forall k \in \langle -\pi/2h, \pi/2h \rangle$

\* Schwartz fce  
 $f \in C^\infty, \|f\|_{L^p} < \infty$   
 $\|f\|_{L^p} = \sup_x |x^d f|$

+ aplikace vety a hladosti u a odrazani  $\hat{u} \rightarrow 0; k \rightarrow \infty$ :

- $u$  ( $p-1$ ) naj deriv  $\in L^2$  a  $p$ -ta deriv. omezena  $\Rightarrow |\hat{v}(k) - \hat{u}(k)| = O(h^{p+1})$   $h \rightarrow 0$
- $u \in C^\infty$  a deriv  $\in L^2 \Rightarrow |\hat{v}(k) - \hat{u}(k)| = O(h^m)$  pro  $h \rightarrow 0$  a  $\forall m$
- $u$  anal. rozli. na pas kolem  $\mathbb{R}$  do  $\mathbb{C} \setminus \{Im \mathbb{Z}\} \setminus \{a\} \dots \Rightarrow |\hat{v}(k) - \hat{u}(k)| = O(e^{-\pi(a-\epsilon)/h})$   $h \rightarrow 0$
- $u$  anal. r.  $u(z) = O(e^{a|z|})$  pro nejake  $a > 0 \Rightarrow \hat{v}(\eta) = \hat{u}(\eta)$  libovolny interval.

③ Interpolace & sinc. funkce:

$\delta_j$  .. kroneker delta  $\xrightarrow{F_h} \hat{\delta}_j(k) = h \xrightarrow{F_h^{-1}} \delta_j = \frac{\sin(\pi x_j / h)}{\pi x_j / h}$   
 ... grid - sinc funkce ...  $S_h(x) \equiv \frac{\sin(\pi x / h)}{\pi x / h} = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx} dx$

Lze zapakovat pro libovol. funkci  $v_j \rightarrow$  Band limited interpolation

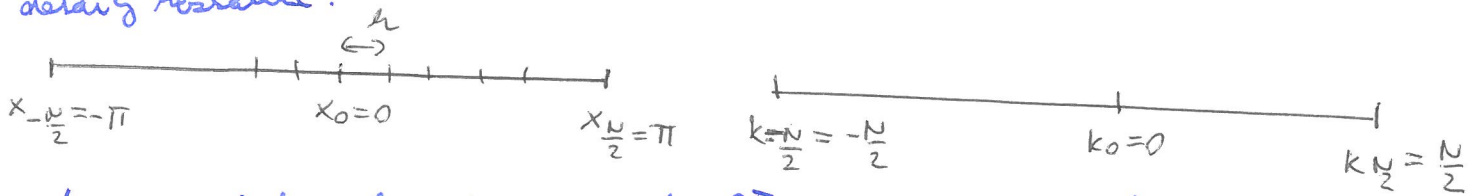
$v_j = \sum_m v_m \delta_{m-j} = \sum_m v_m S_h(x_j - x_m) \dots$  interpolace  
 ... zamekne  $x_j \rightarrow x$

Gibbsov jev: ... band-limited interpolation ...

rekonstruovani  $\theta(x)$  na gridu  $h + \delta$  ... priblizuje 1 a  $\int_1^2 \frac{\sin \pi y}{\pi y} dy \approx 1.089$   
 zmeny pro  $h \rightarrow 0$  ale staly je se (nabazuje)

# 4) DFT

detaily rezolúcie:



kráča na intervale  $(-\pi, \pi)$   $h = \frac{2\pi}{N}$   $x_j = h j$   $j = -\frac{N}{2}, \dots, 0, \dots, \frac{N}{2} - 1$

norma  $\|v\| = \sqrt{h \sum_j |v_j|^2}$

DFT:  $\hat{v}(k) = (F_N v)(k) = h \sum_j e^{i k x_j} v_j$   $k \in \mathbb{Z}$  ale perióda  $N$   
 $\dots k = -N/2, \dots, 0, \dots, N/2 - 1$

