LIE ALGEBRA as left-invariant vector field on Lie group

Def: Curve $\gamma(t)$ on a manifold M is smooth (C^∞) mapping

$$\gamma : \Omega \to M$$

where $\Omega \subset \mathbb{R}$ is open interval.

Def: Two curves γ_1 & γ_2 are tangent at $p \in M$, if

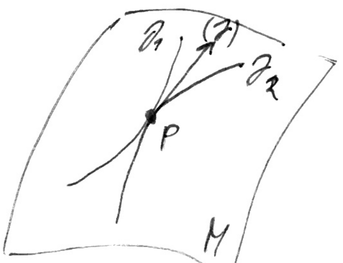
a) $\gamma_1(0) = \gamma_2(0) = p$

b) in some loc. coordinates ϕ on $U(p)$ it holds

$$\left.\frac{dx^\mu(\gamma_1(t))}{dt}\right|_{t=0} = \left.\frac{dx^\mu(\gamma_2(t))}{dt}\right|_{t=0} \quad \forall \mu = 1, \dots, \dim M$$

$$\phi : g \mapsto (x^1, \dots, x^m)$$

Def: Tangent vector (γ) at $p \in M$ is a class of equivalence of curves mutually tangent at p.



- direction of (γ) is given by direction of $\dot\gamma(t)$ & length by the magnitude of $|\dot\gamma(t)|$
- tangent vectors at $p \in M$ form linear vector space:

→ assuming the chart ϕ on $U(p)$ satisfies $\phi(p) = (0, \dots, 0)$, the operations can be defined as

$$\alpha \gamma_1 + \beta \gamma_2 = \phi^{-1} \circ \left[\alpha \phi \circ \gamma_1 + \beta \phi \circ \gamma_2\right]$$

Def: Tangent space T_pM is vect. space of tangent vectors at $p \in M$.

Def: Let $f : M \to \mathbb{R}$ be differentiable function defined on $U(p) \subset M$ and $v = (\gamma) \in T_pM$ vector at $p \in M$. Directional derivative of f along v is

$$v(f) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = \lim_{\Delta t = 0} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\Delta t}$$

- vector components: $(\phi, \mathcal{U}(p))$ is local chart & $\gamma(t) = (x^1(t), \dots, x^m(t))$ is a curve

$$\Rightarrow v(f) = \frac{d}{dt} f(x^1(t), \dots, x^m(t)) = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial f}{\partial x^\mu} \equiv v^\mu \frac{\partial}{\partial x^\mu}(f)$$

$$\Rightarrow v(\cdot) = v^\mu \frac{\partial}{\partial x^\mu} \quad \text{with} \quad \boxed{v^\mu = \frac{d}{dt} \underbrace{x^\mu \circ \gamma(t)}_{\mathbb{R} \to \mathbb{R}} \Big|_{t=0}}$$

- operators $v(\cdot)$ form lin. vector space D_pM of dir. derivatives at $p \in M$
- coordinate basis of D_pM is $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$
- obviously $\dim D_pM = \dim M$
- by construction it holds $T_pM \sim D_pM$:

<u>Theorem:</u> The mapping $c: T_pM \to D_pM$

$$c(\gamma)(f) = v(f) = \frac{df(\gamma(t))}{dt}\Big|_{t=0} \quad \text{for } (\gamma) = v$$

is an isomorphism.

$$\Rightarrow \boxed{\dim T_pM = \dim M}$$

$\Rightarrow$ it is possible to identify T_pM with D_pM & work with $\frac{\partial}{\partial x^\mu}$ as with coordinate basis of T_pM

Def: <u>Tangent bundle</u> $TM \equiv \bigcup_{p \in M} T_pM$

Def: <u>Vector field</u> V on a manifold M is smooth (C^∞) assignment of of a tangent vector $V_p \in T_pM$ at each $p \in M$:
$(Vf) \circ \phi^{-1} \in C^\infty$ $\forall f \in C^\infty(M)$, where ϕ is local chart on $\mathcal{U}(p)$ &

$$Vf: M \to \mathbb{R} \qquad p \mapsto (Vf)(p) = V_p(f) = \frac{df(\gamma_p(t))}{dt}\Big|_{t=0}$$

