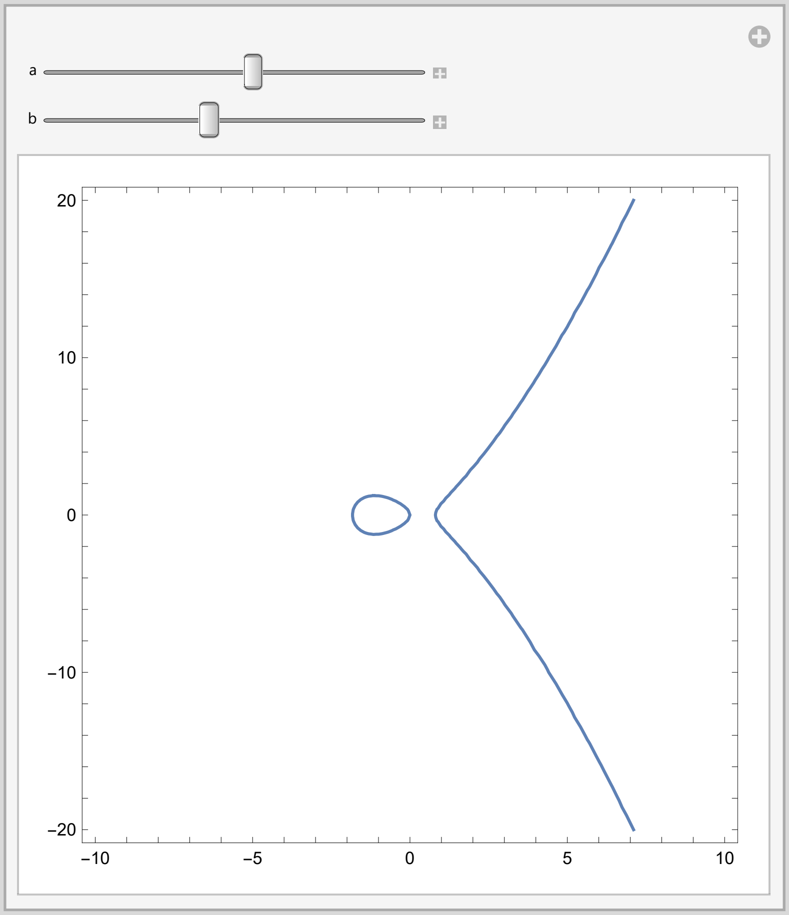
Shapes of elliptic curves

In[2]:=

```
Manipulate[ContourPlot[y^2 == x^3 + a x^2 + b x, {x, -10, 10}, {y, -20, 20}],
 {a, -10, 10}, {b, -10, 10}]
```

Out[2]=



Finding a solution of the problem $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=M$ using theory of elliptic curves

Equivalence with a cubic curve in the Weierstrass form

Homogeneous cubic curve C in the projective plane (all points $\{ta,tb,tc\}$ for $t \neq 0$ are treated as one $\{a,b,c\}$) which has at least one rational solution

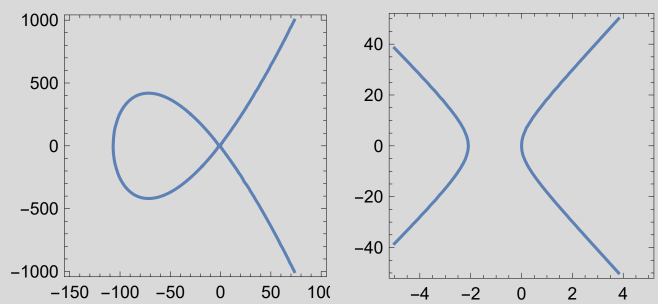
In[3]:=
```
OrigCubic[a_, b_, c_, M_] :=
 M (a + b) (b + c) (c + a) - a (a + b) (c + a) - b (a + b) (b + c) - c (b + c) (c + a)
```

is equivalent to a cubic curve W in the regular plane in the Weierstrass form

In[4]:=
```
Weierstrass[x_, y_, M_] := y^2 - x^3 - (4 M^2 + 12 M - 3) x^2 - 32 (M + 3) x
Print["For M = 4 we have the curve ", Weierstrass[x, y, 4] == 0];
GraphicsRow[{ContourPlot[Weierstrass[x, y, 4] == 0, {x, -150, 100}, {y, -1000, 1000}],
  ContourPlot[Weierstrass[x, y, 4] == 0, {x, -5, 5}, {y, -50, 50}]}, ImageSize → Medium]
```

