CZECHOSLOVAK JOURNAL OF PHYSICS

Volume 54 · 2004 Number 3 · pp. 273–388

Covariant derivative on non-linear fiber bundles

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Received 4 August 2003

A gauge field is usually described as a connection on a principal bundle. It induces a covariant derivative on associated vector bundles, sections of which represent matter fields. In general, however, it is not possible to define a covariant derivative on non-linear fiber bundles, i.e. on those which are not vector bundles. We define *logarithmic covariant derivatives* acting on two special non-linear fiber bundles — on the principal bundle and on the local gauge group bundle. The logarithmic derivatives map from sections of these bundles to the sections of the local gauge algebra bundle. Some properties of the logarithmic derivatives are formulated.

PACS: 02.40.Hw, 11.15.–q, 03.50.Kk

Key words: differential geometry, gauge fields

1 Motivation and summary

The standard geometrical representation of a gauge field is a covariant derivative on a vector bundle over a spacetime (cf. [1,2]). Sections of such a bundle describe a matter field which interacts with other matter through the gauge field. However, because the gauge field can interact with different matter fields, it is necessary to have *the same* covariant derivatives on different vector bundles.

A natural way to define the same covariant derivatives on different vector bundles is to start with a principal bundle $\mathbf{P}M$ over spacetime M with a structure (gauge) group $\mathbf{G}$ (which represents a gauge symmetry of a theory). The gauge field is then represented by a connection on the principal bundle. Vector bundles of matter fields are defined as fiber bundles associated with the principal bundle. The connection on the principal bundle allows to define covariant derivatives on these vector bundles. These are used to construct the Lagrangian of matter fields interacting with the gauge field.

The only problem of this construction is the necessity of manipulating objects from tangent spaces of the principal bundle. The usual mathematical description of the connection on the principal bundle is the connection form ω (which defines horizontal subspaces in the tangent space of the principal bundle), and the "strength" of the gauge field is the curvature form Ω. The connection form is a gauge algebra

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valued 1-form on the principal bundle $\mathbf{P}M$ and the curvature form is a gauge algebra valued 2-form on $\mathbf{P}M$. However, it is not very common in physics to work with objects which live on such complicated spaces. Usually one works with fields (tensor valued functions or sections of vector bundles) on spacetime or with covariant derivatives in vector bundles with spacetime as a base manifold.

A standard way how to avoid working with objects from tangent space of $\mathbf{P}M$ is a trivialization — a choice of a section of $\mathbf{P}M$, i.e. a choice of coordinates in the principal bundle and in associated fiber bundles. Then all objects can be represented as matrix or vector valued functions on spacetime. However, since there is no canonical choice of trivialization, it is an input alien to the geometrical description of the theory. From the geometrical point of view, we would like to avoid any specific choice of coordinates.

Fortunately, for a semisimple structure group[1]), it is possible to represent the connection on the principal bundle by a covariant derivative on the local gauge algebra bundle $\mathbf{g}M$ and the strength of the gauge field by a curvature tensor $\mathcal{F}$ — a $\mathbf{g}M$-valued tensor field on spacetime (see Definition 3). Hence, it is possible to substitute the covariant derivative for the connection form and the curvature tensor for the curvature form. Therefore, it is not necessary to work with objects defined on the principal bundle but only with objects localized on spacetime.

However, there are still some operations and quantities where it is necessary to use a trivialization. A typical example are the transformation properties of the gauge field under a gauge transformation. This is usually written as[2])

$$\tilde{A}_{\boldsymbol{\alpha}}{}^{\mathfrak{m}}{}_{\mathfrak{n}} = A_{\boldsymbol{\alpha}}{}^{\mathfrak{m}}{}_{\mathfrak{n}} - g^{-1}{}^{\mathfrak{m}}{}_{\mathfrak{p}}\, \mathrm{d}_{\boldsymbol{\alpha}} g^{\mathfrak{p}}{}_{\mathfrak{n}} \; . \tag{1}$$

Here $A_{\boldsymbol{\alpha}}{}^{\mathfrak{m}}{}_{\mathfrak{n}}$, $\tilde{A}_{\boldsymbol{\alpha}}{}^{\mathfrak{m}}{}_{\mathfrak{n}}$ are vector potentials describing the gauge field before and after a gauge transformation, expressed in a trivialization, i.e., represented as matrices on the gauge algebra $\mathbf{g}$ with a spacetime index $\boldsymbol{\alpha}$ and algebra indices $\mathfrak{m},\mathfrak{n}$. $g^{\mathfrak{m}}{}_{\mathfrak{n}}$ is a gauge group element represented as a matrix on the gauge algebra, which defines the gauge transformation, and d is the usual gradient of a (matrix valued) function on the spacetime M. However, we would like to avoid choosing the trivialization, i.e., we would like to work with abstract objects — with the element of the local gauge group $g \in \mathrm{Sect}\,\mathbf{G}M$, and with the vector potential as an element of local gauge algebra $\mathcal{A} \in \mathrm{Sect}\,\mathbf{g} \otimes \mathbf{T}^*M$ instead of their matrix representation. Unfortunately, we cannot write an abstract relation like

$$\tilde{A}_{\boldsymbol{\alpha}} = A_{\boldsymbol{\alpha}} - g^{-1}\, \mathcal{D}_{\boldsymbol{\alpha}} g \; , \tag{2}$$

because a covariant derivative $\mathcal{D}_{\boldsymbol{\alpha}} g$ on the local gauge group bundle $\mathbf{G}M$ is not defined. The connection on the principal bundle allows us to define the covariant

[1]) The semisimplicity of the structure group $\mathbf{G}$ is necessary to guarantee that the adjoint representation of the structure group on the gauge algebra $\mathbf{g}$ is faithful. Another standard case is a commutative structure group (e.g. a Maxwell field) where the local gauge algebra bundle $\mathbf{g}M$ is trivial. However, it is always possible to describe the connection as a covariant derivative on a properly chosen vector bundle with faithful representation of the structure group.

[2]) Any constant factor (a charge or maybe the imaginary unit in the case of complex representation of a unitary group) which is often written in front of the vector potential is included into the vector potential here.

derivative only on a fiber bundle with a linear structure (see Definition 1), i.e., only when the standard bundle is a vector space. However, in the case of the local gauge group bundle $\mathbf{G}M$, the standard fiber is the structure group $\mathbf{G}$. Therefore, we have to use the adjoint representation and trivialization to represent the structure group as matrices on the gauge algebra. Then, we can use the gradient to take a derivative of the local gauge group element g.

Another way often used to write the transformation properties of the gauge field is

$$\tilde{\mathcal{A}}^{\mathtt{m}}_{\boldsymbol{\alpha}} \approx \mathcal{A}^{\mathtt{m}}_{\boldsymbol{\alpha}} - \mathrm{d}_{\boldsymbol{\alpha}} a^{\mathtt{m}} \ . \tag{3}$$

Here ${\mathcal{A}_{\boldsymbol{\alpha}}}^{\mathtt{m}}$, ${\tilde{\mathcal{A}}_{\boldsymbol{\alpha}}}^{\mathtt{m}}$ are gauge algebra valued vector potentials[3]) ($\mathbf{g}$-valued 1-forms on the spacetime) and $a^{\mathtt{m}}$ is a generator of the gauge transformation (i.e., $g = \exp(a)$). This relation is generally valid only to the first order in a. It is usually called the "infinitesimal gauge transformation".

