Lecture contents

NTMF061: Group theory and its Applications in Physics

Winter term 2019/20

Week 1: October 4^{th}

- **group definition, order of the group**, examples of the groups, Abelian group
- multiplication table, **rearrangement theorem:** *Each row and each column of the multiplication table contains each element of the group once and only once*
- **subgroup**, order of an element, cyclic subgroup, **theorem:** *Intersection of two subgouprs of $\mathcal{G}$ is again a subgroup of $\mathcal{G}$*
- **left and right cosets with respect to a subgroup**
- **Lagrange theorem:** *Order of a subgroup of $\mathcal{G}$ divides $\#\mathcal{G}$*, index of a subgroup

Tutorial: Classification of point groups, symmetry elements and operations

Week 2: October 11^{th}

- **conjugacy classes**, **theorem:** *Number of elements in any class (g) is a divisor of $\#\mathcal{G}$.*
- **normal (invariant) subgroup**, center of a group, simple and semi-simple groups
- **theorem:** $\mathcal{H} \triangleleft \mathcal{G} \Leftrightarrow \mathcal{H}$ *consists entirely of complete classes of $\mathcal{G}$.*
- product of left/right cosets, **theorem (factor group):** *The set of all distinct cosets with respect to an invariant subgroup $\mathcal{H} \triangleleft \mathcal{G}$ forms a factor (quotient) group.*
- **homomoprhic mapping**, kernel and image of a homomorphism
- surjective, injective and bijective (**isomorphic**) mappings
- **theorem:** *Let $\Phi : \mathcal{G} \to \mathcal{G}'$ be a homomorphism. Then* $\mathrm{Im}\,\phi$ *is a subgroup of* $\mathcal{G}'$, $\mathrm{Ker}\,\phi$ *is invariant subgroup of $\mathcal{G}$ and $Im\,\phi \sim G/\mathrm{Ker}\,\phi$.*, canonical projection of $\mathcal{G}$ onto $\mathcal{G}/\mathcal{H}$
- **direct and semi-direct product groups**, Euler group as a semi-direct group

Week 3: October 18 (only 2/4 units)

- (left/right) **group action on a set**, orbit, stabilizer (isotropy) group, **theorem:** *Let $\mathcal{G}$ be a finite group acting on a set $\mathcal{M}$. Then $(\#\mathcal{G}\cdot m)(\#\mathcal{G}_m) = \#\mathcal{G}$.*
- Group action on itself: left/right translation, conjugation
- **representation of a group as an action on a vector space** (homomorphism to the group of all automorphisms on the vector space), dimension of the representation, faithful representation, **matrix representation**
- **equivalent representations**, intertwining mapping

Week 4: October 25

- equivalent matrix representations are related by similarity transformation
- invariant subspace under group action, **reducible and irreducible representation**, subrepresentation, reducibility of matrix representations
- **completely reducible representation**, block-diagonal form of completely reducible matrix representation
- **theorem:** *Every irreducible representation of a finite group is finite-dimensional.*
- **unitary representation**, **theorem:** *Every finite-dimensional reducible unitary representation of a group $\mathcal{G}$ is completely reducible.*
- **theorem:** *Every finite-dimensional representation of a finite or compact Lie group is equivalent to some unitary representation.*
- **theorem (Maschke):** *Every finite-dimensional reducible representation of a finite or compact Lie group is completely reducible.*
- **Schur lemma I:** *Intertwining mapping between two irreducible representations is either isomorphic (and the two representations are equivalent) or null mapping.*
- **Schur lemma II:** *Let (ρ, V) be a complex finite-dimensional irreducible representation of a group $\mathcal{G}$ and S an intertwining operator on V commuting with all operators $T(g)$ $\forall g \in \mathcal{G}$ from the representation. Then $S = \lambda \mathbb{1}$ for $\lambda \in \mathbb{C}$.*
- **theorem:** *Complex finite-dimensional irreducible representations of an Abelian group are one-dimensional.*
- **theorem:** *Orthogonality relations for irreducible matrix representations*

$$\sum_{g\in\mathcal{G}} [D^\mu(g)_i^j]^* D^\nu(g)_l^k = \frac{\#G}{d_\mu}\delta_{\mu\nu}\delta_{jk}\delta_{il}$$

Week 5: November 1

- **character of a representation**
- **theorem:** *Orthogonality relations for characters*

$$\sum_{g\in\mathcal{G}} \chi^\mu(g)^*\chi^\nu(g) = \#\mathcal{G}\delta_{\mu\nu}$$

- **theorem:** *For finite or compact Lie group, equality of characters of two representations is sufficient condition for their equivalence.*
- decomposition of a reducible representation ρ (of a finite or compact Lie group):

$$\rho = \oplus_\mu n_\mu \rho^\mu \Longrightarrow n_\mu = \frac{1}{\#G}\sum_g \chi^\mu(g)^*\chi(g)$$

  with summation running over all non-equivalent IRREPs ρ^μ.