For M = 4 we have the curve $-224\,x - 109\,x^2 - x^3 + y^2 == 0$

Out[6]=



To get from C to W or the other way, one can use relations

In[7]:=
```
substXY = {x → -4 (a + b + 2 c) (M + 3) / (2 a + 2 b - c + (a + b) M),
  y → 4 (a - b) (M + 3) (2 M + 5) / (2 a + 2 b - c + (a + b) M)}
substABC = {a → (8 (M + 3) - x + y) / (2 (4 - x) (M + 3)),
  b → (8 (M + 3) - x - y) / (2 (4 - x) (M + 3)), c → (-4 (M + 3) - (M + 2) x) / ((4 - x) (M + 3))}
```

Out[7]=

$$\left\{x \to -\frac{4\,(a+b+2\,c)\,(3+M)}{2\,a+2\,b-c+(a+b)\,M},\ y \to \frac{4\,(a-b)\,(3+M)\,(5+2\,M)}{2\,a+2\,b-c+(a+b)\,M}\right\}$$

Out[8]=

$$\left\{a \to \frac{8\,(3+M)-x+y}{2\,(3+M)\,(4-x)},\ b \to \frac{8\,(3+M)-x-y}{2\,(3+M)\,(4-x)},\ c \to \frac{-4\,(3+M)-(2+M)\,x}{(3+M)\,(4-x)}\right\}$$

Because transition from {*x*,*y*} to {*a*,*b*,*c*} and back is given by rational functions, rational solutions of W are transformed to rational solutions of C (and vice versa)

In[9]:=
```
substValuesABCM = {a → 5, b → -9, c → -11, M → 4};
Print["The solution {a,b,c} = ", {a, b, c} /. substValuesABCM,
  " gives the solution {x,y} = ", ({x, y} /. substXY) /. substValuesABCM];
Print["Checking W(x,y) = 0: ",
 Simplify[Weierstrass[x, y, M] /. substXY] /. substValuesABCM]
Print["Checking C(a,b,c) = 0: ", OrigCubic[a, b, c, M] /. substValuesABCM]
```

```
The solution {a,b,c} = {5, -9, -11} gives the solution {x,y} = {-56, -392}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0
```

In[13]:=
```
substValuesXYM = {x → -100, y → 260, M → 4};
Print["The solution {x,y} = ", {x, y} /. substValuesXYM,
  " gives the solution {a,b,c} = ", ({a, b, c} /. substABC) /. substValuesXYM];
Print["Checking W(x,y) = 0: ", Weierstrass[x, y, M] /. substValuesXYM]
Print["Checking C(a,b,c) = 0: ",
 Simplify[OrigCubic[a, b, c, M] /. substABC] /. substValuesXYM]
```

The solution {x,y} = {-100, 260} gives the solution {a,b,c} = $\left\{\frac{2}{7}, -\frac{1}{14}, \frac{11}{14}\right\}$

```
Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0
```

Starting from W and using the substitutions in the general form, we get C multiplied by a factor

In[17]:=
```
Factor[Simplify[Weierstrass[x, y, M] /. substXY]]
Factor[OrigCubic[a, b, c, M]]
```

Out[17]=

$$-\frac{1}{(2a+2b-c+aM+bM)^3} 64 (3+M)^2 (5+2M)^2 \left(-a^3-a^2 b-a b^2-b^3-a^2 c-3 a b c-b^2 c-a c^2-b c^2- c^3+a^2 b M+a b^2 M+a^2 c M+2 a b c M+b^2 c M+a c^2 M+b c^2 M\right)$$

Out[18]=

$$-a^3-a^2 b-a b^2-b^3-a^2 c-3 a b c-b^2 c-a c^2- b c^2-c^3+a^2 b M+a b^2 M+a^2 c M+2 a b c M+b^2 c M+a c^2 M+b c^2 M$$

Positive solution using group addition on the cubic curve

Starting with a known rational solution S on W, we can get a new rational solution using the formula for adding a point S to itself by constructing a tangent to W at S, finding another point where this tangent crosses W and then reflecting it along the axis *x*.
Then we can continue simply by adding S to this new solution, etc.

A function to get a new solution of W:
parameters: p, q, and r are the right-hand site of W written as $y^2 = f(x) = x^3 + p x^2 + q x + r$
S1 and S2 are two known solutions, they can be the same point

In[19]:=

```
GetNewSolutionOfW[p_, q_, r_, S1_, S2_] := Module[
   {xS, yS, x, y, k, y0},
   xS = S1[[1]]; yS = S1[[2]];
   (* a line is given by y = k*x + y0 *)
   If[xS == S2[[1]],
    (* if points are the same we use a tangent line,
    where k is obtained from the derivative of W *)
    k = (3 xS^2 + 2 p * xS + q) / (2 yS),
    (* otherwise k is given simply by *)
    k = (yS - S2[[2]]) / (xS - S2[[1]])
   ];
   y0 = yS - k * xS;
   x = k^2 - p - xS - S2[[1]];
   y = k * x + y0;
   Return[{x, -y}]
  ];
```