$\|\hat{v}\| = \sqrt{\sum_k |v_k|^2}$

konvolúcie:  $(v * w)_m = h \sum_j v_{m-j} w_j = \sum_j v_j w_{m-j}$

Inverzia:  $v \in \ell_N^2 \rightarrow \hat{v}_k \in L_N^2$  a  
 $v_j = \frac{1}{2\pi} \sum_k e^{i k x_j} \hat{v}_k \equiv (F_N^{-1} \hat{v})_j$

- Parseval  $\|\hat{v}\| = \sqrt{2\pi} \|v\|$  + lineárna, translačná, modulačná, redukčná, bez strát
- konvolúcie  $\widehat{(v * w)}_k = \hat{v}_k \hat{w}_k$
- Inverzia:  $F_N^{-1}(v)(k) = \frac{1}{2\pi h} \hat{v}(-k)$

Rýchla FT  $\equiv$  FFT .. Gauss 1805 ale exponenciály < Danielson, Lanczos lemma 1942  
 Cooley & Tukey 1965!

pomala (mimo výpočtu dle formule) ...  $\sim N^2$  operácií  
 rychle  $\sim N \log N$  operácií

pozn: Band limited interp. pro period. fce  $S_N(x) = \frac{\sin(\pi x/h)}{\frac{2\pi}{h} \tan(\frac{x}{2})}$

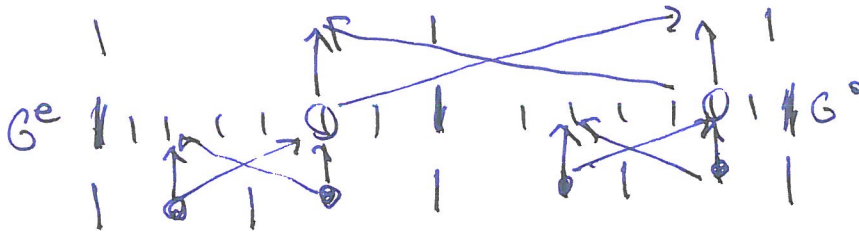
$p(x) = \sum_{m=1}^N v_m S_N(x - x_m)$





# Algoritmus FFT:

- 1) bitová reverse ... předvedení .. jen sáňka elementů ... Noperná
- 2) povrch vore \* schvívá



## Další vylepšení:

udělat se jen do určité úrovně např.  $N=4$  či  $N=8$  a pak se využít efektivnější výpočet sín a cos (kluč pro  $N=4$  jsou jen  $0$  a  $\pm 1$ )  
~~... vyjádření ...~~ ... využít řád  $\approx 10\% - 20\%$   
 + speciální triky.

## Pár reálné a NymRee:

- Dvě reálné (neché) .. pro reál. pí:  $G_{N-m} = G_m^*$   
 pro imag. pí:  $G_{N-m} = -G_m^*$

$$\rightarrow h = g^{(u)} + j g^{(i)} \rightarrow G_m^{(u)} = \frac{1}{2} (G_m + G_{N-m}^*)$$

$$G_m^{(i)} = \frac{1}{2} (G_m - G_{N-m}^*)$$

- Jedna reáln. pí:  $h_m = g_{2m} + j g_{2m+1} \rightarrow$  poloviční délka

$$\rightarrow H_m = G_m^{(u)} + j G_m^{(i)} \dots$$

zde

$$G_m^{(u)} = \sum_{k=0}^{N/2-1} g_{2k} e^{2\pi i k m / (N/2)}$$

$$G_m^{(i)} = \sum_{k=0}^{N/2-1} g_{2k+1} e^{2\pi i k m / (N/2)}$$

potom

$$G_m = G_m^{(u)} + e^{2\pi i m / N} G_m^{(i)} = \frac{1}{2} (H_m + H_{N/2-m}^*) - \frac{j}{2} (H_m - H_{N/2-m}^*) e^{2\pi i m / N}$$