The aim of this paper is to define the *logarithmic covariant derivative on the local gauge group bundle* $\mathbf{G}M$ (Definition 5) — a tool that allows us to write the relations above without specifying coordinates and without any approximation — see Lemma 12 and Theorem 9. The logarithmic covariant derivative $\mathcal{D}\ln g$ acts on sections of the local gauge group bundle $\mathbf{G}M$ and its result is a section of the local gauge algebra bundle $\mathbf{g}M$.

We will also define the *logarithmic covariant derivative on the principal bundle* $\mathbf{P}M$ (Definition 4) which allows a generalization of the following construction[4]): It is often possible to represent the principal bundle as a space of frames in some vector bundle, typically in the local gauge algebra bundle $\mathbf{g}M$. That is, an element $E \in \mathbf{P}M$ can be represented as $E = \{\theta_{\mathtt{m}}\}_{\mathtt{m}=1\ldots N}$, where $\theta_{\mathtt{m}} \in \mathbf{g}M$ are linearly independent, and N is the dimension of the gauge algebra. The trivialization means a choice of a section $E \in \operatorname{Sect}\mathbf{P}M$, i.e. a choice of base vectors in all fibers of $\mathbf{g}M$. The vector potential A of the covariant derivative can then be written as

$$A_{\boldsymbol{\alpha}} = \big(\mathcal{D}_{\boldsymbol{\alpha}}\theta_{\mathtt{m}}\big)\theta^{\mathtt{m}} \ , \tag{4}$$

$\theta^{\mathtt{m}}$ being the dual base in $\mathbf{g}^*M$[5]). However, we do not always wish or can represent the principal bundle as a space of frames and we usually want to work with the vector potential $\mathcal{A}$ from $\mathbf{g}\otimes\mathbf{T}^*M$, not with its adjoint representation A. With the help of the logarithmic covariant derivative on the principal bundle introduced in this paper we will be able to write the equivalent of Eq. (4) in a general case (cf. expression (19)). The logarithmic covariant derivative $\mathcal{D}\ln E$ acts on sections of the principal bundle $\mathbf{P}M$ and gives as a result a section of the local gauge algebra bundle $\mathbf{g}M$.

The plan of the paper is the following. In Section 2 we shortly review the geometrical setting of the gauge field theory. There are no new results here — the

[3]) Matrices A with antisymmetric multiplication given by commutator are adjoint representation of $\mathcal{A}$ with multiplication given by Lie bracket of the gauge algebra.

[4]) See also the discussion of this construction in the case of the principal bundle of the tangent bundle in the next section.

[5]) Of course, the tensor multiplication is assumed here — $\theta_{\mathtt{m}}$ are vectors from $\mathbf{g}M$, $\theta^{\mathtt{m}}$ are forms from $\mathbf{g}^*M$, and the vector potential is from $\mathbf{g}^1_1\otimes\mathbf{T}^*M$.

main aim of this section is to fix the notation and to review facts needed later. The logarithmic covariant derivatives on the principal and on the local gauge group bundles are defined in Section 3. Some important properties of these derivatives are also mentioned there — mainly generalizations of the chain rule, the relation between two different logarithmic covariant derivatives, and the expressions for the curvature tensor in terms of the logarithmic derivatives. The proofs of the lemmas and theorems of Section 3 are relegated to the Appendix.

All constructions in this article are local; we will not study global topological properties of the principal bundle and their consequences for the derivatives defined. The definition of logarithmic derivatives does not introduce new issues in this area. Due to the essential locality of all definitions, we will not refer to any transformation properties under change of a map from an atlas of the spacetime manifold or the principal bundle. All this can be done in standard way. Similarly, we will not discuss issues of differentiability — we assume sufficient smoothness of all objects under consideration.

The logarithmic derivatives on the principal and local gauge group bundles do not bring any essentially new information that could not, in some manner, be expressed without using them. However, they are useful tools when expressing certain quantities in a natural, coordinate-free geometrical way. They are a natural generalization of the notion of covariant derivative in case of non-linear fiber bundles. This generalization has its limitations — it works only for the case of two special non-linear bundles that are closely related with a vector bundle, namely, with the local gauge algebra bundle $\mathbf{g}M$. The logarithmic derivatives have a number of useful properties formulated in Theorems 6, 9–11, in Lemma 12, and in Eq. (19), that extend the standard covariant derivative calculus to the new area. An application of this formalism can be found, e.g., in the proofs of Theorem 10(ii), Theorem 11, and Lemma 12.

2 Geometry of gauge fields

Fiber bundles

Fiber bundles are standard kinematical area for geometrical description of gauge and matter fields. A gauge field is represented by a connection on the principal bundle $\mathbf{P}M$ with a structure group $\mathbf{G}$ and matter fields are sections of the associated vector bundles. Gauge symmetry is invariance of the theory under a "rotation" of the inner degrees of freedom, i.e. under the action of the local gauge group.

In this section, we shortly review some basic facts from the theory of fiber bundles — to fix the notation and for later references. For details see, e.g., [3].

We denote the tangent space of a manifold X by $\mathbf{T}X$, the tangent tensor space $\mathbf{T}^k_l X$, and the space of its sections $\mathfrak{T}^k_l X$. We use abstract indices (bold letters; see, e.g., [4]) to indicate tensor character of tangent tensors and coordinate indices (plain letters) to label components of a tensor in a special frame. However, we try to avoid the use of indices; instead, we use a dot to indicate contraction between vectors and forms.

Fiber bundles describing the inner degrees of freedom are built above a spacetime manifold M. We use Greek indices for tensors from the tangent space $\mathbf{T}M$ and we denote a contraction by dot "$\cdot$", i.e., $a{\cdot}\omega = a^{\boldsymbol{\mu}}\omega_{\boldsymbol{\mu}}$.

The *principal bundle* is denoted by $\mathbf{P}M$, and the bundle projection by π. The intuitive meaning of a fiber $\mathbf{P}_x M$ of the principal bundle is the space of "frames" in the space of inner degrees of freedom at a base manifold point x. We will use letters $E, F, \ldots$ for elements of $\mathbf{P}M$. The structure group $\mathbf{G}$ acts on the principal bundle from the right: e.g., $F = \mathrm{R}_{\bar{g}}\, E = E\bar{g}$. It corresponds to a "change of frame". We assume that $\mathbf{G}$ is a semisimple Lie group and thus an adjoint representation is faithful.

If a manifold $\mathbf{A}$ with an action T of the structure group $\mathbf{G}$ is given (i.e., $\mathrm{T} : \mathbf{G} \times \mathbf{A} \to \mathbf{A}$, $\mathrm{T}_{\bar{g}\bar{h}} = \mathrm{T}_{\bar{g}}\,\mathrm{T}_{\bar{h}}$), we can define an *associated fiber bundle* with the standard fiber $\mathbf{A}$. Intuitively, this is a space of objects ϕ with "coordinates" $\bar{\phi}$ from the standard fiber $\mathbf{A}$ taken with respect to a frame E from the principal bundle $\mathbf{P}M$. We will write[6])

$$\phi = \mathrm{T}_E\, \bar{\phi}\ , \quad \bar{\phi} = \phi[E]\ . \tag{5}$$

The coordinates change by the action T under a change of frame, i.e.,

$$\phi = \mathrm{T}_E\, \bar{\phi} = \mathrm{T}_{E\bar{g}}\, \mathrm{T}_{\bar{g}^{-1}}\, \phi\ , \quad \phi[E\bar{g}] = \mathrm{T}_{\bar{g}^{-1}}\, \phi[E]\ . \tag{6}$$

We call such an associated fiber bundle $(\mathrm{T}_{\mathbf{P}}\mathbf{A})M$, or shortly $\mathbf{A}M$.