Now, start with the solution S = {-100, 260} for *M* = 4 on W obtained from {a,b,c} = {1,-4,11} (found by direct tests for small integers) and iterate until the solution of C is positive (hopefully)

In[20]:=

```
valM = 4;
{x0, y0} = {-100, 260};
{xN, yN} = {x0, y0};
Do[
  {xN, yN} = Simplify[
    GetNewSolutionOfW[4 M^2 + 12 M - 3, 32 (M + 3), 0, {x0, y0}, {xN, yN}] /. {M → valM}];
  substValuesXYM = {x → xN, y → yN, M → valM};
  newABC = ({a, b, c} /. substABC) /. substValuesXYM;
  commonDenominator = LCM@Delete[Denominator@newABC, 0];
  newABC = commonDenominator * newABC;
  Print["The solution {x,y} = ",
   {x, y} /. substValuesXYM, " gives a new solution {a,b,c} = ", newABC];
  Print["Checking W(x,y) = 0: ", Weierstrass[x, y, M] /. substValuesXYM];
  Print["Checking C(a,b,c) = 0: ",
   Simplify[OrigCubic[a, b, c, M] /. substABC] /. substValuesXYM];
  If[Positive@Min@newABC,
   Print["We found a solution of the original problem with ",
    IntegerLength[newABC], " digits after ", i, " iterations. Hooray!"];
   Break[]
  ],
  {i, 1, 15}
 ];
```

The solution {x,y} = $\left\{\frac{8836}{25}, -\frac{950716}{125}\right\}$ gives a new solution {a,b,c} = {9499, -8784, 5165}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{731\,025}{11\,881}, \frac{527\,529\,870}{1\,295\,029}\right\}$
gives a new solution {a,b,c} = {679 733 219, -375 326 521, 883 659 076}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{561\,561\,391\,876}{8\,356\,702\,225}, -\frac{687\,837\,762\,272\,090\,924}{763\,927\,933\,898\,375}\right\}$
gives a new solution {a,b,c} = {6 696 085 890 501 216, -6 531 563 383 962 071, 6 334 630 576 523 495}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
$\left\{-\frac{425\,869\,857\,827\,702\,500}{15\,192\,076\,294\,211\,881}, \frac{448\,412\,887\,098\,983\,162\,164\,732\,300}{1\,872\,516\,697\,802\,137\,088\,411\,221}\right\}$ gives a new solution {a,b,c} =
{5 824 662 475 191 962 424 632 819, -2 798 662 276 711 559 924 688 956, 5 048 384 306 267 455 380 784 631}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{252\,785\,840\,525\,963\,937\,198\,721}{13\,225\,347\,684\,085\,115\,955\,600}, -\frac{343\,764\,653\,760\,831\,645\,784\,970\,282\,294\,394\,569}{1\,520\,934\,975\,898\,868\,459\,000\,385\,442\,296\,000}\right\}$
gives a new solution {a,b,c} = {287 663 048 897 224 554 337 446 918 344 405 429,
-399 866 258 624 438 737 232 493 646 244 383 709, 434 021 404 091 091 140 782 000 234 591 618 320}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{1\,872\,773\,018\,543\,093\,075\,805\,479\,817\,148\,900}{163\,350\,615\,997\,049\,698\,719\,631\,653\,803\,161},\right.$
$\left.\frac{211\,390\,151\,297\,491\,981\,533\,823\,798\,000\,243\,842\,339\,853\,402\,161\,620}{2\,087\,762\,847\,145\,230\,771\,938\,050\,768\,331\,341\,412\,318\,712\,353\,341}\right\}$
gives a new solution {a,b,c} = {3 386 928 246 329 327 259 763 849 184 510 185 031 406 211 324 804,
-678 266 970 930 133 923 578 916 161 648 350 398 206 354 101 381,
1 637 627 722 378 544 613 543 242 758 851 617 912 968 156 867 151}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{8\,304\,830\,821\,343\,520\,148\,948\,729\,081\,523\,501\,917\,420\,036}{1\,462\,178\,082\,526\,064\,533\,227\,321\,540\,886\,890\,051\,021\,025},\right.$
$\left.-\frac{124\,668\,752\,376\,211\,382\,766\,812\,793\,520\,103\,329\,799\,089\,543\,359\,986\,485\,854\,396\,841\,484}{1\,768\,073\,864\,797\,815\,348\,625\,142\,542\,347\,514\,156\,118\,825\,216\,532\,230\,801\,892\,732\,625}\right\}$
gives a new solution {a,b,c} =
{343 258 303 254 635 343 211 175 484 588 572 430 575 289 938 927 656 972 201 563 791,
-2 054 217 703 980 198 940 765 993 621 567 260 834 791 816 664 149 006 217 306 067 776,
2 110 760 649 231 325 855 047 088 974 560 468 667 532 616 164 397 520 142 622 104 465}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{66\,202\,368\,404\,229\,585\,264\,842\,409\,883\,878\,874\,707\,453\,676\,645\,038\,225}{13\,514\,400\,292\,716\,288\,512\,070\,907\,945\,002\,943\,352\,692\,578\,000\,406\,921}, \frac{58\,800\,835\,157\,308\,083\,307\,376\,751\,727\,347\,181\,330\,085\,672\,850\,296\,730\,351\,871\,748\,713\,307\,988\,700\,611\,210}{1\,571\,068\,668\,597\,978\,434\,556\,364\,707\,291\,896\,268\,838\,086\,945\,430\,031\,322\,196\,754\,390\,420\,280\,407\,346\,469}\right\}$ gives a new solution {a,b,c} = {154 476 802 108 746 166 441 951 315 019 919 837 485 664 325 669 565 431 700 026 634 898 253 202 035 277 999, 36 875 131 794 129 999 827 197 811 565 225 474 825 492 979 968 971 970 996 283 137 471 637 224 634 055 579, 4 373 612 677 928 697 257 861 252 602 371 390 152 816 537 558 161 613 618 621 437 993 378 423 467 772 036}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

