Asymptotic directional structure of radiation for fields of algebraic type D *)

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The directional behavior of dominant components of algebraically special spin-s fields near a spacelike, timelike or null conformal infinity is studied. By extending our previous general investigations, we concentrate on fields which admit a pair of equivalent algebraically special null directions, such as the Petrov type-D gravitational fields or algebraically general electromagnetic fields. We introduce and discuss a canonical choice of the reference tetrad near infinity in all possible situations, and we present the corresponding asymptotic directional structures using the most natural parametrizations.

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1 Introduction

In the series of papers [1–4] we analyzed the asymptotic directional properties of electromagnetic and gravitational fields in spacetimes with a nonvanishing cosmological constant Λ. It had been known for a long time [5–7] that — contrary to the asymptotically flat spacetimes — the dominant (radiative) component of the fields is not unique since it substantially depends on the direction along which a null geodesic approaches a given point at conformal infinity $\mathcal{I}$. We demonstrated that, somewhat surprisingly, such directional structure of radiation can be described in closed explicit form. It has a universal character that is essentially determined by the algebraic type of the field, i.e., by the specific local degeneracy and orientation of the principal null directions.

Our results were summarized and thoroughly discussed in the recent topical review [8]. They apply not only to electromagnetic or gravitational fields but to any field of spin s. In addition, the expression representing the directional behavior of radiation can be written in a unified form which covers all three possibilities $\Lambda > 0$, $\Lambda < 0$ or $\Lambda = 0$, corresponding to a spacelike, timelike or null character of $\mathcal{I}$, respectively.

This paper further elaborates and supplements some aspects of our work reviewed in [8]. It extends possible definitions of the canonical reference tetrad for the algebraically simple fields — those which admit an equivalent pair of distinct (degenerate) principal null directions. We systematically describe the most natural choices of the reference tetrad for all possible algebraic structures of type D fields, and for any value of Λ.

*) Dedicated to Prof. Jiří Horáček on the occasion of his 60th birthday

The notation used in the present paper is the same as in [8]; in fact, we shall frequently employ the material already described and derived there. For brevity, we shall refer directly to the equations and sections of the review [8] by prefixing the letter 'R' in front of the reference: equation (R.1.1), section R.2.1, etc.

2 Summary of general results

We wish to study the behavior of radiative component of fields near conformal infinity $\mathcal{I}$. An overview of the concept of conformal infinity can be found in textbooks (e.g., [9]; our notation is described in section R.2 of the work [8]).

Let us only recall that it is possible to define a normalized vector $\mathbf{n}$ normal to the conformal infinity. The causal character of the infinity — spacelike, null, or timelike — is given by the sign of the square of this vector, $\sigma = \mathbf{n}\cdot\mathbf{n} = -1, 0, +1$ (see also Fig. 1). Here and in the following, the dot '$\cdot$' denotes the scalar product, defined using the spacetime metric $\mathbf{g}$. Typically, the causal character of $\mathcal{I}$ is correlated with the sign of the cosmological constant, $\sigma = -\operatorname{sign}\Lambda$ (see section R.2.2 for details).

2.1 Null tetrads

To study various components of the fields, we introduce suitable orthonormal, and associated with them null tetrads. We denote the vectors of an *orthonormal tetrad* as $\mathbf{t}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, where $\mathbf{t}$ is a future-oriented unit timelike vector. With this tetrad we associate a *null tetrad* of null vectors $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\bar{\mathbf{m}}$ by

$$\mathbf{k} = \tfrac{1}{\sqrt{2}}(\mathbf{t}+\mathbf{q})\ , \quad \mathbf{l} = \tfrac{1}{\sqrt{2}}(\mathbf{t}-\mathbf{q})\ , \quad \mathbf{m} = \tfrac{1}{\sqrt{2}}(\mathbf{r}-\mathrm{i}\,\mathbf{s})\ , \quad \bar{\mathbf{m}} = \tfrac{1}{\sqrt{2}}(\mathbf{r}+\mathrm{i}\,\mathbf{s})\ . \tag{1}$$

The normalization conditions for these tetrads are

$$-\mathbf{t}\cdot\mathbf{t} = \mathbf{q}\cdot\mathbf{q} = \mathbf{r}\cdot\mathbf{r} = \mathbf{s}\cdot\mathbf{s} = 1\ , \quad -\mathbf{k}\cdot\mathbf{l} = \mathbf{m}\cdot\bar{\mathbf{m}} = 1\ , \tag{2}$$

respectively, with all other scalar products being zero.

The crucial tetrad in our study is the *interpretation tetrad* $\mathbf{k}_\mathrm{i}$, $\mathbf{l}_\mathrm{i}$, $\mathbf{m}_\mathrm{i}$, $\bar{\mathbf{m}}_\mathrm{i}$. It is a tetrad which is parallelly transported along a null geodesic $z(\eta)$, the vector $\mathbf{k}_\mathrm{i}$ being tangent to the geodesic. With respect to this tetrad we define the radiative component of the field. The precise definition and description of asymptotic behavior of the interpretation tetrad was thoroughly presented in sections R.3.3 and R.3.4, where more details can be found.

Here, we are going to concentrate on the *reference tetrad* $\mathbf{k}_\mathrm{o}$, $\mathbf{l}_\mathrm{o}$, $\mathbf{m}_\mathrm{o}$, $\bar{\mathbf{m}}_\mathrm{o}$. It is a tetrad conveniently defined near the conformal infinity. It serves as the reference frame for parametrization of directions near $\mathcal{I}$. It can be defined using special features of the spacetime geometry (e.g., the Killing vectors, direction toward sources, etc.). Alternatively, it can be adapted to the studied fields — namely, it can be 'aligned' with algebraically special directions of the fields under consideration. A general situation was discussed in sections R.5.4 and R.5.5. In the present work we will offer other possible privileged definitions of the reference tetrad for algebraically simple fields of type D.

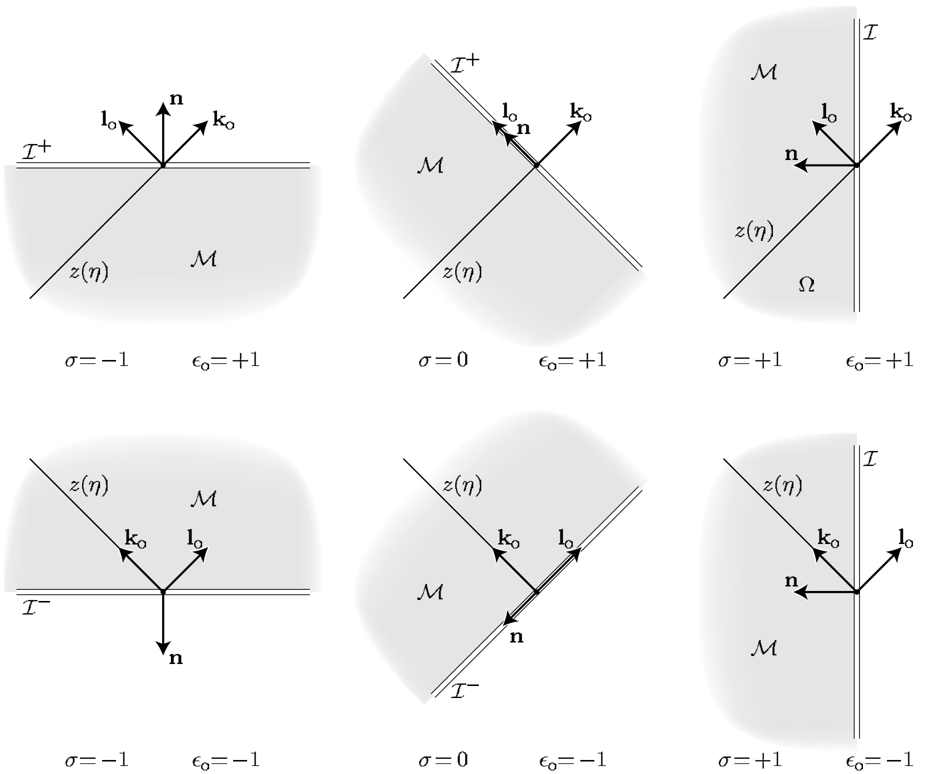

Fig. 1. The reference tetrad adjusted to conformal infinity $\mathcal{I}$ of various character, determined by σ which is a norm of the vector $\mathbf{n}$ normal to $\mathcal{I}$. If the vector $\mathbf{k}_o$ is oriented along an outgoing direction (outward from the physical spacetime $\mathcal{M}$) we have $\epsilon_o = +1$, if it is ingoing (oriented inward to $\mathcal{M}$) then $\epsilon_o = -1$.

Following [8], we require that the reference tetrad is *adjusted to conformal infinity*, i.e., that the vectors $\mathbf{k}_o$ and $\mathbf{l}_o$ satisfy the relation

$$\mathbf{n} = \epsilon_o \tfrac{1}{\sqrt{2}}(-\sigma \mathbf{k}_o + \mathbf{l}_o) \,, \tag{3}$$

where the sign $\epsilon_o = \pm 1$ indicates the outgoing/ingoing orientation of the vector $\mathbf{k}_o$ with respect to $\mathcal{I}$ (see also below). This adjustment condition guarantees that the vectors $\mathbf{k}_o$ and $\mathbf{l}_o$ are collinear with the normal $\mathbf{n}$ to $\mathcal{I}$, and normalized such that

$$\mathbf{n} = \begin{cases} \epsilon_o\, \mathbf{t}_o & \text{for a spacelike infinity } (\sigma = -1)\,, \\ -\epsilon_o\, \mathbf{q}_o & \text{for a timelike infinity } (\sigma = +1)\,, \\ \epsilon_o\, \mathbf{l}_o/\sqrt{2} & \text{for a null infinity } (\sigma = 0)\,. \end{cases} \tag{4}$$

All possible orientations of the reference tetrad with respect to $\mathcal{I}$ are shown in Fig. 1.

The normalization and adjustment conditions do not fix the reference tetrad uniquely. Additional necessary conditions — the alignment with the algebraically special directions — will be specified in Sect. 3.