Def: $VFld(M)$ is real vector space of (smooth) vect. fields on M

- $\dim VFld(M) = +\infty$!
- in general, we can define V of class C^k

Def: Integral curve for vect. field $V \in \mathrm{Vfld}(M)$ passing through $p \in M$ is a curve

$$\gamma_V : (-\epsilon, \epsilon) \to M \,, \quad \gamma_V(0) = p$$

which is ~~for~~ each $t \in (-\epsilon, \epsilon)$ tangent to $V_{\gamma(t)}$:

$$\left.\frac{dx^\mu(\gamma_V(t))}{dt}\right|_{t=t_0} = v^\mu(\gamma(t_0))$$

Def: Vec. field is complete if every integ. curve passing through $p \in M$ for every $p \in M$ can be extended to $\forall t \in \mathbb{R}$

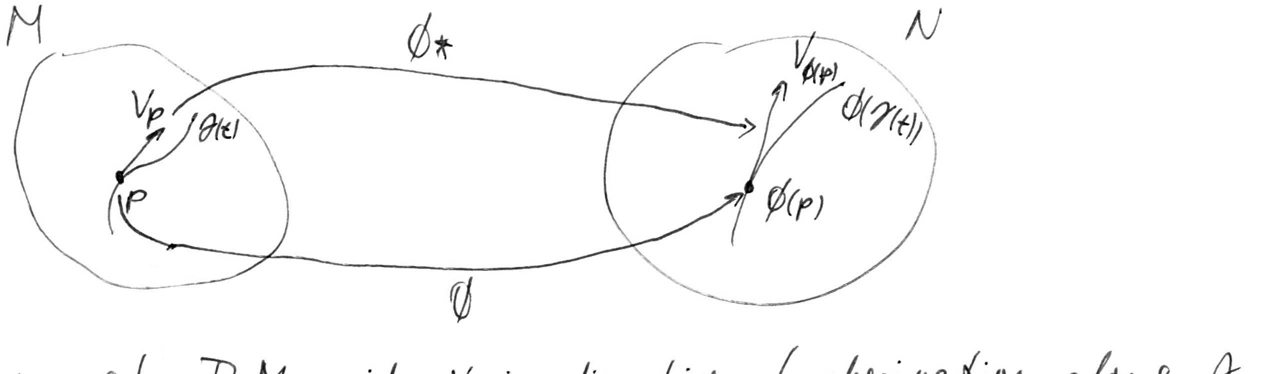
Note:
- for compact M, every $V \in \mathrm{VFld}(M)$ is complete
- integral curves for a complete V cover the whole M & can only intersect at singular points $V = 0$

Def: Push-forward of a vector $V = (\gamma)$ induced by a mapping $\phi : M \to N$ between ~~two~~ manifolds is the mapping

$$\phi_* : T_p G \to T_{\phi(p)} N$$

$$\phi_* \left(\left.\frac{df(\gamma(t))}{dt}\right|_{t=0} \right) = \left.\frac{df(\phi(\gamma(t)))}{dt}\right|_{t=0}$$

- ϕ_* maps tangent vector defined by $\gamma(t)$ in $T_{\gamma(0)}M$ to tangent vector def. by $\phi(\gamma)$ in $T_{\phi(\gamma(0))}N$



- in terms of $D_p M$, if V is directional derivative along γ, $\phi_* V$ is derivative along $\phi \circ \gamma$

- coordinate representation of push-forward

$$\phi : x \to y(x) \qquad V = v^{\mu} \frac{\partial}{\partial x^{\mu}} \quad \Rightarrow \quad \phi_* V = w^{\nu} \frac{\partial}{\partial y^{\nu}}$$

$$\Rightarrow \phi_* V(f) = \frac{df(\phi(\gamma(t)))}{dt} = \frac{\partial f}{\partial y^{\nu}} \frac{\partial y^{\nu}}{\partial x^{\mu}} \frac{dx^{\mu}(\gamma(t))}{dt} = v^{\mu} \frac{\partial y^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial y^{\nu}} f$$

$$\Rightarrow \boxed{w^{\nu} = v^{\mu} \frac{\partial y^{\nu}}{\partial x^{\mu}}} \qquad \frac{\partial y^{\nu}}{\partial x^{\mu}} = J^{\nu}_{\mu} \text{ - Jacobian matrix}$$

$\Leftrightarrow$ vector transformation corresponding to a change of coord.

