

Deformed and twisted black holes with NUTs

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2016 Class. Quantum Grav. 33 115016

(<http://iopscience.iop.org/0264-9381/33/11/115016>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

This content was downloaded by: krtous

IP Address: 195.113.23.37

This content was downloaded on 04/05/2016 at 13:12

Please note that [terms and conditions apply](#).

Deformed and twisted black holes with NUTs

Pavel Krtouš^{1,4}, David Kubizňák², Valeri P Frolov³ and Ivan Kolář^{1,3}

¹Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, Prague, 18000, Czech Republic

²Perimeter Institute, 31 Caroline Street North, Waterloo, ON, N2L 2Y5, Canada

³Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2G7

E-mail: Pavel.Krtous@utf.mff.cuni.cz, dkubiznak@perimeterinstitute.ca, frolov@phys.ualberta.ca and Ivan.Kolar@utf.mff.cuni.cz

Received 17 November 2015

Accepted for publication 11 March 2016

Published 3 May 2016



Abstract

We construct a new class of vacuum black hole solutions whose geometry is deformed and twisted by the presence of NUT charges. The solutions are obtained by ‘unspinning’ the general Kerr-NUT-(A)dS spacetimes, effectively switching off some of their rotation parameters. The resulting geometry has a structure of warped space with the Kerr-like Lorentzian part warped to a Euclidean metric of a deformed and/or twisted sphere, with the deformation and twist characterized by the ‘Euclidean NUT’ parameters. In the absence of NUTs, the solution reduces to a well known Kerr-(A)dS black hole with several rotations switched off. New geometries inherit the original symmetry of the Kerr-NUT-(A)dS family, namely, they possess the full Killing tower of hidden and explicit symmetries. As expected, for vanishing NUT, twist, and deformation parameters, the symmetry is further enlarged.

Keywords: black holes, higher dimensional gravity, NUT charges, explicit and hidden spacetime symmetries

1. Introduction

In four dimensions, a large family of solutions of vacuum Einstein equations with the cosmological constant of type D have been constructed by Plebański and Demiański [1], generalizing the Carter–Plebański form [2, 3] of non-accelerating solutions. Various known

⁴ Author to whom any correspondence should be addressed.

metrics, for example the Kerr metric [4], the Taub-NUT solution [5–8], or the (accelerating) C-metric, e.g. [9] are contained in this family as ‘special cases’. However, to obtain such special cases explicitly is not an easy task, as one has to perform certain (often singular) limits, scaling both the coordinates and the metric parameters. It is also for this reason that the physical meaning of the parameters of the Plebański–Demiański solution, describing, for example, the rotation or a NUT charge, is often obscure and difficult to interpret, see, e.g. [10, 11]. On the other hand, the Carter–Plebański frame is directly linked to the underlying explicit and hidden symmetries of the geometry and hence is invaluable for the study of mathematical properties of the spacetime. For example, it is in this frame that the test scalar field equations do separate [2]. Moreover, being able to describe a large family of spacetimes, the Carter–Plebański (or more generally Plebański–Demiański) metric allows one to generate (by taking special limits) new types of solutions, see, e.g. [12–14] for recent examples.

Although a generalization of the Plebański–Demiański family to higher dimensions remains elusive (see however [15]), a higher-dimensional generalization of the Carter–Plebański class, the so-called *Kerr-NUT-(A)dS spacetime*, has been found by Chen, Lü and Pope [16]. The geometry describes a large family of metrics of various signatures that solve the Einstein equations with the cosmological constant and contain a set of free parameters that can be related to mass, rotations, and NUT parameters. In particular, the general rotating black holes of Myers and Perry [17], their cosmological constant generalizations due to Gibbons *et al* [18, 19], or the higher-dimensional Taub-NUT spaces, e.g. [20–23], are expected to emerge as certain limits of Kerr-NUT-(A)dS spacetimes.

Surprisingly, apart from its ‘integrability properties’ not much is known about the Kerr-NUT-(A)dS geometry and its special subcases. On one hand, it has been shown [24, 25] that, similar to the four-dimensional Carter–Plebański case, the Kerr-NUT-(A)dS spacetime represents a unique Einstein space that admits a completely non-degenerate *closed conformal Killing–Yano* (CCKY) 2-form \mathbf{h} [26, 27]. This tensor uniquely determines *canonical coordinates* in which the metric is written. In these coordinates the Hamilton–Jacobi, Klein–Gordon, and Dirac equations fully separate [28–34]—making in particular the geodesic motion completely integrable [35–39]. Such integrability properties can be further linked to the existence of a full *Killing tower* of explicit and hidden symmetries that can be generated from \mathbf{h} , see, e.g. [40, 41] for a review. On the other hand, again similar to the Carter–Plebański case, it is not straightforward to precisely identify the exact physical meaning of the metric parameters, nor is it particularly simple to obtain special subcases contained in the Kerr-NUT-(A)dS family. For example, the non-rotating Schwarzschild–Tangherlini black hole [42] cannot be simply recovered by merely setting some of the parameters equal to zero.

The goal of this paper is to proceed towards a better understanding of the higher-dimensional Kerr-NUT-(A)dS family [16] as well as to uncover some of its special subcases. We start by performing a basic analysis of the Kerr-NUT-(A)dS spacetime, discussing its signature, coordinate ranges, scaling properties, and the number of independent metric parameters. In particular, we identify the maximally symmetric space, the Euclidean instanton, and the black hole solution. Concentrating on the black hole case, we then show how to switch off some of the rotation parameters, which are constructed so that a new rich class of vacuum (with a possible cosmological constant) black holes can be distinguished not only by mass and rotation parameters but also by NUT charges.

More concretely, we want to study black hole spacetimes that are generated from the Kerr-NUT-(A)dS geometry in the limit where some of the rotation parameters vanish. Unfortunately, one cannot just substitute zero values of the chosen rotational parameters directly in the Kerr-NUT-(A)dS metric, since the ranges of some coordinates would degenerate. This is related to a well known ambiguity in constructing the limiting spaces when

some of the parameters limit to zero [43]: there is always a possibility of making a coordinate transformation depending on the chosen parameters, before taking the limit. To escape the pathology and achieve the limit of vanishing rotations, we have to properly *rescale both the metric parameters and the coordinates*.

As we will demonstrate, the resulting black hole spacetimes have one parameter less than their Kerr-NUT-(A)dS precursor and inherit from it nice geometrical properties. In the absence of NUTs, they reduce to the Kerr-(A)dS black holes [18, 19] with several rotations switched off, in particular we obtain the Schwarzschild–Tangherlini-(A)dS solution [42]. However, when the NUT parameters are maintained, the black hole spacetimes are more interesting and, we believe, new. The metric acquires a warp product structure, where both components of the product have a form similar to the Kerr-NUT-(A)dS metric. Namely, it takes the following form:

$$\mathbf{g} = \tilde{\mathbf{g}} + \tilde{w}^2 \bar{\mathbf{g}}, \quad (1.1)$$

where the Euclidean *seed metric* of the product, $\tilde{\mathbf{g}}$, is exactly the lower dimensional Kerr-NUT-(A)dS metric while the Lorentzian *base metric* $\bar{\mathbf{g}}$ has the same form but modified radial dependence. In general, NUT parameters remain in both components.

We use this as an opportunity to partially rehabilitate the ‘bad reputation’ of NUT parameters which is based on the fact that in four dimensions they typically lead to causally ill-behaved spacetimes [5–7]; see, however, recent discussion in [8]. Similar pathology happens in higher dimensions when the NUT charges maintain their ‘Lorentzian character’. However, when present only in the spatial part of the metric, the ‘Euclidean NUT’ parameters just label deformations of the angular part of the metric and do not introduce regions with closed time-like curves. Such NUTs typically ‘smoothen’ the curvature singularity, although the ‘axes of rotation’ may suffer from non-regularity.

This is precisely what happens with the surviving NUT parameters after we take our limit. Those surviving in the base part of the metric maintain their Lorentzian character and lead to standard pathologies. Those surviving in the seed part of the metric, which is Euclidean, play a different geometric role: they do not lead to pathologies with closed time-like curves and tend to smoothen the curvature singularity. Such NUTs control deformation and also a *global twist* of the Euclidean geometry. Namely, in higher dimensions we have a freedom in making linear combinations and various periodic identifications of Killing coordinates, which we refer to as *global twisting* of the geometry. Since the metric contains non-diagonal terms, the freedom in twisting cannot be easily resolved. The presence of NUT parameters additionally causes non-regularity on the axes of rotation and thus enters non-trivially into the discussion of twisting.

Obviously, the twist is non-trivial only in the case when the seed metric is sufficiently ‘multi-dimensional’, that is, when it possesses more than one azimuthal Killing coordinate. Since the number of Killing coordinates of the resulting seed metric equals (in the studied even-dimensional case) the number of rotation parameters that were *simultaneously* set equal to zero, to have a non-trivial twist, two or more rotations have to be switched off simultaneously. We call the corresponding limit a *multiple-spin-zero limit*. Such a limit leads to the *deformed and twisted* black holes. On the other hand, when the rotation parameters are switched off *successively* one by one, that is when we apply several *single-spin-zero limits*; the resulting seed metrics are two-dimensional and their twist vanishes. In particular, we shall consider a case where all rotations and twists are eventually switched off. We call the corresponding solutions the *deformed* (non-twisted) black holes. Although they are static, their geometry is still non-trivially deformed by the presence of NUT charges. As we shall

show in section 4, the same geometry can be obtained by a successive application of the multiple-spin-zero limit, each of which leaves the base metric only two-dimensional.

After obtaining the limiting metrics we discuss their symmetries. In particular, we show that the resulting warped geometry admits a *degenerate* CCKY 2-form \tilde{h} , inherited from the base metric \tilde{g} . The resulting geometry therefore belongs to a wide class of *generalized Kerr-NUT-(A)dS spacetimes* constructed in [44, 45] and further studied in [32]; see appendix A for their review. Despite the degeneracy of this tensor, we show that the new geometry inherits the full original symmetry structure of the Kerr-NUT-(A)dS family, namely, it possesses the full Killing tower of hidden and explicit symmetries that can be obtained through the scaling limits of the original Killing tower of symmetries. (These symmetries can also be built up by a proper combination of the Killing towers of both seed and base geometries, see [46].) For this reason our new metrics represent a very special sub-class of generalized Kerr-NUT-(A)dS spacetimes, characterized by a property that they maintain the full Killing tower. In particular, this implies that the new class of solutions inherits the integrability properties of the original Kerr-NUT-(A)dS geometry.

As expected, for vanishing NUT, twist, and deformation parameters, the symmetry is further enlarged. For example, the Schwarzschild–Tangherlini-(A)dS solution is spherically-symmetric and its group of isometries is larger than the symmetry group of the initial metric. Similar ‘symmetry enlargement’ also happens when one puts some smaller set of the rotation and NUT parameters equal to zero.

The plan of the paper is as follows. In the next section we introduce the Kerr-NUT-(A)dS geometry and analyze its basic properties. In section 3 the multiple-spin-zero limit (in which several rotation parameters are set equal to zero simultaneously) is exploited to obtain the new twisted and deformed black hole solutions. In section 4 we discuss successive application of the multiple-spin-zero limits and arrive thus to the deformed black holes without twists and rotations. In section 5 we present specific examples in four and six dimensions and section 6 is devoted to discussion. In appendix A we review the generalized Kerr-NUT-(A)dS spacetimes and show that the obtained metrics form their special subclass. Appendix B contains additional technical results supporting the main text.

2. An analysis of Kerr-NUT-(A)dS spacetimes

2.1. The metric

In what follows, we concentrate on analyzing the even-dimensional Kerr-NUT-(A)dS spacetimes⁵ with the spacetime dimensions parameterized as $D = 2N$. The metric of such spacetime takes the following form [16]⁶:

$$\mathbf{g} = \sum_{\mu} \left[\frac{U_{\mu}}{X_{\mu}} \mathbf{d}x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_k A_{\mu}^{(k)} \mathbf{d}\psi_k \right)^2 \right]. \quad (2.1)$$

A variety of the *off-shell* metrics is hidden in the metric functions, X_{μ} , each of which depends just on the corresponding coordinate x_{μ} ,

⁵ A generalization to odd dimensions, where the Kerr-NUT-(A)dS metrics contain extra ‘odd’ terms, is technically straightforward.

⁶ The Greek indices take values $\alpha, \beta, \mu, \dots = 1, \dots, N$, the Latin indices from the middle of the alphabet run through $i, j, k, \dots = 0, \dots, N - 1$. We do not use the Einstein summation convention for these indices, but we do not indicate limits in the sums and products explicitly. We do employ the Einstein summation convention for general spacetime indices $a, b, c, \dots = 1, \dots, 2N$ used later in the paper.

$$X_\mu = X_\mu(x_\mu). \quad (2.2)$$

The vacuum Einstein equations restrict these metric functions to be particular polynomials $X_\mu = \mathcal{X}_\mu^{(0)}$, where

$$\mathcal{X}_\mu^{(p)} = \lambda \left[\mathcal{J}(x_\mu^2) - 2\beta_\mu a_\mu^2 \mathcal{U}_\mu \left(\frac{x_\mu}{a_\mu} \right)^{1-2p} \right]. \quad (2.3)$$

With this choice of X_μ we call the Kerr-NUT-(A)dS metric (2.1) *on-shell*, highlighting the fact that it solves the Einstein equations. (Polynomials $\mathcal{X}_\mu^{(p)}$ with general p will emerge in the limiting process studied in the next section.)

The remaining metric functions $A_\mu^{(k)}$, U_μ and auxiliary functions $J_\mu(a^2)$ are explicit polynomials of coordinates x_μ , defined by the following relations⁷:

$$J_\mu(a^2) = \prod_{\nu \neq \mu} (x_\nu^2 - a^2) = \sum_k A_\mu^{(k)} (-a^2)^{N-1-k}, \quad (2.4)$$

$$U_\mu = J_\mu(x_\mu^2). \quad (2.5)$$

We also define auxiliary functions $J(a^2)$ and $A^{(k)}$,

$$J(a^2) = \prod_\nu (x_\nu^2 - a^2) = \sum_{k=0, \dots, N} A^{(k)} (-a^2)^{N-k}. \quad (2.6)$$

Properties of these functions are listed in appendix B.1.