2.2 Parametrization of null directions

In the following, it will be necessary to parametrize a general null direction $\mathbf{k}$ near $\mathcal{I}$. This can be done with respect to the reference tetrad by a complex *directional parameter* R:

$$\mathbf{k} \propto \mathbf{k}_o + \bar{R}\,\mathbf{m}_o + R\,\bar{\mathbf{m}}_o + R\bar{R}\,\mathbf{l}_o \,. \tag{5}$$

The value $R = \infty$ is also permitted — it corresponds to $\mathbf{k}$ oriented along $\mathbf{l}_o$.

In addition, we introduce the *orientation parameter* ϵ which indicates whether the null direction $\mathbf{k}$ is an outgoing direction (pointing outside the spacetime), $\epsilon = +1$, or if it is an ingoing direction (pointing inside the spacetime), $\epsilon = -1$.

When the infinity $\mathcal{I}$ has a *spacelike* character, it is also possible to parametrize the null direction $\mathbf{k}$ using *spherical angles*, which specify its normalized spatial projection into $\mathcal{I}$. We define the angles θ, ϕ by

$$\mathbf{q} = \cos\theta\;\mathbf{q}_o + \sin\theta\,(\cos\phi\;\mathbf{r}_o + \sin\phi\;\mathbf{s}_o)\;, \tag{6}$$

where $\mathbf{q}$ is the unit vector pointing into the spatial ($\mathbf{q}\cdot\mathbf{n} = 0$) direction given by $\mathbf{k}$, see (R.5.5). The complex parameter R of (5) is actually a stereographic representation of the spatial direction $\mathbf{q}$:

$$R = \tan\left(\tfrac{1}{2}\theta\right)\,\exp(-\mathrm{i}\phi)\;. \tag{7}$$

Near a *timelike* infinity $\mathcal{I}$ we can analogously describe null direction $\mathbf{k}$ by *pseudospherical parameters* ψ, ϕ of its projection into $\mathcal{I}$. If we label the normalized projection of $\mathbf{k}$ by $\mathbf{t}$, cf. (R.5.8), ψ and ϕ are given by

$$\mathbf{t} = \cosh\psi\;\mathbf{t}_o + \sinh\psi\,(\cos\phi\;\mathbf{r}_o + \sin\phi\;\mathbf{s}_o)\;. \tag{8}$$

These parameters have to be supplemented by the orientation ϵ of $\mathbf{k}$ with respect to $\mathcal{I}$. The parameter R is the pseudostereographic representation of $\mathbf{t}$:

$$R = \tanh^{\epsilon\epsilon_o}\left(\tfrac{1}{2}\psi\right)\,\exp(-\mathrm{i}\phi)\;. \tag{9}$$

In fact, we can introduce both these parametrizations simultaneously, independently of the causal character of the infinity, just with respect to the reference tetrad — the angles θ, ϕ using a projection onto the 3-space orthogonal to the time vector $\mathbf{t}_o$ of the reference tetrad, and the parameters ψ, ϕ using a projection to the 2+1-space orthogonal to $\mathbf{q}_o$. In such a case they are related by expressions

$$\tanh\psi = \sin\theta\;,\quad \sinh\psi = \tan\theta\;,\quad \cosh\psi = \cos^{-1}\theta\;,\quad \tanh\left(\tfrac{1}{2}\psi\right) = \tan\left(\tfrac{1}{2}\theta\right)\;. \tag{10}$$

If the infinity has a timelike character, all future-oriented null directions at one point at $\mathcal{I}$ naturally split into two families of outgoing and ingoing directions. The directions of each of these families form a hemisphere which can be projected onto a unit circle, parametrized by ρ and ϕ, such that

$$\rho = \tanh\psi = \sin\theta\ , \tag{11}$$

see also figure R.3.

2.3 Asymptotic directional structure of radiation

The main result of paper [8] is derivation of the explicit dependence of the radiative component of the field on a direction along which the infinity is approached — we call this dependence the *asymptotic directional structure of radiation.*

In [8] we investigated a general spin-s field, and in more detail gravitational ($s = 2$) and electromagnetic ($s = 1$) fields. These fields can be characterized by $2s + 1$ complex components $\varUpsilon_j$, $j = 0, \dots, 2s$, evaluated with respect to a null tetrad. Relation of these components to a spinor representation of the fields, and their transformation properties can be found in R.4.1 and appendix R.B. The components of gravitational and electromagnetic fields are traditionally called $\varPsi_j$, $j = 0, \dots, 4$, and $\varPhi_j$, $j = 0, 1, 2$, respectively — see [10] or equations (R.4.1) and (R.4.2).

To study the asymptotic behavior, we evaluated the field with respect to the interpretational tetrad — the tetrad which is parallelly transported along a null geodesic $z(\eta)$. It turned out that these field components satisfy the standard *peeling-off* property, namely that they exhibit a different fall-off in η when approaching $\mathcal{I}$, η being the affine parameter of the geodesic. The leading component of the field is the component $\varUpsilon^{\mathrm{i}}_{2s}$ with the fall-off of order η^{-1}, and we call it the *radiative component.* In sections R.4.3 and R.4.4 we found that the radiative field component depends on the direction R of the null geodesic along which a fixed point at the infinity is approached. This directional structure is determined mainly by the algebraic structure of the field, and it reads

$$\varUpsilon^{\mathrm{i}}_{2s} \approx \frac{1}{\eta}\ \epsilon_{\mathrm{o}}^{s} \varUpsilon^{\mathrm{o}}_{2s*}\ \frac{\big(1 - \sigma R_1 \bar{R}\big)\big(1 - \sigma R_2 \bar{R}\big) \dots \big(1 - \sigma R_{2s} \bar{R}\big)}{\big(1 - \sigma R \bar{R}\big)^{s}}\ . \tag{12}$$

The complex constants $R_1, \dots, R_{2s}$ represent the *principal null directions* $\mathbf{k}_1, \dots, \mathbf{k}_{2s}$ of the spin-s field, the sign $\sigma = \pm 1, 0$ specifies the causal character of the conformal infinity, ϵ_{o} denotes orientation of the reference tetrad, and $\varUpsilon^{\mathrm{o}}_{2s*}$ is a constant normalization factor of the field evaluated with respect to the reference tetrad, $\varUpsilon^{\mathrm{o}}_{2s} \approx \varUpsilon^{\mathrm{o}}_{2s*} \eta^{-s-1}$ (cf. section R.4.4).

The principal null directions (PNDs) are special directions along which some of the field components vanish — see [9, 10] or section R.4.2 for a precise definition. The field of spin s has $2s$ PNDs. However, these can be degenerate and this degeneracy (or, more generally, mutual relations of all the PNDs) is called the *algebraic structure* of the field. Distinct PNDs are also called *algebraically special directions* of the field. The classification according to the degeneracy of PNDs for a gravitational field is the well-known Petrov classification.

3 Fields of type D

In this paper we wish to discuss the situation when the field has two distinct and equivalent algebraically special directions. This may occur only for fields of an *integer* spin, $s \in \mathbb{N}$, with PNDs having the degeneracy

$$\mathbf{k}_1 = \cdots = \mathbf{k}_s \quad \text{and} \quad \mathbf{k}_{s+1} = \cdots = \mathbf{k}_{2s} \; . \tag{13}$$

The directional structure (12) of such a field takes the form

$$\varUpsilon^{\mathrm{i}}_{2s} \approx \frac{1}{\eta}\; \epsilon_{\mathrm{o}}^{s} \varUpsilon^{\mathrm{o}}_{2s*}\; \frac{\big(1-\sigma R_1 \bar{R}\big)^s \big(1-\sigma R_{2s} \bar{R}\big)^s}{\big(1-\sigma R \bar{R}\big)^s} \; , \tag{14}$$

with the constants R_1 and R_{2s} parametrizing the two distinct PNDs. The directional dependence of the magnitude of a gravitational type-D field and of an algebraically general electromagnetic field are thus quite similar. This similarity is even closer if we recall that the square of the electromagnetic component $\varPhi^{\mathrm{i}}_2$ is proportional to the magnitude of the Poynting vector with respect to the interpretation tetrad, $|\mathbf{S}_{\mathrm{i}}| \approx (1/4\pi) \left|\varPhi^{\mathrm{i}}_2\right|^2$. Indeed, we have

$$\left|\varPsi^{\mathrm{i}}_4\right| \approx \frac{1}{|\eta|}\; |\varPsi^{\mathrm{o}}_{4*}|\; \frac{\left|1-\sigma R_1 \bar{R}\right|^2 \left|1-\sigma R_4 \bar{R}\right|^2}{\left|1-\sigma R \bar{R}\right|^2} \; , \tag{15}$$

$$4\pi\, |\mathbf{S}_{\mathrm{i}}| \approx \left|\varPhi^{\mathrm{i}}_2\right|^2 \approx \frac{1}{\eta^2}\; |\varPhi^{\mathrm{o}}_{2*}|^2\; \frac{\left|1-\sigma R_1 \bar{R}\right|^2 \left|1-\sigma R_2 \bar{R}\right|^2}{\left|1-\sigma R \bar{R}\right|^2} \; . \tag{16}$$

As discussed in section R.4.5, the form of the directional structure (12) depends on the choice of the normalization factor $\varUpsilon^{\mathrm{o}}_{2s*}$. Other choices can sometimes be more convenient, in particular if the factor $\varUpsilon^{\mathrm{o}}_{2s*}$ vanishes, which happens when one of the PNDs points along the direction $\mathbf{l}_{\mathrm{o}}$ of the reference tetrad. For the type-D fields there exists a more natural 'symmetric' choice of normalization of the directional structure of radiation which is guaranteed to be non-degenerate. For these fields we can define a *canonical field component*, namely, the only nonvanishing component with respect of the null tetrad associated with the PNDs (13).

