Theorem XXI: $\sum_\mu d_\mu^2 = \#G$ (sum over non-equiv. IRREPs)

Proof: • Maschke $\Rightarrow$ D^r is completely reducible

$$\Rightarrow D^r = \bigoplus_\mu n_\mu^r D^\mu$$

$$\Rightarrow n_\mu^r = \frac{1}{\#G}\sum_g \chi^\mu(g)^* \chi^r(g) = \frac{1}{\#G}\overbrace{\chi^\mu(e)^*}^{d_\mu}\chi^r(e) = d_\mu$$

$$\Rightarrow D^r = \bigoplus_\mu d_\mu D^\mu \Rightarrow \chi^r(g) = \sum_\mu d_\mu \chi^\mu(g) \quad / g = e$$

$$\Rightarrow \#G = \sum_\mu d_\mu^2 \qquad \boxtimes$$

• next we want to proove $\#IRREP_s = \#(\text{classes})$

Lemma:

Let C be a set of elements from G (each element can be incl. multiple - times).

Then $gCg^{-1} = C \quad \forall g \in G \Leftrightarrow C = \sum_{(g_k)} a_k (g_k)$.

- C need not be a group
- $\sum_{(g_k)}$ runs over distinct classes
- $a_k \geq 0$ $\Rightarrow$ elements of G might be included multiple-times in C
- note: $H \triangleleft G$ $\Rightarrow$ H consists of complete classes.

Proof: $\Leftarrow|$ $g(g_k)g^{-1} = (g_k)$ (rearr.) : $g(hg_kh^{-1})g^{-1} = \hat{h} g_k \hat{h}^{-1}$

$\Rightarrow|$ • $gCg^{-1} = C$ & let $C = \sum_{(g_k)} a_k(g_k) + R$ such that $\exists a \in R: (a) \not\subset R$

• since $g(g_k)g^{-1} = (g_k)\ \forall (g_k)\ \forall g \in G \Rightarrow gRg^{-1} = R\ \forall g \in G$

$\Rightarrow g a g^{-1} \in R \Rightarrow (a) \subset R$ ↯ $\boxtimes$

Def: (Multiplication of classes)

$$(g_k)(g_l) \equiv \{ g_i g_j \mid g_i \in (g_k)\ g_j \in (g_l) \}$$ incl. multiplicity

Example: $C_{3v} = \{E\} + \{C_3, C_3^2\} + \{\sigma_v, \sigma_v', \sigma_v''\}$

$$(C_3)(C_3) = \{C_3^2, E, E, C_3\} = 2(E) + (C_3)$$

$$(C_3)(\sigma_v) = 2(\sigma_v)$$

Def: (class const.) $\displaystyle (g_i)(g_j) = \sum_{(g_k)} c_{ij}^k (g_k)$

- sum runs over all N_c distinct classes
- c_{ij}^k are class constants
- it is consistent:

$$g(g_i)(g_j)g^{-1} = g(g_i)g^{-1}g(g_j)g^{-1} = (g_i)(g_j) = /\text{lemma}/$$

$$= \sum_{(g_k)} a_k (g_k)$$

Lemma: Let (ρ^μ, V) be IRREP of a gr. G on a d_μ-dim vect. space. Then

$$m_i m_j \chi^\mu(g_i) \chi^\mu(g_j) = d_\mu \sum_{(g_k)} c_{ij}^k m_k \chi^\mu(g_k)$$

with $m_i = \#(g_i)$.

Proof: • $\displaystyle A_k = \sum_{h \in (g_k)} D^\mu(h)$ & $g(g_k) = (g_k)g$ (! not element-by-el.) $\Downarrow \sum_h$

$$\Rightarrow D^\mu(g) A_k = A_k D^\mu(g)$$

$\Rightarrow /SL2/ \Rightarrow A_k = \lambda \mathbb{1}$ for D^μ matrix IRREP

• $\displaystyle \operatorname{Tr} A_k = \lambda d_\mu = m_k \chi^\mu(g_k) \Rightarrow A_k = \frac{m_k}{d_\mu} \chi^\mu(g_k) \mathbb{1}$

$$\cdot\ (g_i)(g_j) = \sum_k C_{ij}^k (g_k) \Rightarrow A_i A_j = \sum_k C_{ij}^k A_k$$

$$= \sum_k C_{ij}^k \frac{m_k}{d_\mu} \chi^\mu(g_k) \mathbb{1}$$

$$\Rightarrow Tr(A_i A_j) = \frac{m_i}{d_\mu}\frac{m_j}{d_\mu} \chi^\mu(g_i)\chi^\mu(g_j)\, \cancel{Tr(\mathbb{1})} = \sum_k C_{ij}^k \frac{m_k}{d_\mu} \chi^\mu(g_k) \cancel{Tr(\mathbb{1})}$$

