Homework #5

Assigned: 23.12.2021 **Due: 5.1.2022**

Homomorphism $SL(2,\mathbb{C})$ onto $L_+^\uparrow$

Let ϕ be the homomorphism

$$SL(2,\mathbb{C}) \to L_+^\uparrow$$

constructed in the class. The homomorphism is based on the one-to-one association of a four-vector from Minkowski space with a Hermitean 2×2 matrix,

$$x = (x_0, x_1, x_2, x_3)^T \quad \leftrightarrow \quad X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

Action of $SL(2,\mathbb{C})$ on the space of Hermitean matrices (and, therefore, Minkowski space) is then defined through

$$X \longmapsto \tilde{X} = AXA^\dagger, \qquad A \in SL(2,\mathbb{C}).$$

For more details, see the lecture notes.

1. *(3 points)* Find the matrix $A \in SL(2,\mathbb{C})$ that is mapped to the rotation around the $\langle x \rangle$-axis by an angle α.

2. *(5 points)* Show that the kernel of the homomorphism is $\mathrm{Ker}_\phi = \{\mathbb{1}, -\mathbb{1}\}$. Don't forget to prove that there is no other matrix in $SL(2,\mathbb{C})$ that is mapped to identity transformation.

3. *(5 points)* Show that the preimage of every element of $L_+^\uparrow$ contains exactly 2 elements and, therefore, the ϕ represents double covering. *Hint: Use the fact that* Ker_ϕ *contains two elements.*

Group $GA(1,\mathbb{R})$ reloaded

1. *(7 points)* Find the Killing-Cartan metric for the $GA(1,\mathbb{R})$ group and compute its eigenvalues.