Having two distinct algebraic directions $\mathbf{k}_1$ and $\mathbf{k}_{2s}$, we can define *algebraically special null tetrad* $\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$, $\mathbf{m}_{\mathrm{s}}$, $\bar{\mathbf{m}}_{\mathrm{s}}$ (and associated orthonormal tetrad $\mathbf{t}_{\mathrm{s}}$, $\mathbf{q}_{\mathrm{s}}$, $\mathbf{r}_{\mathrm{s}}$, $\mathbf{s}_{\mathrm{s}}$) by requiring that $\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$ are *proportional* to the PNDs and future-oriented, and that the spatial vector $\mathbf{s}_{\mathrm{s}}$ is *tangent* to $\mathcal{I}$,

$$\mathbf{k}_{\mathrm{s}} \propto \mathbf{k}_1 \; , \quad \mathbf{l}_{\mathrm{s}} \propto \mathbf{k}_{2s} \; , \quad \mathbf{s}_{\mathrm{s}} \cdot \mathbf{n} = 0 \; . \tag{17}$$

For PNDs, which are not tangent to the conformal infinity, the normalization of null vectors $\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$ can be fixed by condition

$$\epsilon_1\, \mathbf{k}_{\mathrm{s}} \cdot \mathbf{n} = \epsilon_{2s}\, \mathbf{l}_{\mathrm{s}} \cdot \mathbf{n} \; , \tag{18}$$

where ϵ_1, $\epsilon_{2s} = \pm 1$ parametrize orientations of the PNDs with respect to $\mathcal{I}$. The special case of PNDs tangent to $\mathcal{I}$ will be discussed below.

Using the definition of PNDs (see section R.4.2), we find that the field components with respect to the algebraically special tetrad have very special form — only the component Υ_s^{s} is nonvanishing (Ψ_2^{s} for gravitational and Φ_1^{s} for electromagnetic fields). This component is, in fact, independent of the choice of the spatial vectors $\mathbf{r}_{\mathrm{s}}$, $\mathbf{s}_{\mathrm{s}}$, and thus it does not depend on the normal vector $\mathbf{n}$, which we used in the definition (17). It also does not depend on the normalization (18), provided that the normalization (2) is satisfied. We shall use the privileged component Υ_s^{s} for the normalization of the directional structure of radiation. However, the algebraically special tetrad is *not* adjusted to the infinity (cf. condition (3)) since $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ are not in general collinear with $\mathbf{n}$ and thus it cannot be used as a reference tetrad.

Nevertheless, we can define privileged reference tetrad which is 'somehow aligned' with the algebraically special tetrad, and which shares some of the symmetries of the geometric situation. We shall always assume that the reference tetrad satisfies the normalization and adjustment conditions (2), (3), and we set $\mathbf{s}_{\mathrm{o}} = \mathbf{s}_{\mathrm{s}}$. This is, however, still not sufficient to completely fix the reference tetrad. The remaining necessary condition cannot be prescribed in general — we have to discuss separately several possible cases, depending on the character of the infinity $\mathcal{I}$, and on the orientation of the PNDs with respect to $\mathcal{I}$.

Below we shall define the reference tetrad for all possible cases. We shall present the relation between the component $\Upsilon_{2s}^{\mathrm{o}}$ and the canonical component Υ_s^{s}, which can be substituted into the directional structure (14). Finally, we shall rewrite the results in terms of the angular variables introduced with respect to the reference tetrad.

3.1 Spacelike $\mathcal{I}$

We begin with a spacelike conformal infinity, $\sigma = -1$. In this case the two distinct future-oriented algebraically special directions are either both ingoing or both outgoing, and we accordingly set the orientation ϵ_{o} of the reference tetrad. We define the reference tetrad by conditions

$$\mathbf{q}_{\mathrm{o}} = \mathbf{q}_{\mathrm{s}} \;, \quad \mathbf{s}_{\mathrm{o}} = \mathbf{s}_{\mathrm{s}} \;, \quad \epsilon_{\mathrm{o}} = \epsilon_1 = \epsilon_{2s} \;, \tag{19}$$

and by adjustment condition (4). It follows that the algebraically special directions $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ are parametrized with respect to the reference tetrad by a single parameter θ_{s} as

$$\begin{aligned} \mathbf{k}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}} \cos^{-1}\theta_{\mathrm{s}} \left(\mathbf{t}_{\mathrm{o}} + \cos\theta_{\mathrm{s}}\, \mathbf{q}_{\mathrm{o}} + \sin\theta_{\mathrm{s}}\, \mathbf{r}_{\mathrm{o}}\right) \;, \\ \mathbf{l}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}} \cos^{-1}\theta_{\mathrm{s}} \left(\mathbf{t}_{\mathrm{o}} - \cos\theta_{\mathrm{s}}\, \mathbf{q}_{\mathrm{o}} + \sin\theta_{\mathrm{s}}\, \mathbf{r}_{\mathrm{o}}\right) \;. \end{aligned} \tag{20}$$

The algebraically special and reference tetrads are thus related by

$$\mathbf{t}_{\mathrm{s}} = \cos^{-1}\theta_{\mathrm{s}}\, \mathbf{t}_{\mathrm{o}} + \tan\theta_{\mathrm{s}}\, \mathbf{r}_{\mathrm{o}} \;, \quad \mathbf{q}_{\mathrm{s}} = \mathbf{q}_{\mathrm{o}} \;, \quad \mathbf{r}_{\mathrm{s}} = \cos^{-1}\theta_{\mathrm{s}}\, \mathbf{r}_{\mathrm{o}} + \tan\theta_{\mathrm{s}}\, \mathbf{t}_{\mathrm{o}} \;, \quad \mathbf{s}_{\mathrm{s}} = \mathbf{s}_{\mathrm{o}} \;, \tag{21}$$

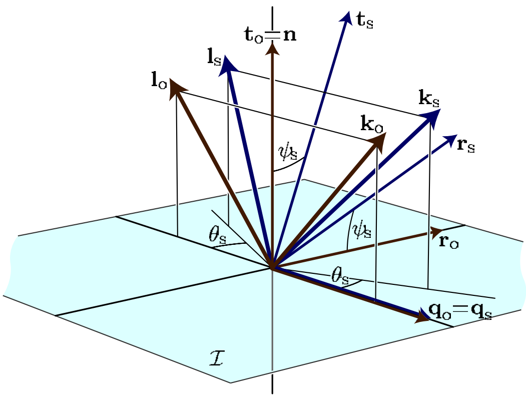

Fig. 2. Algebraically special and reference tetrads at a spacelike infinity. Vectors $\mathbf{k}_\mathrm{o}$, $\mathbf{l}_\mathrm{o}$ ($\mathbf{k}_\mathrm{s}$, $\mathbf{l}_\mathrm{s}$, respectively) of the null tetrad, and $\mathbf{t}_\mathrm{o}$, $\mathbf{q}_\mathrm{o}$, $\mathbf{r}_\mathrm{o}$ ($\mathbf{t}_\mathrm{s}$, $\mathbf{q}_\mathrm{s}$, $\mathbf{r}_\mathrm{s}$) of the orthonormal reference (algebraically special, respectively) tetrad are shown; the direction $\mathbf{s}_\mathrm{o}=\mathbf{s}_\mathrm{s}$ tangent to $\mathcal{I}$ and orthogonal to PNDs is hidden. The vectors $\mathbf{k}_\mathrm{s}$, $\mathbf{l}_\mathrm{s}$ are aligned with algebraically special directions (degenerate PNDs), $\mathbf{t}_\mathrm{o}$ is normal to the infinity $\mathcal{I}$, and $\mathbf{q}_\mathrm{o}$, $\mathbf{r}_\mathrm{o}$ are tangent to $\mathcal{I}$. The relation of both the tetrads is parametrized by the angle θ_s between $\mathbf{q}_\mathrm{o}$ and the projection of $\mathbf{k}_\mathrm{s}$ onto $\mathcal{I}$. The special tetrad can be obtained from the reference tetrad by a boost in $\mathbf{t}_\mathrm{o}$-$\mathbf{r}_\mathrm{o}$ plane with rapidity parameter ψ_s given by $\sinh\psi_\mathrm{s}=\tan\theta_\mathrm{s}$, cf. (21).

which is actually a boost in $\mathbf{t}_\mathrm{o}$-$\mathbf{r}_\mathrm{o}$ plane with rapidity parameter ψ_s related to θ_s by (10), see Fig. 2.

Inspecting the spatial projections of $\mathbf{k}_\mathrm{s}$ and $\mathbf{l}_\mathrm{s}$ onto conformal infinity $\mathcal{I}$, we find that their angular coordinates with respect to the reference tetrad are $\theta_1=\theta_\mathrm{s}$, $\phi_1=0$, and $\theta_{2s}=\pi-\theta_\mathrm{s}$, $\phi_{2s}=0$, respectively. It means that the complex parameters R_1 and R_{2s} of both these algebraic special directions are

$$R_1=\tan\left(\tfrac{1}{2}\theta_\mathrm{s}\right)\ ,\quad R_{2s}=\cot\left(\tfrac{1}{2}\theta_\mathrm{s}\right)\ . \tag{22}$$

Straightforward calculation shows that the transformation (21) from the algebraically special to the reference tetrad can be decomposed into boost (R.3.5), subsequent null rotation with $\mathbf{k}$ fixed (R.3.4), and null rotation with $\mathbf{l}$ fixed (R.3.3), given by the parameters $B=2(1+\cos^{-1}\theta_\mathrm{s})^{-1}$, $L=-\frac{1}{2}\tan\theta_\mathrm{s}$, and $K=-\tan(\theta_\mathrm{s}/2)$. Applying these transformations to the field components (relations (R.4.6), (R.4.5), and (R.4.4)) we easily find that

$$\varUpsilon^\mathrm{o}_{2s}=\binom{2s}{s}\bar{L}^s\,\varUpsilon^\mathrm{s}_s=(-1)^s\frac{(2s)!}{2^s(s!)^2}\tan^s\theta_\mathrm{s}\,\varUpsilon^\mathrm{s}_s\ . \tag{23}$$

For gravitational and electromagnetic fields we can write explicit expressions for all field components with respect to the reference tetrad as

$$\Psi_0^{\mathrm{o}} = \Psi_4^{\mathrm{o}} = \frac{3}{2}\tan^2\theta_{\mathrm{s}}\,\Psi_2^{\mathrm{s}}\;,\quad \Psi_1^{\mathrm{o}} = \Psi_3^{\mathrm{o}} = -\frac{3}{2}\frac{\tan\theta_{\mathrm{s}}}{\cos\theta_{\mathrm{s}}}\,\Psi_2^{\mathrm{s}}\;,\quad \Psi_2^{\mathrm{o}} = \Big(1+\frac{3}{2}\tan^2\theta_{\mathrm{s}}\Big)\Psi_2^{\mathrm{s}}\;, \tag{24}$$

$$\Phi_0^{\mathrm{o}} = \Phi_2^{\mathrm{o}} = -\tan\theta_{\mathrm{s}}\,\Phi_1^{\mathrm{s}}\;,\quad \Phi_1^{\mathrm{o}} = \cos^{-1}\theta_{\mathrm{s}}\,\Phi_1^{\mathrm{s}}\;. \tag{25}$$

Following the discussion in section R.4.4 we assume asymptotic behavior (R.4.18), i.e., $\Upsilon_j^{\mathrm{o}} \approx \Upsilon_{j*}^{\mathrm{o}}\,\eta^{-s-1}$ with constant coefficients $\Upsilon_{j*}^{\mathrm{o}}$. Similarly, we introduce the coefficient $\Upsilon_{s*}^{\mathrm{s}}$ by $\Upsilon_s^{\mathrm{s}} \approx \Upsilon_{s*}^{\mathrm{s}}\,\eta^{-s-1}$. Clearly, the relations (23)–(25) hold also in their 'stared' forms.