Let us mention some special cases of the associated fiber bundles. First, the principal bundle $\mathbf{P}M$ itself can be viewed as an associated bundle with the structure group $\mathbf{G}$ as the standard fiber and with left multiplication as the action on it, i.e., $\mathbf{P}M \cong (\mathrm{L}_{\mathbf{P}}\mathbf{G})M$. Here, we identify

$$\mathrm{L}_E\, \bar{g} = \mathrm{R}_{\bar{g}}\, E\ , \tag{7}$$

i.e., $F[E] = \bar{g}$ if $F = E\bar{g}$. The *fundamental vector field* $\mathfrak{f}_{\bar{a}} \in \mathfrak{T}\,\mathbf{P}M$ on the principal bundle $\mathbf{P}M$ associated with an element $\bar{a} \in \mathbf{g}$ of the gauge algebra (Lie algebra of the structure group $\mathbf{G}$) is defined by

$$\mathfrak{f}_{\bar{a}}\big|_E = \frac{D}{d\alpha}\,\mathrm{R}_{\bar{g}_\alpha}\, E\Big|_{\alpha=0}\ , \quad \text{where} \quad \bar{a} = \frac{D\bar{g}_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0}\ , \quad \bar{g}_0 = \bar{e}\ . \tag{8}$$

The *fundamental form* θ on the principal bundle $\mathbf{P}M$ at a point $E \in \mathbf{P}_x M$ is a linear mapping from the tangent space of the fiber $\mathbf{P}_x M$ to the gauge algebra $\mathbf{g}$ "inverse" to the fundamental vector field:

$$\theta \bullet \mathfrak{f}_{\bar{a}} = \bar{a}\ . \tag{9}$$

Here, the bold dot "$\bullet$" represents contraction in the tangent space to the fiber $\mathbf{P}_x M$. We use it also for contraction in the tangent space to the entire principal bundle $\mathbf{P}M$.

[6]) Here we use the same symbol T as for the action of the group $\mathbf{G}$ on the standard fiber $\mathbf{A}$. These are different operations, of course, but our notation is justified by, for example, the formal associativity $\mathrm{T}_{E\bar{g}}\, \bar{\phi} = \mathrm{T}_E(\mathrm{T}_{\bar{g}}\, \bar{\phi})$ — see (6).

Another example of an associated fiber bundle is the *local gauge group bundle* $\mathbf{G}M$. Its standard fiber is again the structure group $\mathbf{G}$ but now with the adjoint representation Ad as the action: $\mathbf{G}M = (\mathrm{Ad}_{\mathbf{P}}\mathbf{G})M$. Each fiber of the local gauge group bundle has a group structure — group operations are:

$$gh = T_E(g[E]h[E]) \;, \quad g^{-1} = T_E g[E]^{-1} \;, \quad e = T_E \bar{e} \;. \tag{10}$$

It is easy to check that this is independent on the choice of $E \in \mathbf{P}M$. This bundle is called the local gauge group bundle because its sections form the local gauge group. The whole theory should be invariant under the action of this group. The action of the local gauge group on any associated fiber bundle $(\mathrm{T}_{\mathbf{P}}\mathbf{A})M$ is defined as

$$\mathrm{T}_g\, \phi = \mathrm{T}_E(\mathrm{T}_{g[E]}\, \phi[E]) \;. \tag{11}$$

Again, it does not depend on the choice of E. As a special case, we have "left" action of $\mathbf{G}M$ on $\mathbf{P}M$ (we write $\mathrm{L}_g\, E = gE$) and the adjoint representation Ad of $\mathbf{G}M$ on itself.

Next example of an associated fiber bundle is the *local gauge algebra bundle* $\mathbf{g}M$. The standard fiber $\mathbf{g}$ is the Lie algebra of the structure group $\mathbf{G}$, which we identify with the tangent space $\mathbf{T}_e\mathbf{G}$ at the unit element e. The action of the structure group on $\mathbf{G}$ is adjoint representation ad, i.e., $\mathbf{g}M = (\mathrm{ad}_{\mathbf{P}}\mathbf{g})M$. Again, each fiber $\mathbf{g}_x M$ has the structure of a Lie algebra, where the Lie brackets and exponential map are defined by

$$[a, b] = \mathrm{ad}_E[a[E], b[E]] \;, \quad \exp a = \mathrm{ad}_E \exp(a[E]) \;. \tag{12}$$

Contraction in the algebra bundle will be denoted by a square dot "$\centerdot$".

Finally, we mention the case of the standard fiber $\mathbf{E}$ having a linear structure and the action of the structure group being a representation. In this case, in addition to the representation T of $\mathbf{G}M$, we can also define a representation t of the local gauge algebra $\mathbf{g}M$. Let T be the representation of the structure group $\mathbf{G}$ on the linear standard fiber $\mathbf{E}$ and t its generator — the representation of the Lie algebra $\mathbf{g}$:

$$\mathrm{T}_{\exp \bar{a}} = \exp(\mathrm{t}_{\bar{a}}) \;. \tag{13}$$

Then the representation of $\mathbf{g}_x M$ on $\mathbf{E}_x M$ is given by

$$\mathrm{t}_a \phi = \mathrm{T}_E\, \mathrm{t}_{a[E]}\, \phi[E] \;. \tag{14}$$

The dependence of t_a is linear in a; thus we can write $\mathrm{t}_a = a \centerdot \mathrm{t}$ with $\mathrm{t} \in \mathbf{E}^1_1 \otimes \mathbf{g}^* M$.

The principal bundle $\mathbf{F}M$ of tangent spaces of the base manifold (spacetime) M can be viewed as a space of vector bases in $\mathbf{T}M$, i.e., we can write $E = \{e^{\boldsymbol{\alpha}}_{\mu}\}_{\mu=1,\ldots,d}$. The structure group is the group $\mathbf{GL}(d)$ of nondegenerated matrices, the local gauge group bundle $\mathbf{GL}\, M$ can be represented as a space of nondegenerated (1,1)-tensors (i.e., $g = g^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$), and the local gauge algebra bundle as a space of all (1,1)-tensors: $\mathbf{gl}\, M = \mathbf{T}^1_1 M$. Clearly,

$$E\bar{g} = \mathrm{R}_{\bar{g}}\, E = \{e^{\boldsymbol{\alpha}}_{\nu}\bar{g}^{\nu}{}_{\mu}\}\ , \quad gE = \mathrm{L}_g\, E = \{g^{\boldsymbol{\alpha}}{}_{\boldsymbol{\kappa}} e^{\boldsymbol{\kappa}}_{\mu}\}\ , \tag{15}$$

$$(gh)^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = g^{\boldsymbol{\alpha}}{}_{\boldsymbol{\kappa}} h^{\boldsymbol{\kappa}}{}_{\boldsymbol{\beta}}\ , \quad [a,b]^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = a^{\boldsymbol{\alpha}}{}_{\boldsymbol{\kappa}} b^{\boldsymbol{\kappa}}{}_{\boldsymbol{\beta}} - b^{\boldsymbol{\alpha}}{}_{\boldsymbol{\kappa}} a^{\boldsymbol{\kappa}}{}_{\boldsymbol{\beta}}\ , \tag{16}$$

($E \in \mathbf{F}M$, $g, h \in \mathbf{GL}M$, $a, b \in \mathbf{gl}M$ and $\bar{g} \in \mathbf{GL}(d)$). Finally, tangent tensor spaces are associated bundles with the standard fiber given by a tensor power of $\mathbb{R}^d$ with the standard **GL**-action on it.