Inspecting the polynomials (2.3), we see that the on-shell Kerr-NUT-(A)dS metric contains N parameters a_μ , these are related to *rotations*. They can be combined into polynomials $\mathcal{J}_\mu(x^2)$, $\mathcal{A}_\mu^{(k)}$, \mathcal{U}_μ , $\mathcal{J}(x^2)$, and $\mathcal{A}^{(k)}$ defined in a manner analogous to functions $J_\mu(a^2)$, $A_\mu^{(k)}$, U_μ , $J(a^2)$, and $A^{(k)}$, (2.4)–(2.6), with x_μ and a_μ ‘interchanged’. For example,

$$\mathcal{J}_\mu(x^2) = \prod_{\nu \neq \mu} (a_\nu^2 - x^2) = \sum_k \mathcal{A}_\mu^{(k)} (-x^2)^{N-1-k}, \quad (2.7)$$

$$\mathcal{U}_\mu = \mathcal{J}_\mu(a_\mu^2), \quad (2.8)$$

see appendix B.1.

The on-shell metric also contains the parameter λ related to the *cosmological constant* according to

$$\Lambda = (2N - 1)(N - 1)\lambda, \quad (2.9)$$

and parameters β_μ which encode *mass* and what is usually vaguely called *NUT parameters* or *NUT charges*.

Let us note here that we took advantage of the ‘vagueness’ of NUT charges β_μ and introduced a particular combination of parameters $\beta_\mu \mathcal{U}_\mu a_\mu^{2p+1}$ in (2.3) instead of writing just a simple parameter b_μ , which is common practice. The reason is that parameters β_μ introduced in this way will have a trivial scaling behavior under the limit performed below whereas parameters b_μ would have to be nontrivially rescaled to obtain the desired result.

⁷ For brevity, we do not indicate the dependence of $J_\mu(a^2)$, $A_\mu^{(k)}$, and U_μ on coordinates x_μ . A completely explicit notation should be $J_\mu(a^2; x_1, \dots, x_N)$, $A_\mu^{(k)}(x_1, \dots, x_N)$, etc. A similar remark also applies to functions $\mathcal{J}_\mu(x^2)$, $\mathcal{A}_\mu^{(k)}$, and \mathcal{U}_μ defined below.

The metric (2.1) can also be expressed using a different set of angular coordinates ϕ_α ,

$$\phi_\alpha = \lambda a_\alpha \sum_k A_\alpha^{(k)} \psi_k, \quad \psi_k = \sum_\alpha \frac{(-a_\alpha^2)^{N-1-k}}{\mathcal{U}_\alpha} \frac{\phi_\alpha}{\lambda a_\alpha}. \quad (2.10)$$

Since ϕ 's are just constant linear combinations of ψ 's, they are also Killing coordinates. In the maximally symmetric case, they are related to N independent planes of rotations. Using these angles, the metric takes the form

$$\mathbf{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_\alpha \frac{J_\mu(a_\alpha^2)}{\mathcal{U}_\alpha} \frac{1}{\lambda a_\alpha} d\phi_\alpha \right)^2 \right]. \quad (2.11)$$

2.2. The gauge freedom and counting of parameters

Summarizing the previous subsection, the on-shell Kerr-NUT-(A)dS geometry is given by the metric (2.1) or (2.11) with X_μ given by $\mathcal{X}_\mu^{(0)}$, (2.3). The geometry is labeled by parameters a_μ and β_μ . The remaining parameter λ is fixed by the cosmological constant through (2.9).

However, the parameters a_μ and β_μ are not independent. There exist a one-parametric gauge freedom in rescaling coordinates, metric functions, and parameters which leaves the metric in the same form. Namely, it reads

$$\begin{aligned} x_\mu &\rightarrow s x_\mu, & \phi_\alpha &\rightarrow \phi_\alpha, & \psi_k &\rightarrow s^{-(k+1)} \psi_k, \\ a_\mu &\rightarrow s a_\mu, & \beta_\mu &\rightarrow \beta_\mu, \\ X_\mu &\rightarrow s^{2N} X_\mu, & U_\mu &\rightarrow s^{2(N-1)} U_\mu, & A_\mu^{(k)} &\rightarrow s^{2k} A_\mu^{(k)}. \end{aligned} \quad (2.12)$$

One of the parameters a_μ can thus be set to a suitable value using this gauge freedom. We will use this freedom in the Lorentzian case, requiring gauge condition (2.28) below. Thus, for a fixed cosmological constant the on-shell Kerr-NUT-(A)dS metrics form $(2N - 1)$ -parametric family of solutions of the Einstein equations.

Inspecting metric (2.11) with $X_\mu = \mathcal{X}_\mu^{(0)}$, we can also observe that the dependence on the cosmological parameter λ is just a global scaling. Indeed, one can easily see that

$$\lambda \mathbf{g} = \mathbf{g}|_{\lambda=1}. \quad (2.13)$$

We will use this property mainly in the Euclidean case to eliminate a global scale in one part of the warped metric, see (3.16).

2.3. The orthonormal frame and signature

It is useful to introduce the following orthonormal frame of 1-forms:

$$\begin{aligned} \mathbf{e}^\mu &= \left(\frac{U_\mu}{X_\mu} \right)^{\frac{1}{2}} dx_\mu, \\ \hat{\mathbf{e}}^\mu &= \left(\frac{X_\mu}{U_\mu} \right)^{\frac{1}{2}} \sum_k A_\mu^{(k)} d\psi_k = \left(\frac{X_\mu}{U_\mu} \right)^{\frac{1}{2}} \sum_\alpha \frac{J_\mu(a_\alpha^2)}{\mathcal{U}_\alpha} \frac{d\phi_\alpha}{\lambda a_\alpha}, \end{aligned} \quad (2.14)$$

and the dual frame of vectors:

$$\mathbf{e}_\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \partial_{x_\mu}, \quad \hat{\mathbf{e}}_\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_k \frac{(-x_\mu^2)^{N-1-k}}{U_\mu}, \quad \partial_{\psi_k} = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_\alpha \frac{\mathcal{J}_\alpha(x_\mu^2)}{U_\mu} \lambda a_\alpha \partial_{\phi_\alpha}. \quad (2.15)$$

The form (2.1) of the metric indicates that there is a natural splitting of the tangent space into 2-planes spanned on \mathbf{e}_μ and $\hat{\mathbf{e}}_\mu$. These planes, which we call the μ -planes, are characterized by antisymmetric 2-forms

$$\omega^\mu = \mathbf{e}^\mu \wedge \hat{\mathbf{e}}^\mu. \quad (2.16)$$

The metric and the corresponding Levi-Civita tensor are then written as

$$\mathbf{g} = \sum_\mu (\mathbf{e}^\mu \mathbf{e}^\mu + \hat{\mathbf{e}}^\mu \hat{\mathbf{e}}^\mu), \quad \varepsilon = \omega^1 \wedge \dots \wedge \omega^N. \quad (2.17)$$

Although the expression (2.17) suggests that the (off-shell) Kerr-NUT-(A)dS metric is positive definite, it can actually describe geometries of various signatures. The normalization in (2.14) and (2.15) can be only ‘artificial’ and the positively normalized vectors and 1-forms can be, in fact, imaginary. The real signature is given by signs of the metric functions X_μ/U_μ and these depend on a particular choice of coordinate ranges. The signature can also be affected if we choose some of the coordinates to be Wick-rotated, i.e. purely imaginary.

We will be mainly interested in (i) geometries with the Euclidean signature and positive curvature and (ii) physical geometries of the Lorentzian signature and arbitrary sign of the curvature. Before specifying the coordinate ranges and Wick rotations in these cases, let us first identify the maximally symmetric geometry.

2.4. Maximally symmetric spaces

It has been already observed in [47] that the class of Kerr-NUT-(A)dS geometries contains also a trivial case of the maximally symmetric spaces. It is obtained by setting the NUT and mass parameters equal to zero, $\beta_\mu = 0$, while the parameters a_μ remain unrestricted. The metric functions X_μ then simplify into a common polynomial $\lambda \mathcal{J}(x^2)$ in the corresponding variable, $X_\mu = \lambda \mathcal{J}(x_\mu^2)$. The roots of this polynomial are exactly the parameters a_μ^2 .

With this choice and employing the orthogonality relations (B23), the metric (2.11) can be transformed into

$$\mathbf{g} = \sum_\mu \left[\frac{U_\mu}{\lambda \mathcal{J}(x_\mu^2)} dx_\mu^2 - \frac{J(a_\mu^2)}{U_\mu} \frac{1}{\lambda a_\mu^2} d\phi_\mu^2 \right]. \quad (2.18)$$

Introducing $N + 1$ coordinates ρ_μ , $\mu = 0, 1, \dots, N$, instead of N coordinates x_μ , and employing the Jacobi transformation⁸

$$\lambda \rho_\mu^2 = \frac{J(a_\mu^2)}{-a_\mu^2 U_\mu}, \quad \lambda \rho_0^2 = \frac{J(0)}{\mathcal{J}(0)} = \frac{A^{(N)}}{\mathcal{A}^{(N)}}, \quad (2.19)$$

⁸ The expression for ρ_0 could be written in a way similar to other ρ_μ , provided we extended the set of parameters a_μ by one additional parameter $a_0 = 0$.

one can show that the new coordinates are restricted by the following constraint:

$$\lambda \sum_{\mu=0}^N \rho_{\mu}^2 = 1, \quad (2.20)$$

while the metric takes a simple form

$$\mathbf{g} = d\rho_0^2 + \sum_{\mu} [d\rho_{\mu}^2 + \rho_{\mu}^2 d\phi_{\mu}^2]. \quad (2.21)$$

Clearly, $\rho_0, \rho_{\mu}, \phi_{\mu}$ are multi-cylindrical coordinates on a $2N$ -dimensional (pseudo-)sphere, given by the constraint (2.20), embedded into a $(2N + 1)$ -dimensional flat space.

2.5. An Euclidean instanton

Assuming that all coordinates x_{μ}, ψ_k are real and parameters $a_{\mu}, \beta_{\mu}, \lambda$ are positive, the signature of the metric (2.1) is determined by signs of the metric functions X_{μ}/U_{μ} . To determine these signs we need to specify ranges of coordinates x_{μ} . All these coordinates, as well as corresponding parameters a_{μ} and β_{μ} , enter our metric in an equivalent way. However, it will be useful to relabel these quantities in such a way that parameters a_{μ} are ordered as follows,

$$0 < a_1 < a_2 < \dots < a_N. \quad (2.22)$$

In the maximally symmetric case $\beta_{\mu} = 0$, parameters a_{μ} coincide with the roots of the polynomials X_{μ} , and determine axes as well as restrict the ranges of coordinates x_{μ} . Zeros in functions U_{μ} can be avoided by choosing

$$-a_1 < x_1 < a_1 < x_2 < a_2 < \dots < x_N < a_N. \quad (2.23)$$

With angular coordinates restricted to a circle

$$-\pi < \phi_{\mu} < \pi, \quad (2.24)$$

the metric (2.11) describes a homogeneous metric on the Euclidean sphere with radius $\ell = 1/\sqrt{\lambda}$, that is written in ‘multi-elliptic’ coordinates parameterized by constants a_{μ} .

For sufficiently small positive values of parameters β_{μ} , the roots of metric functions X_{μ} change only slightly; they determine a more narrow range, now different for each latitude coordinate x_{μ} . Namely, we require

$${}^{(-)}x_{\mu} < x_{\mu} < {}^{(+)}x_{\mu}, \quad (2.25)$$

where ${}^{(\pm)}x_{\mu}$ are roots of X_{μ} such that $a_{\mu-1} < {}^{(-)}x_{\mu} < {}^{(+)}x_{\mu} < a_{\mu}$ (with ${}^{(-)}x_1 < -a_1 < {}^{(+)}x_1 < a_1$ for $\mu = 1$). With this choice the metric (2.1) represents the *Euclidean instanton* of signature $(+ + \dots +)$.

Parameters β_{μ} encode deformations of the geometry, namely how it deviates from the geometry of the sphere. Parameters a_{μ} are for non-vanishing β_{μ} also non-trivial—they do not label just a choice of coordinates, as in the maximally symmetric case, but they also change the geometry. For non-trivial values of β_{μ} the geometry does not have a curvature singularity—the singularity of the Riemann tensor occurs outside the coordinate range (2.23) of the Euclidean instanton.

The global definition and regularity of the geometry described by metrics (2.1) or (2.11) has to be concluded by specifying which Killing angles should be identified. Any linear

combination of Killing coordinates (with constant coefficients) forms again a Killing coordinate and it is not *a priori* clear which of the Killing coordinates should be periodic. Typically, angles ψ_k are *not* those which should be periodic. Since Killing coordinates are non-trivially coupled in the metric (the metric is not diagonal in these directions), a particular choice of the periodicity of Killing coordinates can introduce a non-trivial twisting of the geometry, as well as possible irregularities on the axes.

We will not discuss these characteristics in more detail and for our purposes we will simply remember that the Euclidean instanton describes a deformed and twisted spherical-like geometry. (For more details we refer the interested reader to a vast literature on the subject of gravitational instantons, e.g. [20–23, 48–55].)