We found a solution of the original problem with {81, 80, 79} digits after 8 iterations. Hooray!

In[24]:=
```
GCD[Delete[newABC, 0]]
```

Out[24]=
```
1
```

But what about to start with the solution S = {-56,-392} for *M* = 4 on W obtained from {a,b,c} = {5,-9,-11} (also found by direct tests for small integers) ?

In[25]:=
```
valM = 4;
{x0, y0} = {-56, -392};
{xN, yN} = {x0, y0};
Do[
  {xN, yN} = Simplify[
    GetNewSolutionOfW[4 M^2 + 12 M - 3, 32 (M + 3), 0, {x0, y0}, {xN, yN}] /. {M → valM}];
  substValuesXYM = {x → xN, y → yN, M → valM};
  newABC = ({a, b, c} /. substABC) /. substValuesXYM;
  commonDenominator = LCM@Delete[ Denominator@newABC, 0];
  newABC = commonDenominator * newABC;
  Print["The solution {x,y} = ",
   {x, y} /. substValuesXYM, " gives a new solution {a,b,c} = ", newABC];
  Print["Checking W(x,y) = 0: ", Weierstrass[x, y, M] /. substValuesXYM];
  Print["Checking C(a,b,c) = 0: ",
   Simplify[OrigCubic[a, b, c, M] /. substABC] /. substValuesXYM];
  If[Positive@Min@newABC,
   Print["We found a solution of the original problem with ",
    IntegerLength[newABC], " digits after ", i, " iterations. Hooray!"];
   Break[]
  ],
  {i, 1, 15}
 ];
```

The solution {x,y} = $\left\{\frac{676}{49}, \frac{55\,796}{343}\right\}$ gives a new solution {a,b,c} = {-8784, 5165, 9499}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{2\,661\,344}{731\,025}, -\frac{15\,064\,525\,504}{625\,026\,375}\right\}$

gives a new solution {a,b,c} = {396 650 011, 934 668 779, −137 430 135}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{102\,131\,236}{9\,534\,155\,449}, -\frac{1\,445\,821\,255\,910\,884}{930\,943\,540\,506\,707}\right\}$

gives a new solution {a,b,c} = {6 334 630 576 523 495, 6 696 085 890 501 216, −6 531 563 383 962 071}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
$\left\{-\frac{107\,686\,807\,024\,829\,816}{24\,381\,163\,902\,906\,721}, \frac{123\,411\,115\,329\,073\,216\,055\,343\,688}{3\,806\,989\,255\,077\,396\,146\,554\,769}\right\}$ gives a new solution {a,b,c} =
{6 311 022 082 244 686 913 565 321, 1 903 482 249 063 500 625 874 475, −203 834 328 157 024 668 573 221}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{252\,785\,840\,525\,963\,937\,198\,721}{13\,225\,347\,684\,085\,115\,955\,600}, -\frac{343\,764\,653\,760\,831\,645\,784\,970\,282\,294\,394\,569}{1\,520\,934\,975\,898\,868\,459\,000\,385\,442\,296\,000}\right\}$

