Homework #5

Assigned: 20.12.2019 **Deadline: bring for examination**

Homomorphism $SL(2,\mathbb{C})$ onto $L_+^\uparrow$

Let ϕ be the homomorphism

$$SL(2,\mathbb{C}) \to L_+^\uparrow$$

constructed in the class. The homomorphism is based on the one-to-one association of a four-vector from Minkowski space with a Hermitean 2×2 matrix,

$$x = (x_0, x_1, x_2, x_3)^T \quad \leftrightarrow \quad X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

Action of $SL(2,\mathbb{C})$ on the space of Hermitean matrices (and, therefore, Minkowski space) is then defined through

$$X \longmapsto \tilde{X} = AXA^\dagger, \qquad A \in SL(2,\mathbb{C}).$$

For more details, see the lecture notes.

Show that

1. *(5 points)* matrix

   $$M_\tau = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} \in SL(2,\mathbb{C})$$

   is mapped to the boost in the $\langle z\rangle$ direction with velocity $v = \tanh(2\tau)$,

2. *(5 points)* matrices

   $$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in SL(2,\mathbb{C})$$

   and

   $$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in SL(2,\mathbb{C})$$

   are mapped to rotations around $\langle z\rangle$ by an angle 2θ and around $\langle y\rangle$ by an angle 2α, respectively,

3. *(5 points)* kernel of the homomorphism is $\mathrm{Ker}_\phi = \{\mathbb{1}, -\mathbb{1}\}$ (Don't forget to show that there is no other matrix in $SL(2,\mathbb{C})$ that is mapped to identity transformation).
   Try to show that if the kernel of ϕ contains n elements, then the preimage of every element of $L_+^\uparrow$ contains exactly n elements and, therefore, the ϕ represents double covering.