Substituting (22), the 'stared' version of (23), (7), and $\sigma = +1$ into (14) we finally obtain the asymptotic directional structure of radiation for type-D fields

$$\Upsilon_{2s}^{\mathrm{i}} \approx \frac{(-\epsilon_{\mathrm{o}})^s}{\eta}\,\frac{(2s)!}{2^s(s!)^2}\,\Upsilon_{s*}^{\mathrm{s}}\left[\frac{\exp(\mathrm{i}\phi)}{\cos\theta_{\mathrm{s}}}\Big(\sin\theta + \sin\theta_{\mathrm{s}}\cos\phi - \mathrm{i}\sin\theta_{\mathrm{s}}\cos\theta\sin\phi\Big)\right]^s\;. \tag{26}$$

The null direction along which the field is measured is parametrized by angles θ, ϕ, the field itself is characterized by the normalization component $\Upsilon_{s*}^{\mathrm{s}}$ and by the parameter θ_{s} which encodes the directions of the algebraically special directions with respect to a spacelike infinity $\mathcal{I}$. As discussed in section R.4.5, only the magnitude of this radiative component has a physical meaning. For the magnitude

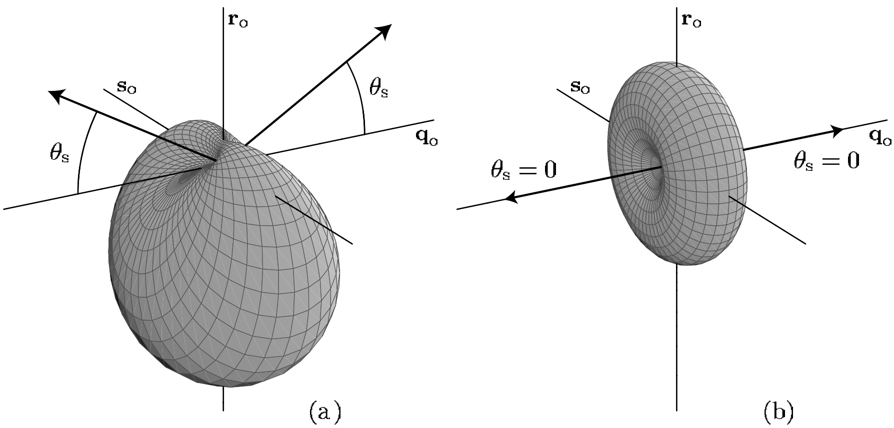

Fig. 3. Directional structure of radiation near a spacelike infinity. Directions in the diagrams correspond to spatial directions (projections onto $\mathcal{I}$) of null geodesics along which the infinity is approached. The diagrams show the directional dependence of the magnitude of the radiative-field component (27) (or (28)). The arrows depict the directions which are spatially opposite to algebraically special directions (PNDs); the radiative component evaluated along the geodesics in these directions is asymptotically vanishing. The diagram (a) shows a general orientation of algebraically special directions, the diagram (b) corresponds to the case, when both distinct PNDs are spatially opposite.

of the Ψ_4^{i} component of the gravitational field and for the Poynting vector of the electromagnetic field we thus obtain

$$\left|\Psi_4^{\mathrm{i}}\right| \approx \frac{1}{|\eta|}\,\frac{3}{2}\,\frac{|\Psi_{2*}^{\mathrm{s}}|}{\cos^2\theta_{\mathrm{s}}}\left|\sin\theta + \sin\theta_{\mathrm{s}}\cos\phi - \mathrm{i}\sin\theta_{\mathrm{s}}\cos\theta\sin\phi\right|^2 , \tag{27}$$

$$4\pi\left|\mathbf{S}_{\mathrm{i}}\right| \approx \left|\Phi_2^{\mathrm{i}}\right|^2 \approx \frac{1}{\eta^2}\,\frac{\left|\Phi_{1*}^{\mathrm{s}}\right|^2}{\cos^2\theta_{\mathrm{s}}}\left|\sin\theta + \sin\theta_{\mathrm{s}}\cos\phi - \mathrm{i}\sin\theta_{\mathrm{s}}\cos\theta\sin\phi\right|^2 . \tag{28}$$

These are exactly the expressions for the asymptotic directional structure of radiation as derived in [11] for test electromagnetic field of accelerated charges in de Sitter spacetime, and for gravitational field and electromagnetic fields of the C-metric spacetime with $\Lambda > 0$, as presented in [1]. This directional structure is illustrated in Fig. 3.

3.2 Timelike $\mathcal{I}$ with non-tangent PNDs, $\epsilon_1 \neq \epsilon_{2s}$

Now we shall study the situation near a timelike conformal infinity ($\sigma = -1$), when both distinct algebraic directions are not tangent to $\mathcal{I}$, such that one of them is outgoing and the other ingoing, $\epsilon_1 \neq \epsilon_{2s}$. In this case we require that the orientation ϵ_{o} of the reference tetrad is adjusted to ϵ_1, and that $\mathbf{t}_{\mathrm{s}}$ is aligned along $\mathbf{t}_{\mathrm{o}}$,

$$\mathbf{t}_{\mathrm{o}} = \mathbf{t}_{\mathrm{s}} , \quad \mathbf{s}_{\mathrm{o}} = \mathbf{s}_{\mathrm{s}} , \quad \epsilon_{\mathrm{o}} = \epsilon_1 = -\epsilon_{2s} , \tag{29}$$

together with the adjustment condition (4). Again, the algebraically special directions $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ are parametrized with respect to the reference tetrad by a single parameter θ_{s}:

$$\begin{aligned} \mathbf{k}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\mathrm{o}} + \cos\theta_{\mathrm{s}}\,\mathbf{q}_{\mathrm{o}} + \sin\theta_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}}\right) , \\ \mathbf{l}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\mathrm{o}} - \cos\theta_{\mathrm{s}}\,\mathbf{q}_{\mathrm{o}} - \sin\theta_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}}\right) . \end{aligned} \tag{30}$$

The algebraically special and reference tetrads are thus related by

$$\mathbf{t}_{\mathrm{s}} = \mathbf{t}_{\mathrm{o}} , \quad \mathbf{q}_{\mathrm{s}} = \cos\theta_{\mathrm{s}}\,\mathbf{q}_{\mathrm{o}} + \sin\theta_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}} , \quad \mathbf{r}_{\mathrm{s}} = -\sin\theta_{\mathrm{s}}\,\mathbf{q}_{\mathrm{o}} + \cos\theta_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}} , \quad \mathbf{s}_{\mathrm{s}} = \mathbf{s}_{\mathrm{o}} , \tag{31}$$

which is a spatial rotation in $\mathbf{q}_{\mathrm{o}}$-$\mathbf{r}_{\mathrm{o}}$ plane by angle θ_{s}, see Fig. 4a.

To read out the normalized projection into $\mathcal{I}$ of the null vectors $\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$, it is useful to rewrite (30) in a different way

$$\begin{aligned} \mathbf{k}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}}\cosh^{-1}\psi_{\mathrm{s}}\left(\mathbf{q}_{\mathrm{o}} + \cosh\psi_{\mathrm{s}}\,\mathbf{t}_{\mathrm{o}} + \sinh\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}}\right) , \\ \mathbf{l}_{\mathrm{s}} &= \tfrac{1}{\sqrt{2}}\cosh^{-1}\psi_{\mathrm{s}}\left(-\mathbf{q}_{\mathrm{o}} + \cosh\psi_{\mathrm{s}}\,\mathbf{t}_{\mathrm{o}} - \sinh\psi_{\mathrm{s}}\,\mathbf{r}_{\mathrm{o}}\right) , \end{aligned} \tag{32}$$

where we have used the parameter ψ_{s} instead of θ_{s} related by (10). Comparing the normalized projections of $\mathbf{k}_1 = \mathbf{k}_{\mathrm{s}}$ and $\mathbf{k}_{2s} = \mathbf{l}_{\mathrm{s}}$ with (8), we find that the pseudospherical parameters ψ, ϕ, ϵ of the algebraically special directions are $\psi_1 = \psi_{\mathrm{s}}$, $\phi_1 = 0$, $\epsilon_1 = \epsilon_{\mathrm{o}}$, and $\psi_{2s} = \psi_{\mathrm{s}}$, $\phi_{2s} = \pi$, $\epsilon_{2s} = -\epsilon_{\mathrm{o}}$ respectively. The corresponding complex parameters are

$$R_1 = \tanh\left(\tfrac{1}{2}\psi_{\mathrm{s}}\right) , \quad R_{2s} = -\coth\left(\tfrac{1}{2}\psi_{\mathrm{s}}\right) . \tag{33}$$