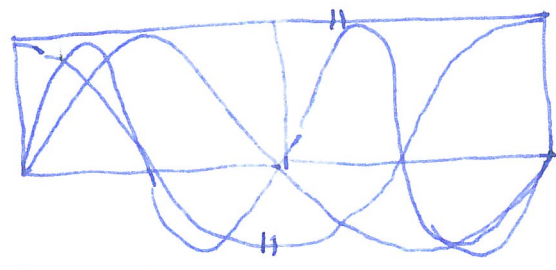
- Rychlá transformace pro sín a cos. bází

... pro okraj úlohy s pedu.  $\varphi(k)=0$  nebo  $\varphi'(k)=0$  na hraně

Sinová transform:  $G_k = \sum_{j=0}^{N-1} g_j \sin(\pi j k / N)$   $j=0 \dots N-1$ ; ale  $g_0 = 0$

norm. Standard.F.T.:  $G_k = \sum_{j=0}^{N-1} g_j e^{2\pi i j k / N}$  .. liči se fakt. 2 v exp.

norm... nepři. N=



.. tritě dodělanost  $g_{2N-j} = -g_j$   $j=0, \dots, N-1$

.. dvojnás. interval s její doděf. liči kolem bodu  $j=N$

jele fakt:  $G_k = \sum_{j=0}^{2N-1} g_j e^{2\pi i j k / 2N} = \sum_{j=0}^{N-1} \dots \sum_{j'=0}^{N-1} g_{j'} e^{-2\pi i j' k / (2N)}$   
 $= 2i \sum_{j=0}^{N-1} g_j \sin(\pi j k / N)$

→ dvojnás faktor 2 ...  $G_k$  má 0 uhl. část.

.. vypláči se dvojčet součinitele ve všeč dim. ... NonRec

2D - FFT

$$G(n_1, n_2) = \sum_{k_1} \sum_{k_2} e^{2\pi i n_1 k_1 / N_1} e^{2\pi i n_2 k_2 / N_2} g(k_1, k_2)$$

$$= \text{FFT}_{k_1} [\text{FFT}_{k_2} [g_{k_1, k_2}]]$$

.. složitost  $N^2 \log_2 N$   
 $\{N_1 N_2 [\log_2 N_1 + \log_2 N_2]\}$   
 stejě jako sin/cos transf.

→ v Nonrec subrutina pro oběč dim

.. invence jako v 1D ale prefaktor  $\frac{1}{N_1 N_2}$

Aplikace: řešení BW problému ..  $g'(t) = h \sum g_n(t - i k_n) e$



D. Foreword:

"big three" technologies for numerical sol. PDE:

- 1950s : finite differences
- 1960s : finite element meths
- 1970s : spectral meths

→ references  
see book

SPECTRAL METHODS:  
(when to use them)