gives a new solution {a,b,c} = {287 663 048 897 224 554 337 446 918 344 405 429,
−399 866 258 624 438 737 232 493 646 244 383 709, 434 021 404 091 091 140 782 000 234 591 618 320}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{2\,721\,248\,174\,248\,487\,782\,331\,750\,590\,553\,336}{40\,476\,087\,012\,277\,321\,721\,986\,154\,677\,921},\right.$
$\left.\frac{107\,335\,468\,101\,079\,959\,460\,936\,040\,706\,324\,839\,082\,873\,332\,219\,192}{257\,512\,183\,458\,285\,591\,362\,681\,702\,356\,474\,632\,289\,186\,649\,969}\right\}$

gives a new solution {a,b,c} = {2 483 374 184 359 796 574 501 041 038 118 372 847 330 545 855 435,
−1 350 035 390 678 773 406 246 674 701 393 228 548 486 358 866 679,
3 452 374 377 128 426 321 811 335 889 716 695 573 305 695 482 939}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{6\,478\,053\,524\,552\,625\,623\,839\,628\,465\,124\,452\,884\,145\,956}{1\,245\,636\,875\,525\,200\,464\,539\,184\,202\,153\,614\,229\,609},\right.$
$\left.\frac{16\,659\,913\,412\,846\,099\,183\,252\,330\,384\,510\,918\,009\,297\,731\,314\,186\,105\,439\,712\,688\,964}{43\,962\,986\,240\,372\,060\,709\,927\,105\,816\,763\,012\,422\,999\,312\,489\,817\,805\,553\,227}\right\}$