Connection

The gauge field is represented by a connection on the principal bundle. Let us recall that a *connection* $\mathbf{H}$ defines *horizontal subspaces* $\mathbf{H}_E$ of the tangent fibers $\mathbf{T}_E(\mathbf{P}M)$ that are invariant under the action of the structure group. We denote by hor and ver the projectors on the horizontal and vertical subspaces. The connection can be given by a *connection form* ω (**g**-valued 1-form on the principal bundle). The connection defines a horizontal lift ("parallel transport") in the principal bundle and it can be extended to associated bundles by the rule of keeping the coordinates constant with respect to a parallelly transported frame.

This allows us to define a covariant derivative on associated vector bundles:

Definition 1 (Covariant derivative on vector bundle)
Let $\mathbf{E}M = (\mathrm{T}_{\mathbf{P}}\mathbf{E})M$ be a vector bundle associated with the principal bundle $\mathbf{P}M$, and let a connection $\mathbf{H}$ be given. We define *covariant derivative*

$$\mathcal{D}\phi \in \mathrm{Sect}(\mathbf{E} \otimes \mathbf{T}^* M)$$

of a section $\phi \in \mathrm{Sect}\,\mathbf{E}M$ as

$$\xi{\cdot}\mathcal{D}\phi \overset{\text{def}}{=} \mathcal{D}_\xi \phi\ ,$$

$$\mathcal{D}_\xi \phi \overset{\text{def}}{=} \mathrm{T}_{E_0} \frac{d}{d\varepsilon} \phi_\varepsilon[E_\varepsilon]\Big|_{\varepsilon=0}\ .$$

$\mathcal{D}_\xi\phi$ is the *covariant derivative in a direction* $\xi \in \mathfrak{T}M$, E_ε is a horizontal lift in $\mathbf{P}M$ of a curve x_ε with tangent vector ξ, and $\phi_\varepsilon = \phi|_{x_\varepsilon}$. ∘

The principal condition here is the linear structure of the standard fiber — it is only thanks to this structure that the derivative $(d/d\varepsilon)\phi_\varepsilon[E_\varepsilon]\big|_{\varepsilon=0}$ is again an element of the standard fiber. It is not possible to use a similar definition to introduce a covariant derivative on a non-linear fiber. Only if we were able to map a result of the derivative $(d/d\varepsilon)\phi_\varepsilon[E_\varepsilon]\big|_{\varepsilon=0}$ to a standard fiber of some associated bundle (maybe a different one) we could define a covariant derivative. This will be done for the principal bundle and for the local gauge group bundle in the next section.

Later we will need the following:

Lemma 1
Let ϕ be a section of a fiber bundle $\mathbf{A}M = (\mathrm{T}_{\mathbf{P}}\,\mathbf{A})M$ and $\xi \in \mathfrak{T}M$. We can choose a curve x_ε such that $\xi = Dx_\varepsilon/d\varepsilon\big|_{\varepsilon=0}$. We denote the values of ϕ along x_ε

by ϕ_ε and a horizontal lift of x_ε in $\mathbf{P}M$ by E_ε. Then

$$\operatorname{ver}(\xi \cdot D\phi|_{x_0}) = \frac{\mathrm{D}}{\mathrm{d}\varepsilon}\big(\mathrm{T}_{E_0}\, \phi_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\ ,$$
$$\operatorname{hor}(\xi \cdot D\phi|_{x_0}) = \frac{\mathrm{D}}{\mathrm{d}\varepsilon}\big(\mathrm{T}_{E_\varepsilon}\, \phi_0[E_0]\big)\Big|_{\varepsilon=0}\ ,$$

where $D\phi$ is a differential of the map $\phi : M \to \mathbf{A}M$. □

It is well known that all connections form an affine space and that a difference between two connections can be characterized by a vector potential:

Definition 2 (Vector potential)
Let $\mathbf{H}$ and $\tilde{\mathbf{H}}$ be two connections on the principal bundle $\mathbf{P}M$ given by fundamental forms ω and $\tilde{\omega}$. The *vector potential* $\mathcal{A} \in \operatorname{Sect} \mathbf{g} \otimes \mathbf{T}^* M$ of the connection $\tilde{\mathbf{H}}$ with respect of $\mathbf{H}$ is defined by equation

$$\pi_* \mathcal{A}[E] = (\tilde{\omega} - \omega)|_E\ ,$$

where π is the projection from $\mathbf{P}M$ to the base manifold M and $E \in \operatorname{Sect} \mathbf{P}M$. We will write

$$\mathcal{A} = \tilde{\mathbf{H}} \ominus \mathbf{H}\ .$$ ∘

Remark
The consistency (independence on the choice of E) follows from the transformation properties of $\tilde{\omega}$ and ω under the action of the structure group and from the fact that both these forms act as the fundamental form on the vertical vectors — see, e.g., [3]. ∘

Finally, let us recall the definitions of the curvature form and curvature tensor:

Definition 3 (Curvature)
Let $\mathbf{H}$ be a connection on the principal bundle $\mathbf{P}M$ given by a connection form ω. The *curvature form* Ω ($\mathbf{g}$-valued 2-form on $\mathbf{P}M$) is given by

$$\Omega \stackrel{\text{def}}{=} \operatorname{hor}_* \mathrm{d}\omega\ .$$

The *curvature tensor* $\mathcal{F} \in \operatorname{Sect}(\mathbf{g} \otimes \mathbf{\Lambda}^2 M)$ (i.e. $\mathbf{g}M$-valued 2-form on M) is defined by equation

$$\pi_* \mathcal{F}[E] = \Omega|_E\ ,$$

where E is an arbitrary section of the principal bundle $\mathbf{P}M$. ∘

Remark
Independence on the choice of E follows from the transformation properties of Ω — see again [3]. ∘

The relation between curvature tensors of two connections follows from Cartan's structure equations and definitions of the vector potential:

Theorem 2
Let $\tilde{\mathbf{H}}$, $\mathbf{H}$ be two connections on the principal bundle $\mathbf{P}M$ with curvature tensors $\tilde{\mathcal{F}}$, $\mathcal{F}$ and let $\mathcal{A} = \tilde{\mathbf{H}} \ominus \mathbf{H}$. Then

$$\tilde{\mathcal{F}} - \mathcal{F} = \mathrm{d}^{\mathcal{D}}\mathcal{A} + [\mathcal{A},\mathcal{A}] = \mathrm{d}^{\tilde{\mathcal{D}}}\mathcal{A} - [\mathcal{A},\mathcal{A}]\ . \qquad \square$$

Here $\mathrm{d}^{\mathcal{D}}$ is the *covariant external derivative*; it acts on $\mathrm{Sect}\,\mathbf{g}\otimes\mathbf{\Lambda}^p M$ as the external derivative d in tangent tensor indices, and as the covariant derivative $\mathcal{D}$ in gauge algebra indices. If we extend the action of the covariant derivative $\mathcal{D}$ on the tangent bundle $\mathbf{T}M$ using any torsion-free connection we can write

$$\mathrm{d}^{\mathcal{D}}\omega = \mathcal{D}\wedge\omega\ . \tag{17}$$

3 Logarithmic covariant derivative

Now we can define a covariant derivative on the principal bundle. The derivative of a section $F(x)$ is essentially a vertical part of a change of the section projected to the local gauge algebra $\mathbf{g}M$.