2.6. Black hole solutions

The Lorentzian signature can be achieved by Wick-rotating some of the coordinates and parameters. Different choices can lead to different interpretations of the metric. We restrict ourselves to the case where the coordinate x_N is Wick-rotated to a radial coordinate r and the angular coordinate ϕ_N to the time coordinate t . The remaining coordinates retain their original character. Namely, we set

$$x_N^2 = -r^2, \quad \phi_N = \lambda a_N t, \quad (2.26)$$

with r and t real, and introduce a real mass parameter m by

$$\beta_N = m\lambda^{N-\frac{3}{2}}. \quad (2.27)$$

Notice that all coordinates ψ_k remain real. We also use the invariance (2.12) of the metric to correlate a_N with λ ,

$$a_N^2 = -\frac{1}{\lambda}. \quad (2.28)$$

Other parameters a_μ remain restricted as in the Euclidean case

$$0 < a_1 < a_2 < \dots < a_{N-1} < a_N, \quad (2.29)$$

where the last inequality applies only for a_N real (i.e. for $\lambda < 0$). The ranges of coordinates $x_{\bar{\mu}}$, $\bar{\mu} = 1, \dots, N-1$, are given again by roots of functions $X_{\bar{\mu}}$, as in (2.25).

With these choices metric (2.11) has the physical signature $(-+ \dots +)$. More explicitly, it reads

$$\begin{aligned} g = & -\frac{\Delta}{\Sigma} \left(\prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} dt - \sum_{\bar{\nu}} \frac{\bar{J}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \bar{U}_{\bar{\nu}}} d\phi_{\bar{\nu}} \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \sum_{\bar{\mu}} \frac{(r^2 + x_{\bar{\mu}}^2) \bar{U}_{\bar{\mu}}}{-\mathcal{X}_{\bar{\mu}}^{(0)}} dx_{\bar{\mu}}^2 \\ & + \sum_{\bar{\mu}} \frac{-\mathcal{X}_{\bar{\mu}}^{(0)}}{(r^2 + x_{\bar{\mu}}^2) \bar{U}_{\bar{\mu}}} \left(\frac{1 - \lambda r^2}{1 + \lambda x_{\bar{\mu}}^2} \prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} dt + \sum_{\bar{\nu}} \frac{(r^2 + a_{\bar{\nu}}^2) \bar{J}_{\bar{\mu}}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \bar{U}_{\bar{\nu}}} d\phi_{\bar{\nu}} \right)^2, \quad (2.30) \end{aligned}$$

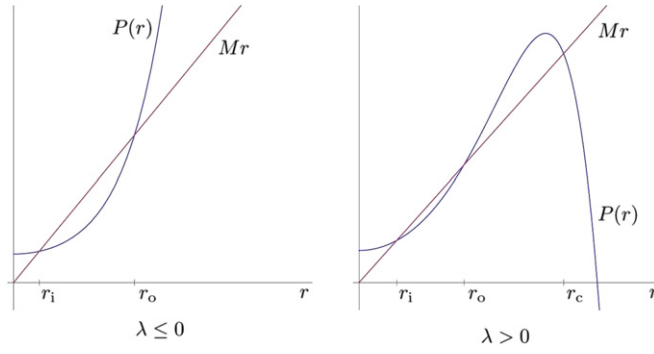


Figure 1. Black hole horizons. The zeros of Δ , (2.31), determine the horizons of the black hole. Δ has the following structure: $\Delta = P(r) - Mr$, where $P(r)$ is an even polynomial and M is a constant. The polynomial $P(r)$ has no real roots for $\lambda \leq 0$ and two real roots for $\lambda > 0$. The zeros of Δ correspond to intersections of the polynomial and the straight halfline. For $\lambda \leq 0$ there can be two intersections (outer and inner horizons), one touching the intersection (extremal horizon) or no intersections (naked singularity). For $\lambda > 0$ there is one additional intersection corresponding to the cosmological horizon.

with

$$\begin{aligned} \Delta &= -\mathcal{X}_N^{(0)} = (1 - \lambda r^2) \prod_{\bar{\nu}} (r^2 + a_{\bar{\nu}}^2) - 2m r \prod_{\bar{\nu}} (1 + \lambda a_{\bar{\nu}}^2), \\ -\mathcal{X}_{\bar{\mu}}^{(0)} &= (1 + \lambda x_{\bar{\mu}}^2) \bar{\mathcal{J}}(x_{\bar{\mu}}^2) - 2\beta_{\bar{\mu}} a_{\bar{\mu}} (1 + \lambda a_{\bar{\mu}}^2) \bar{\mathcal{U}}_{\bar{\mu}} x_{\bar{\mu}}, \\ \Sigma &= U_N = \prod_{\bar{\nu}} (r^2 + x_{\bar{\nu}}^2). \end{aligned} \quad (2.31)$$

Here, the barred indices take values $\bar{\mu}, \bar{\nu} = 1, \dots, N - 1$ and functions $\bar{\mathcal{J}}_{\bar{\mu}}, \bar{\mathcal{U}}_{\bar{\mu}}, \bar{\mathcal{U}}_{\bar{\mu}}$ and $\bar{\mathcal{J}}$ are defined by expressions (2.4) with restricted sets of coordinates $x_{\bar{\mu}}$ and parameters $a_{\bar{\mu}}$ (without x_N and a_N).

The roots of the metric function Δ determine horizons. For $\lambda \leq 0$ there can be two roots (outer and inner horizons), one double root (extremal horizon) or no roots (the case of naked singularity). For $\lambda > 0$ there is always one additional root representing the cosmological horizon. See figure 1.

For $m \neq 0$ and $\beta_{\bar{\mu}} = 0$, $\bar{\mu} = 1, \dots, N - 1$, the parameters $a_{\bar{\mu}}$, $\bar{\mu} = 1, \dots, N - 1$, can be identified with the rotation parameters of the black hole. However, for non-trivial NUT parameters $\beta_{\bar{\mu}}$ the interpretation is more unclear. Similar to the Euclidean instanton, both sets of parameters $\beta_{\bar{\mu}}$ and $a_{\bar{\mu}}$, as well as the specification of the periodicity of the Killing coordinates, affect the deformation of the geometry. Since the Lorentzian signature is involved, the geometry has in general NUT-like irregular behavior including the existence of closed time-like curves around the axes.

2.7. The Killing tower

Let us conclude this section by a brief review of the symmetry structure of the (off-shell) Kerr-NUT-(A)dS spacetimes; see [36, 37, 39, 40] for more details. As mentioned in the introduction, the *canonical form* of Kerr-NUT-(A)dS spacetimes (2.1) is uniquely determined

by the existence of a non-degenerate CCKY 2-form \mathbf{h} [24, 25, 27]⁹. This 2-form reads

$$\mathbf{h} = \sum_{\mu} x_{\mu} \omega^{\mu} = \sum_{\mu} x_{\mu} dx_{\mu} \wedge \sum_k A_{\mu}^{(k)} d\psi_k, \tag{2.32}$$

and satisfies the CCKY equation

$$\nabla^a h_{bc} = g_{ab} \xi_c - g_{ac} \xi_b, \tag{2.33}$$

with the *primary Killing–Yano 1-form* ξ given by

$$\xi = \frac{1}{D-1} \nabla \cdot \mathbf{h} = \sum_{\mu} \frac{X_{\mu}}{U_{\mu}} \sum_k A_{\mu}^{(k)} d\psi_k. \tag{2.34}$$

Raising the index, ξ turns into a *primary Killing vector*

$$\xi = l_{(0)} = \partial_{\psi_0}. \tag{2.35}$$

The non-degenerate CCKY 2-form \mathbf{h} generates a full *Killing tower* of symmetries [36, 56]. Namely, it defines N CCKY forms $\mathbf{h}^{(k)}$, $k = 0, \dots, N-1$, of rank $2k$,

$$\mathbf{h}^{(k)} = \frac{1}{k!} \frac{1}{\sqrt{\mathcal{A}^{(k)}}} \mathbf{h}^{\wedge k}. \tag{2.36}$$

Each of these CCKY forms then defines the following Killing–Yano (KY) forms $f^{(k)}$ of rank $2(N-k)$:

$$f^{(k)} = * \mathbf{h}^{(k)}, \tag{2.37}$$

and the following rank-2 Killing tensors $\mathbf{k}^{(k)}$:

$$\mathbf{k}^{(k)} = f^{(k)} \underset{2N-2k-1}{\bullet} f^{(k)}. \tag{2.38}$$

Here \bullet denotes the partial contraction of two antisymmetric forms in the first p indices divided by $p!$,

$$(\alpha \underset{p}{\bullet} \beta)_{a\dots b\dots} = \frac{1}{p!} \alpha_{n_1\dots n_p a\dots} \beta^{n_1\dots n_p} b\dots, \tag{2.39}$$

where the upper indices were raised using the metric. If no index p is indicated, the full contraction in all indices is assumed. The primary Killing vector ξ and the above Killing tensors then define N Killing vectors $l_{(j)}$ according to

$$l_{(j)} = \mathbf{k}_{(j)} \cdot \xi. \tag{2.40}$$

In particular, we find the following expressions for the Killing vectors and Killing tensors:

$$l_{(j)} = \frac{1}{\mathcal{A}^{(j)}} \partial_{\psi_j}, \tag{2.41}$$

⁹ Roughly speaking, while coordinates x_{μ} are related to the eigenvalues of \mathbf{h} , coordinates ψ_k are the Killing coordinates corresponding to the Killing vectors generated from \mathbf{h} by formula (2.40) below. The functional independence of all these coordinates is guaranteed by a ‘non-degeneracy’ condition imposed on \mathbf{h} .

$$\begin{aligned}
 \mathbf{k}_{(j)} &= \frac{1}{\mathcal{A}^{(j)}} \sum_{\mu} A_{\mu}^{(j)} (\mathbf{e}_{\mu} \mathbf{e}_{\mu} + \hat{\mathbf{e}}_{\mu} \hat{\mathbf{e}}_{\mu}) \\
 &= \sum_{\mu} \frac{A_{\mu}^{(j)}}{\mathcal{A}^{(j)}} \left[\frac{X_{\mu}}{U_{\mu}} \partial_{x_{\mu}}^2 + \frac{U_{\mu}}{X_{\mu}} \left(\sum_k \frac{(-x_{\mu}^2)^{N-1-k}}{U_{\mu}} \partial_{\psi_k} \right)^2 \right]. \tag{2.42}
 \end{aligned}$$

Note that, in expressions (2.41) and (2.42) we have a normalization prefactor $1/\mathcal{A}^{(j)}$ that originates from a definition (2.36). Such normalization is not used in the literature. Since this prefactor is just a constant (a combination of parameters a_{μ}), it does not influence the Killing property of the defined objects. However, it will be useful in the limiting procedure below.

The above Killing tower of symmetries stands behind many of the remarkable integrability properties of Kerr-NUT-(A)dS spacetimes, see [28–39] for more details. In particular, the N Killing vectors $\mathbf{l}_{(j)}$ together with the N rank-2 Killing tensors $\mathbf{k}_{(j)}$ define $2N$ integrals for geodesic motion that are independent and in involution, making this motion completely integrable in the Liouville sense [35–39].

This property remains true for the twisted and deformed black holes obtained by a limiting procedure in the next section. Namely, as we shall see, all the objects from the Killing tower do survive the limit and correspondingly obey the same mutual commutation relations that guarantee the above integrability.

3. Twisted and deformed black holes

3.1. Multiple-spin-zero limit

Let us now concentrate on a situation when the Kerr-NUT-(A)dS spacetimes describe the Lorentzian black hole solution, studied in section 2.6. In this case we have the following ordering of coordinates x_{μ} and parameters a_{μ} :

$$x_1 < a_1 < x_2 < a_2 < \dots < x_{N-1} < a_{N-1}. \tag{3.1}$$

Parameters a_{μ} are closely related to rotations: for vanishing NUT parameters they are exact rotations [18, 19]—in the presence of NUTs a proper identification is more delicate. Nevertheless, in both cases it seems that ‘unspinning’ the rotations is equivalent to sending these parameters to zero.

However, it is obvious from (3.1) that our metrics (2.1) or (2.30) are not suitable for a straightforward limit of vanishing parameters a_{μ} , or for a limit when such parameters coincide. If such limits were performed naively, some of the coordinates x_{μ} would degenerate. Since x_{μ} are eigen-values of the principal CCKY tensor \mathbf{h} , the latter becomes degenerate in this limit. But what is worse, the metric itself would not be well defined. In the following we show that a well defined limit $a_{\mu} \rightarrow 0$ can be achieved by a proper simultaneous scaling of corresponding coordinates x_{μ} . We first perform the limit for the canonical metric (2.1), then specialize to the black hole case.

Namely, let us consider a *multiple-spin-zero* limit in which the first \bar{N} parameters $a_{\bar{\mu}}$ and the corresponding coordinates $x_{\bar{\mu}}$ are homogeneously rescaled to zero, while the remaining $\tilde{N} = N - \bar{N}$ parameters and coordinates are left unscaled but renamed to be indexed from 1 to \tilde{N} . The Killing coordinates ϕ_{μ} , NUT parameters β_{μ} , and λ remain unscaled but they also get appropriately renamed. We denote by $\tilde{\mathcal{M}}$ a $2\tilde{N}$ -dimensional submanifold which emerges in the limiting procedure—it is spanned on the first \tilde{N} μ -planes. Since we are sending the corresponding rotations to zero, we call it the *unspinned sector*. The complementary

$2\tilde{N}$ -dimensional submanifold $\tilde{\mathcal{M}}$ emerging as a space spanned on the unscaled μ -planes will be called the *regular sector*. Similarly, we say that the scaled (unscaled) coordinates and parameters belong to the unspinned (regular) sector, respectively.