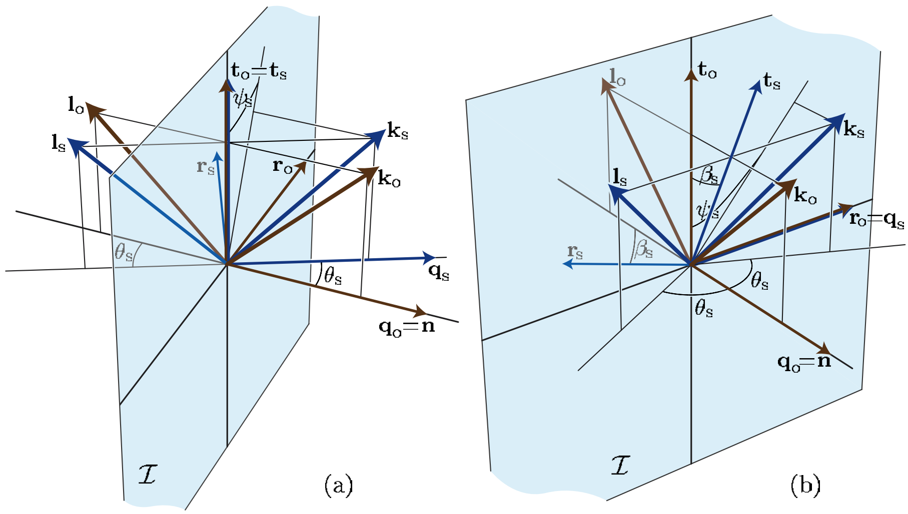

Fig. 4. Algebraically special and reference tetrads near a timelike infinity. Vectors $\mathbf{k}_o$, $\mathbf{l}_o$ ($\mathbf{k}_s$, $\mathbf{l}_s$) of the null reference (algebraically special, respectively) tetrad are shown. The axes correspond to the timelike direction $\mathbf{t}_o$ and the spatial directions $\mathbf{q}_o$, $\mathbf{r}_o$ of the reference tetrad. The direction $\mathbf{s}_o = \mathbf{s}_s$ tangent to $\mathcal{I}$ and orthogonal to PNDs is hidden. The vectors $\mathbf{k}_s$, $\mathbf{l}_s$ are aligned with the algebraically special directions (degenerate PNDs), $\mathbf{q}_o$ is normal to infinity $\mathcal{I}$ and $\mathbf{t}_o$, $\mathbf{r}_o$ are tangent to it. The vectors $\mathbf{t}_s$, $\mathbf{q}_s$, $\mathbf{r}_s$ of the algebraically special tetrad are drawn only in diagrams (a), (b) and (d); for simplicity they are omitted in the diagram (c), but see (53). Different diagrams correspond to different orientations of PNDs with respect to the infinity $\mathcal{I}$: in the diagram (a) one PND is ingoing and one is outgoing (cf. Subsect. 3.2), in (b) both PNDs are outgoing (or ingoing, respectively, cf. Subsect. 3.3), the diagram (c) shows the situation when one PND is tangent to $\mathcal{I}$ (Subsect. 3.4), and, finally, both PNDs are tangent in (d) (Subsect. 3.5). The relation of the reference and algebraically special tetrads in the generic cases (a) and (b) can be parametrized by the angle θ_s (the angle between $\mathbf{q}_o$ and the projection of $\mathbf{k}_s$ to the space normal to $\mathbf{t}_o$), or by pseudospherical parameter ψ_s (the lorenzian angle between $\mathbf{t}_o$ and the projection of $\mathbf{k}_s$ to $\mathcal{I}$). These parameters are related by (10). In the case (a) the special tetrad can be obtained from the reference tetrad by a spatial rotation in $\mathbf{q}_o$-$\mathbf{r}_o$ plane by θ_s, cf. (31); in the case (b) the special tetrad is the reference tetrad boosted by rapidity β_s given by $\sinh\beta_s = \cot\theta_s = \sinh^{-1}\psi_s$, cf. (42). *(continued)*

Again, the transformation (31) can be decomposed into boost, null rotation with $\mathbf{k}$ fixed, and null rotation with $\mathbf{l}$ fixed, given by $B = 2\cosh\psi_s(1+\cosh\psi_s)^{-1}$, $L = \frac{1}{2}\tanh\psi_s$, and $K = -\tanh(\psi_s/2)$. Applying these transformations to the field components we obtain

$$\Upsilon^{o}_{2s} = \frac{(2s)!}{2^s(s!)^2}\tanh^s\!\psi_s\,\Upsilon^{s}_{s}\ , \tag{34}$$

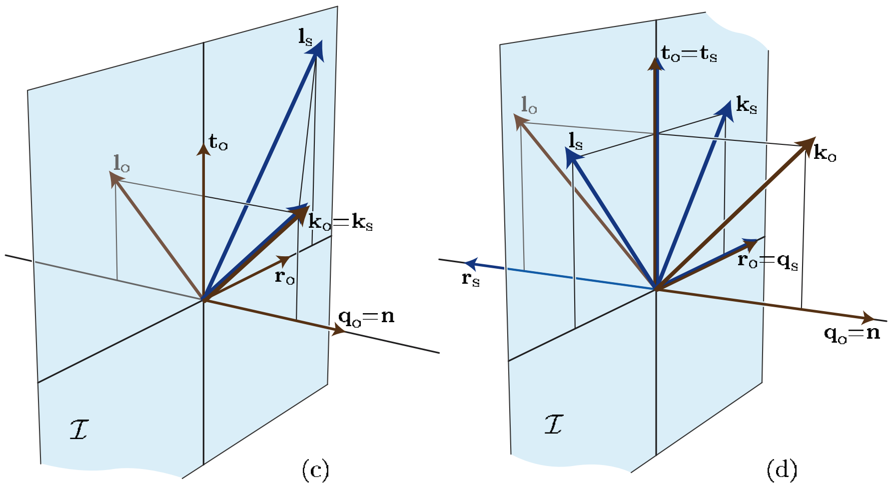

Fig. 4. (*continued*)

and, in more detail, for gravitational and electromagnetic fields

$$\begin{gathered}\Psi^{\rm o}_0=\Psi^{\rm o}_4=\tfrac{3}{2}\tanh^2\!\psi_{\rm s}\,\Psi^{\rm s}_2\;,\qquad -\Psi^{\rm o}_1=\Psi^{\rm o}_3=\frac{3}{2}\frac{\tanh\psi_{\rm s}}{\cosh\psi_{\rm s}}\,\Psi^{\rm s}_2\;,\\ \Psi^{\rm o}_2=\Big(1-\tfrac{3}{2}\tanh^2\!\psi_{\rm s}\Big)\,\Psi^{\rm s}_2\;,\end{gathered}\tag{35}$$

$$-\Phi^{\rm o}_0=\Phi^{\rm o}_2=\tanh\psi_{\rm s}\;\Phi^{\rm s}_1\;,\quad \Phi^{\rm o}_1=\cosh^{-1}\!\psi_{\rm s}\;\Phi^{\rm s}_1\;.\tag{36}$$

Substituting (33), 'stared' version of (34), (9), and $\sigma=+1$ into expression (14), we obtain the asymptotic directional structure of radiation in the form

$$\Upsilon^{\rm i}_{2s}\approx\frac{\epsilon^s}{\eta}\,\frac{(2s)!}{2^s(s!)^2}\,\Upsilon^{\rm s}_{s*}\,\bigg[\frac{\exp({\rm i}\phi)}{\cosh\psi_{\rm s}}\Big(\sinh\psi+\epsilon\epsilon_{\rm o}\sinh\psi_{\rm s}\cos\phi-{\rm i}\sinh\psi_{\rm s}\cosh\psi\sin\phi\Big)\bigg]^s\;.\tag{37}$$

The direction is given by pseudospherical parameters ψ, ϕ, ϵ, the field is characterized by the component $\Upsilon^{\rm s}_{s*}$ and by the parameter $\psi_{\rm s}$, which fixes the orientation of the algebraically special directions with respect to infinity $\mathcal{I}$. Again, only the magnitude of component $\Upsilon^{\rm i}_{2s}$ has a physical meaning so that

$$\big|\Psi^{\rm i}_4\big|\approx\frac{1}{|\eta|}\,\frac{3}{2}\,\frac{|\Psi^{\rm s}_{2*}|}{\cosh^2\psi_{\rm s}}\,\big|\sinh\psi+\epsilon\epsilon_{\rm o}\sinh\psi_{\rm s}\cos\phi-{\rm i}\sinh\psi_{\rm s}\cosh\psi\sin\phi\big|^2\;,\tag{38}$$

$$4\pi\,|\mathbf{S}_{\rm i}|\approx\big|\Phi^{\rm i}_2\big|^2\approx\frac{1}{\eta^2}\,\frac{|\Phi^{\rm s}_{1*}|^2}{\cosh^2\psi_{\rm s}}\,\big|\sinh\psi+\epsilon\epsilon_{\rm o}\sinh\psi_{\rm s}\cos\phi-{\rm i}\sinh\psi_{\rm s}\cosh\psi\sin\phi\big|^2\;.\tag{39}$$

This directional structure is illustrated in Fig. 5a.

3.3 Timelike $\mathcal{I}$ with non-tangent PNDs, $\epsilon_1 = \epsilon_{2s}$

In the previous case we have studied the directional structure of radiation near a timelike $\mathcal{I}$ with two PNDs oriented in opposite directions with respect to the infinity. Now, we shall discuss the situation when both PNDs are outgoing, or both ingoing, $\epsilon_1 = \epsilon_{2s}$. The derivation of the directional structure is similar to the previous case and we shall thus sketch it only briefly.