$$[x^{\mu}, U(p)] \to [y^{\nu}, U(\phi(p))]$$

Def: Lie bracket of vec. fields $V, W \in VFld(M)$ is vector field $[V, W] \in VFld(M)$ defined at $p \in M$ as

$$[V, W]_p : C^{\infty}(M) \to \mathbb{R} \qquad [V, W]_p(f) = V_p(Wf) - W_p(Vf)$$

- it can be shown that $[V, W]_p \in T_pM$
- simple "product" of two vec. fields, $VW(f) \equiv V_p(Wf)$ can't be vector - it's dif. operator of 2^{nd} order
- $[\cdot, \cdot]$ is def. only for vec. fields ($\Leftarrow$ contains derivative of the field)

- coordinate repre:

$$[V, W] = (v^{\nu} w^{\mu}{}_{,\nu} - w^{\nu} v^{\mu}{}_{,\nu}) \frac{\partial}{\partial x^{\mu}}$$

- anti-sym.: $[W, V] = -[V, W]$
- satisfies Jacobi identity: $[V, [W, Z]] + [W, [Z, V]] + [Z, [V, W]] = 0$

$\Rightarrow$ it is good candidate for the "commutator" in Lie algebra if the manifold M is a Lie group

but dim $VFld(M) = +\infty >$ dim M $\Rightarrow$ $VFld(M)$ can't be LA of LG M

NOTE: • $[\cdot,\cdot]$ is closely connected to Lie derivative of a vec. field Y along X:

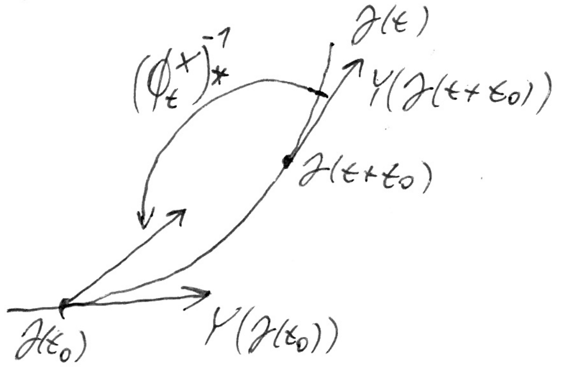
$$\mathcal{L}_X Y = [X,Y]$$

$$\mathcal{L}_X Y\Big|_p = \lim_{t\to 0} \frac{1}{t}\left[\left(\phi_t^X\right)^{-1}_* Y\left(\phi_t^X(p)\right) - Y(p)\right]$$

• here, ϕ_t^X is flow generated by the vec. field X:

$$\phi_t^X : M \to M \qquad \gamma(t_0) \mapsto \gamma(t_0+t)$$

where $\gamma(t)$ is integral curve to X passing through $\gamma(t_0)$



Def: Left & right translations of a Lie group G are diffeomorphisms of G def. by

$$R_h : G \to G \qquad g \mapsto gh$$

$$L_h : G \to G \qquad g \mapsto hg$$

Def: diffeomorphism – smooth isomorphism between two dif. manifolds (cf. homeomorphism)

• R_h, L_h are C^∞ by def. of the group op. on the Lie group

Def: Vec. field $V \in VFld(G)$ on a LG G is left-invariant if

$$\forall g \in G : (L_g)_* V = V \quad \Leftrightarrow \quad (L_g)_* V_{g'} = V_{gg'}$$

• terminology: V is L_g-related with itself

- coordinate repre: $g'(y) = h\,g(x) \Leftrightarrow x \to y(x)$

$$\underset{\uparrow}{v^\nu(y)}\frac{\partial}{\partial y^\nu} = \underbrace{v^\mu(x)\frac{\partial y^\nu}{\partial x^\mu}}\frac{\partial}{\partial y^\nu}$$

field in $g'(y)$ field translated from $g(x)$ to $g'(y)$ - cf. page 18

$\Rightarrow$ condition for left-invariance is $\boxed{v^\nu(y) = v^\mu(x)\frac{\partial y^\nu}{\partial x^\mu}}$

- $\mathcal{L}(G)$ ... set of all left-invariant vec. fields
- $V \in \mathcal{L}(G) \Rightarrow$ <u>V is fully determined by the corresponding vector from T_eV</u> (or any other T_gV)

  $\to$ the whole field for any $g \in G$ can be constructed using left-translation:

$$V_g = (l_g)_* V_e \qquad (+)$$

  $\to$ such constructed field is smooth due to smoothness of l_g

<u>Theorem:</u> The set $\mathcal{L}(G)$ forms real linear vec. space isomorphic to T_eG.