- ODE or PDE to high accuracy
- simple domain
- smooth data defining the problem

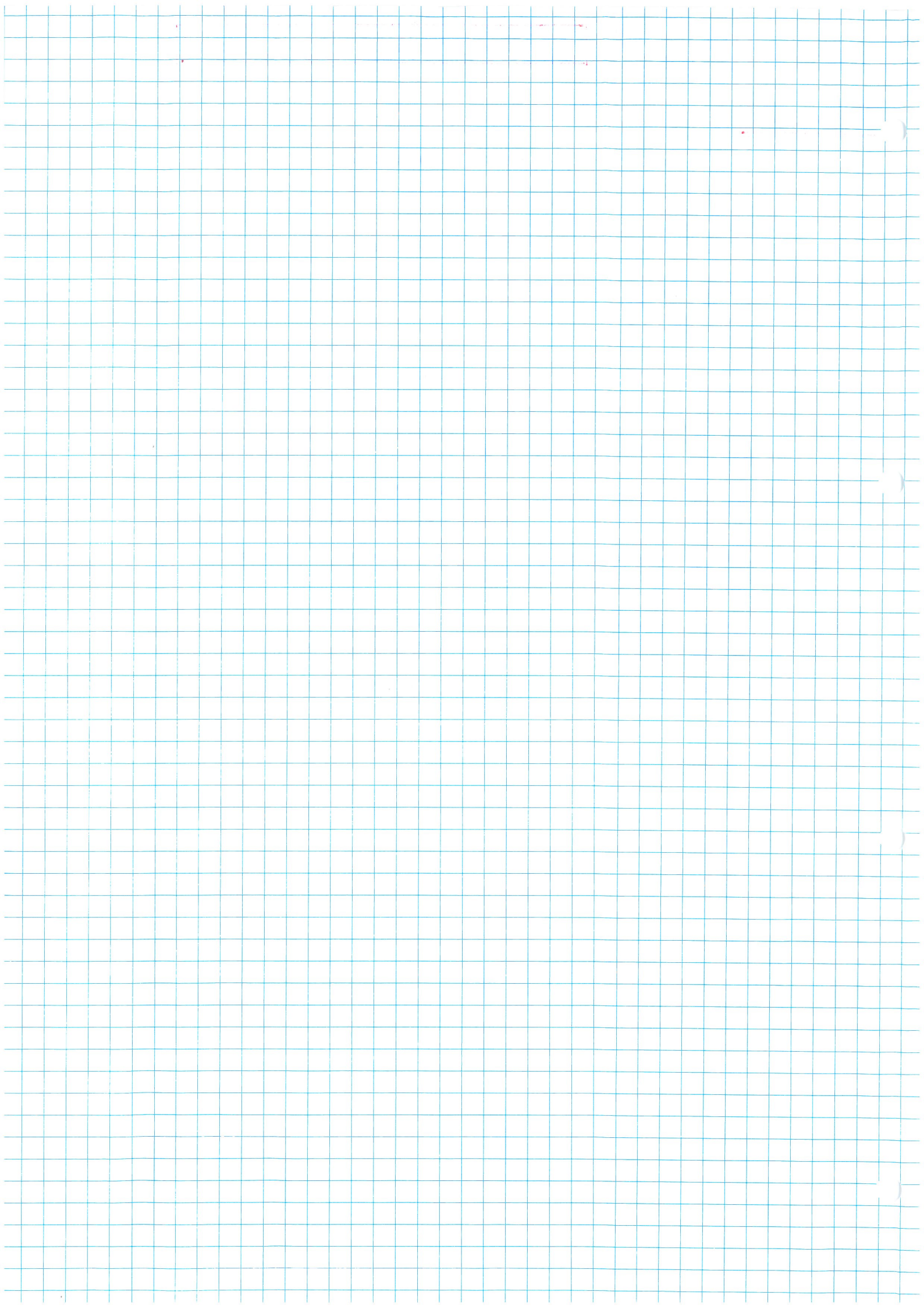
often 10 digits where fin. dif. or FEM get 2-3  
and for low accuracy required → lower memory requirements

Fundamental ideas: Ch 4: connection between smoothness ↔ rate of decay of Fourier  
→ understanding aliasing error

Ch 5: analogy .. roots of poly & electric charges in plane  
→ explains Chebyshev grid

Ch 8: Link: Chebyshev on  $(-1, 1)$  ↔ Fourier on  $(-\pi, \pi)$  ↔ Laurent series on  $\mathbb{T}$  circle  
→ explains how to compute Chebyshev using FFT

this book: emphasis on collocation ("pseudospectral") methods





• finite grid

... choice of interpolant  $p(x_j) = u_j$  + differentiation  $w_j = \dot{p}_j$

we will see: Fourier  $\rightarrow$

- circulant
- Toeplitz
- skew-symmetric

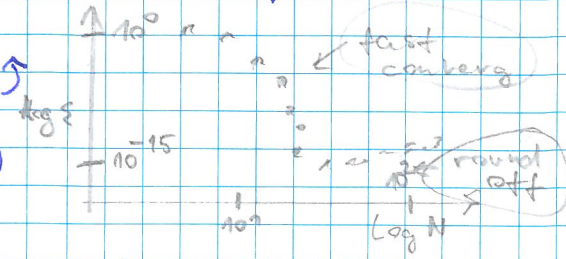
$$D_N = \begin{pmatrix} -\frac{1}{2} \cot \frac{3}{2}h & \frac{1}{2} \cot \frac{3}{2}h & & \\ \frac{1}{2} \cot \frac{1}{2}h & & & \\ & -\frac{1}{2} \cot \frac{1}{2}h & & \\ & & \ddots & \end{pmatrix}$$