If σ is a scaling parameter which goes to zero, $\sigma \rightarrow 0$, the limit is characterized by

$$\begin{array}{ll} \text{unspinned} & \text{regular} \\ a_{\bar{\mu}} = \sigma \bar{a}_{\mu}, & a_{\tilde{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}, \\ \beta_{\bar{\mu}} = \bar{\beta}_{\mu}, & \beta_{\tilde{N}+\tilde{\mu}} = \tilde{\beta}_{\tilde{\mu}}, \end{array} \quad (3.2)$$

$$\begin{array}{ll} x_{\bar{\mu}} = \sigma \bar{x}_{\mu}, & x_{\tilde{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}, \\ \phi_{\bar{\mu}} = \bar{\phi}_{\mu}, & \phi_{\tilde{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}. \end{array} \quad (3.3)$$

We use the barred indices $\bar{\mu}, \bar{\nu}, \dots$ and \bar{k}, \bar{l}, \dots for the unspinned quantities, and the tilded indices $\tilde{\mu}, \tilde{\nu}, \dots$ and $\tilde{k}, \tilde{l}, \dots$ for regular quantities. They take the following values:

$$\begin{array}{ll} \bar{\mu}, \bar{\nu}, \dots = 1, \dots, \tilde{N}, & \tilde{\mu}, \tilde{\nu}, \dots = 1, \dots, \tilde{N}, \\ \bar{k}, \bar{l}, \dots = 0, \dots, \tilde{N} - 1, & \tilde{k}, \tilde{l}, \dots = 0, \dots, \tilde{N} - 1. \end{array} \quad (3.4)$$

Let \approx denotes the equality valid up to the terms of the higher order in σ . Employing relations (2.10) between angular coordinates ϕ_{μ} and ψ_k , we find

$$\psi_{\tilde{k}} \approx \tilde{\psi}_{\tilde{k}}, \quad \psi_{\tilde{N}+\tilde{k}} \approx \sigma^{-(2\tilde{k}+1)} \frac{1}{\tilde{\mathcal{A}}^{(\tilde{N})}} \bar{\psi}_{\tilde{k}}. \quad (3.5)$$

Note that, comparing equations (3.3) and (3.5), the new coordinates are ‘enumerated’ differently for Latin and Greek indices. Namely, for Latin indices the unspinned sector corresponds to ‘lower’ values and the regular sector to ‘upper’ values of indices, whereas this is opposite for the Greek indices.

At the same time, metric functions (2.4) and (2.6) give

$$\begin{array}{ll} U_{\bar{\mu}} \approx \sigma^{2(\tilde{N}-1)} \tilde{\mathcal{A}}^{(\tilde{N})} \bar{U}_{\bar{\mu}}, & U_{\tilde{N}+\tilde{\mu}} \approx (-\tilde{x}_{\tilde{\mu}}^2)^{\tilde{N}} \tilde{U}_{\tilde{\mu}}, \\ A_{\bar{\mu}}^{(\bar{k})} \approx \bar{A}^{(\bar{k})}, & A_{\tilde{\mu}}^{(\tilde{N}+\bar{k})} \approx \sigma^{2\bar{k}} \tilde{\mathcal{A}}^{(\tilde{N})} \bar{A}_{\tilde{\mu}}^{(\bar{k})}, \\ A_{\tilde{N}+\tilde{\mu}}^{(\bar{k})} \approx \tilde{A}_{\tilde{\mu}}^{(\bar{k})}, & A_{\tilde{N}+\tilde{\mu}}^{(\tilde{N}+\bar{k})} \approx \sigma^{2(\bar{k}+1)} \tilde{\mathcal{A}}^{(\tilde{N}-1)} \bar{A}^{(\bar{k}+1)}, \\ A^{(\bar{k})} \approx \bar{A}^{(\bar{k})}, & A^{(\tilde{N}+\bar{k})} \approx \sigma^{2\bar{k}} \tilde{\mathcal{A}}^{(\tilde{N})} \bar{A}^{(\bar{k})}, \end{array} \quad (3.7)$$

and analogous relations hold for \mathcal{U}_{μ} , $\mathcal{A}_{\mu}^{(k)}$, and $\mathcal{A}^{(k)}$. Starting from (2.3), we also have

$$\mathcal{X}_{\bar{\mu}}^{(p)} \approx \sigma^{2\tilde{N}} \tilde{\mathcal{A}}^{(\tilde{N})} \bar{\mathcal{X}}_{\bar{\mu}}^{(p)}, \quad \mathcal{X}_{\tilde{N}+\tilde{\mu}}^{(p)} \approx (-\tilde{x}_{\tilde{\mu}}^2)^{\tilde{N}} \tilde{\mathcal{X}}_{\tilde{\mu}}^{(p+\tilde{N})}. \quad (3.8)$$

Hence, we have obtained the following relations for various subterms of the metric (2.1):

$$\begin{array}{ll} \frac{\mathcal{X}_{\bar{\mu}}^{(p)}}{U_{\bar{\mu}}} \approx \sigma^2 \frac{\tilde{\mathcal{A}}^{(\tilde{N})}}{\bar{\mathcal{A}}^{(\tilde{N})}} \frac{\bar{\mathcal{X}}_{\bar{\mu}}^{(p)}}{\bar{U}_{\bar{\mu}}}, & \frac{\mathcal{X}_{\tilde{N}+\tilde{\mu}}^{(p)}}{U_{\tilde{N}+\tilde{\mu}}} \approx \frac{\tilde{\mathcal{X}}_{\tilde{\mu}}^{(p+\tilde{N})}}{\tilde{U}_{\tilde{\mu}}}, \\ \sum_k A_{\bar{\mu}}^{(k)} d\psi_k \approx \frac{1}{\sigma} \frac{\tilde{\mathcal{A}}^{(\tilde{N})}}{\bar{\mathcal{A}}^{(\tilde{N})}} \sum_{\bar{k}} \bar{A}_{\bar{\mu}}^{(\bar{k})} d\bar{\psi}_{\bar{k}}, & \sum_k A_{\tilde{N}+\tilde{\mu}}^{(k)} d\psi_k \approx \sum_{\tilde{k}} \tilde{A}_{\tilde{\mu}}^{(\tilde{k})} d\tilde{\psi}_{\tilde{k}}. \end{array} \quad (3.9)$$

In all these expressions barred (or tilded) functions are defined in terms of $\bar{a}_{\bar{\mu}}$ and $\bar{x}_{\bar{\mu}}$ (or $\tilde{a}_{\tilde{\mu}}$ and $\tilde{x}_{\tilde{\mu}}$) in a way analogous to how unbarred functions are defined in terms of a_{μ} and x_{μ} . In other words, in both regular and unspinned sectors we use the same functional definitions, using the appropriate coordinates and parameters for a given sector.

3.2. Twisted and deformed black holes

Putting everything together, we find that in the leading order of σ expansion the on-shell Kerr-NUT-(A)dS metric decouples to a warped product of the base and seed metrics in the regular and unspinned sectors,

$$\mathbf{g} \approx \tilde{\mathbf{g}} + \tilde{w}^2 \bar{\mathbf{g}}. \quad (3.10)$$

In the regular sector, the base metric $\tilde{\mathbf{g}}$ is the off-shell $2\tilde{N}$ -dimensional Kerr-NUT-(A)dS metric spanned on coordinates $\tilde{x}_{\tilde{\mu}}, \tilde{\psi}_{\tilde{k}}$, characterized by parameters $\tilde{a}_{\tilde{\mu}}, \tilde{\beta}_{\tilde{\mu}}$, and metric functions

$$\tilde{X}_{\tilde{\mu}} = \tilde{\mathcal{X}}_{\tilde{\mu}}^{(\tilde{N})}. \quad (3.11)$$

In the unspinned sector, the seed metric $\bar{\mathbf{g}}$ is the on-shell $2\tilde{N}$ -dimensional Kerr-NUT-(A)dS metric spanned on $\bar{x}_{\tilde{\mu}}, \bar{\psi}_{\tilde{k}}$, characterized by parameters $\bar{a}_{\tilde{\mu}}, \bar{\beta}_{\tilde{\mu}}$, and metric functions

$$\bar{X}_{\tilde{\mu}} = \bar{\mathcal{X}}_{\tilde{\mu}}^{(0)}. \quad (3.12)$$

The warp-factor \tilde{w} is

$$\tilde{w}^2 = \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} = \frac{\tilde{x}_1^2 \dots \tilde{x}_{\tilde{N}}^2}{\tilde{a}_1^2 \dots \tilde{a}_{\tilde{N}}^2}. \quad (3.13)$$

It depends only on the tilded variables, i.e. on the regular sector. The Levi-Civita tensor associated with the obtained metric reads

$$\varepsilon \approx \tilde{w}^{2\tilde{N}} \tilde{\varepsilon} \wedge \bar{\varepsilon}. \quad (3.14)$$

One can easily show that the obtained warped geometry (3.10) still satisfies the vacuum Einstein equations with the cosmological constant given by (2.9); for more details see [46]. One can also calculate the Riemann tensor and the Kretschmann scalar for such metrics, as described in appendix B.2.

For the Lorentzian signature, it is useful to eliminate the dependence of the unspinned metric $\bar{\mathbf{g}}$ on the curvature scale λ . After the Wick rotation (2.26) and normalization (2.28), the warp-factor \tilde{w} becomes

$$\tilde{w}^2 = \lambda \tilde{r}^2 \frac{\tilde{x}_1^2 \dots \tilde{x}_{\tilde{N}-1}^2}{\tilde{a}_1^2 \dots \tilde{a}_{\tilde{N}-1}^2}. \quad (3.15)$$

As we discussed in (2.13), metric $\lambda \bar{\mathbf{g}} = \bar{\mathbf{g}}|_{\lambda=1}$ is independent of λ . Thus,

$$\mathbf{g} \approx \tilde{\mathbf{g}} + \tilde{r}^2 \frac{\tilde{x}_1^2 \dots \tilde{x}_{\tilde{N}-1}^2}{\tilde{a}_1^2 \dots \tilde{a}_{\tilde{N}-1}^2} \bar{\mathbf{g}}|_{\lambda=1}. \quad (3.16)$$

The unspinned metric $\bar{\mathbf{g}}$ describes the on-shell Euclidean instanton, i.e. a deformed twisted sphere. In the black hole case, the regular metric $\tilde{\mathbf{g}}$ has a Lorentzian signature and describes a part of the metric which includes a time coordinate, radial coordinate, and rotating directions. Notice that the regular metric belongs only to the off-shell class of Kerr-NUT-(A)dS metrics since the metric functions $\tilde{X}_{\tilde{\mu}}$ are modified and their form depends on the dimensionality of the unspinned sector, see (3.11).

Inspecting metric (3.10), we observe that although we took the limit of vanishing $a_{\tilde{\mu}} = \sigma \bar{a}_{\tilde{\mu}} \rightarrow 0$, the unspinned parameters $\bar{a}_{\tilde{\mu}}$ remain in the resulting metric. The limit only ‘decouples’ both sets of regular and unspinned variables and parameters. Counting parameters in the obtained metric, we have $2\tilde{N}$ parameters $\tilde{a}_{\tilde{\mu}}, \tilde{\beta}_{\tilde{\mu}}$ labeling regular metric $\tilde{\mathbf{g}}$ and $2\tilde{N}$

parameters $\bar{a}_{\bar{\mu}}, \bar{\beta}_{\bar{\mu}}$ labeling unspinned metric \bar{g} . However, now we can use the rescaling freedom (2.12) for both parts of the metric independently. Two of the parameters are therefore superfluous and they can be fixed by gauge fixing conditions. The limiting procedure therefore eliminated only one parameter from the original metric.

We have thus obtained a new rich class of higher-dimensional black hole solutions that solve the vacuum Einstein equations with the cosmological constant and contain one parameter less than the original Kerr-NUT-(A)dS spacetimes. Because of the presence of NUT parameters the new spacetimes represent a non-trivial generalization of Kerr-AdS black holes obtained in [17–19] which no longer belongs, due to the singular character of the scaling limit, to the Kerr-NUT-(A)dS class.

3.3. Non-rotating black holes with twist

An important special case of the limit occurs for $\tilde{N} = 1, \bar{N} = N - 1$. Since we are interested in the physical signature, we employ the normalization (2.28), and the Wick rotation (2.26),

$$\begin{aligned} \lambda r^2 &= \frac{x_{\tilde{N}}^2}{a_{\tilde{N}}^2} = \frac{\tilde{x}_1^2}{\tilde{a}_1^2} = \tilde{w}^2, \\ t &= a_N \phi_N = a_N \tilde{\phi}_1 \approx \psi_0 = \tilde{\psi}_0. \end{aligned} \tag{3.17}$$

Introducing the mass parameter (2.27), the metric function $\tilde{X}_1 = \tilde{\mathcal{X}}_1^{(N)} = -f$ can be rewritten as

$$f = 1 - \lambda r^2 - \frac{2m}{r^{2N-3}}. \tag{3.18}$$

With these definitions, the limiting metric takes the following form:

$$g \approx -fdt^2 + \frac{1}{f}dr^2 + r^2\bar{g}|_{\lambda=1}. \tag{3.19}$$

It describes a deformed and twisted static black hole with an angular part given by the Euclidean instanton $\bar{g}|_{\lambda=1}$.

In particular for $\bar{\beta}_{\bar{\mu}} = 0$, the unspinned metric simplifies to a maximally symmetric geometry, that is, $\bar{g}|_{\lambda=1}$ describes the Euclidean $2\bar{N}$ -dimensional sphere of unit radius. We thus recovered the Schwarzschild–Tangherlini solution describing a non-rotating higher-dimensional black hole. The metric \bar{g} on the sphere is, however, written in multi-elliptical coordinates that are characterized by constants $\bar{a}_{\bar{\mu}}$.

3.4. Limit of the Killing tower

We now turn to the limit of the symmetry objects in the Killing tower. Obviously, since in the limit $\sigma \rightarrow 0$ the first \bar{N} eigenvalues $x_{\bar{\mu}}$ of the CCKY tensor \mathbf{h} vanish, this tensor becomes a degenerate CCKY tensor $\tilde{\mathbf{h}}$ and the resulting metric has to belong to the class of generalized Kerr-NUT-(A)dS spacetimes [44, 45], reviewed in appendix A. However, as we show now, the obtained metric is a very distinguished one in this class as it possesses an enhanced symmetry of the full Killing tower. In particular, we show that the limiting $\tilde{\mathbf{h}}$ can be understood as a non-degenerate CCKY on the submanifold $\tilde{\mathcal{M}}$ and that another 2-form $\tilde{\mathbf{h}}$ emerges in the limit and effectively plays the role of a non-degenerate CCKY tensor on the unspinned submanifold $\bar{\mathcal{M}}$. (See appendix B.3 for more details.)