$\Rightarrow \dim \mathcal{L}(G) = \dim G$

<u>Proof:</u> Isham, page 159; stems directly from $(+)$, we just need to show that $(+)$ is isomorphism.

<u>Theorem:</u> Let $\phi: G \to G'$ be diffeomorphism between two smooth manifolds. Then

$$\phi_*[V,W] = [\phi_* V, \phi_* W] \qquad \text{(Exercise)}$$

$\Rightarrow (l_g)_*[V,W] = [(l_g)_* V, (l_g)_* W] = /V, W \in \mathcal{L}(G)/ = [V,W]$

$\Rightarrow \mathcal{L}(G)$ is closed with resp. to Lie bracket

$\Rightarrow \mathcal{L}(G)$ has the structure of LA of G

- $\mathcal{L}(G) \sim T_e(G)$ & it is possible to define commutator of two vectors from $T_e(G)$ using corresp. left-inv. fields:

$$[V_e, W_e] = [V, W]_e$$

for $V, W \in \mathcal{L}(G)$ generated from V_e, W_e as $V_g = (L_g)_* V_e \dots$

Def: $T_e G$ with the commutator

$$[V_e, W_e] = [V, W]_e$$

forms Lie algebra of a L. group G

MATRIX LIE GROUPS & THEIR ALGEBRAS

Theorem (Ado)

Every finite-dim. abstract LA is isomorphic to some LA of square matrices with std. commutator.

Theorem (Freudenthal, de Vries, Lin. L. groups, 1969, p. 42)

Every real Lie algebra is isomorphic to real LA of some linear Lie group.

$\Rightarrow$ to every LA there exists lin. LG

- fundamental matrix group: $GL(n, \mathbb{R})$
  - $\rightarrow$ invertible $n \times n$ matrices ($\det A \neq 0$) with $\mathbb{R}$ elements
  - $\rightarrow$ $\dim GL(n, \mathbb{R}) = n^2$
  - $\rightarrow$ non-abelian (except $GL(1, \mathbb{R}) \sim (\mathbb{R} \setminus \{0\}, \cdot)$)
  - isomorphic abstract LG: $GL(V)$ ... group of automorphisms on a n-dim. vec. space V

→ disconnected manifold – matrices with positive det form connected subgroup $GL^+(n,\mathbb{R})$

→ non-compact (maximal compact subgroup is $O(n)$)

(→ $GL^+(n,\mathbb{R})$ is doubly-connected for $n>2$)

→ global parametrization: n^2 mat. elements $a^i_j = A^i_j$

• <u>left-invariant vec. fields on $GL(n,\mathbb{R})$</u>

. left-translation: $(L_A X)^i_j = a^i_k x^k_j$

. basis of $T_e G$ is $\partial^j_{0i} = \dfrac{\partial}{\partial x^i_j}$ ... n^2 derivatives along coord. axes at $\mathbb{1}$

→ general vector from $T_e G$ is given by an arbitrary matrix C: $V_C = c^i_j \partial^j_i$ ... lin. combination of basis vec.

→ corresp. left-invariant vec. field is

$$V_C(X) = C(X)^i_j \partial^j_i$$

• $C(X)$ can be found by left-translation of V_C:

1, find expression for push-forward generated by L_Y from X_0 to $X = YX_0$

b, set $X_0 = \mathbb{1}$, $Y = X$ & $V_C(X_0 \to \mathbb{1}) = V_C$

<u>ad 1,</u> • $(L_Y X_0)^i_j = y^i_k x_0{}^k_j$

• $(L_Y)_* C(X_0)^i_j \partial^j_{0i} = C(X_0)^k_l \dfrac{\partial x^i_j}{\partial x_0{}^k_l} \partial^j_i = C(X)^i_j \partial^j_i$

• $\dfrac{\partial x^i_j}{\partial x_0{}^k_l} = \Big/ x^i_j = y^i_k x_0{}^k_j = \delta^l_j y^i_k x_0{}^k_l \Big/ = \delta^l_j y^i_k$