The reference tetrad is fixed by the conditions

$$\mathbf{r}_\mathrm{o} = \mathbf{q}_\mathrm{s}\ , \quad \mathbf{s}_\mathrm{o} = \mathbf{s}_\mathrm{s}\ , \quad \epsilon_\mathrm{o} = \epsilon_1 = \epsilon_{2s}\ , \tag{40}$$

together with the adjustment condition (4). Therefore

$$\begin{aligned}
\mathbf{k}_\mathrm{s} &= \tfrac{1}{\sqrt{2}}\,\sin^{-1}\theta_\mathrm{s}\left(\mathbf{t}_\mathrm{o} + \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} + \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}\right)\\
&= \tfrac{1}{\sqrt{2}}\sinh^{-1}\psi_\mathrm{s}\left(\mathbf{q}_\mathrm{o} + \cosh\psi_\mathrm{s}\,\mathbf{t}_\mathrm{o} + \sinh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}\right)\ ,\\
\mathbf{l}_\mathrm{s} &= \tfrac{1}{\sqrt{2}}\,\sin^{-1}\theta_\mathrm{s}\left(\mathbf{t}_\mathrm{o} + \cos\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o} - \sin\theta_\mathrm{s}\,\mathbf{r}_\mathrm{o}\right)\\
&= \tfrac{1}{\sqrt{2}}\sinh^{-1}\psi_\mathrm{s}\left(\mathbf{q}_\mathrm{o} + \cosh\psi_\mathrm{s}\,\mathbf{t}_\mathrm{o} - \sinh\psi_\mathrm{s}\,\mathbf{r}_\mathrm{o}\right)\ ,
\end{aligned} \tag{41}$$

where parameters θ_s and ψ_s are again related by Eqs. (10). The algebraically special and reference tetrads are thus

$$\mathbf{t}_\mathrm{s} = \sin^{-1}\theta_\mathrm{s}\,\mathbf{t}_\mathrm{o} + \cot\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o}\ , \quad \mathbf{q}_\mathrm{s} = \mathbf{r}_\mathrm{o}\ , \quad -\mathbf{r}_\mathrm{s} = \cot\theta_\mathrm{s}\,\mathbf{t}_\mathrm{o} + \sin^{-1}\theta_\mathrm{s}\,\mathbf{q}_\mathrm{o}\ , \quad \mathbf{s}_\mathrm{s} = \mathbf{s}_\mathrm{o}\ , \tag{42}$$

see Fig. 4(b). Pseudospherical parameters ψ, ϕ, ϵ of projections of the PNDs into $\mathcal{I}$ are $\psi_1 = \psi_\mathrm{s}$, $\phi_1 = 0$, $\epsilon_1 = \epsilon_\mathrm{o}$ and $\psi_{2s} = \psi_\mathrm{s}$, $\phi_{2s} = \pi$, $\epsilon_{2s} = \epsilon_\mathrm{o}$, respectively, i.e.,

$$R_1 = \tanh(\tfrac{1}{2}\psi_\mathrm{s})\ , \quad R_{2s} = -\tanh(\tfrac{1}{2}\psi_\mathrm{s})\ . \tag{43}$$

The transformation from the algebraically special to the reference tetrad can be decomposed into boost (R.3.5), null rotation with $\mathbf{k}$ fixed (R.3.4), and null rotation with $\mathbf{l}$ fixed (R.3.3) with parameters $B = 2\tanh(\psi_\mathrm{s}/2)$, $L = \frac{1}{2}\coth(\psi_\mathrm{s}/2)$, and $K = -\tanh(\psi_\mathrm{s}/2)$. For the field components we obtain

$$\varUpsilon^\mathrm{o}_{2s} = \frac{(2s)!}{2^s(s!)^2}\coth^s\frac{\psi_\mathrm{s}}{2}\,\varUpsilon^\mathrm{s}_s\ , \tag{44}$$

and, in more detail, for gravitational and electromagnetic fields

$$\varPsi^\mathrm{o}_0 = \tfrac{3}{2}\tanh^2\left(\tfrac{1}{2}\psi_\mathrm{s}\right)\,\varPsi^\mathrm{s}_2\ , \quad \varPsi^\mathrm{o}_2 = -\tfrac{1}{2}\varPsi^\mathrm{s}_2\ , \quad \varPsi^\mathrm{o}_4 = \tfrac{3}{2}\coth^2\left(\tfrac{1}{2}\psi_\mathrm{s}\right)\,\varPsi^\mathrm{s}_2\ , \quad \varPsi^\mathrm{o}_1 = \varPsi^\mathrm{o}_3 = 0\ , \tag{45}$$

$$\varPhi^\mathrm{o}_0 = -\tanh\left(\tfrac{1}{2}\psi_\mathrm{s}\right)\,\varPhi^\mathrm{s}_1\ , \quad \varPhi^\mathrm{o}_1 = 0\ , \quad \varPhi^\mathrm{o}_2 = \cot\left(\tfrac{1}{2}\psi_\mathrm{s}\right)\,\varPhi^\mathrm{s}_1\ . \tag{46}$$

Substituting into the expression (14) we finally obtain

$$\begin{aligned}
\varUpsilon^\mathrm{i}_{2s} \approx{}& \frac{\epsilon^s}{\eta}\,\frac{(2s)!}{2^s(s!)^2}\,\varUpsilon^\mathrm{s}_{s*}\\
&\times\left[\frac{\exp(\mathrm{i}\phi)}{\sinh\psi_\mathrm{s}}\Big((\cosh\psi + \epsilon\epsilon_\mathrm{o}\cosh\psi_\mathrm{s})\cos\phi - \mathrm{i}(\epsilon\epsilon_\mathrm{o} + \cosh\psi_\mathrm{s}\cosh\psi)\sin\phi\Big)\right]^s .
\end{aligned} \tag{47}$$

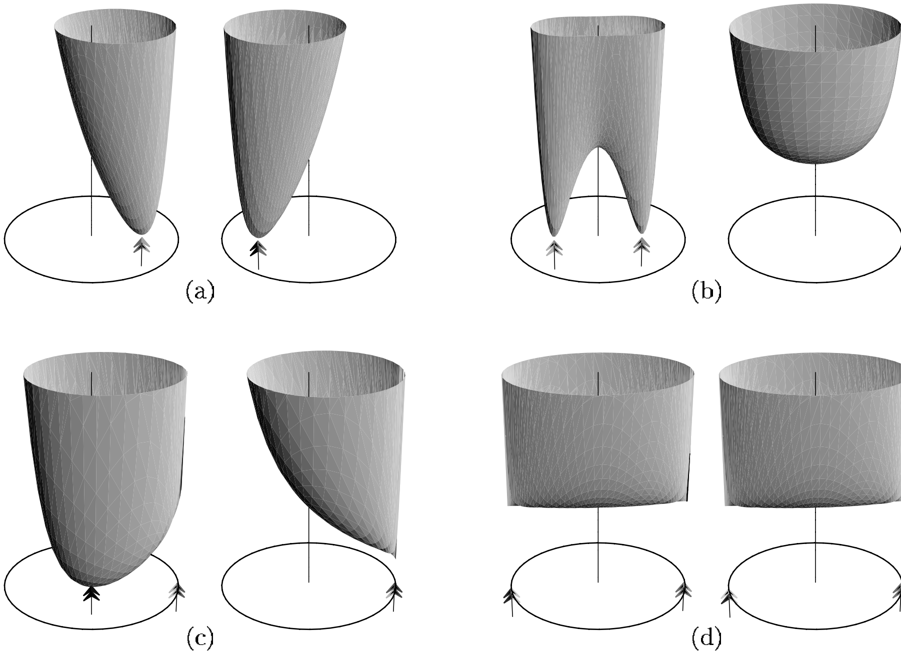

Fig. 5. Directional structure of gravitational radiation near a timelike infinity. The four diagrams, each consisting of a pair of radiation patterns, correspond to different orientation of algebraically special directions with respect to the infinity $\mathcal{I}$: (a) one PND outgoing and one ingoing (cf. Subsect. 3.2), (b) both PNDs outgoing (Subsect. 3.3; the case with both PNDs ingoing is analogous), (c) one PND tangent to $\mathcal{I}$ and one outgoing (Subsect. 3.4), (d) both PNDs tangent to $\mathcal{I}$ (Subsect. 3.5). In each diagram the circles in horizontal plane represent spatial projections of hemispheres of ingoing (left circle) and outgoing (right circle) directions. The circles are parametrized by coordinates ρ, ϕ defined in Subsect. 2.2, cf. Eq. (11). On the vertical axis the magnitude of the radiative field component is plotted (cf. (38), (49), (59) and (69)). The arrows indicate mirror reflections with respect to $\mathcal{I}$ of the algebraically special directions (PNDs). The radiative component evaluated along the geodesics in these directions is (for non-tangent PNDs) asymptotically vanishing. The radiative component diverges for unphysical geodesics tangent to $\mathcal{I}$ (the border of the circles) due to fixed normalization of the null directions — see section R.5.5.

The phase of this component is unphysical, its magnitude can be put into the form

$$|\Upsilon^{\mathrm{i}}_{2s}| \approx \frac{1}{|\eta|}\,\frac{(2s)!}{2^s(s!)^2}\,|\Upsilon^{\mathrm{s}}_{s*}|\,\Big(\sinh^{-2}\psi_{\mathrm{s}}\,\big(\cosh\psi_{\mathrm{s}}+\epsilon\epsilon_{\mathrm{o}}\cosh\psi\big)^2+\sinh^2\psi\sin^2\phi\Big)^{s/2}\,. \tag{48}$$

For gravitational and electromagnetic fields it gives

$$|\Psi^{\mathrm{i}}_4| \approx \frac{1}{|\eta|}\,\frac{3}{2}|\Psi^{\mathrm{s}}_{2*}|\,\Big(\sinh^{-2}\psi_{\mathrm{s}}\,\big(\cosh\psi_{\mathrm{s}}+\epsilon\epsilon_{\mathrm{o}}\cosh\psi\big)^2+\sinh^2\psi\sin^2\phi\Big)\,, \tag{49}$$

$$4\pi\,|\mathbf{S}_{\rm i}| \approx \left|\Phi^{\rm i}_2\right|^2 \approx \frac{1}{\eta^2}\,|\Phi^{\rm s}_{1*}|^2 \left(\sinh^{-2}\psi_{\rm s}\left(\cosh\psi_{\rm s} + \epsilon\epsilon_{\rm o}\cosh\psi\right)^2 + \sinh^2\psi\sin^2\phi\right)\,, \tag{50}$$

which is illustrated in Fig. 5b.