First, employing (3.3) and (3.9), we obtain the following expressions for 2-forms ω^μ :

$$\omega^{\tilde{\mu}} \approx \tilde{w}^2 \tilde{\omega}^{\tilde{\mu}}, \quad \omega^{\tilde{N}+\tilde{\mu}} \approx \tilde{\omega}^{\tilde{\mu}}. \quad (3.20)$$

The unspinned 2-forms $\tilde{\omega}^{\tilde{\mu}}$ are normalized in the sense of metric \tilde{g} . The warp factor guarantees the normalization of $\omega^{\tilde{\mu}}$ with respect to g .

Starting from expression (2.32) for h , only terms with regular coordinates $x_{\tilde{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$ survive in the leading order and we get

$$h \approx \tilde{h} = \sum_{\tilde{\mu}} \tilde{x}_{\tilde{\mu}} \tilde{\omega}^{\tilde{\mu}}. \quad (3.21)$$

The limit of the primary KY form ξ is straightforward, see (2.34) and (3.9),

$$\xi \approx \tilde{\xi}. \quad (3.22)$$

The limit (3.21) of the CCKY form h is degenerate: it vanishes in the unspinned sector. Therefore the limiting metric does not have the canonical form (2.1) and the obtained \tilde{h} cannot be used for a direct construction of the Killing tower. (This tensor can, however, be understood as a non-degenerate CCKY 2-form on the base manifold $\tilde{\mathcal{M}}$, see (3.21)). Nevertheless, we can find the limits of all objects $h^{(k)}$, $f^{(k)}$, $k_{(k)}$, and $l_{(k)}$ directly.

Had we used the standard definitions of the Killing tower objects without a prefactor given by suitable powers of $\mathcal{A}^{(k)}$ (see the discussion below expressions (2.41) and (2.42)), we would find that some of the objects in the limit vanished. However, it would be possible to extract the coefficient of the leading term in the σ expansion. Instead, this can be simply achieved by including the discussed prefactors into the definitions of Killing objects—under the limit these prefactors supply necessary powers of the scaling parameter σ to obtain a finite result.

After the limit, the Killing tower naturally splits into two sets, one associated with the metric \tilde{g} of the unspinned sector and the other associated with the regular metric \tilde{g} . Here, we present only the resulting expressions; details of the limiting process can be found in appendix B.3. The limits are

$$\begin{aligned} h^{(\tilde{k})} &\approx \tilde{h}^{(\tilde{k})}, \\ h^{(\tilde{N}+\tilde{k})} &\approx \tilde{w}^{2\tilde{k}+1} \tilde{\varepsilon} \wedge \tilde{h}^{(\tilde{k})}. \end{aligned} \quad (3.23)$$

Similarly, for the KY forms we obtain

$$\begin{aligned} f^{(\tilde{k})} &\approx \tilde{w}^{2\tilde{N}} \tilde{f}^{(\tilde{k})} \wedge \tilde{\varepsilon}, \\ f^{(\tilde{N}+\tilde{k})} &\approx \tilde{w}^{2(\tilde{N}-\tilde{k})+1} \tilde{f}^{(\tilde{k})}. \end{aligned} \quad (3.24)$$

The Killing tensors are quadratic in KY forms according to (2.38). The limit gives

$$\begin{aligned} k_{(\tilde{k})} &\approx \tilde{k}_{(\tilde{k})} + \frac{\tilde{A}^{(\tilde{k})}}{\tilde{\lambda}^{(\tilde{k})}} \tilde{w}^{-2} \tilde{g}^{-1}, \\ k_{(\tilde{N}+\tilde{k})} &\approx \tilde{k}_{(\tilde{k})}. \end{aligned} \quad (3.25)$$

Finally, we perform the limit of the Killing vectors. In this case, however, even after employing the normalization factor $\mathcal{A}^{(k)}$ in (2.41), the limit of the Killing vectors related to the unspinned coordinates is not finite¹⁰

$$\begin{aligned} \mathbf{l}_{(\tilde{k})} &\approx \tilde{\mathbf{l}}_{(\tilde{k})} = \frac{1}{\tilde{\mathcal{A}}^{(\tilde{k})}} \partial_{\tilde{\psi}_{\tilde{k}}}, \\ \mathbf{l}_{(\tilde{N}+\tilde{k})} &\approx \sigma \tilde{\mathbf{l}}_{(\tilde{k})} = \sigma \frac{1}{\tilde{\mathcal{A}}^{(\tilde{k})}} \partial_{\tilde{\psi}_{\tilde{k}}}. \end{aligned} \quad (3.26)$$

Nevertheless, the leading term of the Killing vectors $\mathbf{l}_{(\tilde{N}+\tilde{k})}$ is proportional to the Killing vector $\tilde{\mathbf{l}}_{(\tilde{k})}$ of the unspinned part $\tilde{\mathbf{g}}$ of the metric. The limiting metric thus admits at least the same number of explicit Killing vectors as the original one, only their relation to the original Killing vectors is slightly more complicated.

It is possible to check that both expressions (2.37) and (2.38), relating CCKY forms $\mathbf{h}^{(j)}$, KY forms $\mathbf{f}^{(j)}$, and Killing tensors $\mathbf{k}_{(j)}$ remain valid after the limit. The relation (2.40) between the Killing vectors $\mathbf{l}_{(j)}$ and Killing tensors $\mathbf{k}_{(j)}$ holds also in the leading order. However, with the normalization (3.26) it remains non-degenerate only for vectors related to regular coordinates, i.e. for $j = \tilde{j} = 0, \dots, \tilde{N} - 1$. For $j = \tilde{N} + \tilde{j} = \tilde{N}, \dots, N - 1$ this relation degenerates in the zeroth order and one has to study next order terms in ξ and $\mathbf{k}_{(\tilde{N}+\tilde{k})}$ to establish its validity.

To summarize, we have found that although the limiting metric \mathbf{g} does not admit a *non-degenerate* CCKY form, it possesses the full Killing tower of symmetries that is similar to the Killing tower of the original metric. It splits into two sectors which are closely related to the Killing towers of regular metric $\tilde{\mathbf{g}}$ and unspinned metric $\bar{\mathbf{g}}$; see relations (3.23)–(3.26). Since the resulting Killing tower is obtained by the limiting procedure, the surviving symmetries satisfy the same mutual relations as the original symmetries, namely, they mutually commute. This then guarantees integrability and separability properties similar to the those of the original Kerr-NUT-(A)dS spacetimes [28–40]. The integrability and separability properties based on the Killing tower of a warped space (3.10) has been also recently discussed in a slightly broader context of metrics that admit complete separability of the Klein–Gordon equation [30].

4. Deformed black holes

Our aim in this section is to completely switch off all the rotations and twists. In such a way we construct so the static deformed non-twisted black holes whose deformations are caused by the presence of non-trivial NUT parameters.

In the previous section, we have studied the multiple-spin-zero limit in which several unspinned coordinates x_μ and their corresponding parameters were simultaneously rescaled to zero. We have found that the unspinned parameters survived this limit, but they decoupled from the regular coordinates and parameters. It would be natural to study now a limit in which we switch off just one rotation but perform such a limit successively several times, finally switching-off all the rotations. Since we have seen that NUT parameters survive the limit, the resulting metric should contain remnants of them. However, to see details of such a procedure we would need to generalize the limit discussed in the previous section to the case of general metric polynomial $\mathcal{X}_\mu^{(p)}$, see equation (2.3). Indeed, already after the first $\tilde{N} = 1$ limit, the base metric $\tilde{\mathbf{g}}$ on $2\tilde{N} = 2N - 2$ dimensional manifold $\tilde{\mathcal{M}}$ is an off-shell metric with metric

¹⁰ One could have included an additional prefactor for $\mathbf{l}_{(\tilde{N}+\tilde{k})}$ which would behave as σ . However, we do not see any natural simple candidate for this, so we leave the explicit σ -dependence in final expressions.

functions given by $\mathcal{X}_\mu^{(1)}$. To perform the next $\tilde{N} = 1$ limit we would need to generalize the limit from the previous section to this more general case (we only studied the on-shell case with $\mathcal{X}_\mu^{(0)}$).

Such a generalization is possible and the successive application of the single-spin-zero limit can be performed. Interestingly, it turns out that the resulting geometry can be obtained also by a different limiting procedure in which we repeatedly rescale all coordinates except one. That is, we repeatedly perform the multiple-spin-zero limit $\tilde{N} = N - 1$, $\tilde{N} = 1$: in the first step applied to the original metric, in the second step to the unspinned metric $\tilde{\mathbf{g}}$, etc. In the following we will present details of such a limit.

The metric obtained by this combined limiting process will be denoted by $\mathring{\mathbf{g}}$. In the Lorentzian case, it describes the deformed (but non-twisted) black holes. Let $\mathring{\mathbf{h}}^{(j)}$, $\mathring{\mathbf{f}}^{(j)}$, and $\mathring{\mathbf{k}}_{(j)}$ be limits of CCKY forms $\mathbf{h}^{(j)}$, KY forms $\mathbf{f}^{(j)}$, and Killing tensors $\mathbf{k}_{(j)}$, respectively. They turn out to be finite and they are CCKY forms, KY forms, and Killing tensors of the limiting metric $\mathring{\mathbf{g}}$.

4.1. Atomic 2-metrics

To write down explicitly the limiting metric $\mathring{\mathbf{g}}$ and its symmetries, we first introduce some auxiliary objects. Let ${}^\alpha\mathbf{g}$ denote the canonical Kerr-NUT-(A)dS metric (2.11) in 2α number of dimensions, spanned on coordinates x_1, \dots, x_α and $\phi_1, \dots, \phi_\alpha$. Let ${}^\alpha\boldsymbol{\varepsilon}$ represent the corresponding Levi-Civita tensor and ${}^\alpha\mathbf{h}^{(j)}$, ${}^\alpha\mathbf{f}^{(j)}$, ${}^\alpha\mathbf{k}_{(j)}$, $j = 0, \dots, \alpha$ the symmetry objects in the corresponding Killing tower. We introduce ‘auxiliary’ 2α -dimensional metrics ${}^\alpha\mathring{\mathbf{g}}$ and ${}^\alpha\mathring{\mathbf{g}}$ and the corresponding Levi-Civita tensors ${}^\alpha\mathring{\boldsymbol{\varepsilon}}$ and ${}^\alpha\mathring{\boldsymbol{\varepsilon}}$ by the following expressions:

$$\begin{aligned} {}^\alpha\mathring{\mathbf{g}} &= \mathbf{q}_\alpha + \xi_\alpha^2 \mathbf{q}_{\alpha-1} + \xi_\alpha^2 \xi_{\alpha-1}^2 \mathbf{q}_{\alpha-2} + \dots + \xi_\alpha^2 \dots \xi_2^2 \mathbf{q}_1, \\ {}^\alpha\mathring{\boldsymbol{\varepsilon}} &= \boldsymbol{\varepsilon}_\alpha \wedge \xi_\alpha^2 \boldsymbol{\varepsilon}_{\alpha-1} \wedge \xi_\alpha^2 \xi_{\alpha-1}^2 \boldsymbol{\varepsilon}_{\alpha-2} \wedge \dots \wedge \xi_\alpha^2 \dots \xi_2^2 \boldsymbol{\varepsilon}_1, \end{aligned} \quad (4.1)$$

$$\begin{aligned} {}^\alpha\mathring{\mathbf{g}} &= \mathbf{q}_N + \xi_N^2 \mathbf{q}_{N-1} + \xi_N^2 \xi_{N-1}^2 \mathbf{q}_{N-2} + \dots \\ &\quad + \xi_N^2 \dots \xi_{N-\alpha+2}^2 \mathbf{q}_{N-\alpha+1}, \\ {}^\alpha\mathring{\boldsymbol{\varepsilon}} &= \boldsymbol{\varepsilon}_N + \xi_N^2 \boldsymbol{\varepsilon}_{N-1} \wedge \xi_N^2 \xi_{N-1}^2 \boldsymbol{\varepsilon}_{N-2} \wedge \dots \\ &\quad \wedge \xi_N^2 \dots \xi_{N-\alpha+2}^2 \boldsymbol{\varepsilon}_{N-\alpha+1}. \end{aligned} \quad (4.2)$$

The above definitions are expressed in terms of rescaled coordinates $\xi_\mu = \frac{x_\mu}{a_\mu}$ and ϕ_μ , through *atomic 2-metrics* \mathbf{q}_μ and 2-forms $\boldsymbol{\varepsilon}_\mu$

$$\begin{aligned} \mathbf{q}_\mu &= \frac{1}{\lambda} \left(\frac{1}{\Delta_\mu} d\xi_\mu^2 + \Delta_\mu d\phi_\mu^2 \right), \\ \boldsymbol{\varepsilon}_\mu &= \frac{1}{\lambda} d\xi_\mu \wedge d\phi_\mu, \end{aligned} \quad (4.3)$$

where

$$\Delta_\mu = 1 - \xi_\mu^2 - 2\beta_\mu \xi_\mu^{\alpha-2\mu+3}. \quad (4.4)$$

4.2. Deformed black holes

Now we summarize the results of the combined limiting process—the derivation is delegated to appendix B.4. The limiting metric is $\hat{g} = {}^N\hat{g} = {}^N\hat{g}$. The explicit expressions for it and the Levi-Civita tensors are

$$\begin{aligned}\hat{g} &= \mathbf{q}_N + \xi_N^2 \mathbf{q}_{N-1} + \cdots + \xi_N^2 \cdots \xi_2^2 \mathbf{q}_1, \\ \hat{\epsilon} &= \epsilon_N \wedge \xi_N^2 \epsilon_{N-1} \wedge \cdots \wedge \xi_N^2 \cdots \xi_2^2 \epsilon_1.\end{aligned}\quad (4.5)$$

If we perform the Wick rotation (2.26), and assume normalization (2.28), we can write the limit of the Lorentzian metric as

$$\begin{aligned}\hat{g} &= -f dt^2 + \frac{1}{f} dr^2 \\ &\quad + r^2 [\mathbf{q}_{N-1} + \xi_{N-1}^2 (\mathbf{q}_{N-2} + \cdots + \xi_3^2 (\mathbf{q}_2 + \xi_2^2 \mathbf{q}_1))] |_{\lambda=1},\end{aligned}\quad (4.6)$$

with f given by (3.18). Here, coordinates ξ_μ , $\mu > 1$, run between roots of Δ_μ , (4.4), which are for small positive β_μ inside interval $(0, 1)$. This restriction does not apply to ξ_N in the Lorentzian case, since in that case ξ_N is Wick rotated to radial coordinate r . The range of ξ_1 is inside $(-1, 1)$ and the metric \mathbf{q}_1 is trivial; see the discussion of metric \hat{g} in section 5.1.