$\boxtimes$

<u>Theorem XXII</u>: Number of non-equivalent IRREPs of a finite group is equal to the number of distinct classes.

<u>Proof</u>: • N_R – #IRREPs, N_C – number of classes; $m_k = \#(g_k)$

1, $\sum_{(g_k)} m_k \chi^\mu(g_k)^* \chi^\nu(g_k) = \#G\, \delta_{\mu\nu} \Rightarrow N_R \le N_C$

2, Lemma $\Rightarrow N_C \le N_R$:

• $m_i m_j \chi^\mu(g_i)\chi^\mu(g_j) = d_\mu \sum_{(g_k)} C_{ij}^k m_k \chi^\mu(g_k)$ for ρ^μ IRREP

• reg. repre $\Rightarrow \chi^r(g_k) = \delta_{k1} \#G = \sum_\alpha d_\alpha \chi^\alpha(g_k)$; $(g_1) = (e)$

$$\Rightarrow \boxed{m_i m_j \sum_\mu \chi^\mu(g_i)\chi^\mu(g_j)} = \sum_{(g_k)} C_{ij}^k m_k \delta_{k1} \#G = \underline{C_{ij}^1 \#G}$$

• what is C_{ij}^1 ?

$$(g_i)(g_j) = C_{ij}^1 (e) + \sum_{k=2}^{N_C} C_{ij}^k (g_k)$$

$\Rightarrow C_{ij}^1(e) = m_i \delta_{ij'}$ where $(g_{j'})$ is the class of elements inverse to (g_j)

• $\underline{m_j = m_{j'}}$: $a \sim b \in (g_j)$, $a^{-1} \in (g_{j'})$ $\Rightarrow b = g a g^{-1}$

$\Rightarrow b^{-1} = g a^{-1} g^{-1} \Rightarrow b \in (g_{j'})$

• $\chi^\mu(g_{j'}) = \chi^\mu(g_j)^*$ from equivalence with an unitary repre

$$\Rightarrow u_i u_j \sum_\mu \chi^\mu(g_i)^* \chi^\mu(g_j) = u_j \#G \delta_{ij}$$

$$\Rightarrow \sum_\mu \chi^\mu(g_i)^* \chi^\mu(g_j) \sim \delta_{ij} \qquad \Rightarrow N_C \le N_R \quad \Rightarrow N_C = N_R$$

∎

• orthogonality of N_C vectors of length N_R

Theorem XXIII (Frobenius irreducibility criterion)

Representation (ρ, V) of a finite group is irreducible

$$\Leftrightarrow \sum_g \chi(g)^* \chi(g) = \sum_{(g_u)} u_u \chi(g_u)^* \chi(g_u) = \#G$$

Proof: • $\rho = \bigoplus_\mu \alpha_\mu \rho^\mu \Rightarrow \alpha_\mu = \frac{1}{\#G} \sum_g \chi^\mu(g)^* \chi(g) \in \mathbb{N}_0$!

$$\Rightarrow \chi(g) = \alpha_\mu \chi^\mu(g)$$

$$\Rightarrow \sum_g \chi(g)^* \chi(g) = \sum_{\mu\nu} \alpha_\mu \alpha_\nu \sum_g \chi^\mu(g)^* \chi^\nu(g) = \#G \sum_\mu \alpha_\mu^2$$

• for $\rho = \rho^\nu$ IRREP is $\alpha_\mu = \delta_{\mu\nu} = \sum_\mu \alpha_\mu^2 = 1$

• for ρ reducible is $\sum_\mu \alpha_\mu^2 > 1$ ∎