gives a new solution {a,b,c} =
{−2 054 217 703 980 198 940 765 993 621 567 260 834 791 816 664 149 006 217 306 067 776,
2 110 760 649 231 325 855 047 088 974 560 468 667 532 616 164 397 520 142 622 104 465,
343 258 303 254 635 343 211 175 484 588 572 430 575 289 938 927 656 972 201 563 791}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{3\,027\,225\,665\,568\,448\,626\,703\,883\,379\,680\,659\,311\,003\,137\,472\,091\,150\,304}{66\,202\,368\,404\,229\,585\,264\,842\,409\,883\,878\,874\,707\,453\,676\,645\,038\,225},\right.$
$\left.-\frac{5\,951\,043\,690\,090\,397\,469\,452\,596\,820\,042\,230\,118\,740\,428\,276\,310\,771\,488\,013\,914\,566\,097\,407\,924\,545\,851\,456}{17\,033\,750\,828\,350\,273\,864\,302\,184\,324\,987\,268\,233\,756\,155\,527\,802\,269\,272\,046\,755\,420\,002\,651\,509\,010\,375}\right\}$
gives a new solution {a,b,c} =
{-75 325 962 525 379 922 497 547 802 683 539 615 501 464 274 037 547 624 568 031 470 107 460 639 959 387 061,
137 211 312 120 705 701 411 473 512 317 968 603 024 979 592 973 551 357 146 751 192 967 446 785 917 250 491,
149 873 336 803 401 113 247 625 861 581 739 438 102 261 891 754 592 894 935 710 516 699 882 406 130 045 705}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
$\left\{\frac{1\,177\,968\,534\,627\,534\,858\,907\,092\,707\,072\,326\,838\,094\,925\,492\,304\,421\,371\,193\,426\,620\,836}{118\,409\,605\,600\,680\,022\,431\,773\,967\,481\,898\,535\,553\,439\,578\,539\,416\,848\,574\,000\,951\,809},\right.$
4 821 157 371 082 391 021 295 995 028 170 203 814 089 234 429 647 768 105 662 015 947 974 177 674 395 060 450 919-
920 251 817 511 404 /
40 745 570 395 353 799 422 613 237 927 450 391 080 285 643 367 157 149 931 665 708 503 121 532 780 073 320 422-
789 057 705 530 623} gives a new solution {a,b,c} =
{-837 195 266 509 174 235 125 746 309 036 231 647 159 179 965 839 046 831 731 856 095 997 939 696 389 509 603 637-
933 974 969 828 075,
368 094 076 261 423 520 198 252 448 006 319 306 363 128 641 572 895 194 683 647 890 995 604 722 209 255 509 092-
046 087 984 549 776,
893 239 764 490 691 457 892 841 485 218 511 543 529 685 600 814 390 821 492 330 243 770 049 637 905 147 591 198-
476 929 291 616 899}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
$\left\{-\frac{176\,613\,819\,055\,907\,365\,496\,948\,802\,029\,570\,221\,099\,110\,645\,595\,727\,042\,609\,430\,146\,822\,817\,663\,007\,685\,816}{57\,568\,767\,352\,691\,538\,421\,127\,256\,939\,004\,273\,576\,606\,383\,789\,031\,812\,135\,330\,398\,146\,280\,573\,352\,780\,481},\right.$
-(7 688 314 001 745 193 401 849 447 646 114 706 166 175 451 275 537 102 649 559 138 889 428 207 302 496 968 819-
972 017 432 671 652 509 129 758 453 860 898 931 448 /
436 797 753 254 722 947 600 251 359 259 316 854 893 468 021 587 881 937 523 727 254 933 071 395 515 525 536-
709 990 883 181 560 976 683 695 525 987 354 905 121)} gives a new solution {a,b,c} =
{323 435 738 711 883 341 462 116 931 375 544 262 267 024 197 847 542 210 485 267 190 464 229 468 383 302 016 201-
294 026 963 491 333 592 996 489 432 106 417 289,
598 018 381 631 354 534 385 311 490 165 355 196 773 290 314 831 010 162 255 236 436 515 236 872 043 908 045 486-
008 935 273 193 208 919 059 291 355 709 950 555,
-149 646 070 251 971 063 718 697 141 125 722 359 373 197 204 571 150 951 365 432 552 253 130 099 138 571 607 106-
063 707 940 185 441 815 707 153 108 715 559 509}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

```
The solution {x,y} =
 {611 121 966 546 695 353 616 417 118 812 817 017 504 066 459 474 579 394 361 552 465 636 737 752 874 941 795 934 -
     328 097 281 /
   6 251 576 205 640 678 744 757 231 732 285 129 578 756 532 216 941 601 382 455 512 610 872 781 364 622 299 863 -
     183 936 846 400,
  -(2 367 431 843 916 880 896 843 959 536 447 794 867 011 596 658 521 065 851 038 315 771 969 760 394 056 494 216 -
       734 928 371 164 654 729 097 421 606 442 990 405 023 665 700 085 580 241 929 121 /
      494 292 811 181 173 421 775 688 980 934 320 622 451 059 346 730 456 160 079 493 196 155 413 871 417 411 024 -
       350 042 849 173 680 833 701 545 926 481 729 927 489 127 820 384 117 541 312 000)}
  gives a new solution {a,b,c} =
 {25 264 646 056 899 142 163 824 865 804 362 754 960 227 923 064 081 373 425 705 473 390 151 113 050 945 791 310 -
   584 331 430 793 802 674 878 938 200 409 599 803 912 277 154 762 238 244 691 399,
  29 999 509 744 732 903 957 512 784 877 258 344 694 251 116 381 123 505 127 782 104 934 090 633 839 058 779 744 -
   054 188 173 123 112 133 073 781 413 295 580 613 959 608 554 933 398 728 549 641,
  -28 260 231 730 101 974 324 675 674 030 458 815 217 496 967 748 542 813 216 945 576 855 690 817 051 847 799 399 -
    000 076 574 938 158 087 009 116 796 468 304 704 853 738 981 463 748 235 689 760}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
 {-(11 652 587 687 687 381 269 723 020 106 269 205 549 866 553 957 706 088 214 594 715 121 262 361 379 130 136 583 -
       764 346 760 362 550 010 611 455 331 256 /
      2 136 402 557 134 050 053 154 772 005 576 450 982 953 528 376 224 203 239 024 298 845 527 839 269 557 253 040 -
       701 446 930 640 250 336 624 778 050 881),
  4 257 206 618 473 135 680 248 952 319 362 787 322 448 234 925 362 697 815 621 153 351 099 943 195 680 417 230 546 -
    528 151 966 485 692 733 564 884 256 687 338 141 075 735 393 361 651 319 472 184 814 653 823 568 815 122 551 288 /
   98 747 164 467 834 990 428 994 292 366 327 740 468 634 974 367 652 027 605 990 800 003 231 230 780 976 171 147 -
    268 475 029 286 198 245 653 461 962 094 790 771 845 357 423 771 546 850 564 623 152 019 001 307 668 564 234 -
    721} gives a new solution {a,b,c} =
 {184 386 514 670 723 295 219 914 666 691 038 096 275 031 765 336 404 340 516 686 430 257 803 895 506 237 580 602 -
   582 859 039 981 257 570 380 161 221 662 398 153 794 290 821 569 045 182 385 603 418 867 509 209 632 768 359 835,
  32 343 421 153 825 592 353 880 655 285 224 263 330 451 946 573 450 847 101 645 239 147 091 638 517 651 250 940 -
   206 853 612 606 768 544 181 415 355 352 136 077 327 300 271 806 129 063 833 025 389 772 729 796 460 799 697 289,
  16 666 476 865 438 449 865 846 131 095 313 531 540 647 604 679 654 766 832 109 616 387 367 203 990 642 764 342 -
   248 100 534 807 579 493 874 453 954 854 925 352 739 900 051 220 936 419 971 671 875 594 417 036 870 073 291 371}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

We found a solution of the original problem with
 {168, 167, 167} digits after 12 iterations. Hooray!
```