The curvature of \hat{g} can be obtained by repetitive application of the warped splitting while employing relations (B24)–(B27), to get the following expression for the Riemann tensor:

$$\begin{aligned}\mathbf{R} &= -\frac{\lambda}{4} \sum_{\mu=1}^N \Delta_\mu'' \left(\prod_{\nu=\mu+1}^N \xi_\nu^2 \right) \mathbf{q}_\mu \llcorner \mathbf{q}_\mu \\ &\quad - \frac{\lambda}{2} \sum_{\mu=2}^N \frac{\Delta_\mu'}{\xi_\mu} \sum_{\nu=1}^{\mu-1} \left(\prod_{\alpha=\nu+1}^N \xi_\alpha^2 \right) \mathbf{q}_\mu \llcorner \mathbf{q}_\nu \\ &\quad - \frac{\lambda}{2} \sum_{\mu=2}^N \Delta_\mu \sum_{\alpha,\beta=1}^{\mu-1} \left(\prod_{\kappa=\alpha+1}^N \xi_\kappa^2 \right) \left(\prod_{\lambda=\beta+1}^{\mu-1} \xi_\lambda^2 \right) \mathbf{q}_\alpha \llcorner \mathbf{q}_\beta,\end{aligned}\quad (4.7)$$

where for two 2-forms \mathbf{p} , \mathbf{q} we defined

$$(\mathbf{p} \llcorner \mathbf{q})_{abcd} = 4 p_{[a|[c} q_{d]|b]}.\quad (4.8)$$

Substituting the expressions for Δ_μ , (4.4), we can analyze the singular behavior of the curvature. In our case, it is sufficiently represented by the singularities of the Kretschmann scalar, which simplifies to

$$\mathcal{K} = 4N(2N-1)\lambda^2 + \lambda^2 \sum_{\mu=2}^N \frac{16(\mu-1)^2(2\mu-1)(2\mu-3)\beta_\mu^2}{\xi_\mu^{4\mu-2} (\prod_{\alpha=\mu+1}^N \xi_\alpha^2)^2}.\quad (4.9)$$

For non-trivial parameters β_μ we thus obtained a deformed geometry which is, however, not twisted as the angular Killing coordinates ϕ_μ do not mix among themselves (the metric is diagonal). For the Lorentzian signature the geometry has a curvature singularity at $r = 0$. Inspecting the expression for the Kretschmann scalar (4.9), we see that the singularity occurs for $\xi_\mu = 0$, $\mu > 1$, which is out of the coordinate ranges, except for the radial coordinate r .

For vanishing NUT parameters, $\beta_\mu = 0$, $\mu = 1, \dots, N-1$, one can introduce latitude angles θ_μ ,

$$\xi_\mu = \cos \theta_\mu \quad (4.10)$$

and the atomic 2-metric (4.3) simplifies to the homogeneous metric on a 2-sphere

$$q_\mu = d\theta_\mu^2 + \sin^2 \theta_\mu d\phi_\mu^2. \quad (4.11)$$

The limits of the CCKY forms, KY forms and Killing tensors are

$$\mathring{h}^{(j)} = j_W j \mathring{e}, \quad (4.12)$$

$$\mathring{f}^{(j)} = j_W^{2(N-j)+1} N^{-j} \mathring{e}, \quad (4.13)$$

$$\mathring{k}_{(j)} = N^{-j} \mathring{g}^{-1}. \quad (4.14)$$

They satisfy relations (2.37) and (2.38). Relation (2.40) between Killing vectors and Killing tensors degenerates since in the limit Killing vectors $\mathbf{l}_{(j)}$ vanish except for $j = 0$. However, we can identify immediately another set of N explicit Killing vectors of the limiting metric \mathring{g} , namely vectors ∂_{ϕ_μ} .

5. Explicit examples

After presenting our general limiting procedure, it is helpful to demonstrate it on a couple of simple examples. We first apply the formalism in physical dimension $D = 4$, where we just recover the standard solutions. In the next subsection we study the $D = 6$ case, where we can identify more interesting alternatives.

5.1. $D = 4$ dimensions

Our starting point is the four-dimensional on-shell Kerr-NUT-(A)dS metric with Lorentzian signature $(-+++)$. Since in lower dimensions it is more natural to label coordinates by letters instead of indexing, we use the following notation: $x_1 \rightarrow x$, $a_1 \rightarrow a$, $\psi_0 \rightarrow t$, $\psi_1 \rightarrow \psi$. The metric (2.30) then reads

$$\mathbf{g} = -\frac{\Delta}{\Sigma} (dt + x^2 d\psi)^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma}{S} dx^2 + \frac{S}{\Sigma} (dt - r^2 d\psi)^2, \quad (5.1)$$

where

$$\begin{aligned} \Delta &= -X_2 = (1 - \lambda r^2)(a^2 + r^2) - 2m(1 + \lambda a^2)r, \\ S &= -X_1 = (1 + \lambda x^2)(a^2 - x^2) - 2\alpha a(1 + \lambda a^2)x, \\ \Sigma &= U_2 = -U_1 = r^2 + x^2. \end{aligned} \quad (5.2)$$

This is the standard Kerr-NUT-(A)dS solution written in Carter's coordinates [2, 3]. It is labeled by parameter λ fixed by the cosmological constant, mass m , NUT parameter α , and rotation parameter a . The exact interpretation of parameters for nonvanishing α is, however, nontrivial, see [10, 11].

We perform the limit $N = 2 \rightarrow \bar{N} = 1$, $\tilde{N} = 1$ which leads to the warped metric

$$\mathbf{g} = \tilde{\mathbf{g}} + \tilde{r}^2 \tilde{\mathbf{g}}|_{\lambda=1}. \quad (5.3)$$

The regular sector is characterized by $\tilde{N} = 1$ and signature $(-+)$. Coordinates \tilde{r} , \tilde{t} and mass parameter \tilde{m} are trivially related to the original ones: $\tilde{r} = r$, $\tilde{t} = t$, and $\tilde{m} = m$. The regular metric $\tilde{\mathbf{g}}$ is given by

$$\tilde{g} = -f d\tilde{t}^2 + \frac{1}{f} d\tilde{r}^2, \tag{5.4}$$

$$f = 1 - \lambda r^2 - \frac{2m}{r}. \tag{5.5}$$

The unspinned sector is characterized by $\tilde{N} = 1$, signature $(++)$, and the following notation dictionary: $\tilde{x}_1 \rightarrow \bar{x}$, $\tilde{\psi}_0 \rightarrow \bar{\psi}$, $\tilde{a}_1 \rightarrow \bar{a}$, $\tilde{\beta}_1 \rightarrow \bar{\alpha}$. Coordinates and parameters are related to those before the limit as $\bar{x} \approx \frac{1}{\sigma}x$, $\bar{\psi} \approx -\sigma\frac{1}{\lambda}\psi$, $\bar{a} \approx \frac{1}{\sigma}a$, $\bar{\alpha} = \alpha$. The metric reads

$$\bar{g}|_{\lambda=1} = \frac{1}{\bar{X}} d\bar{x}^2 + \bar{X} d\bar{\psi}^2, \tag{5.6}$$

$$\bar{X} = \bar{a}^2 - \bar{x}^2 - 2\bar{\alpha}\bar{a}\bar{x}. \tag{5.7}$$

By rescaling coordinates,

$$\xi = \frac{\bar{x}}{\bar{a}}, \quad \phi = \bar{a}\bar{\psi}, \tag{5.8}$$

it is possible to eliminate parameter \bar{a} , obtaining the following metric (setting $\bar{\alpha} = \alpha$):

$$\bar{g}|_{\lambda=1} = \frac{1}{X} d\xi^2 + X d\phi^2, \tag{5.9}$$

$$X = 1 - \xi^2 - 2\alpha\xi. \tag{5.10}$$

This metric is an Euclidean version of the two-dimensional black hole solution of the $D \rightarrow 2$ Einstein–dilaton gravity with a cosmological constant, constructed in [57]; see also [58] for the interpretation of parameters and its thermodynamics.

Coordinate ξ runs between two roots of X , $-\alpha$ being the center of this interval. An introduction of the angular coordinate $\vartheta \in (0, \pi)$,

$$\xi = -\alpha + \sqrt{1 + \alpha^2} \cos \vartheta, \tag{5.11}$$

brings the metric to its spherical form

$$\bar{g}|_{\lambda=1} = d\vartheta^2 + (1 + \alpha^2) \sin^2 \vartheta d\phi^2, \tag{5.12}$$

with α affecting the angular deficit at poles of the sphere. We see that in this case, the NUT parameter α is rather trivial: it can be absorbed into conicity of the symmetry axis of the geometry.

Putting everything together, the obtained metric reads

$$g = -f d\tilde{t}^2 + \frac{1}{f} d\tilde{r}^2 + \tilde{r}^2 (d\vartheta^2 + (1 + \alpha^2) \sin^2 \vartheta d\phi^2), \tag{5.13}$$

with the metric function f given by (5.5). By switching-off the rotation we have recovered the Schwarzschild-(A)dS solution.

This examples illustrates that although the NUT parameter α survived in this case, it belongs to the unspinned sector and hence completely decouples from temporal and radial coordinates. Therefore, after the limit it does not play a role of real NUT charge—we have not recovered the Taub–NUT solution [5, 6]. Since our limiting procedure completely decouples regular and unspinned sectors, to maintain a nontrivial Lorentzian NUT parameter, we would need to have $\tilde{N} \geq 2$. Even in the regular sector the surviving parameter α is rather trivial. For $\tilde{N} = 1$, it contributes only to the conicity of the geometry. This is not the case for all surviving NUT parameters in the unspinned sector in higher dimensions $\tilde{N} > 1$ as we are going to see in the next subsection.

5.2. $D = 6$ dimensions

For the $N = 3$ Lorentzian on-shell Kerr-NUT-(A)dS metric we employ the following notation dictionary: $x_1, x_2 \rightarrow x, y$, $\psi_0, \psi_1, \psi_2 \rightarrow t, \varphi, \psi$, $a_1, a_2 \rightarrow a, b$, and $\beta_1, \beta_2 \rightarrow \alpha, \beta$. The metric (2.30) then reads

$$\begin{aligned} \mathbf{g} = & -\frac{\Delta}{\Sigma}(\mathbf{d}t + (x^2 + y^2)\mathbf{d}\varphi + x^2y^2\mathbf{d}\psi)^2 + \frac{\Sigma}{\Delta}\mathbf{d}r^2 \\ & + \frac{U}{X}\mathbf{d}x^2 + \frac{X}{U}(\mathbf{d}t + (y^2 - r^2)\mathbf{d}\varphi - y^2r^2\mathbf{d}\psi)^2 \\ & + \frac{V}{Y}\mathbf{d}y^2 + \frac{Y}{V}(\mathbf{d}t + (x^2 - r^2)\mathbf{d}\varphi - x^2r^2\mathbf{d}\psi)^2, \end{aligned} \quad (5.14)$$

where

$$\begin{aligned} \Delta &= (1 - \lambda r^2)(a^2 + r^2)(b^2 + r^2) - 2m(1 + \lambda a^2)(1 + \lambda b^2)r, \\ X &= -(1 + \lambda x^2)(a^2 - x^2)(b^2 - x^2) + 2\alpha a(1 + \lambda a^2)(b^2 - a^2), \\ Y &= -(1 + \lambda y^2)(a^2 - y^2)(b^2 - y^2) + 2\beta b(1 + \lambda b^2)(a^2 - b^2), \\ \Sigma &= U_3 = (x^2 + r^2)(y^2 + r^2), \\ U &= U_1 = (x^2 - y^2)(x^2 + r^2), \\ V &= U_2 = (y^2 - x^2)(y^2 + r^2). \end{aligned} \quad (5.15)$$

5.2.1. Black hole with one rotation. Let us first perform a single-spin-zero limit $N = 3 \rightarrow \tilde{N} = 1$, $\tilde{N} = 2$, switching-off one rotation parameter, $a_1 \rightarrow 0$. We obtain the warped metric, see (3.16),

$$\mathbf{g} = \tilde{\mathbf{g}} + \tilde{r}^2 \frac{\tilde{y}^2}{\tilde{b}^2} \tilde{\mathbf{g}}|_{\lambda=1}. \quad (5.16)$$

In the regular sector we have a four-dimensional metric $\tilde{\mathbf{g}}$ with signature $(-+++)$ for which we employ the following notation dictionary: $\tilde{x}_1 \rightarrow \tilde{y}$, $\tilde{\psi}_1 \rightarrow \tilde{\varphi}$, $\tilde{a}_1 \rightarrow \tilde{b}$, $\tilde{\beta}_1 \rightarrow \tilde{\beta}$. As usual, in the regular sector coordinates and parameters after the limit are trivially related to those before the limit: $\tilde{y} = y$, $\tilde{r} = r$, $\tilde{t} = t$, $\tilde{\varphi} = \varphi$, $\tilde{m} = m$, $\tilde{\beta} = \beta$, and $\tilde{b} = b$. The metric resembles the Kerr-NUT-(A)dS solution (5.1) in four dimensions,

$$\tilde{\mathbf{g}} = -\frac{\Delta}{\tilde{\Sigma}}(\mathbf{d}\tilde{t} + \tilde{y}^2\mathbf{d}\tilde{\varphi})^2 + \frac{\tilde{\Sigma}}{\Delta}\mathbf{d}\tilde{r}^2 + \frac{\tilde{\Sigma}}{\tilde{\Delta}}\mathbf{d}\tilde{y}^2 + \frac{\tilde{\Delta}}{\tilde{\Sigma}}(\mathbf{d}\tilde{t} - \tilde{r}^2\mathbf{d}\tilde{\varphi})^2, \quad (5.17)$$