not sparse  
 $\rightarrow$  dense

**EXAMPLE**

the same as previous but with  $\uparrow$

.. spectral convergence ... faster than  $O(N^{-m})$   
 $\forall m > 1$



Further speeding: circulant matrixes  $\equiv$  convolution

- calculation using FFT

SUMMARY:

• Spectral method: - global interpolation

+ formula linear  $\rightarrow$   
daji se psat jako mat. násob.  
nebo FFT

- calculation of derivative as derivative of interpolant

• normally - trigonometric interpolants on periodic grids

- polynomial interpolants on bounded grids

Spectral differentiation of function v

Equivalent description: (1) -  $\{v_j\} \rightarrow$  find band limited interpolant  $p(x)$   
 - set  $w_j = p'(x_j)$   
 $\Leftrightarrow$  (2) -  $\{v_j\} \rightarrow$  (sFT)  $\rightarrow \hat{v}(k)$   
 - def  $\hat{w}(k) = ik \hat{v}(k)$   
 - Compute  $\hat{w}(k) \rightarrow$  (sFT) $^{-1} \rightarrow w_j$

Another description ... construction of differentiation matrix

trick - use linearity ...  $w_j = \delta_{j0}$  .. kronecker delta

(sFT):  $\hat{w}(k) = h \sum e^{-ikx_j} \delta_{j0} = h$

(sFT $^{-1}$ ):  $p(x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx} dk = \frac{h}{2i\pi x} [e^{ix\frac{\pi}{h}} - e^{-ix\frac{\pi}{h}}] = \frac{\sin \frac{\pi x}{h}}{\frac{\pi x}{h}} \equiv S_h(x)$

Sinc function:  $S_h(x) = \frac{\sin \frac{\pi x}{h}}{\frac{\pi x}{h}}$

v knize odkazy na datost viti Sinc fun

pozn: transl. invar: .. interpolant  $\delta_{j,m}$  is  $S_h(x-x_m)$

$\Rightarrow$  different formulation of interpolation (band limited)

$v_j = \sum_m v_m \delta_{j,m} \Rightarrow p(x) = \sum_{m=-\infty}^{\infty} v_m S_h(x-x_m)$

$\Rightarrow$  differentiation:  $w_j = p'(x_j) = \sum_m v_m S'_h(x_j-x_m)$

matrix multiplication!

$\rightarrow$  doubly infinite Toeplitz matrix D defined by  $m=0$  column:

$S'_h(x_j) = \frac{\pi}{h} \left( \frac{\sin y}{y} \right)'_{y=\frac{\pi x_j}{h} = \pi j} = \frac{\pi}{h} \left( \frac{\cos y}{y} - \frac{\sin y}{y^2} \right) = \begin{cases} 0 & j=0 \\ \frac{(-1)^j}{jh} & j \neq 0 \end{cases}$

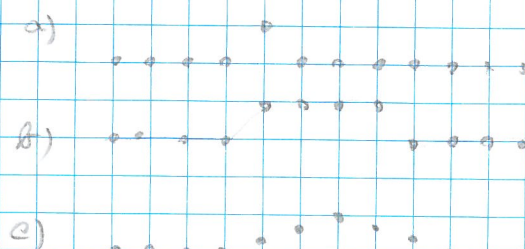
note: similar for all spectral collocation methods

.. mth column of matrix D given by deriv of interpolant  $p'(x_j)$

**EXAMPLE** .. band limited interpol for:

$\rightarrow$  oscillations near discontinuity:

Gibbs phenomenon



Second derivative:  $S''_h(x_j) = \begin{cases} -\frac{\pi^2}{3h^2} & j=0 \\ 2 \frac{(-1)^j}{j^2 h^2} & j \neq 0 \end{cases}$