- regular representation of a finite group, **theorem:** $\#G = \sum_\mu d_\mu^2$
- multiplication of conjugacy classes, class constants $C_iC_j = \sum_k c_{ij}^k C_k$
- **theorem:** *Number of non-equivalent IRREPs of a finite group is equal to the number of distinct conjugacy classes.*
- **theorem (Frobenius):** *Representation (ρ, V) of a finite group $\mathcal{G}$ is irreducible $\Leftrightarrow \sum_{(g_k)} n_k\chi(g_k)^*\chi(g_k) = \#\mathcal{G}$.*

Tutorial:

1. Vector and pseudo-vector representation of $O(3)$
2. Character table for D_{3h}

Week 6: November 8

- **direct product representation**, symmetric and anti-symmetric products of equivalent representations
- **symmetrization (projection) operators** (complete and incomplete), construction of a basis of irreducible (sub)representation
- **symmetries in quanum mechanics**
  - transformation of a wave function (group action on a Hilbert space $\mathcal{L}^2(\mathbb{R}^3)$
  - transformation of an operator

- symmetry group as a group of transformations leaving invariant the Hamiltonian of a system
- eigenfunctions of the Hamiltonian as bases of irreducible representations of the symmetry group, degeneration of energy levels (normal and accidental, hidden symmetries)

Tutorial

1. Character table for D_{3h} part II (transformation of quadratic functions)

Week 7: November 15

- relations between representations of a group and its subgroups
  - **subduced and induced representations**, decomposition to irreducible representations
  - **theorem (Frobenius reciprocity):** $\alpha_{\mu}^{\nu\uparrow G} = \alpha_{\nu}^{\mu\downarrow H}$

Tutorial

1. Induced representations and Frobenius reciprocity
2. MO-LCAO for H_3^{2+}

Week 8: November 22

- decomposition of direct product representation – **Clebsh-Gordan series**
- basis of direct product representation – **Clebsh-Gordan coefficients**
- selection rules for **matrix elements of invariant scalar operators**
- **irreducible tensor operators**, **Wigner-Eckart theorem**
- molecular vibrations and optical transitions
  - **normal coordinates** (vibrational modes) as bases of IRREPs of the symmetry group
  - activity of vibrational modes in infrared spectrum and in Raman scattering

Tutorial

1. Normal coordinates for diatomic molecule
2. Optical transitions in CO_3^{2+} ion
3. (FOR SELF-STUDY) Splitting of energy levels of atoms in a crystal lattice

Week 9: November 29

- **Symmetric (permutation) group $\mathcal{S}_n$**
  - composition rule, decomposition to disjoint cycles, composition of cycles
  - classes (elements with the same cycle structure)
  - transpositions
  - even permutations as invariant subgroup, generators of $\mathcal{S}_n$
  - irreducible representations of $\mathcal{S}_n$: Young diagrams, Young tableaux, hook rule (dimensions of IRREPs)
  - bases of IRREPs of $\mathcal{S}_n$ – subduction chain $\mathcal{S}_n \downarrow \mathcal{S}_{n-1} \downarrow \cdots \downarrow \mathcal{S}_1$
  - characters of IRREPs of $\mathcal{S}_n$
  - orthogonal matrix representations

Tutorial:

1. Character table of $\mathcal{S}_4$

Week 10: December 12

LIE GROUPS

- **$SO(3)$ as a group of orthogonal matrices 3×3 with unit determinant**
  - **linearization** – antisymmetic matrices as generators of infinitezimal rotations
  - general rotation as **exponential** of the generators
  - group $O(3)$ has the same generators but exponential mapping covers only the connected subgroup $SO(3)$
  - generators of rotations form **Lie algebra** $\mathfrak{so}(3)$ with **structure constants**

$$[J_i, J_j] = i c_{ij}^k J_k, \qquad c_{ij}^k = \varepsilon_{ijk}$$

  - $(J_i)_{jk} = -i c_{ij}^k$ is **adjoing representation** of the Lie algebra $\mathfrak{so}(3)$