However, the metric functions have the $\tilde{N} = 1$ off-shell form (3.11) instead of the usual (5.2),

$$\begin{aligned} \Delta &= -\tilde{X}_2 = (1 - \lambda \tilde{r}^2)(\tilde{b}^2 + \tilde{r}^2) - 2\tilde{m}(1 + \lambda \tilde{b}^2)\frac{1}{\tilde{r}}, \\ \tilde{\Delta} &= -\tilde{X}_1 = (1 + \lambda \tilde{y}^2)(\tilde{b}^2 - \tilde{y}^2) - 2\tilde{\beta}\tilde{b}^3(1 + \lambda \tilde{b}^2)\frac{1}{\tilde{y}}, \\ \tilde{\Sigma} &= \tilde{U}_2 = -\tilde{U}_1 = \tilde{r}^2 + \tilde{y}^2. \end{aligned} \quad (5.18)$$

The unspinned sector leads to the $\tilde{N} = 1$ metric $\tilde{\mathbf{g}}|_{\lambda=1}$ with signature $(++)$. With the notation dictionary $\tilde{x}_1 \rightarrow \tilde{x}$, $\tilde{\psi}_0 \rightarrow \tilde{\psi}$, $\tilde{a}_1 \rightarrow \tilde{a}$, and $\tilde{\beta}_1 \rightarrow \tilde{\alpha}$, it takes exactly the form (5.6) studied in the previous section; a discussion leading to (5.12) applies again. New coordinates

and parameters are related to those before the limit as follows: $\bar{x} \approx \frac{1}{\sigma}x$, $\bar{\psi} \approx -\sigma\frac{b^2}{\lambda}\psi$, $\bar{a} \approx \frac{1}{\sigma}a$, and $\bar{\alpha} = \alpha$.

We can observe that the Killing coordinate $\bar{\psi}$ is decoupled from the time coordinate \bar{t} (there are no off-diagonal terms $g_{\bar{t}\bar{\psi}}$), i.e. there is no rotation in this direction. However, there remains a nontrivial rotation in the $\bar{\varphi}$ direction. The surviving NUT parameter $\tilde{\beta}$ belongs to the regular sector and as such it is coupled to the temporal coordinate. Consequently, its presence has familiar NUT-like consequences on the geometry, including the existence of closed time-like curves around the axis. The case with vanishing $\tilde{\beta}$ has been discussed in [59].

5.2.2. Twisted black hole. Another alternative in six dimensions is to perform the multiple-spin-zero limit $N = 3 \rightarrow \bar{N} = 2$, $\tilde{N} = 1$. The resulting metric reads (3.19),

$$\mathbf{g} = -f dt^2 + \frac{1}{f} dr^2 + r^2 \bar{\mathbf{g}}|_{\lambda=1} \quad (5.19)$$

with the metric function¹¹

$$f = 1 - \lambda r^2 - \frac{2m}{r^3}. \quad (5.20)$$

We recognize the Schwarzschild–Tangherlini-like dependence on the radial coordinate in six dimensions. However, the geometry reduces to the Schwarzschild–Tangherlini solution only for the vanishing NUT parameters in the unspinned sector.

Indeed, an unspinned metric $\bar{\mathbf{g}}|_{\lambda=1}$ is the $\bar{N} = 2$ Euclidean instanton with signature $(++++)$. With the notation dictionary $\bar{x}_1, \bar{x}_2 \rightarrow \bar{x}, \bar{y}$, $\bar{\psi}_0, \bar{\psi}_1 \rightarrow \bar{\varphi}, \bar{\psi}$, $\bar{\beta}_1, \bar{\beta}_2 \rightarrow \alpha, \beta$, and $\bar{a}_1, \bar{a}_2 \rightarrow \bar{a}, \bar{b}$, it reads

$$\begin{aligned} \bar{\mathbf{g}}|_{\lambda=1} = & \frac{\bar{y}^2 - \bar{x}^2}{\bar{X}} d\bar{x}^2 + \frac{\bar{X}}{\bar{y}^2 - \bar{x}^2} (d\bar{\varphi} + \bar{y}^2 d\bar{\psi})^2 \\ & + \frac{\bar{y}^2 - \bar{x}^2}{-\bar{Y}} d\bar{y}^2 + \frac{-\bar{Y}}{\bar{y}^2 - \bar{x}^2} (d\bar{\varphi} + \bar{x}^2 d\bar{\psi})^2, \end{aligned} \quad (5.21)$$

where

$$\begin{aligned} \bar{X} &= (\bar{a}^2 - \bar{x}^2)(\bar{b}^2 - \bar{x}^2) - 2\bar{\alpha}\bar{a}(\bar{b}^2 - \bar{a}^2)\bar{x}, \\ \bar{Y} &= (\bar{a}^2 - \bar{y}^2)(\bar{b}^2 - \bar{y}^2) - 2\bar{\beta}\bar{b}(\bar{a}^2 - \bar{b}^2)\bar{y}. \end{aligned} \quad (5.22)$$

The ranges of \bar{x} and \bar{y} are given by intervals between the roots of \bar{X} and \bar{Y} , respectively, such that $-\bar{a} < \bar{x} < \bar{a} < \bar{y} < \bar{b}$; see the general discussion near (2.25). The metric is not diagonal, it mixes Killing directions (we have a non-vanishing term $g_{\bar{\varphi}\bar{\psi}}$), which we refer to as local twisting of the geometry. For $\bar{\alpha}, \bar{\beta} > 0$, the curvature is singular only at $r = 0$. For demonstration, the Kretschmann scalar is

$$\begin{aligned} \mathcal{K} = & 100\lambda^2 + \frac{1600m^2}{r^{10}} + \frac{80(\bar{a}^2 - \bar{b}^2)^2}{r^4(\bar{y}^2 - \bar{x}^2)^6} \\ & \times ((\bar{a}^2\bar{\alpha}^2 + \bar{b}^2\bar{\beta}^2)(\bar{x}^2 + \bar{y}^2)(\bar{x}^4 + 14\bar{x}^2\bar{y}^2 + \bar{y}^4) + 4\bar{a}\bar{b}\bar{x}\bar{y}(3\bar{x}^2 + \bar{y}^2)(\bar{x}^2 + 3\bar{y}^2)), \end{aligned} \quad (5.23)$$

¹¹ We have eliminated tildes in the regular sector in this case, since tilded quantities are trivially related to those before the limit: $\tilde{r} = r$, $\tilde{t} = t$, $\tilde{m} = m$.

$\bar{x} = \bar{y}$ is out of the allowed coordinate range. There can exist additional non-regularities on the axes of symmetry. They depend on a choice of identification of Killing coordinates, which also introduces global twisting of geometry in these directions.

The limiting metric (3.19) thus represents a generalization of the static Schwarzschild–Tangherlini solution with an angular part given by the twisted four-dimensional Euclidean instanton (5.21).

5.2.3. Deformed black hole. After ‘decoupling’ the four-dimensional Euclidean instanton, by ‘unspinning’ it from the radial and temporal directions, we can also ‘untwist’ its Killing angular direction. Performing so the second single-spin-zero limit $\tilde{N} = 2 \rightarrow \tilde{N} = 1, \tilde{N} = 1$, we arrive at the ‘combined metric’ (4.5). Using the following notations: $\xi_1, \xi_2 \rightarrow \xi, v, \phi_1, \phi_2 \rightarrow \phi, \psi, \beta_1, \beta_2 \rightarrow \alpha, \beta$, the metric reads

$$\begin{aligned} \mathbf{g} &= \mathbf{q}_r + r^2 \mathbf{q}_v + r^2 v^2 \mathbf{q}_\xi \\ &= -f dt^2 + \frac{1}{f} dr^2 + r^2 \left(\frac{1}{\Delta_v} dv^2 + \Delta_v d\psi^2 \right) + r^2 v^2 \left(\frac{1}{\Delta_\xi} d\xi^2 + \Delta_\xi d\phi^2 \right), \end{aligned} \quad (5.24)$$

where

$$\begin{aligned} f &= 1 - \lambda r^2 - \frac{2m}{r^3}, \\ \Delta_v &= 1 - v^2 - \frac{2\beta}{v}, \\ \Delta_\xi &= 1 - \xi^2 - 2\alpha\xi. \end{aligned} \quad (5.25)$$

Metric $\mathbf{q}_r = -f dt^2 + f^{-1} dr^2$ describes a geometry spanning the radial and temporal directions and has the familiar Schwarzschild–Tangherlini form \mathbf{d} . Metric $\mathbf{q}_\xi = \Delta_\xi^{-1} d\xi^2 + \Delta_\xi d\phi^2$ is spherical, (5.9). The middle part $\mathbf{q}_v = \Delta_v^{-1} dv^2 + \Delta_v d\psi^2$ has the Euclidean signature for $\frac{1}{\sqrt{27}} > \beta \geq 0$, when Δ_v has two roots which specify the allowed range of v . For $\beta \neq 0$ this metric describes a non-spherical geometry.

The solution (5.24) thus describes a static deformed Schwarzschild–Tangherlini black hole. Since the metric is diagonal, its Killing directions are not twisted. The curvature is singular for $r = 0$, as can be seen, e.g. from the Kretschmann scalar

$$\mathcal{K} = 60\lambda^2 + \frac{960m^2}{r^{10}} + \frac{48\beta^2}{r^4 v^6}. \quad (5.26)$$

Remember that for $\beta > 0$, value $v = 0$ is outside the allowed coordinate range.

6. Discussion

In the present paper we have analyzed the basic properties of the higher-dimensional Kerr-NUT-(A)dS metrics. In particular, we discussed the signature and coordinate ranges, and identified the maximally symmetric space, the Euclidean instanton, and the black hole solution. We have then constructed a new class of black hole solutions of the Einstein equations with a cosmological constant by taking a special (singular) limit in the space of the Kerr-NUT-(A)dS metrics.

As shown by Geroch [43], limiting procedures of this kind are generally non-unique. A procedure we adopted in the paper is singled out by the following properties. The Kerr-NUT-(A)dS metric in $2N$ dimensions possesses, besides the cosmological constant, $2N - 1$ free

physical parameters. The limiting metric has one parameter less. Geodesic equations in the Kerr-NUT-(A)dS geometry admit $2N$ independent integrals of motion in involution: N of them are of the first order in momentum, while the other N are quadratic in momentum. The limiting space preserves this property. Moreover, no new independent integrals of motion are in general present in the obtained space. The Kerr-NUT-(A)dS metric admits a non-degenerate CCKY tensor \mathbf{h} , responsible for the integrability properties of this space. In our limiting procedure this tensor becomes degenerate, $\mathbf{h} \approx \tilde{\mathbf{h}}$. As a result, the metric belongs to a wider class of metrics called the generalized Kerr-NUT-(A)dS spacetimes (see appendix A). Such metrics do not necessarily admit the full Killing tower of symmetries. However, our limiting procedure was constructed in such a way that the adopted ‘enhancement’ of the vanishing part of the CCKY tensor results in the appearance of a new additional two-form $\tilde{\mathbf{h}}$. This object ‘takes care’ of the additional integrals of motion, which ‘disappeared’ because of the degeneracy of the surviving CCKY tensor $\tilde{\mathbf{h}}$. In other words, the full Killing tower of symmetries still exists, though it cannot be generated from a single non-degenerate CCKY tensor.

The obtained black hole solutions generalize the Kerr-(A)dS spacetimes constructed in [17–19] by including the Euclidean and Lorentzian NUT charges. The resulting geometry has a structure of warped space with the Kerr-like Lorentzian part warped to a Euclidean metric of a deformed and/or twisted sphere, with the deformation and twist characterized by the Euclidean NUT parameters. The presence of Euclidean NUTs is less severe than that of their Lorentzian counterparts; typically Euclidean NUT charges smoothen the curvature singularity and introduce a global twisting of the geometry (the axes of symmetry may suffer from non-regularity due to identifications of Killing angles). As expected, for vanishing NUTs the metric reduces to the Kerr-(A)dS black hole spacetime, written in Jacobi-type coordinates [32], with several rotations switched off.

In the present paper we have limited ourselves to even dimensions and the limit of vanishing rotation parameters. A few interesting possible developments immediately come to mind. The most obvious one is the extension of the present results to odd dimensions. Next is to consider another interesting limit where some of the rotation parameters are set equal but non-zero. One may expect that the resulting metric also belongs to the generalized Kerr-NUT-(A)dS class and possesses some enhanced symmetry. In particular, it is known that a very special subcase of such a limit leads to the ‘most general known’ Einstein–Kähler manifolds [60]. One might even be able to obtain the recently constructed super-entropic black holes [61].

Although we studied the limit under which the NUT parameters survive and characterize a deformation of the resulting geometry, our limiting procedure is significantly different from the four-dimensional limit in which the Carter–Plebański generic metric is turned to the Taub–NUT solution, see, e.g. [11]. In our limit, the metric functions $\mathcal{X}_\mu^{(p)}$ are approximately given by the even polynomials $\mathcal{J}(x_\mu^2)$ with roots $\pm a_\mu$, and these roots get only slightly modified by the linear NUT term. By sending $\bar{a}_\mu \rightarrow 0$ we send some of these roots to zero. As a result the spacetime assumes a warp structure with ‘minimal coupling’ between the regular and unspinned sectors. On other hand, the Taub–NUT limit corresponds to the case when the even polynomial $\mathcal{J}(x_\mu^2)$ does not have a maximal number of real roots and the real roots appear only due to the NUT term. The Taub–NUT solution is then obtained when two of these roots coincide. A detailed discussion of such a limit in higher dimensions awaits proper study.