Definition 4 (Logarithmic covariant derivative on $\mathbf{P}M$)
Let $F \in \mathrm{Sect}\,\mathbf{P}M$. The *logarithmic covariant derivative* $\mathcal{D}\!\ln F \in \mathrm{Sect}(\mathbf{g}\otimes\mathbf{T}^*M)$ is defined as

$$\mathcal{D}\!\ln F \stackrel{\mathrm{def}}{=} \mathrm{ad}_F\,\omega \bullet \mathrm{D}F = \mathrm{ad}_F\,\theta \bullet \mathrm{ver}(\mathrm{D}F)\ .$$

Here, $\mathrm{D}F$ is a differential of the mapping $F : M \to \mathbf{P}M$. We will use notation

$$\mathcal{D}_\xi\!\ln F \stackrel{\mathrm{def}}{=} \xi\cdot\mathcal{D}\!\ln F$$

for *logarithmic covariant derivative in a direction* $\xi \in \mathfrak{T}M$. $\circ$

Similarly, we can define a covariant derivative on the local gauge group $\mathbf{G}M$. Again, it is the vertical part of the change of a section g, now "left-shifted" to the local gauge algebra:

Definition 5 (Logarithmic covariant derivative on $\mathbf{G}M$)
Let $g \in \mathrm{Sect}\,\mathbf{G}M$. The *logarithmic covariant derivative* $\mathcal{D}\!\ln g \in \mathrm{Sect}(\mathbf{g}\otimes\mathbf{T}^*M)$ is defined as

$$\mathcal{D}\!\ln g \stackrel{\mathrm{def}}{=} \mathrm{L}^*_{g^{-1}}\,\mathrm{ver}(\mathrm{D}g)\ .$$

We will use notation

$$\mathcal{D}_\xi\!\ln g \stackrel{\mathrm{def}}{=} \xi\cdot\mathcal{D}\!\ln g\ .$$

for *logarithmic covariant derivative in a direction* $\xi \in \mathfrak{T}M$. $\circ$

The derivatives $\mathcal{D}\!\ln E$ and $\mathcal{D}\!\ln g$ are called *logarithmic* because they substitute heuristic expressions $(\mathcal{D}E)E^{-1}$ and $\mathcal{D}\!\ln g = g^{-1}\mathcal{D}g$. This can be made rigorous for a principal bundle represented as a space of frames in a vector bundle — e.g., for the principal bundle $\mathbf{F}M$ of tangent space $\mathfrak{T}M$ (see Section 2). In this case we have

Lemma 3

Let $E = \{e^{\boldsymbol{\alpha}}_{\mu}\} \in \mathrm{Sect}\,\mathbf{F}M$, $g = g^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} \in \mathrm{Sect}\,\mathbf{GL}\,M$, and $\nabla\mathrm{ln}$, ∇ are covariant derivatives associated with a connection on $\mathbf{F}M$. Then

$$(\nabla_{\boldsymbol{\mu}}\mathrm{ln}\, E)^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = (\nabla_{\boldsymbol{\mu}} e^{\boldsymbol{\alpha}}_{\kappa})\, e^{\kappa}_{\boldsymbol{\beta}}\ ,$$
$$(\nabla_{\boldsymbol{\mu}}\mathrm{ln}\, g)^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = g^{-1\boldsymbol{\alpha}}{}_{\boldsymbol{\kappa}} \nabla_{\boldsymbol{\mu}}\, g^{\boldsymbol{\kappa}}{}_{\boldsymbol{\beta}}\ ,$$

where $e^{\mu}_{\boldsymbol{\alpha}}$ is a dual base in $\mathbf{T}^{*}M$. □

Now we return to the general situation. We write down two lemmas that could also serve as definitions in a similar way as Definition 1 determines the covariant derivative on vector bundles:

Lemma 4

Let $F \in \mathrm{Sect}\,\mathbf{P}M$, $\xi \in \mathfrak{T}M$. Let x_ε, E_ε be as in Lemma 1 with $E_0 = F|_{x_0}$; we denote the values of F along x_ε by F_ε. Then

$$\mathcal{D}_{\xi}\mathrm{ln}\, F|_{x_0} = \mathrm{ad}_{E_0} \frac{D}{d\varepsilon} F_\varepsilon[E_\varepsilon]\Big|_{\varepsilon=0}\ . \qquad \square$$

Lemma 5

Let $g \in \mathrm{Sect}\,\mathbf{G}M$, $\xi \in \mathfrak{T}M$. Let x_ε and E_ε be as in Lemma 1; we denote $g_\varepsilon = g|_{x_\varepsilon}$. Then

$$\mathcal{D}_{\xi}\mathrm{ln}\, g|_{x_0} = \mathrm{ad}_{E_0} \frac{D}{d\varepsilon}\Big(g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon]\Big)\Big|_{\varepsilon=0}\ . \qquad \square$$

Next, we formulate some important properties of the covariant derivatives defined above — the analogues of the usual chain rule for the derivative of a product.

Theorem 6 (Chain rules)

Let $E \in \mathrm{Sect}\,\mathbf{P}M$, $g, h \in \mathrm{Sect}\,\mathbf{G}M$, $a, b \in \mathrm{Sect}\,\mathbf{g}M$, $\phi, \psi \in \mathrm{Sect}\,\mathbf{E}M$, where $\mathbf{E}M = (\mathrm{T}_{\mathbf{P}}\mathbf{E})M$ is a vector bundle associated with $\mathbf{P}M$. Then

$$\mathcal{D}\mathrm{ln}\,(gE) = \mathrm{ad}_g(\mathcal{D}\mathrm{ln}\, E + \mathcal{D}\mathrm{ln}\, g) = \mathrm{ad}_g\, \mathcal{D}\mathrm{ln}\, E - \mathcal{D}\mathrm{ln}\, g^{-1}\ , \tag{i}$$
$$\mathcal{D}\mathrm{ln}\,(gh) = \mathrm{ad}_{h^{-1}}\, \mathcal{D}\mathrm{ln}\, g + \mathcal{D}\mathrm{ln}\, h = \mathrm{ad}_{h^{-1}}(\mathcal{D}\mathrm{ln}\, g - \mathcal{D}\mathrm{ln}\, h^{-1})\ , \tag{ii}$$
$$\mathcal{D}(\mathrm{T}_g\,\phi) = \mathrm{T}_g\big(\mathcal{D}\phi + (\mathcal{D}\mathrm{ln}\, g)\cdot \mathrm{t}\phi\big)\ , \tag{iii}$$
$$\mathcal{D}(\mathrm{ad}_g\, a) = \mathrm{ad}_g\big(\mathcal{D}a + [\mathcal{D}\mathrm{ln}\, g, a]\big)\ , \tag{iv}$$
$$\mathcal{D}(a\cdot \mathrm{t}\phi) = (\mathcal{D}a)\cdot \mathrm{t}\phi + a\cdot \mathrm{t}(\mathcal{D}\phi)\ , \tag{v}$$
$$\mathcal{D}[a, b] = [\mathcal{D}a, b] + [a, \mathcal{D}b]\ , \tag{vi}$$
$$\mathcal{D}(\phi\psi) = (\mathcal{D}\phi)\psi + \phi(\mathcal{D}\psi)\ . \tag{vii}$$

□

The straightforward consequence of Theorem 6(ii) is

Lemma 7

Let $g \in \operatorname{Sect}\mathbf{G}M$, then

$$\mathcal{D}\ln g^{-1} = -\operatorname{ad}_g \mathcal{D}\ln g \ . \qquad \square$$