3.4 Timelike $\mathcal{I}$, one PND tangent to $\mathcal{I}$

Until now we have concentrated on a generic orientation of algebraically special directions with respect to the conformal infinity. In this and the next sections we are going to study the special cases when the PNDs are *tangent* to $\mathcal{I}$. This can only occur for timelike or null conformal infinity, the latter case will be discussed in Subsect. 3.6.

First, we assume that only one of two distinct PNDs, say $\mathbf{k}_{2s}$, is tangent to $\mathcal{I}$. Let us note that in such a case we require normalization (18) only for the vector $\mathbf{k}_{\rm s} \propto \mathbf{k}_1$, the normalization of the other PND $\mathbf{l}_{\rm s}$ is fixed by the condition $\mathbf{k}_{\rm s}\cdot\mathbf{l}_{\rm s} = -1$. We use the PND $\mathbf{k}_{\rm s}$ as the vector $\mathbf{k}_{\rm o}$, i.e., we define the reference tetrad by conditions

$$\mathbf{k}_{\rm o} = \mathbf{k}_{\rm s}\,,\quad \mathbf{s}_{\rm o} = \mathbf{s}_{\rm s}\,,\quad \epsilon_{\rm o} = \epsilon_1\,, \tag{51}$$

together with the condition (4). The algebraically special directions $\mathbf{k}_{\rm s}$ and $\mathbf{l}_{\rm s}$ are then given in terms of the reference tetrad as

$$\mathbf{k}_{\rm s} = \frac{1}{\sqrt{2}}\left(\mathbf{t}_{\rm o} + \mathbf{q}_{\rm o}\right)\,,\quad \mathbf{l}_{\rm s} = \sqrt{2}\left(\mathbf{t}_{\rm o} + \mathbf{r}_{\rm o}\right)\,, \tag{52}$$

see Fig. 4c. The tetrads are related by

$$\mathbf{t}_{\rm s} = \tfrac{3}{2}\mathbf{t}_{\rm o} + \tfrac{1}{2}\mathbf{q}_{\rm o} + \mathbf{r}_{\rm o}\,,\quad \mathbf{q}_{\rm s} = -\tfrac{1}{2}\mathbf{t}_{\rm o} + \tfrac{1}{2}\mathbf{q}_{\rm o} - \mathbf{r}_{\rm o}\,,\quad \mathbf{r}_{\rm s} = \mathbf{t}_{\rm o} + \mathbf{q}_{\rm o} + \mathbf{r}_{\rm o}\,,\quad \mathbf{s}_{\rm s} = \mathbf{s}_{\rm o}\,. \tag{53}$$

Complex parametrizations of $\mathbf{k}_1$ and $\mathbf{k}_{2s}$ are then

$$R_1 = 0\,,\quad R_{2s} = +1\,. \tag{54}$$

The transformation from the algebraically special to the reference tetrad is just the null rotation with $\mathbf{k}$ fixed with $L = -1$. Applying this transformation, we obtain

$$\Upsilon^{\rm o}_{2s} = (-1)^s\frac{(2s)!}{(s!)^2}\,\Upsilon^{\rm s}_s\,, \tag{55}$$

specifically for $s = 2,\,1$,

$$\Psi^{\rm o}_0 = \Psi^{\rm o}_1 = 0\,,\quad \Psi^{\rm o}_2 = \Psi^{\rm s}_2\,,\quad \Psi^{\rm o}_3 = -3\Psi^{\rm s}_2\,,\quad \Psi^{\rm o}_4 = 6\Psi^{\rm s}_2\,, \tag{56}$$

$$\Phi^{\rm o}_0 = 0\,,\quad \Phi^{\rm o}_1 = \Phi^{\rm s}_1\,,\quad \Phi^{\rm o}_2 = -2\Phi^{\rm s}_1\,. \tag{57}$$

Substituting into (14), we get the asymptotic directional structure of radiation

$$\Upsilon^{\rm i}_{2s} \approx \frac{\epsilon^s}{\eta}\,\frac{(2s)!}{2^s(s!)^2}\,\Upsilon^{\rm s}_{s*}\left(\epsilon\epsilon_{\rm o} + \cosh\psi - \sinh\psi\,\exp({\rm i}\phi)\right)^s\,, \tag{58}$$

i.e.,

$$\left|\Psi_4^{\mathrm{i}}\right| \approx 3\frac{|\Psi_{2*}^{\mathrm{s}}|}{|\eta|}\left(\epsilon\epsilon_{\mathrm{o}}+\cosh\psi\right)\left(\cosh\psi-\sinh\psi\cos\phi\right) , \tag{59}$$

$$4\pi\left|\mathbf{S}_{\mathrm{i}}\right| \approx \left|\Phi_2^{\mathrm{i}}\right|^2 \approx 2\frac{|\Phi_{1*}^{\mathrm{s}}|^2}{\eta^2}\left(\epsilon\epsilon_{\mathrm{o}}+\cosh\psi\right)\left(\cosh\psi-\sinh\psi\cos\phi\right) , \tag{60}$$

for gravitational and electromagnetic field, see Fig. 5c and (R.5.37).

3.5 Timelike $\mathcal{I}$, two PNDs tangent to $\mathcal{I}$

Next, let both the PNDs be tangent to a timelike conformal infinity. In such a situation there exists no natural normalization of both PNDs analogous to the condition (18) used above. This is related to an ambiguity in the choice of the timelike unit vector $\mathbf{t}_{\mathrm{s}}$ — we can choose any of the (future-oriented) unit vectors in the plane $\mathbf{k}_1$–$\mathbf{k}_{2s}$. However, despite the fact that we cannot fix the algebraically special tetrad uniquely, the nonvanishing component Υ_s^{s} is independent of this ambiguity: different choices of the special tetrad only differ by a boost in $\mathbf{k}_1$–$\mathbf{k}_{2s}$ plane, and Υ_s^{s} does not change under such a boost. In the following, we arbitrarily choose one particular algebraically special tetrad with respect to which we define the reference tetrad. The reference tetrad thus shares the same ambiguity as the algebraically special tetrad.

The reference tetrad is simply fixed by conditions

$$\mathbf{t}_{\mathrm{o}}=\mathbf{t}_{\mathrm{s}} , \quad \mathbf{s}_{\mathrm{o}}=\mathbf{s}_{\mathrm{s}} , \quad \epsilon_{\mathrm{o}}=\epsilon_1 , \tag{61}$$

and (4). PNDs $\mathbf{k}_{\mathrm{s}}$ and $\mathbf{l}_{\mathrm{s}}$ are given by

$$\mathbf{k}_{\mathrm{s}}=\tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\mathrm{o}}+\mathbf{r}_{\mathrm{o}}\right) , \quad \mathbf{l}_{\mathrm{s}}=\tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\mathrm{o}}-\mathbf{r}_{\mathrm{o}}\right) , \tag{62}$$

so that the algebraically special and reference tetrads are related by

$$\mathbf{t}_{\mathrm{s}}=\mathbf{t}_{\mathrm{o}} , \quad \mathbf{q}_{\mathrm{s}}=\mathbf{r}_{\mathrm{o}} , \quad \mathbf{r}_{\mathrm{s}}=-\mathbf{q}_{\mathrm{o}} , \quad \mathbf{s}_{\mathrm{s}}=\mathbf{s}_{\mathrm{o}} . \tag{63}$$

It is just a simple spatial rotation by $\pi/2$ in $\mathbf{q}_{\mathrm{o}}$-$\mathbf{r}_{\mathrm{o}}$ plane, as illustrated in Fig. 4d. The complex directional parameters of $\mathbf{k}_1$ and $\mathbf{k}_{2s}$ are

$$R_1=+1 , \quad R_{2s}=-1 . \tag{64}$$

The transformation can be decomposed into boost $B=2$, null rotation with $\mathbf{k}$ fixed $L=-1/2$, and null rotation with $\mathbf{l}$ fixed $K=1$. Applying them, we obtain

$$\Upsilon_{2s}^{\mathrm{o}}=(-1)^s\,\frac{(2s)!}{2^s(s!)^2}\,\Upsilon_s^{\mathrm{s}} , \tag{65}$$

and

$$\Psi_0^{\mathrm{o}}=\Psi_4^{\mathrm{o}}=\tfrac{3}{2}\,\Psi_2^{\mathrm{s}} , \quad \Psi_2^{\mathrm{o}}=-\tfrac{1}{2}\Psi_2^{\mathrm{s}} , \quad \Psi_1^{\mathrm{o}}=\Psi_3^{\mathrm{o}}=0 , \tag{66}$$

$$\Phi_0^{\mathrm{o}}=-\Phi_2^{\mathrm{o}}=\Phi_1^{\mathrm{s}} , \quad \Phi_1^{\mathrm{o}}=0 . \tag{67}$$

Substituting into (14), we get

$$\begin{aligned} \varUpsilon^{\mathrm{i}}_{2s} &\approx \frac{(-\epsilon_{\mathrm{o}})^s}{\eta}\,\frac{(2s)!}{2^s(s!)^2}\,\varUpsilon^{\mathrm{s}}_{s*}\,\Big(\exp(\mathrm{i}\phi)\big(\cos\phi-\mathrm{i}\,\epsilon\epsilon_{\mathrm{o}}\cosh\psi\,\sin\phi\big)\Big)^s\,, \\ \big|\varUpsilon^{\mathrm{i}}_{2s}\big| &\approx \frac{1}{|\eta|}\,\frac{(2s)!}{2^s(s!)^2}\,\big|\varUpsilon^{\mathrm{s}}_{s*}\big|\,\big(1+\sinh^2\psi\,\sin^2\phi\big)^{s/2}\,, \end{aligned} \tag{68}$$

which for gravitational and electromagnetic fields gives

$$\big|\varPsi^{\mathrm{i}}_4\big| \approx \frac{3}{2}\frac{1}{|\eta|}\,\big|\varPsi^{\mathrm{s}}_{2*}\big|\,\big(1+\sinh^2\psi\,\sin^2\phi\big)\,, \tag{69}$$

$$4\pi\,|\mathbf{S}_{\mathrm{i}}| \approx \big|\varPhi^{\mathrm{i}}_2\big|^2 \approx \frac{1}{\eta^2}\,\big|\varPhi^{\mathrm{s}}_{1*}\big|^2\,\big(1+\sinh^2\psi\,\sin^2\phi\big)\,, \tag{70}$$

see Fig. 5d and (R.5.36).