- **Review of differential geommetry**
  - **topological space**, open and closed sets, neighborhood of a point, **continuous mapping**, homeomorphism
  - **connected**, **path-connected** and **simply-connected** topological spaces, **compactness**
  - **topological manifold**, **coordinate map**, atlas, **differentiable manifolds** (smooth, analytical)

- **Lie groups as smooth manifolds**
  - topological group, **smooth mapping**, **real Lie group**, linear group
  - global topological properties of Lie groups – $E(2)$, $SO(2)$, $SO(3)$, $SU(2)$

<u>**Week 11: December 13**</u>

LIE ALGEBRAS – left-invariant vector fields on Lie groups

- **tangent vectors** as a class of equivalence of tangent curves, tangent space T_pM, directional derivative, **isomorphism of T_pM and the space of derivatives** D_pM, tangent bundle
- **vector field**, **integral curve** of a vect. field
- **push-forward mapping**
- **Lie bracket**
- **left-invariant vector field**, **isomorphism of T_eG and the space** $\mathcal{L}(G)$ of left-invariant fields on a Lie group G
- push-forward of Lie bracket, commutator of vectors from T_eG using Lie bracket of the corresponding fields from $\mathcal{L}(G)$, T_eG **as Lie algebra of** G

EXPONENTIAL MAPPING

- **one-parameter subgroup** of a LG
- **theorem:** Every one-parameter subgroup of G is integral curve of some left-invariant vector field and every integral curve of a left-invariant vector field is one-parameter subgroup.
- **theorem:** Left-invariant vector fields on G are complete.
- **exponential mapping** from LA $\mathcal{G}$ to LG G
- **theorem:** Exponential mapping is local diffeomorphism between T_eG and $U(e) \subset G$
- connected subgroup, **theorem:** Let G be compact LG, then every element of its connected subgroup can be written as $g = \exp(X)$ for some $X \in \mathcal{G}$.
- **theorem:** Every connected component of a LG is a right coset with respect to connected subgroup.
- **theorem:** Every point from the connected subgroup of G can be written as a finite product of exponential elements.

Tutorial:

1. **matrix groups** and their algebras (left-invariant fields, structure constants and commutator on T_eG) – $\mathfrak{gl}(n, \mathbb{R})$

Week 12: December 20

Relations between Lie groups and their Lie algebras

- **homomoprhism and isomorphism** between LAs
- derived homomoprhism of LAs, LA of a subgroup of G is subalgebra of the LA of G
- **theorem:** "Let Φ be isomorphism between two LGs. Then derived homomoprhism Φ_* is isomorphism between corresponding LAs."
- discrete subgroup, **theorem:** "If the kernel of a surjective homomoprhism Φ between two LGs is discrete, then the derived homomoprhism Φ_* is isomorphism between corresponding LAs."
- relation between non-isomorphic LGs with isomorphic LAs, **universal covering group**

Structure of simple- and semi-simple LA

- invariant subalgebra, **simple** and **semi-simple LA a LG**
- **adjoint representation** of LA
- **Killing-Cartan form** and metric, properties of K-C form
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- **theorem:** Cartan criteria for solvable and **semi-simple** LA
- **theorem:** LG G is compact if and only if the C-K form on corresponding LA is negative definite.

Tutorial:

1. Homomorphism (double covering) $SL(2,\mathbb{C}) \to L_+^\uparrow$ and $SU(2) \to SO(3)$
2. Killing-Cartan form on $\mathfrak{sl}(2,\mathbb{R})$

Week 13: 10.1.

Representations of Lie algebras

- representation of LA on V as a homomoprhism $\mathcal{G} \to \mathrm{End(V)}$, **matrix representation of LA**
- analytical representation of a LG, **theorem:** "Relations between analytical representation of LG and corresponding representation of LA"

- relations of representations of $SO(3)$ and $\mathfrak{so}(3) \sim \mathfrak{su}(2)$, multivalued representations and universal covering group
- adjoint representation of LA and LG
- **theorem:** "Adjoint representation of a semi-simple LA is faithful, adjoint representation of a simple LA is irreducible."
- **complexification of LA**, relations of representations of real and complexified LAs

Irreducible representations of semi-simple LA

- **Casimir operator**, quadratic Casimir operator in semi-simple LA
- rank LG/LA, **Racah theorem**
- IRREPs of $\mathfrak{su}(2)$ and $\mathfrak{so}(1,3) \sim \mathfrak{sl}(2,\mathbb{C})$