→ $C(X)^i_j = C(X_0)^k_l \delta^l_j y^i_k = C(X_0)^k_j y^i_k$

ad b) $Y \to X$, $C(X_0) \to C$

$$C(X)^i_j = X^i_k C^k_j$$

$$\Rightarrow \boxed{V_C(X) = X^i_k C^k_j \partial^j_i = Tr(XC\partial) \in \mathcal{L}(GL(n,\mathbb{R}))}$$

- Weyl basis of $n\times n$ matrices

$$(E^l_k)^i_j = \delta^i_k \delta^l_j$$

$\Rightarrow$ basis of $\mathcal{L}(G)$: $e^l_k(X) = V_{E^l_k}(X) = X^i_m (E^l_k)^m_j \partial^j_i$

$$\Rightarrow \boxed{e^l_k(X) = X^i_k \partial^l_i}$$

$$\Rightarrow V_C(X) = C^k_l e^l_k(X)$$

- structure constants of $gl(n,\mathbb{R})$ in Weyl basis

$$[e^i_j, e^l_k](f) = X^\mu_j \partial^i_\mu (X^\nu_k \partial^l_\nu) f - X^\nu_k \partial^l_\nu (X^\mu_j \partial^i_\mu) f$$

$= /$ 2nd derivatives of f cancel $/ =$

$$= X^\mu_j \delta^\nu_\mu \delta^i_k \partial^l_\nu f - X^\nu_k \delta^l_j \delta^\mu_\nu \partial^i_\mu f$$

$$= (\delta^i_k X^\nu_j \partial^l_\nu - \delta^l_j X^\mu_k \partial^i_\mu) f$$

$$\Rightarrow \boxed{[e^i_j, e^l_k] = \delta^i_k e^l_j - \delta^l_j e^i_k} \quad (*)$$

$\Rightarrow$ we can define $gl(n,\mathbb{R})$ as $T_e(GL(n,\mathbb{R}))$

- $V_C = V_C(X = \mathbb{1}) = C^i_j \partial^j_i = Tr(C\partial)$
- basis vectors: $e^j_i = \partial^j_i$

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- commutator is derived from the Lie bracket (*)

$$[V_C, V_D] = [c^j_i e^i_j, d^k_\ell e^\ell_k] = c^j_i d^k_\ell (\delta^i_k e^\ell_j - \delta^\ell_j e^i_k)$$

$$= c^j_k d^k_\ell e^\ell_j - d^k_\ell c^\ell_i e^i_k = (c^k_\ell d^\ell_i - d^k_\ell c^\ell_i) e^i_k$$

$$= [C, D]^k_i e^i_k = V_{[C,D]}$$

$\Rightarrow$ $T_e(GL(n,\mathbb{R}))$ with std. matrix commutator is LA isomorphic to $gl(n,\mathbb{R})$

= LA of all $n\times n$ $\mathbb{R}$ matrices with mat. commutator

- here we have explicitly constructed the isomorphisms

$$\mathcal{L}(GL(n,\mathbb{R})) \sim T_e(GL(n,\mathbb{R})) \sim End(\mathbb{R}^n)$$

$End(\mathbb{R})$ ... endomorphisms of $\mathbb{R}^n$ ($n\times n$ matrices incl. non-invertible)

- regularity of matrices from $GL(n,\mathbb{R})$ is ensured by the exp. mapping

INTERMEZZO

Def: A Lie subgroup of G is a subset $H\subset G$ that is
V1
a, a subgroup of G in the algebraic sense
b, a submanifold of G
c, a topological group with respect to the subspace topology

Def: A subgroup H of a LG G is called Lie subgroup
V2 if it is a Lie group (with respect to induced group operation), and the inclusion map

$$\iota_H : H \hookrightarrow G \quad \iota(x) = x$$

is a smooth imersion (and therefore a Lie group homomorphism)

- $f: M\to N$ is an immersion if $D_p f : T_pM \to T_{f(p)}N$ is injective for all $p\in M$ ($D_p f$ is derivative of f)

<u>Proposition</u>: Let $H < GL(n,\mathbb{R})$ be a subgroup, then the LA h is subalgebra of $gl(n,\mathbb{R})$

$\Rightarrow$ study of linear L. groups and their L. algebras can be restricted to matrix subalgebras

<u>Def</u>: Subalgebra $h \subset g$ of a LA g is a subset of g which is closed under all operations on g (it is a vector subspace closed under commutator)

