# ***Group theory and its aplication in physics***

Exam requirements

The first part of the course (general theory, representations of finite groups) will be tested including the majority of the proofs. For Lie groups, emphasis is put on the understanding of the concepts and relations. The examination is oral only. In the case of an ambiguous classification, a simple computational problem can be given as an additional question (usually chosen among the derivations left as an exercise).

## Basic terms

- definition of a group, "rearrangement theorem (proof), subgroup", "intersection of two subgroups is a subgroup (proof)"
- left/right cosets, Lagrange theorem (proof), conjugacy classes and their properties
- normal subgroup, "normal subgroups consists of complete classes (proof)", factor group
- direct product group – definition, theorem about its subgroups (proof)
- group mappings – homomorphism, izomorphism, kernel an image, "$\mathrm{Im}\,\varphi$ is subgroup $G'$, $\mathrm{Ker}\,\varphi$ is normal subgroup $G$ and $\mathrm{Im}\,\varphi \sim G/\mathrm{Ker}\,\varphi$"
- group action on a set, orbit, stabilizer, "number of elements in an orbit times number of elements of the stabilizer equals the order of the group (proof)", left and right translation, conjugation

## Representations of a finite groups

- representation of a group as an action on a vector space, matrix representations, equivalent representations
- reducibility and irreducibility, subrepresentation, symmetrization operators
- "Every irreducible representation of a finite group is finite-dimensional (proof)"
- unitary representation, "Every finite-dimensional unitary reducible representation is completely reducible (proof)", "Every finite-dimensional reducible representation of a finite or compact Lie group is completely reducible (proof)"
- Schur lemmas (proof)
- "complex finite-dimensional irreducible representations of an abelian group are one-dimensional (proof)"
- orthogonality relations for matrix representations (proof)

- character of a representation, orthogonality relation for characters (proof), "for a finite group, the equality of characters is sufficient condition for the equivalence of two representations (proof)"
- number of inequivalent irreducible representations of finite groups, dimensions of irreducible representations, Frobenius irreducibility criterion (proof)
- relations between representations of a group and its subgroups (subduction, induction)
- direct product representations and its characters, Clebsh-Gordan series, C-G coefficients
- Wigner-Eckart theorem (proof), selection rules for physical observables/processes

## Symmetries in Quantum Mechanics

- action of a group on a Hilbert space, transformation of operators
- classification of eigenvalues and eigenfunctions of the operators according to irreducible representations of the symmetry group of the Hamiltonian
- degeneracy of the energy levels, splitting of energy levels in the presence of a perturbation
- activity of optical transitions

## Symmetric group

- cycles, composition of cycles, classes
- irreducible representations: Young diagram and tableau, subduction chain $S_n \downarrow S_{n-1} \downarrow \ldots \downarrow S_1$

## Lie groups

- Lie group as a smooth manifold, linear Lie group
- Real Lie algebra of a Lie groups (matrix and geometrical formulation), matrix groups and their algebras
- one-parameter subgroup, exponential mapping, derived homomorphism fo Lie algebras
- "exponential mapping is a local diffeomorphism between the tangent space $T_eG$ and the neigborhood of identity $U(e) \subset G$ in the Lie group (proof principle)", "every element from the connected subgroup of a Lie group $G$ can be expressed as a finite product of exponential elements"

- relationship between Lie groups with isomorphic Lie algebras, universal covering group, double covering of $SO(3)$ by $SU(2)$

**Representations of Lie algebras (and groups)**

- analytical representation of a Lie group, representation of a Lie algebra, relationship between analytical matrix representation of LG and corresponding representation of LA
- adjoint representation of LA, Killing-Cartan metric, Cartan criterion for semi-simple Lie algebras/groups
- complexification of a real Lie algebra, relationship between representations of real and complexified Lie algebras (proof)
- irreducible representations of semi-simple Lie algebras – Casimir operator, Racah theorem
- direct sum of Lie algebras, "Lie algebra of a direct-product group is isomorphic to the direct sum of Lie algebras of its subgroups"