In[29]:=

```
GCD[Delete[newABC, 0]]
```

Out[29]=

```
1
```

Solution for $M = 6$:

In[30]:=
```
valM = 6;
substValuesABCM = {a → 3, b → -7, c → -23, M → valM};
{x0, y0} = ({x, y} /. substXY) /. substValuesABCM;
{xN, yN} = {x0, y0};
Do[
  {xN, yN} = Simplify[
    GetNewSolutionOfW[4 M^2 + 12 M - 3, 32 (M + 3), 0, {x0, y0}, {xN, yN}] /. {M → valM}];
  substValuesXYM = {x → xN, y → yN, M → valM};
  newABC = ({a, b, c} /. substABC) /. substValuesXYM;
  commonDenominator = LCM@Delete[ Denominator@newABC, 0];
  newABC = commonDenominator * newABC;
  Print["The solution {x,y} = ",
   {x, y} /. substValuesXYM, " gives a new solution {a,b,c} = ", newABC];
  Print["Checking W(x,y) = 0: ", Weierstrass[x, y, M] /. substValuesXYM];
  Print["Checking C(a,b,c) = 0: ",
   Simplify[OrigCubic[a, b, c, M] /. substABC] /. substValuesXYM];
  If[Positive@Min@newABC,
   Print["We found a solution of the original problem with ",
    IntegerLength[newABC], " digits after ", i, " iterations. Hooray!"];
   Break[]
  ],
  {i, 1, 15}
 ];
```