<u>Example</u>: • $so(3)$ as subalg. of $gl(n,\mathbb{R})$ ... antisym. matrices

• in general, the condition defining the subalgebra corresp. to a subgroup of $GL(n,\mathbb{R})$ is obtained by derivative of a curve $X(t) \subset H \subset GL(n,\mathbb{R})$ passing through $\mathbb{1}_n$ & setting $t=0$:

$$X(t) = \mathbb{1} + Ct \in H \Rightarrow C \in h$$

<u>Example</u>: <u>$su(2) \sim so(3)$:</u> [NOTE: moving to $GL(n,\mathbb{C})$]

• $SU(2)$: $U^\dagger U = 1$ & $\det U = 1$

$\to U = \mathbb{1} + iAt \Rightarrow A^\dagger = A$

$\to \det U = \exp(\mathrm{Tr}\log U) \sim \exp(\mathrm{Tr}(iAt)) = 1 \Rightarrow \mathrm{Tr}A = 0$

$\Rightarrow su(2)$ are traceless hermitian matrices

$\Rightarrow$ basis is given by Pauli matrices σ_i : $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

$\eta = \mathrm{diag}_n(1,\dots,1;-1,\dots,-1)$ $n = r+s$, r-times 1, s-times -1

	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$O(r,s)$	$SO(r,s)$	$GL(n,\mathbb{C})$	$U(n)$	$SU(n)$
$A \in G$	$\det A \neq 0$	$\det A = 1$	$A^T\eta A = \eta$	& $\det A = 1$	$\det A \neq 0$	$A^\dagger A = 0$	& $\det A = 1$
$\mathcal{G}$	$gl(n,\mathbb{R})$	$sl(n,\mathbb{R})$	$o(r,s)$ $\sim$	$so(r,s)$	$gl(n,\mathbb{C})$	$u(n)$	$su(n)$
$a \in \mathcal{G}$	—	$\mathrm{Tr}\,a = 0$	$(\eta a)^T = -\eta a$	— " —	—	$a^\dagger = -a$	" & $\mathrm{Tr}\,a = 0$
dim	n^2	n^2-1	$\frac{n(n-1)}{2}$	— " —	$2n^2$	n^2	n^2-1

• dim $u(n)$: diag imaginary $\to n$, Re & Im offdiag $\Rightarrow 2 \times \frac{1}{2}(n(n-1))$

ONE-PARAMETER SUBGROUPS & EXPONENTIAL MAPPING

(Fecko 11.3 → ... p. 225 onwards)

- $G \to \mathcal{G}$ ... linearization around $e \in G$
- $\mathcal{G} \to G$ ... exp. mapping
  → need not cover the whole G (if non-compact, disconnected, ...)
  → $\exists$ non-isomorphic groups with the same LA

<u>Def:</u> One-parametric subgroup of a LG G is a curve satisf.

$$i,\ \gamma(t+s) = \gamma(t)\gamma(s) \qquad \forall t,s \in \mathbb{R}$$

$$ii,\ \gamma(0) = e$$

<u>Note:</u>
- cf. flow ϕ_t^X generated by a vec. field X (page 19)
- every element of γ can be written as

$$\gamma(t) = \underbrace{\gamma\left(\tfrac{t}{N}\right)\cdots\gamma\left(\tfrac{t}{N}\right)}_{N\text{-times}}$$

$$\left(\to \text{cf. } R(\varphi) = \lim_{N\to\infty}\left(R\left(\tfrac{\varphi}{N}\right)\right)^N \text{ for } SO(3)\right)$$

⇒ the whole subgroup is determined by an element $\gamma(\epsilon) \in \mathcal{U}(e)$

⇒ 1-param subgroups determined by elements of $\mathcal{G}$?

<u>Theorem:</u> Let $V \in \mathcal{L}(G)$ is left-inv. vec. field generated by $V_e \in \mathcal{G}$. Then

1, its integral curve $\gamma^V(t)$ starting from e is a 1-par. subgroup;

2, if, in turn, $\gamma(t)$ is an arbitrary 1-par. subgroup, then it is the integral curve of the left-invar. vec. field $V \in \mathcal{L}(G)$ defined by $V_e = \dot{\gamma}(0)$.