3.6 Null $\mathcal{I}$

Finally, we investigate the case of conformal infinity $\mathcal{I}$ of a null character, $\sigma = 0$. It can be easily observed from (12) (cf. section R.5.1 for more detail) that the directional structure of radiation near the null infinity is *independent* of the direction

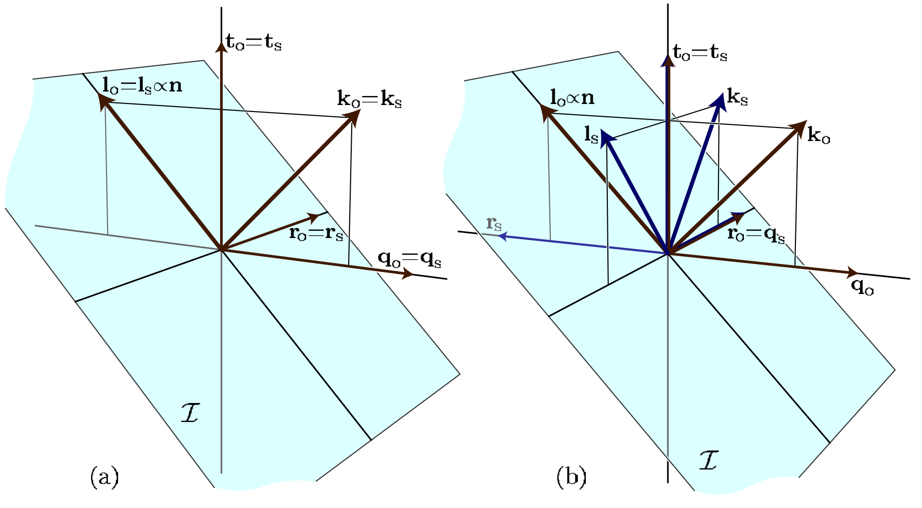

Fig. 6. Algebraically special and reference tetrads at a null infinity. Vectors $\mathbf{k}_{\mathrm{o}}$, $\mathbf{l}_{\mathrm{o}}$ ($\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$) of the null tetrad, and $\mathbf{t}_{\mathrm{o}}$, $\mathbf{q}_{\mathrm{o}}$, $\mathbf{r}_{\mathrm{o}}$ ($\mathbf{t}_{\mathrm{s}}$, $\mathbf{q}_{\mathrm{s}}$, $\mathbf{r}_{\mathrm{s}}$) of the orthonormal reference (algebraically special, respectively) tetrad are shown; the direction $\mathbf{s}_{\mathrm{o}}=\mathbf{s}_{\mathrm{s}}$ tangent to $\mathcal{I}$ and orthogonal to PNDs is not plotted. The vectors $\mathbf{k}_{\mathrm{s}}$, $\mathbf{l}_{\mathrm{s}}$ are aligned with algebraically special directions. The diagram (a) depicts the situation with one PND tangent to $\mathcal{I}$. In this case the algebraically special tetrad can be used as the reference tetrad. In the diagram (b) neither of both distinct PNDs is tangent to $\mathcal{I}$. The algebraically special tetrad can be then obtained from the reference tetrad by a spatial rotation in $\mathbf{q}_{\mathrm{o}}$–$\mathbf{r}_{\mathrm{o}}$ plane by $\pi/2$.

along which the infinity is approached. The only interesting question is whether the dominant field component is vanishing or not — in other words: whether the field is radiative or nonradiative. It follows from the definition of PNDs that the normalization factor $\Upsilon^{\rm o}_{2s*}$ in (12) is vanishing if and only if one of the PNDs is tangent to $\mathcal{I}$. Thus, the tangency of PNDs to $\mathcal{I}$ serves as the geometrical characterization of radiative/nonradiative fields.

Let us first assume that one of the PNDs, say $\mathbf{l}_{\rm s}$, is tangent to $\mathcal{I}$. As in the previous section we cannot fix the normalization of algebraically special tetrad using the condition (18); the algebraically special tetrad cannot be selected uniquely. Nevertheless, the field component $\Upsilon^{\rm s}_{s}$ is still unique. If we choose one algebraically special tetrad, we can use it also as the reference tetrad — it satisfies the adjustment condition (4), cf. Fig. 6. As mentioned above, the component $\Upsilon^{\rm o}_{2s}$ is then vanishing:

$$\Upsilon^{\rm i}_{2s} \approx 0 \ . \tag{71}$$

If both distinct PNDs are not tangent to $\mathcal{I}$, we may normalize them by (18) and fix the reference tetrad by the condition $\mathbf{t}_{\rm o} = \mathbf{t}_{\rm s}$, namely,

$$\mathbf{t}_{\rm o} = \mathbf{t}_{\rm s}\ , \quad \mathbf{s}_{\rm o} = \mathbf{s}_{\rm s}\ , \quad \epsilon_{\rm o} = \epsilon_1\ , \tag{72}$$

together with the adjustment condition (4), see Fig. 6. In terms of the reference tetrad the PNDs $\mathbf{k}_{\rm s}$ and $\mathbf{l}_{\rm s}$ are given by

$$\mathbf{k}_{\rm s} = \tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\rm o} + \mathbf{r}_{\rm o}\right)\ , \quad \mathbf{l}_{\rm s} = \tfrac{1}{\sqrt{2}}\left(\mathbf{t}_{\rm o} - \mathbf{r}_{\rm o}\right) \tag{73}$$

and their complex parameters are

$$R_1 = +1\ , \quad R_{2s} = -1\ . \tag{74}$$

The relation between the algebraically special and reference tetrads is thus the same as in Subsect. 3.5. Using (12), $\sigma = 0$, and relation (65), we find that the radiative component has no directional structure:

$$\Upsilon^{\rm i}_{2s} \approx \epsilon_{\rm o}^{s}\,\frac{(2s)!}{(s!)^2}\,\Upsilon^{\rm s}_{s*}\,\frac{1}{\eta}\ . \tag{75}$$

In particular, for gravitational and electromagnetic field we obtain

$$\left|\Psi^{\rm i}_4\right| \approx 3\,|\Psi^{\rm s}_{2*}|\,\frac{1}{|\eta|}\ , \tag{76}$$

$$4\pi\,|\mathbf{S}_{\rm i}| \approx \left|\Phi^{\rm i}_2\right|^2 \approx 2|\Phi^{\rm s}_{1*}|^2\,\frac{1}{\eta^2}\ . \tag{77}$$

4 Conclusions

We have analyzed the asymptotic directional structure of fields, that are characterized by the existence of two distinct, but equivalent algebraicaly special null directions. This involves a generic electromagnetic field ($s = 1$), the Petrov type D gravitational fields ($s = 2$) having double-degenerate principal null directions, and other possible fields of an integer spin s, which admit a pair of s-degenerate PNDs.

The structure of such fields near the conformal infinity depends on the specific orientation of these algebraically special directions with respect to $\mathcal{I}$, and on the causal character of $\mathcal{I}$. In the case of a spacelike conformal infinity ($\Lambda > 0$) there is essentially only one possibility which is described in Subsect. 3.1, whereas for a timelike conformal infinity ($\Lambda < 0$) four different situations may occur that have to be discussed separately, see Subsects. 3.2–3.5. For the conformal infinity having a null character ($\Lambda = 0$), the asymptotic directional structure disappears: when one of the (degenerate) PNDs is tangent to $\mathcal{I}$, the radiative component vanishes, otherwise the radiation is present and it is independent of the direction along which the infinity is approached, cf. Subsect. 3.6.

In all such cases we have introduced the privileged 'symmetric' reference tetrad which is naturally adapted to the algebraically special directions and to $\mathcal{I}$. These are illustrated in Figs. 2, 4, and 6. With respect to these reference tetrads it is possible to characterize any null direction by standard (pseudo)spherical parameters. The corresponding explicit directional structure of radiation for a spacelike $\mathcal{I}$ is presented in expression (26), and the four possibilities for a timelike $\mathcal{I}$ are given by (37), (47), (58), (68).

These results generalize our previous study of the asymptotic directional structure of gravitational and electromagnetic radiation in the C-metric spacetimes [1, 2] to other fields which are of the type D. On the other hand, the expressions presented here are more detailed and more explicit than those given in the review article [8]. It would now be an interesting task to apply them on particular exact model spacetimes of type D. This may provide a deeper insight into the geometric relation between the structure of the sources and the properties of radiation generated by them, as observed at spacelike or timelike conformal infinities.

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References

[1] P. Krtouš and J. Podolský: Phys. Rev. D **68** (2003) 024005.

[2] J. Podolský, M. Ortaggio, and P. Krtouš: Phys. Rev. D **68** (2003) 124004.

[3] P. Krtouš, J. Podolský, and J. Bičák: Phys. Rev. Lett. **91** (2003) 061101.

[4] P. Krtouš and J. Podolský: Phys. Rev. D **69** (2004) 084023.

[5] R. Penrose: in *Relativity, Groups and Topology*, Les Houches 1963, (Eds. C. DeWitt and B. DeWitt), Gordon and Breach, New York, 1964, p. 563.

[6] R. Penrose: Proc. R. Soc. Lond., A **284** (1965) 159.

[7] R. Penrose: in *The Nature of Time*, (Ed. T. Gold), Cornell University Press, Ithaca, New York, 1967, p. 42.

[8] P. Krtouš and J. Podolský: Class. Quantum Grav. **21** (2004) R233–273.

[9] R. Penrose and W. Rindler: *Spinors and Space-Time*, Cambridge University Press, Cambridge (England), 1984; 1986.

[10] H. Stephani *et al.*: *Exact Solutions of Einstein's Field Equations*, Second Edition, Cambridge University Press, Cambridge (England), 2002.

[11] J. Bičák and P. Krtouš: Phys. Rev. Lett. **88** (2002) 211101.