The solution {x,y} = $\left\{\frac{21316}{25}, \frac{3479764}{125}\right\}$ gives a new solution {a,b,c} = {-24 869, 26 304, 12 605}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{19719200}{149769}, -\frac{67892477120}{57960603}\right\}$
gives a new solution {a,b,c} = {-7 010 997 913, 9 962 121 367, 14 741 015 373}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{11153235405316}{65466898225}, \frac{55998079267790803036}{16750687914339625}\right\}$ gives a new solution {a,b,c} =
{-399 635 339 857 662 336, 423 865 825 845 143 591, 344 600 079 128 906 665}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{55496484804732005000}{848911622338728001}, -\frac{611927353157444401143311173400}{782156608853411408252092001}\right\}$
gives a new solution {a,b,c} = {-3 709 408 926 756 435 263 894 180 473,
5 289 522 737 323 629 458 801 572 077, 5 601 501 004 856 850 884 990 304 823}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
$\left\{\frac{142\,792\,635\,200\,419\,923\,154\,542\,001}{2\,696\,654\,291\,764\,165\,617\,315\,600}, \frac{3\,862\,921\,795\,547\,336\,221\,111\,733\,846\,263\,658\,315\,599}{4\,428\,308\,931\,512\,839\,875\,117\,040\,555\,811\,304\,000}\right\}$
gives a new solution {a,b,c} = {-3 947 273 214 056 112 611 641 932 285 192 198 633 259,
3 778 570 377 038 559 830 581 535 407 335 117 997 939, 4 070 627 436 031 293 758 660 082 617 456 391 013 440}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{15\,506\,420\,257\,032\,978\,694\,027\,850\,462\,380\,449\,800}{548\,104\,712\,214\,243\,236\,197\,019\,479\,252\,596\,001},\right.$
$\left.-\frac{151\,662\,160\,212\,520\,395\,961\,628\,985\,891\,697\,945\,369\,824\,775\,959\,627\,030\,760}{405\,784\,359\,626\,375\,246\,805\,808\,176\,665\,998\,169\,932\,787\,200\,084\,894\,001}\right\}$
gives a new solution {a,b,c} = {-815 923 886 178 656 038 947 477 254 264 840 525 058 302 026 190 395 683,
1 414 401 999 299 585 078 135 301 950 024 835 142 145 003 502 627 648 887,
1 135 765 668 330 055 092 250 323 976 085 973 407 298 527 010 733 979 623}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{\frac{820\,094\,267\,616\,096\,303\,471\,549\,943\,416\,347\,224\,712\,318\,541\,316}{44\,396\,650\,022\,611\,353\,691\,870\,564\,590\,618\,305\,726\,975\,541\,025},\right.$
$\left.\frac{2\,716\,081\,986\,507\,589\,175\,981\,330\,265\,954\,316\,205\,959\,274\,869\,941\,379\,704\,965\,679\,582\,191\,834\,236}{9\,354\,601\,726\,450\,174\,255\,964\,398\,532\,011\,970\,706\,197\,103\,568\,457\,839\,275\,699\,734\,501\,351\,375}\right\}$
gives a new solution {a,b,c} =
{-23 653 053 272 182 405 114 504 253 854 907 737 996 871 323 554 596 696 886 558 694 080 263 471,
16 289 328 882 340 965 120 515 308 879 714 559 149 588 601 003 364 769 362 936 593 893 145 856,
25 281 620 419 320 048 351 768 026 986 562 305 577 512 151 706 265 133 292 320 809 015 221 455}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} = $\left\{-\frac{435\,973\,553\,644\,007\,497\,263\,616\,496\,715\,410\,063\,036\,105\,052\,190\,989\,827\,232\,800}{37\,120\,220\,903\,144\,207\,602\,639\,176\,664\,847\,334\,977\,046\,340\,745\,690\,556\,785\,769},\right.$
-(1 116 671 061 890 488 092 353 007 389 720 816 188 964 142 743 041 322 361 610 468 559 957 881 054 667 198 164 -
994 935 360 /
7 151 807 692 957 789 282 317 920 696 364 735 398 193 423 099 908 491 394 572 239 792 874 765 391 518 416 763 -
158 603)} gives a new solution {a,b,c} =
{-64 717 948 697 678 464 838 747 111 173 776 757 364 272 678 503 865 274 880 411 181 224 711 461 102 283 679 136 -
152 793,
214 449 816 774 943 558 249 504 736 256 427 289 876 763 007 256 465 315 522 205 958 764 758 802 564 515 862 112 -
581 047,
103 628 367 595 578 987 091 419 134 117 399 744 176 328 895 734 801 141 764 805 532 010 539 626 795 539 699 048 -
159 773}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

```
The solution {x,y} =
 {111 513 006 329 223 241 636 247 179 480 316 015 868 092 644 476 336 925 786 426 030 347 526 773 316
  / 17 186 379 442 078 309 900 650 029 886 707 665 284 018 980 667 365 823 866 371 135 497 412 875 625,
  7 509 600 692 213 508 641 434 126 586 786 922 733 721 241 269 483 680 489 738 473 257 911 706 622 105 754 526 128 -
     383 613 616 518 026 041 648 997 364 /
    71 248 642 462 279 399 097 518 431 292 343 940 320 128 810 878 206 800 940 363 402 606 265 793 938 749 483 289 -
     314 527 295 541 204 996 400 796 875} gives a new solution {a,b,c} =
 {-89 538 305 283 456 477 498 663 693 926 361 732 820 926 189 248 425 318 358 147 867 876 247 902 898 624 183 909 -
     917 412 197 607 224 560 490 524 224,
  20 896 999 013 801 002 522 426 402 938 151 836 792 621 476 479 275 865 314 476 738 857 747 782 720 578 088 533 -
    147 052 708 518 040 528 357 255 149,
  92 107 332 003 270 792 067 767 553 725 416 294 654 721 595 750 665 584 113 901 591 308 391 184 024 376 586 589 -
    667 869 747 157 372 710 541 019 275}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

The solution {x,y} =
 {-(368 406 825 189 389 446 206 930 455 620 030 479 941 820 808 363 982 190 123 187 113 815 699 551 953 492 247 -
       529 424 200 /
      73 372 269 638 620 505 626 414 314 453 737 026 841 614 024 979 452 768 244 889 422 235 319 362 169 671 960 -
       594 540 001),
  -(1 224 717 116 975 196 618 887 066 380 874 104 920 452 801 600 815 944 569 514 584 227 312 213 759 085 697 368 -
       672 604 644 706 083 567 491 957 541 230 073 881 717 657 892 941 627 480 /
      19 874 578 645 829 371 568 750 145 538 253 586 763 031 157 448 752 874 460 084 109 694 176 637 586 336 211 -
       208 905 263 246 459 935 540 102 305 463 436 143 982 820 620 241 810 001)}
  gives a new solution {a,b,c} =
 {2 250 324 022 012 683 866 886 426 461 942 494 811 141 200 084 921 223 218 461 967 377 588 564 477 616 220 767 789 -
    632 257 358 521 952 443 049 813 799 712 386 367 623 925 971 447,
  20 260 869 859 883 222 379 931 520 298 326 390 700 152 988 332 214 525 711 323 500 132 179 943 287 700 005 601 -
    210 288 797 153 868 533 207 131 302 477 269 470 450 828 233 936 557,
  1 218 343 242 702 905 855 792 264 237 868 803 223 073 090 298 310 121 297 526 752 830 558 323 845 503 910 071 851 -
    999 217 959 704 024 280 699 759 290 559 009 162 035 102 974 023}

Checking W(x,y) = 0: 0

Checking C(a,b,c) = 0: 0

We found a solution of the original problem with
 {133, 134, 133} digits after 10 iterations. Hooray!
```

In[35]:=

```
GCD[Delete[newABC, 0]]
```

Out[35]=

```
1
```