DODATEK: ZÁKLADNÍ MATEMATICKÉ "TRIKY"

1, Implicitní funkce

$f(x,y,z)=0$ ... implicitní funkční vztah mezi proměnnými x,y,z:

$$x = x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \qquad (1)$$

$$z = z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy \qquad (2)$$

- dosadíme $(2) \to (1)$

$$\left[\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y - 1\right] dx + \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] dy = 0$$

- dx, dy nezávislé $\Rightarrow$

$$\boxed{\begin{aligned} &a)\ \left(\frac{\partial x}{\partial z}\right)_y = \left[\left(\frac{\partial z}{\partial x}\right)_y\right]^{-1} \\ &b)\ \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\frac{\partial z}{\partial y}\right)_x}{\left(\frac{\partial z}{\partial x}\right)_y} \end{aligned}}$$

2, Úplný diferenciál funkcí více proměnných

- $f = f(x_1, \dots, x_n)$ & $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \Leftrightarrow \oint df = 0$$

- integrační faktor:

$dw = \sum_i A_i(x_1, \dots, x_n) dx_i$ není úplný diferenciál

$$\Rightarrow \frac{\partial A_i}{\partial x_j} \neq \frac{\partial A_j}{\partial x_i}$$

- může $\exists\, \mu = \mu(x_1, \dots, x_n)$: $\dfrac{\partial(\mu A_i)}{\partial x_j} = \dfrac{\partial(\mu A_j)}{\partial x_i}$

$\Rightarrow d\sigma = \mu\, dw$ je úplný dif. a $\exists\, \sigma = \sigma(x_1, \dots, x_n)$

! $d\sigma = 0 \Leftrightarrow dw = 0$!

Příklad: $(x^2+y)\,dx - x\,dy = 0$

$$\frac{\partial}{\partial y}(x^2+y) = 1 \neq \frac{\partial}{\partial x}(-x) = -1$$

$$\frac{\partial}{\partial y}\left[\mu(x^2+y)\right] = \frac{\partial \mu}{\partial y}(x^2+y) + \mu = \frac{\partial}{\partial x}\left(-\mu x\right) = -x\frac{\partial \mu}{\partial x} - \mu$$

$\Rightarrow$ mohu volit $\mu = \mu(x)$ $\Rightarrow$ $-x\dfrac{d\mu}{dx} = 2\mu$

$$\Rightarrow \frac{d\mu}{\mu} = -2\frac{dx}{x} \Rightarrow \log\mu = -2\log x \Rightarrow \mu = \frac{1}{x^2}$$

$$\Rightarrow \boxed{d\sigma = \left(1+\frac{y}{x^2}\right)dx - \frac{1}{x}\,dy}$$

integrace

1, $\int dy$ $\Rightarrow$ $\sigma = -\dfrac{y}{x} + \tilde{C}(x)$ $\Rightarrow \left/\dfrac{\partial}{\partial x}\right/ \Rightarrow \dfrac{\partial \sigma}{\partial x} = \dfrac{y}{x^2} + \dfrac{d\tilde{C}}{dx}$

$$\Rightarrow \frac{y}{x^2} + \frac{d\tilde{C}}{dx} = 1 + \frac{y}{x^2} \Rightarrow \frac{d\tilde{C}}{dx} = 1 \Rightarrow \tilde{C} = x + C$$

oprava konstanta

$$\Rightarrow \boxed{\sigma = x - \frac{y}{x} + C}$$

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3) Jacobiho determinanty

$$\frac{\partial(u, v, \dots, w)}{\partial(x, y, \dots, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \dots & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \dots & \\ \vdots & & \ddots & \\ \frac{\partial w}{\partial x} & \dots & \dots & \frac{\partial w}{\partial z} \end{vmatrix}$$

- $u, v, \dots, w$ jsou funkce $x, y, \dots, z$

- parciální derivace: $\left(\dfrac{\partial u}{\partial x}\right)_{y,\dots,z} = \dfrac{\partial(u, y, \dots, z)}{\partial(x, y, \dots, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \dots & \frac{\partial u}{\partial z} \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & \dots & 1 \end{vmatrix}$

- řetízkové pravidlo: $\dfrac{\partial(u, v, \dots, w)}{\partial(x, y, \dots, z)} = \dfrac{\partial(u, v, \dots, w)}{\partial(r, s, \dots, t)} \dfrac{\partial(r, s, \dots, t)}{\partial(x, y, \dots, z)}$

– řetízkové pravidlo $\frac{\partial f_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \frac{\partial f_i}{\partial x_j}$ je "součin matic"

a platí $A = B\cdot C \Rightarrow |A| = |B||C|$

• přímý důsledek: $\frac{\partial(x,y,\dots,z)}{\partial(u,v,\dots,w)} = \left[\frac{\partial(u,v,\dots,w)}{\partial(x,y,\dots,z)}\right]^{-1}$

<u>příklad:</u>

$$\left(\frac{\partial S}{\partial T}\right)_{V,N} = \frac{\partial(S,V,N)}{\partial(T,V,N)} = \frac{\partial(S,V,N)}{\partial(T,p,N)}\,\frac{\partial(T,p,N)}{\partial(T,V,N)} =$$

$$= \begin{vmatrix} \left(\frac{\partial S}{\partial T}\right)_{p,N} & \left(\frac{\partial S}{\partial p}\right)_{T,N} \\ \left(\frac{\partial V}{\partial T}\right)_{p,N} & \left(\frac{\partial V}{\partial p}\right)_{T,N}\end{vmatrix} \left(\frac{\partial p}{\partial V}\right)_{T,N} \overset{\text{N už nepíšu}}{=} \left[\left(\frac{\partial S}{\partial T}\right)_p \left(\frac{\partial V}{\partial p}\right)_T - \left(\frac{\partial S}{\partial p}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\right]\left(\frac{\partial p}{\partial V}\right)_T$$

$$= \left(\frac{\partial S}{\partial T}\right)_p - \left(\frac{\partial S}{\partial p}\right)_T \frac{\left(\frac{\partial V}{\partial T}\right)_p}{\left(\frac{\partial V}{\partial p}\right)_T} = \left(\frac{\partial S}{\partial T}\right)_p + \left(\frac{\partial S}{\partial p}\right)_T\left(\frac{\partial p}{\partial T}\right)_V$$

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4) <u>Homogenní funkce</u> p-tého řádu

$$F(\lambda x_1,\dots,\lambda x_k) = \lambda^p F(x_1,\dots,x_k)$$

$$\Rightarrow p\lambda^{p-1}F(x_1,\dots,x_k) = \frac{dF}{d\lambda} = \sum_j \frac{\partial F}{\partial(\lambda x_j)}\frac{\partial(\lambda x_j)}{\partial\lambda} = \sum_j \frac{\partial F}{\partial(\lambda x_j)}x_j$$

$$\Rightarrow /\lambda = 1/ \Rightarrow \boxed{pF(x_1,\dots,x_k) = \sum_j \frac{\partial F}{\partial x_j}x_j}$$ Eulerův teorém