Rules 6(iii) and 6(v) tell us that representations T and t are covariantly constant:

Lemma 8

Let $g \in \operatorname{Sect}\mathbf{G}M$, $a \in \operatorname{Sect}\mathbf{g}M$, then

$$\mathrm{T}_g^{-1}\mathcal{D}\mathrm{T}_g = (\mathcal{D}\ln g)\cdot \mathrm{t} \ ,$$
$$\mathcal{D}\mathrm{t}_a = \mathrm{t}_{\mathcal{D}a} \ , \quad \text{i.e.,} \quad \mathcal{D}\mathrm{t} = 0 \ . \qquad \square$$

Covariant derivatives on $\mathbf{G}M$ and $\mathbf{g}M$ are, of course, related but this relation is more complicated than one could expect from expression $\mathcal{D}\ln\exp a$ because the gauge group multiplication is not commutative:

Theorem 9

Let $a \in \operatorname{Sect}\mathbf{g}M$, then

$$\mathcal{D}\ln\,\exp(a) = \mathcal{D}a + \frac{1}{2!}[\mathcal{D}a, a] + \frac{1}{3!}[[\mathcal{D}a, a], a] + \ldots \qquad \square$$

Now we formulate the relation between covariant derivatives corresponding to two different connections in terms of the vector potential (see Definition 2):

Theorem 10 (Difference of covariant derivatives)

Let $\mathbf{H}$, $\tilde{\mathbf{H}}$ be two connections on $\mathbf{P}M$, $\mathcal{A} = \tilde{\mathbf{H}} \ominus \mathbf{H}$, and $\mathcal{D}\ln$, $\mathcal{D}$ be covariant derivatives on the principal bundle $\mathbf{P}M$, the local gauge group, and algebra bundles $\mathbf{G}M$ and $\mathbf{g}M$, and on an associated vector bundle $\mathbf{E}M$. Then

$$\tilde{\mathcal{D}}\ln F - \mathcal{D}\ln F = \mathcal{A} \ , \qquad F \in \operatorname{Sect}\mathbf{P}M \ , \qquad \text{(i)}$$
$$\tilde{\mathcal{D}}\ln g - \mathcal{D}\ln g = \operatorname{ad}_{g^{-1}}\mathcal{A} - \mathcal{A} \ , \qquad g \in \operatorname{Sect}\mathbf{G}M \ , \qquad \text{(ii)}$$
$$\tilde{\mathcal{D}}a - \mathcal{D}a = [\mathcal{A}, a] \ , \qquad a \in \operatorname{Sect}\mathbf{g}M \ , \qquad \text{(iii)}$$
$$\tilde{\mathcal{D}}\phi - \mathcal{D}\phi = \mathcal{A}\cdot \mathrm{t}\phi \ , \qquad \phi \in \operatorname{Sect}\mathbf{E}M \ . \qquad \text{(iv)}$$

$\square$

The usual method of handling the covariant derivative is the so-called, *trivialization* (see, e.g., [4]). It consists of a choice of a section E of the principal bundle $\mathbf{P}M$ (i.e. a choice of “frame” in associated fiber bundles) and of expressing all quantities with respect to this “frame”. Such a choice also defines a *coordinate connection* $\mathbf{h}$ by the requirement that E be horizontal. That is, for the corresponding covariant derivative $\eth\ln$ on $\mathbf{P}M$ we get

$$\eth\ln E = 0 \ . \qquad (18)$$

The curvature tensor of the coordinate connection is zero. As a consequence of Theorem 10(i), we conclude that the vector potential $\mathcal{A} = \mathbf{H} \ominus \mathbf{h}$ of the connection

$\mathbf{H}$ with respect to the coordinate connection $\mathbf{h}$ is given exactly by the logarithmic covariant derivative of E:

$$\mathcal{A} = \mathcal{D}\mathrm{ln}\, E\ . \tag{19}$$

Finally we discuss the relation between the logarithmic covariant derivative and the curvature tensor $\mathcal{F}$. The curvature tensor plays a fundamental role in expressions that contain the commutator of covariant derivatives, some of them are summarized in the following theorem:

Theorem 11 (Curvature)
Let $\mathbf{H}$ be a connection, $\mathcal{D}\mathrm{ln}$ and $\mathcal{D}$ be the corresponding covariant derivatives, and $\mathcal{F}$ be the corresponding curvature tensor. Then

$$\begin{aligned}
&\mathcal{F} = \mathrm{d}^{\mathcal{D}}\mathcal{D}\mathrm{ln}\, E - [\mathcal{D}\mathrm{ln}\, E, \mathcal{D}\mathrm{ln}\, E]\ , && E \in \mathrm{Sect}\, \mathbf{P}M\ , && \text{(i)}\\
&\mathrm{ad}_{g^{-1}}\, \mathcal{F} - \mathcal{F} = \mathrm{d}^{\mathcal{D}}\mathcal{D}\mathrm{ln}\, g + [\mathcal{D}\mathrm{ln}\, g, \mathcal{D}\mathrm{ln}\, g]\ , && g \in \mathrm{Sect}\, \mathbf{G}M\ , && \text{(ii)}\\
&[\mathcal{F}, a] = \mathrm{d}^{\mathcal{D}}\mathrm{d}^{\mathcal{D}} a\ , && a \in \mathrm{Sect}\, \mathbf{g}M\ , && \text{(iii)}\\
&\mathcal{F} \cdot \mathrm{t}\phi = \mathrm{d}^{\mathcal{D}}\mathrm{d}^{\mathcal{D}} \phi\ , && a \in \mathrm{Sect}\, \mathbf{E}M\ . && \text{(iv)}
\end{aligned}$$

□

The first relation in the last theorem is essentially a well-known equation for the curvature tensor in terms of the vector potential (19). From the second relation and from Theorem 2 follows how the curvature tensor changes under a gauge transformation. The gauge transformation is given by a "rotation" of the inner degrees of freedom by a local gauge group element g, i.e., for example, $a \to \tilde{a} = \mathrm{ad}_g\, a$ for $a \in \mathrm{Sect}\, \mathbf{g}M$. The gauge field represented by a connection $\mathbf{H}$ transforms to a connection $\tilde{\mathbf{H}}$ and the corresponding covariant derivatives have to satisfy the condition

$$\tilde{\mathcal{D}}\tilde{a} = \widetilde{\mathcal{D}a}\ . \tag{20}$$

This is achieved by a proper choice of the vector potential:

Lemma 12 (Gauge transformation)
The vector potential $\mathcal{A} = \tilde{\mathbf{H}} \ominus \mathbf{H}$ corresponding to a gauge transformation of the connection $\mathbf{H}$ by a local gauge group element $g \in \mathrm{Sect}\, \mathbf{G}M$ is

$$\mathcal{A} = \mathcal{D}\mathrm{ln}\, g^{-1}\ . \tag{i}$$

The curvature tensor transforms as

$$\tilde{\mathcal{F}} = \mathrm{ad}_g\, \mathcal{F}\ . \tag{ii}$$

□

The first equation is the desired coordinate-free relation for a change of a gauge field under a non-infinitesimal gauge transformation mentioned in the introduction (cf. Eq. (2)). The second equation is the standard behaviour of the curvature tensor under a gauge transformation.