Proof: i) $\gamma(t)$ integral to $V \in \mathcal{L}(G) \Rightarrow \gamma(t+s) = \gamma(t)\gamma(s)$:

- consider $\Gamma(s) = \gamma(t+s)$ ... it is obviously tangent to V, only starting from $\gamma(t)$ instead from $\gamma(0) = e$:

$$V_{\Gamma(0)}(f) = \frac{d}{ds} f(\gamma(t+s))\Big|_{s=0} = \frac{d}{d\tau} f(\gamma(\tau))\Big|_{\tau=t} = V_{\gamma(t)}(f)$$

$$= \big/ V \in \mathcal{L}(G) \Rightarrow V_{\gamma(t)} = (L_{\gamma(t)})_* V_e \big/ = [(L_{\gamma(t)})_* V_e](f)$$

$$\equiv /\text{def. of push-forward}/ = \frac{d}{ds} f\big(L_{\gamma(t)}(\gamma(s))\big)\Big|_{s=0}$$

$$= \frac{d}{ds} f(\gamma(t)\gamma(s))\Big|_{s=0} \Rightarrow \gamma(t)\gamma(s) = \gamma(t+s) \quad \boxtimes$$

ii) $\gamma(t)\gamma(s) = \gamma(t+s) \Rightarrow V_\gamma \in \mathcal{L}(G)$ with $V_e = \dot{\gamma}(0)$?

$$V_{\gamma(t)}(f) \equiv \frac{d}{ds} f(\gamma(t+s))\Big|_{s=0} = /\text{assump.}/ = \frac{d}{ds} f(\gamma(t)\gamma(s))\Big|_{s=0}$$

$$\equiv [(L_{\gamma(t)})_* V_e](f)$$

$$\Rightarrow V_{\gamma(t)} = (L_{\gamma(t)})_* V_e \Rightarrow V \text{ is left-invariant} \quad \boxtimes$$

Corollary: $V \in \mathcal{L}(G) \Rightarrow V$ is complete.

Proof: $\gamma(\tau)$ def $\forall \tau \in \mathbb{R}$ since $\gamma(\tau = t+s) = \gamma(t)\gamma(s)$ is defined for any $\gamma(t), \gamma(s) \in G$

Note: there is one-to-one correspondence

$X \in \mathfrak{g} \leftrightarrow$ one-param. subgroup $\leftrightarrow$ left-invariant $V \in \mathcal{L}(G)$ with $V_e = X$

Example: • one-param. subgroups of $GL(n,\mathbb{R})$

$$V_C(X) = Tr(XC\partial) = x^i_k C^k_l \partial^l_i$$

$$\Rightarrow (v_{C(X)})^i_l = x^i_k C^k_l \in \mathcal{L}(GL(n,\mathbb{R}))$$

• $\gamma_C(t) = (x^1_1(t), \dots x^n_n(t))$ integ. curve is defined by

$$\frac{dx^i_l(t)}{dt} = (v_{C(X)})^i_l = x^i_k(t) C^k_l \quad \& \quad x^i_l(0) = \delta^i_l$$

• matrix form:

$$\frac{dX(t)}{dt} = X(t)C \quad \& \quad X(0) = \mathbb{1}_n \Rightarrow X(t) = X(0)\exp(tC)$$

$$\Rightarrow X(t) = \exp(tC)$$

$$\left[\text{cf. } \exp(tC) \equiv \mathbb{1} + tC + \frac{1}{2!}t^2C^2 + \dots \Rightarrow \frac{d}{dt}\exp(tC) = \exp(tC)\cdot C\right]$$

$\Rightarrow$ for $GL(n,\mathbb{R})$, the one-param. subgroups defined by left-invariant fields are given by matrix exp. of the corresp. element from $gl(n,\mathbb{R})$

$\Downarrow$

Def: Let $\gamma^X(t) \subset G$ is one-param. subgroup corresp. to a vector $X \in \mathcal{G}$. Define exponential map by

$$\exp: \mathcal{G} \to G \qquad X \mapsto \exp X \equiv \gamma(1)$$

$\Rightarrow$ image of $X \in \mathcal{G}$ lies on $\gamma^X(t) \subset G$ in a parametric distance $t=1$.