$\rightarrow D^2 v = \frac{2}{h^2} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} v$   
 really D.D matrix multiplication to  $(-\frac{\pi}{h}, \frac{\pi}{h})$

**SUMMARY** - function on grid  $h\mathbb{Z}$  ... unique interpol. band limited  
 - Derivative: .. from semidiskr. transf. ...  $ik \hat{v}$  or derivatives of sine function

II. Unbounded Grids: the Semidiscrete Fourier Transform

infinite grid:  $h\mathbb{Z} : x_j = hj ; j \in \mathbb{Z}$

pozn: nepraktické, ale rovníklíč myšlenky; hodi se jenom pro rychle ubíhající fce

• Review of continuous case:

**FT**:  $u(x); x \in \mathbb{R} \longrightarrow \hat{u}(k); k \in \mathbb{R}$  x - physical variable  
.. k - Fourier variable  $\equiv$  wave numb.

$$\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x) dx ; \quad \text{inversion: } u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k) dk$$

• Semidiscrete F.T.

physical space: discrete unbounded ...  $x \in h\mathbb{Z}$   
 Fourier space: bounded continuous ...  $k \in \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$

Aliasing (důvod, že stačí pozorit se na ...)

- functions  $f_j = e^{ik_1 x_j}$  and  $g_j = e^{ik_2 x_j}$  are different on  $\mathbb{R} x_j \in \mathbb{R}$  for  $k_1 \neq k_2$
- but on  $x_j \in h\mathbb{Z}$   $f_j = g_j$  if  $(k_1 - k_2)$  multiple of  $\frac{2\pi}{h}$

practical examples: - wagon wheel effect: camera 24 s<sup>-1</sup>; wheel 20 spokes/s = -4 spokes/s

Def: semidiscrete FT: **SFT**

$$\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j ; \quad k \in \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$$

pozn:  $v \in \mathcal{L}^2(\mathbb{Z})$   
 $\hat{v} \in \mathcal{L}^2 \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$

inverse:  $v_j = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ikx_j} \hat{v}(k) dk \quad j \in \mathbb{Z}$

for  $h \rightarrow 0$  converges to continuous case

note: identical to Fourier series (backwards)

Spectral interpolation:

... use of formula (SFT<sup>-1</sup>) for  $x_j \rightarrow x \in \mathbb{R}$

orig. ... analyt. fce  $p(x) = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ikx} \hat{v}(k) dk ; \quad \dots$  it is  $p(x_j) = v_j$

pozn: - f analytic at  $z \in \mathbb{C} \Leftrightarrow$  differentiable in  $\mathbb{C}$  ... i.e Taylor ser. converg. to f

- např.  $p(x)$  je analyt. pro  $\hat{v} \in \mathcal{L}^1 \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$  nebo  $\mathcal{L}^2 \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$   
 $\uparrow \Leftrightarrow v \in \mathcal{L}^2(\mathbb{Z})$

note: • FT  $p(x) \rightarrow \hat{p}(k) = \hat{v}(k) \cdot \mathcal{X}_{\langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle}$

•  $p(x)$  is band-limited interpolant of  $v$  (def. ... compact support  $\text{supp } \hat{v} \subseteq \langle -\frac{\pi}{h}, \frac{\pi}{h} \rangle$ )

• SAMPLING THEOREM: There is unique band limited interpolant ... (SFT)<sup>-1</sup> (Whittaker, Shannon, Nyquist)



note: interpolant  $p(x)$  is "trigonometric polynomial"  $\deg \leq \frac{N}{2}$

i.e. lin. komb.  $1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots, \sin \frac{N}{2}x, \cos \frac{N}{2}x$   
for  $\leftarrow$  choice  $\rightarrow$  is not used

Interpolation of Kronecker delta:

$$\delta_{j0} = \begin{cases} 1 & \dots j \equiv 0 \pmod{N} \\ 0 & \dots j \not\equiv 0 \pmod{N} \end{cases}$$