Appendix: Proofs

Proof of Lemma 4
Using Lemma 1 we get

$$\mathcal{D}_\xi \ln F = \mathrm{ad}_F\, \theta \bullet\ \mathrm{ver}(\mathrm{D}F) = \mathrm{ad}_{E_0}\, \theta \bullet \frac{D}{d\varepsilon}\big(E_0 F_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\ .$$

Due to $F_0[E_0] = \bar{e}$, we can use Equation (8) and the last expression transforms to:

$$\mathrm{ad}_{E_0}\, \theta \bullet \mathfrak{f}_{(D/d\varepsilon)} F_\varepsilon[E_\varepsilon]\big|_{\varepsilon=0} = \mathrm{ad}_F\, \frac{D}{d\varepsilon} F_\varepsilon[E_\varepsilon]\Big|_{\varepsilon=0}\ .$$

Here we used relation (9). This completes the proof of Lemma 4. ■

Proof of Lemma 5
Using Lemma 1, we get

$$\mathcal{D}\ln g = \mathrm{L}^*_{g^{-1}}\ \mathrm{ver}(\mathrm{D}g) = \mathrm{L}^*_{g_0^{-1}}\, \frac{D}{d\varepsilon}\big(\mathrm{Ad}_{E_0}\, g_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\ .$$

Including the left shift in the derivative and using (11), the last expression changes to:

$$\frac{D}{d\varepsilon}\big(\mathrm{L}_{g_0^{-1}}\ \mathrm{Ad}_{E_0}\, g_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\big(\mathrm{Ad}_{E_0}\, g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\ .$$

Extracting Ad_{E_0} from the derivative, we obtain Lemma 5. ■

Proof of Theorem 6
Let ξ, E_ε, F_ε, g_ε, h_ε, ϕ_ε be as in Lemma 4 and Lemma 5. It follows from the first Lemma:

$$\begin{aligned} &\mathcal{D}_\xi \ln gF = \\ &\qquad = \mathrm{ad}_{E_0}\, \theta \bullet \frac{D}{d\varepsilon}\big(E_0 g_\varepsilon[E_\varepsilon] F_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} = \\ &\qquad = \mathrm{ad}_{E_0}\, \theta \bullet \Big(\frac{D}{d\varepsilon}\big(E_0 g_0[E_0] F_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} + \theta \bullet \frac{D}{d\varepsilon}\big(E_0 g_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\Big)\ . \end{aligned}$$

Equation (8) gives

$$\begin{aligned} &\mathcal{D}_\xi \ln gF = \\ &\qquad = \mathrm{ad}_{E_0}\, \theta \bullet \Big(\mathfrak{f}_{(D/d\varepsilon) F_\varepsilon[E_\varepsilon]|_{\varepsilon=0}}\Big|_{E_0 g_0[E_0]} + \mathfrak{f}_{(D/d\varepsilon)(g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon])|_{\varepsilon=0}}\Big|_{E_0 g_0[E_0]}\Big) = \\ &\qquad = \mathrm{ad}_{gF}\, \theta \bullet \big(\mathfrak{f}_{(\mathcal{D}_\xi \ln F)[E_0]} + \mathfrak{f}_{(\mathcal{D}_\xi \ln g)[E_0]}\big)\Big|_{gF} = \\ &\qquad = \mathrm{ad}_F\, \mathrm{ad}_{g[F]}\big(\mathcal{D}_\xi \ln F + \mathcal{D}_\xi \ln g\big)[F] = \\ &\qquad = \mathrm{ad}_g\big(\mathcal{D}_\xi \ln F + \mathcal{D}_\xi \ln g\big)\ , \end{aligned}$$

where we used Eqs. (9), (11), and $F|_x = F_0 = E_0$. This proves the first equality in the rule (i).

Using Lemma 5, we get

$$\begin{aligned}
\mathcal{D}_\xi \ln gh &= \mathrm{ad}_{E_0} \frac{D}{d\varepsilon}\big(h_0[E_0]^{-1} g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon] h_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} = \\
&= \mathrm{ad}_{E_0}\Big(\frac{D}{d\varepsilon}\big(h_0[E_0]^{-1} g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon] h_0[E_0]\big)\Big|_{\varepsilon=0} \\
&\qquad + \frac{D}{d\varepsilon}\big(h_0[E_0]^{-1} h_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0}\Big) = \\
&= \mathrm{ad}_{E_0}\, \mathrm{ad}_{h_0[E_0]^{\text{-}1}} \frac{D}{d\varepsilon}\big(g_0[E_0]^{-1} g_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} + \mathcal{D}_\xi \ln h = \\
&= \mathrm{ad}_{h^{\text{-}1}}\, \mathcal{D}\ln g + \mathcal{D}\ln h\ .
\end{aligned}$$

This concludes the proof of the first equality in (ii).

Part (iii) follows from Definition 1, Equation 11, and Lemma 5:

$$\begin{aligned}
\mathcal{D}_\xi \mathrm{T}_g \phi &= \mathrm{T}_{E_0} \frac{d}{d\varepsilon}\big(\mathrm{T}_{g_\varepsilon[E_\varepsilon]} \phi_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} = \\
&= \mathrm{T}_{E_0}\Big(\frac{d}{d\varepsilon}\big(\mathrm{T}_{g_0[E_0]} \phi_\varepsilon[E_\varepsilon]\big)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}\big(\mathrm{T}_{g_\varepsilon[E_\varepsilon]} \phi_0[E_0]\big)\Big|_{\varepsilon=0}\Big) = \\
&= \mathrm{T}_{E_0} \mathrm{T}_{g_0[E_0]}\Big(\frac{d}{d\varepsilon}\phi_\varepsilon[E_\varepsilon]\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}\big(\mathrm{T}_{g_0[E_0]^{\text{-}1} g_\varepsilon[E_\varepsilon]} \phi_0[E_0]\big)\Big|_{\varepsilon=0}\Big) = \\
&= \mathrm{T}_g\Big(\mathcal{D}_\xi \phi + \mathrm{T}_{E_0} \mathrm{t}_{(d/d\varepsilon)\left(g_0[E_0]^{\text{-}1} g_\varepsilon[E_\varepsilon]\right)\big|_{\varepsilon=0}} \phi_0[E_0]\Big) = \\
&= \mathrm{T}_g\Big(\mathcal{D}_\xi \phi + \mathrm{t}_{\mathcal{D}_\xi \ln g}\, \phi\Big)\ .
\end{aligned}$$

Rule (iv) is just the corollary of (iii) in case of $\mathbf{E}M = \mathbf{g}M$, $\mathrm{T} = \mathrm{ad}$.

Rules (v)–(vii) are standard properties of a covariant derivative on a gauge algebra bundle $\mathbf{g}M$ and a vector bundle $\mathbf{E}M$ and we do not prove them here.