Proposition: The one-param. subgroup $\gamma^X(t)$ can be expressed as

$$\gamma^X(t) = \exp(tX)$$

Proof: (exercise)

- $X \in \mathcal{G} \Rightarrow tX \in \mathcal{G} \Rightarrow \exp(tX) = \gamma^{tX}(1) \overset{?}{=} \gamma^X(t)$
- we need to show $\gamma^{tX}(s) = \gamma^X(ts)$

$$\Rightarrow \frac{d}{ds} f(\gamma(ts))\Big/_{s=0} = \frac{d}{d\tau} f(\gamma(\tau))\Big/_{\tau=0} \frac{d(ts)}{ds}\Big/_{s=0}$$

$$= t \frac{d}{d\tau} f(\gamma(\tau))\Big/_{\tau=0} = (tX)f$$

$$\to \gamma^{tX}(0) = e = \gamma^X(t \cdot 0) = \gamma^X(0)$$

$\Rightarrow \gamma^{tX}(s)$ & $\gamma^X(ts)$ are the same curves / 1-param. subgroups

$\boxtimes$

$\Rightarrow \gamma^X(t) \in G$ is an image with resp. to exp of the "straight line" tX in $\mathcal{G}$.

Note:
- $t = 0 \Rightarrow \exp(0X) = \gamma^X(0) = e$
- $t = -1 \Rightarrow \exp(-X) = \gamma(-1)$ & $\gamma(-1)\gamma(1) = \gamma(0) = e$

$$\Rightarrow \exp(-X) = (\exp(X))^{-1}$$

Theorem: Exp. map is local diffeomorphism between the neighborhood of null vector in $\mathcal{G}$ and $U(e) \subset G$.

- $\exists\, U(0) \subset T_e(G)$ $\exp: U(0) \to G$ is bijection & $\exp(X)$ and $\exp^{-1}(X)$ are smooth
- $\forall g \in U(e) \subset G \;\; \exists X \in U(0) \subset \mathcal{G}: \; g = \exp(X)$

Proof: (hint)

- basically, $\mathcal{G}$ is vect. space $\Rightarrow$ $\mathcal{G} \sim \mathbb{R}^n$ & G is locally isomorphic to $\mathbb{R}^n$

$\Rightarrow$ normal coordinates on $U(e) \subset G$

- let e_i be a basis of $\mathcal{G}$ $\Rightarrow$ $X = X^i e_i \in \mathcal{G}$ is some vector def. by coordinates X^i
- define a chart on $U(e) \subset G$ as follows:
  $\rightarrow$ to a point $g = \exp(X) \subset U(e)$ asign coordinates $x^i = X^i$ ... coordinates of X

$\Rightarrow$ one-parameter subgroups are given simply by

$$g^X(t) \Leftrightarrow x^i(t) = X^i t \qquad \text{(exercise)}$$

- in general, exp does not cover the whole group $\Leftrightarrow$ the diffeomorphism does not $\exists$ globally; in particular, if
  i) G is not connected - exp is smooth $\Rightarrow$ maps only onto the connected subgroup (cf. $SO(3) \subset O(3)$)

  ii) G is non-compact (cf. $SL(2,\mathbb{R})$ - homework)

Def: If $G = \exp(\mathcal{G})$, we speak of exponential group.

Theorem: Let G be compact LG, then every element of its connected component can be writen as $g = \exp(X)$ for some $X \in \mathcal{G}$.

Theorem: Let G be LG (comp. or non-comp.), then every element can be written as finite product of exp. elements,

$$g = \prod_i \exp(X_i)$$

for some finite set of $X_i \in \mathcal{G}$.

Proof: (hint)

- let $g \in G$ arb. & $g_v = e^V \in G$ is an element close to g

  $\Rightarrow$ $\mathcal{U}(e^V)$ is mapped by e^{-V} onto $\mathcal{U}(e)$

  (neighborhood of any element is locally diffeomorphic to $\mathcal{U}(e)$)

- any element from $\mathcal{U}(e)$ can be written as $\exp(w)$

  $\Rightarrow$ any element from $\mathcal{U}(e^V)$ can be written as $e^V e^{w_1}$ ... we have left-translated $\mathcal{U}(e)$ onto $\mathcal{U}(e^V)$

$\Rightarrow$ repeat ... any element from $\mathcal{U}(e^V e^{w_1})$ can be written as $e^V e^{w_1} e^{w_2}$ for some $w_2 \in \mathfrak{g}$

$\Rightarrow$ repeat until $g \in \mathcal{U}(e^V \prod_i e^{w_i})$

- prove that this happens in a finite # steps - nontrivial