The remaining parts of (i) and (ii) follow from Lemma 7. ∎

Proof of Lemma 7

The lemma is a straightforward consequence of the first equality in Theorem 6(ii) and the fact that $\mathcal{D}\ln e = 0$. ∎

Proof of Lemma 8

The lemma is a consequence of rules 6(iii), 6(v), and of the tensor character of the representation T_g and the generator t. ∎

Proof of Theorem 9

Let T be a representation of the gauge group on a vector bundle, and t its generator. We denote the generator of the adjoint representation of the gauge algebra by c , i.e.,

$$\mathrm{ad}_{\exp a} = \exp(\mathrm{c}_a)\ , \quad \mathrm{c}_a m = [a, m]\ .$$

Applying repeatedly $\mathrm{t}_m \mathrm{t}_a = \mathrm{t}_a \mathrm{t}_a + \mathrm{t}_{-\mathrm{c}_a m}$, it is straightforward to prove the relation:

$$\mathrm{t}_m \mathrm{t}_m^k = \sum_{l=0}^{k} \binom{k}{l} \mathrm{t}_a^{k-l}\, \mathrm{t}_{(-\mathrm{c}_a)^l m}\ . \tag{*}$$

Expanding $\exp \mathrm{t}_a$ into power series we obtain

$$\mathcal{D}\mathrm{T}_{\exp a} = \mathcal{D}\exp \mathrm{t}_a = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathcal{D}(\mathrm{t}_a^n) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\sum_{k=1}^{n} \mathrm{t}_a^{n-k}\,\mathrm{t}_{\mathcal{D}a}\,\mathrm{t}_a^{k-1}\ ,$$

were we have used Lemma 8. Substituting Eq. (*) and rearranging sums we get

$$\begin{aligned}
\mathcal{D}\mathrm{T}_{\exp a} &= \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\,\sum_{k=1}^{n} \mathrm{t}_a^{n-k} \sum_{l=0}^{k-1} \binom{k-1}{l} \mathrm{t}_a^{k-l-1}\,\mathrm{t}_{(-\mathrm{c}_a)^l\mathcal{D}a} = \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n-1} \left(\frac{1}{n!}\sum_{k=l+1}^{n}\binom{k-1}{l}\right)\mathrm{t}_a^{n-l-1}\,\mathrm{t}_{(-\mathrm{c}_a)^l\mathcal{D}a} = \\
&= \sum_{m=0}^{\infty}\frac{1}{m!}\,\mathrm{t}_a^m \sum_{l=0}^{\infty}\frac{1}{(l+1)!}\mathrm{t}_{(-\mathrm{c}_a)^l\mathcal{D}a} = \\
&= \exp(\mathrm{t}_a)\,\mathrm{t}_{\sum_{l=0}^{\infty}\frac{1}{(l+1)!}(-\mathrm{c}_a)^l\mathcal{D}a}\ ,
\end{aligned}$$

i.e., using Lemma 8,

$$(\mathcal{D}\ln\,\exp a)\,{\scriptstyle\blacksquare}\,\mathrm{t} = \mathrm{T}_{\exp a}^{-1}\mathcal{D}\mathrm{T}_{\exp a} = \Big(\sum_{l=0}^{\infty}\frac{1}{(l+1)!}(-\mathrm{c}_a)^l\mathcal{D}a\Big)\,{\scriptstyle\blacksquare}\,\mathrm{t}\ .$$

If we choose the faithful representation t, Theorem 9 follows. ∎

Proof of Theorem 10

With help of Definitions 4, 2 and thanks to $\pi F = \mathrm{id}$, we get (i):

$$\tilde{\mathcal{D}}\ln F - \mathcal{D}\ln F = \mathrm{ad}_F(\tilde{\omega}-\omega)\bullet \mathrm{D}F = \mathcal{A}{\cdot}\mathrm{D}\pi\bullet\mathrm{D}F = \mathcal{A}{\cdot}\mathrm{D}(\pi F) = \mathcal{A}\ .$$

Using the relation we have just proved and Theorem 6(i), we can write:

$$\begin{aligned}
\mathcal{A} = \tilde{\mathcal{D}}\ln(gE) - \mathcal{D}\ln(gE) &= \mathrm{ad}_g\big(\tilde{\mathcal{D}}\ln E - \tilde{\mathcal{D}}\ln g - \mathcal{D}\ln E + \mathcal{D}\ln g\big) = \\
&= \mathrm{ad}_g\big(\mathcal{A} + \tilde{\mathcal{D}}\ln g - \mathcal{D}\ln g\big)\ .
\end{aligned}$$

Relation (ii) follows immediately.

Relations (iii) and (iv) are well-known properties of covariant derivatives on vector bundles and we will not prove them here. ∎

Proof of Theorem 11

Part (i) follows from Theorem 2 if we choose $\tilde{\mathbf{H}}$ as the coordinate connection given by the section E, i.e., $\tilde{\mathcal{D}} = \eth$, $\tilde{\mathcal{F}} = 0$, $\mathcal{A} = -\mathcal{D}\ln E$ (see (18), (19)).

To prove (ii), we substitute $E = gF$ in the relation we just proved and use Theorem 6(i)

$$\begin{aligned}
\mathrm{ad}_{g^{-1}}\,\mathcal{F} &= \mathrm{ad}_{g^{-1}}\big(\mathrm{d}^{\mathcal{D}}\mathcal{D}\ln(gF) - [\mathcal{D}\ln(gF), \mathcal{D}\ln(gF)]\big) = \\
&= \mathrm{ad}_{g^{-1}}\Big(\mathrm{d}^{\mathcal{D}}\big(\mathrm{ad}_g(\mathcal{D}\ln F + \mathcal{D}\ln g)\big) \\
&\qquad\qquad - \mathrm{ad}_g[\mathcal{D}\ln F + \mathcal{D}\ln g, \mathcal{D}\ln F + \mathcal{D}\ln g]\Big)\ .
\end{aligned}$$

From Theorem 6(iv) and (17), we have

$$\mathrm{d}^{\mathcal{D}}\,\mathrm{ad}_g\,a = \mathrm{ad}_g\big(\mathrm{d}^{\mathcal{D}}a + [\mathcal{D}\!\ln g, a] + [a, \mathcal{D}\!\ln g]\big)\ ,$$

and, therefore,

$$\begin{aligned}\mathrm{ad}_{g^{-1}}\mathcal{F} &= \mathrm{d}^{\mathcal{D}}(\mathcal{D}\!\ln F + \mathcal{D}\!\ln g) + [\mathcal{D}\!\ln g, \mathcal{D}\!\ln F + \mathcal{D}\!\ln g]\\ &\qquad + [\mathcal{D}\!\ln g, \mathcal{D}\!\ln F + \mathcal{D}\!\ln g] - [\mathcal{D}\!\ln F + \mathcal{D}\!\ln g, \mathcal{D}\!\ln F + \mathcal{D}\!\ln g]\\ &= \mathcal{F} + \mathrm{d}^{\mathcal{D}}\mathcal{D}\!\ln g + [\mathcal{D}\!\ln g, \mathcal{D}\!\ln g]\,,\end{aligned}$$

q.e.d. ∎

Proof of Lemma 12

From (20) with help of Theorem 10(iii) and Theorem 6(iv), we get

$$\begin{aligned}0 &= \tilde{\mathcal{D}}\,\mathrm{ad}_g\,a - \mathrm{ad}_g\,\mathcal{D}a = \mathcal{D}\,\mathrm{ad}_g\,a + [\mathcal{A}, \mathrm{ad}_g\,a] - \mathrm{ad}_g\,\mathcal{D}a\\ &= \mathrm{ad}_g[\mathcal{D}\!\ln g, a] + [\mathcal{A}, \mathrm{ad}_g\,a] = [\mathrm{ad}_g\,\mathcal{D}\!\ln g + \mathcal{A}, \mathrm{ad}_g\,a]\ .\end{aligned}$$

This is true for any $a \in \mathrm{Sect}\,\mathbf{g}M$, so due to the fact that the adjoint representation is faithful and due to Lemma 7, we get (i):

$$\mathcal{A} = -\,\mathrm{ad}_g\,\mathcal{D}\!\ln g = \mathcal{D}\!\ln g^{-1}\ .$$

Relation (ii) follows from Theorem 2 and Theorem 11(ii). ∎

The author has been supported by the grants GAČR 202/02/0735 of the Czech Republic and GAUK 166/2003 of the Charles University. The author would like to thank Prof. Jiří Bičák and Dr. Martin Žofka for reading the manuscript.

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