Another future direction would be to generalize the above limit to solutions with non-trivial matter fields. In particular, one may consider a test electromagnetic field [62] and its impact on the integrability of charged geodesic motion.

Surprisingly, not much is known about the higher-dimensional Kerr-NUT-(A)dS geometry. Being a higher-dimensional generalization of the Carter–Plebański class it likely contains a large family of solutions of Einstein’s equations of various signatures and physical interpretations. In the present paper we have slightly uncovered the basic structure of this family. In particular, by studying the signature and ranges of coordinates we identified black hole solutions of the Kerr-(A)dS type. We also were able to find one special subclass contained in this family that arises from switching off one of the ‘rotation parameters’. However, many open tasks remain. Probably the most difficult is to analyze the meaning of free parameters that appear in the Kerr-NUT-(A)dS solution. An analogous task has been accomplished for the Carter–Plebański class only recently [10, 11]. The process of identification of these parameters is intrinsically related to a procedure by which various special limits can be taken and various subclasses identified. The metrics of the generalized Kerr-NUT-(A)dS type are likely to emerge in this process, smearing the gap between the two families. We believe that, similar to its four-dimensional Carter–Plebański counterpart, the (generalized) Kerr-NUT-(A)dS family ‘hides’ many new interesting solutions yet to be discovered.

Acknowledgments

V F thanks the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for financial support. P K was supported by Grant GAČR P203/12/0118. D K is supported in part by the Perimeter Institute for Theoretical Physics and by the Natural Sciences and Engineering Research Council of Canada. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. I K was supported by the Charles University Grant No. SVV-260211. V F and P K appreciate the hospitality of the Perimeter Institute, Waterloo, where this work started.

Appendix A. Generalized Kerr-NUT-(A)dS spacetimes

The generalized Kerr-NUT-(A)dS metric is the most general metric that admits a (not necessarily non-degenerate) rank-2 CCKY tensor [32, 44, 45]. In a general dimension D , the metric possesses a bundle structure, with the fiber being the $2n$ -dimensional Kerr-NUT-(A)dS metric and the base taking a form of the product space $B = M^1 \times M^2 \times \dots \times M^I \times M^0$. Here the manifolds M^i are $2m_i$ -dimensional Kähler manifolds with metrics \mathbf{g}^i and Kähler 2-forms $\omega^i = d\mathbf{B}^i$, and M^0 is an ‘arbitrary’ manifold of dimension m_0 and metric \mathbf{g}^0 ,

$$D = 2n + 2|m| + m_0, \quad |m| = \sum_{i=1}^I m_i. \quad (\text{A1})$$

The generalized Kerr-NUT-(A)dS metric takes the following form:

$$\mathbf{g} = \sum_{\mu=1}^n \frac{dx_\mu^2}{P_\mu(x)} + \sum_{\mu=1}^n P_\mu(x) \left(\sum_{k=0}^{n-1} A_\mu^{(k)} \theta_k \right)^2 + \sum_{i=1}^I \prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) \mathbf{g}^i + A^{(n)} \mathbf{g}^0, \quad (\text{A2})$$

while the CCKY 2-form reads

$$\mathbf{h} = \sum_{\mu=1}^n x_{\mu} \mathbf{d}x_{\mu} \wedge \left(\sum_{k=1}^{n-1} A_{\mu}^{(k)} \boldsymbol{\theta}_k \right) + \sum_{i=1}^I \xi_i \prod_{\mu=1}^n (x_{\mu}^2 - \xi_i^2) \boldsymbol{\omega}^i, \tag{A3}$$

where

$$\begin{aligned} \boldsymbol{\theta}_k &= \mathbf{d}\psi_k - 2 \sum_{i=1}^I (-1)^{n-k} \xi_i^{2(n-k)-1} \mathbf{B}^i, \\ P_{\mu} &= X_{\mu}(x_{\mu}) \left[x_{\mu}^{m_0} \prod_{i=1}^I (x_{\mu}^2 - \xi_i^2)^{m_i} (-1)^n U_{\mu} \right]^{-1}. \end{aligned} \tag{A4}$$

Coordinates x_{μ} are the non-constant functionally independent eigenvalues of \mathbf{h} , whereas parameters ξ_i stand for the non-zero constant eigenvalues of \mathbf{h} , each having multiplicity m_i that determines the dimension of Kähler manifolds M^i . The dimension m_0 of the manifold M^0 equals the multiplicity of the zero value eigenvalue of \mathbf{h} . For $m_0 = 1$, the metric \mathbf{g}^0 can take a special form

$$A^{(n)} \mathbf{g}^0 = \frac{c}{A^{(n)}} \left(\sum_{k=0}^n A^{(k)} \boldsymbol{\theta}_k \right)^2. \tag{A5}$$

Assuming that the base metrics \mathbf{g}^0 and \mathbf{g}^i are Einstein spaces with cosmological constants λ^0 and λ^i , the generalized Kerr-NUT-(A)dS metric solves the vacuum Einstein equations with the cosmological constant, $\mathbf{Ric}_{ab} = \lambda \mathbf{g}_{ab}$, provided the metric functions X_{μ} take the following form:

$$X_{\mu} = x_{\mu} \left(d_{\mu} + \int \chi(x_{\mu}) x_{\mu}^{m_0-2} \prod_{i=1}^I (x_{\mu}^2 - \xi_i^2)^{m_i} dx_{\mu} \right), \tag{A6}$$

where

$$\chi(x) = \sum_{k=-\epsilon}^n \alpha_k x^{2k}, \quad \alpha_n = -\lambda. \tag{A7}$$

For the general type ($\epsilon = 0$) of \mathbf{g}^0 , we have $\alpha_0 = (-1)^{n-1} \lambda^0$, whereas for the special type of \mathbf{g}^0 described by (A5) we have ($\epsilon = 1$)

$$\alpha_0 = (-1)^{n-1} 2c \sum_{i=1}^I \frac{m_i}{\xi_i^2}, \quad \alpha_{-1} = (-1)^{n-1} 2c. \tag{A8}$$

The cosmological constants λ^i are given by $\lambda^i = (-1)^{n-1} \chi(\xi_i)$.

A subfamily of solutions with vanishing NUT charges, describing the Kerr-(A)dS black holes with partially equal and some vanishing angular momenta have been identified in [32]. The corresponding scaling limit is in a way analogous to the one performed in the main text of this paper, with the exception that in [32] the NUT parameters were switched off.

Let us stress that the generalized Kerr-NUT-(A)dS metrics do not necessarily admit the Killing tower of symmetries. The presence of a degenerate CCKY tensor is not enough to generate this full tower and a much smaller subset of symmetries exists in these spacetimes. In particular, metrics \mathbf{g}^i are in general ‘arbitrary’ Kähler metrics without any additional symmetries.

The new metrics obtained in this paper, describing the twisted and deformed black holes, belong to the class of generalized Kerr-NUT-(A)dS spacetimes. In particular, they correspond

to the even-dimensional case for which all the Kähler metrics g^i identically vanish and metric g^0 becomes again the Kerr-NUT-(A)dS spacetime. As discussed in the main text, these solutions are distinguished by the presence of a full Killing tower of symmetries. Moreover, the presence of NUT charges makes them a non-trivial generalization of Kerr-(A)dS black holes discovered in [17–19].

Appendix B. Technical results

B.1. Properties of metric functions

In section 2 we have defined auxiliary functions $J(a^2)$ and $A^{(k)}$ as follows¹²

$$J(a^2) = \prod_{\nu} (x_{\nu}^2 - a^2) = \sum_{k=0, \dots, N} A^{(k)} (-a^2)^{N-k}, \quad (\text{B1})$$

and their complementary functions

$$\mathcal{J}(x^2) = \prod_{\nu} (a_{\nu}^2 - x^2) = \sum_{k=0, \dots, N} \mathcal{A}^{(k)} (-x^2)^{N-k}. \quad (\text{B2})$$

The above definitions imply that

$$A^{(k)} = \sum_{\substack{\mu_1, \dots, \mu_k \\ \mu_1 < \dots < \mu_k}} x_{\mu_1}^2 \dots x_{\mu_k}^2, \quad (\text{B3})$$

$$\mathcal{A}^{(k)} = \sum_{\substack{\mu_1, \dots, \mu_k \\ \mu_1 < \dots < \mu_k}} a_{\mu_1}^2 \dots a_{\mu_k}^2. \quad (\text{B4})$$

Similarly, we define functions $J_{\mu}(a^2)$, $A_{\mu}^{(j)}$, $\mathcal{J}_{\mu}(x^2)$, and $\mathcal{A}_{\mu}^{(j)}$, which skip the μ th variables x_{μ} and a_{μ} as follows

$$J_{\mu}(a^2) = \prod_{\substack{\nu \\ \nu \neq \mu}} (x_{\nu}^2 - a^2) = \sum_k A_{\mu}^{(k)} (-a^2)^{N-1-k}, \quad (\text{B5})$$

$$\mathcal{J}_{\mu}(x^2) = \prod_{\substack{\nu \\ \nu \neq \mu}} (a_{\nu}^2 - x^2) = \sum_k \mathcal{A}_{\mu}^{(k)} (-x^2)^{N-1-k}, \quad (\text{B6})$$

with

$$A_{\mu}^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2, \quad (\text{B7})$$

$$\mathcal{A}_{\mu}^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} a_{\nu_1}^2 \dots a_{\nu_k}^2. \quad (\text{B8})$$

¹² Let us remind readers that, if not indicated otherwise, the sums (and products) run over ‘standard’ ranges of indices:

$$\sum_{\mu} \equiv \sum_{\mu=1}^N, \quad \sum_k \equiv \sum_{k=0}^{N-1}.$$

These functions satisfy

$$\begin{aligned} J(x_\mu^2) &= 0, & \mathcal{J}(a_\mu^2) &= 0, \\ J_\mu(x_\nu^2) &= 0, & \mathcal{J}_\mu(a_\nu^2) &= 0, & \text{for } \nu \neq \mu. \end{aligned} \quad (\text{B9})$$

Finally, we define

$$U_\mu = J_\mu(x_\mu^2) = \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2), \quad (\text{B10})$$

$$\mathcal{U}_\mu = \mathcal{J}_\mu(a_\mu^2) = \prod_{\nu \neq \mu} (a_\nu^2 - a_\mu^2). \quad (\text{B11})$$

The polynomials $A^{(k)}$ and $A_\mu^{(k)}$ satisfy the following identities:

$$A^{(k)} = A_\mu^{(k)} + x_\mu^2 A_\mu^{(k-1)}, \quad (\text{B12})$$

$$\sum_k A_\mu^{(k)} \frac{(-x_\nu^2)^{N-1-k}}{U_\nu} = \delta_\mu^\nu, \quad (\text{B13})$$

$$\sum_\mu A_\mu^{(k)} \frac{(-x_\mu^2)^{N-1-l}}{U_\mu} = \delta_l^k, \quad (\text{B14})$$

$$\sum_\mu A_\mu^{(k)} \frac{(-x_\mu^2)^N}{U_\mu} = -A^{(k+1)}, \quad (\text{B15})$$

$$\sum_\mu A_\mu^{(k)} = (N - k)A^{(k)}, \quad (\text{B16})$$

$$\sum_\mu (N - k)A^{(k)} \frac{(-x_\nu^2)^{N-1-k}}{U_\nu} = 1, \quad (\text{B17})$$

$$\sum_{k=0, \dots, N} A^{(k)} (-x_\nu^2)^{N-k} = 0, \quad (\text{B18})$$

$$\sum_{l=0, \dots, k} A^{(l)} (-x_\nu^2)^{k-l} = A_\mu^{(k)}. \quad (\text{B19})$$

Analogous identities hold also for the complementary polynomials $\mathcal{A}^{(k)}$ and $\mathcal{A}_\mu^{(k)}$.

For the functions $J(a^2)$ and $\mathcal{J}(x^2)$ we can write

$$\begin{aligned} \prod_\mu J(a_\nu^2) &= (-1)^N \prod_\nu \mathcal{J}(x_\mu^2), \\ \prod_{\mu \neq \kappa} J_\kappa(a_\nu^2) &= (-1)^{N-1} \prod_{\nu \neq \kappa} \mathcal{J}_\kappa(x_\mu^2). \end{aligned} \quad (\text{B20})$$

These functions satisfy important orthogonality relations

$$\sum_\alpha \frac{J_\nu(a_\alpha^2)}{\mathcal{U}_\alpha} \frac{\mathcal{J}_\alpha(x_\mu^2)}{U_\mu} = \delta_\nu^\mu, \quad (\text{B21})$$

$$\sum_{\alpha} \frac{J_{\mu}(a_{\alpha}^2) J_{\nu}(a_{\alpha}^2)}{J(a_{\alpha}^2) \mathcal{U}_{\alpha}} = -\frac{U_{\mu}}{\mathcal{J}(x_{\mu}^2)} \delta_{\mu\nu}, \quad (\text{B22})$$

$$\sum_{\mu} J_{\mu}(a_{\alpha}^2) J_{\mu}(a_{\beta}^2) \frac{\mathcal{J}(x_{\mu}^2)}{U_{\mu}} = -J(a_{\alpha}^2) \mathcal{U}_{\alpha} \delta^{\alpha\beta}. \quad (\text{B23})$$

B.2. The curvature tensor

The Riemann tensor of the off-shell Kerr-NUT-(A)dS metric has been calculated in [47]. The obtained limiting metric (3.10) has a warped-product structure with both components belonging to the Kerr-NUT-(A)dS class. The curvature of this metric can thus be obtained by using general formulae relating the curvature of the warped metric to curvatures of its components and derivatives of the warped factor. Following [46, 63], the Riemann and Ricci tensors read

$$\mathbf{R}_{ab}{}^c{}_d = \tilde{\mathbf{R}}_{ab}{}^c{}_d + \frac{2}{\tilde{w}} \tilde{\mathbf{H}}_{d[a} \tilde{\delta}_{b]}^c - 2\tilde{w} \tilde{\mathbf{g}}^{ce} \tilde{\mathbf{H}}_{e[a} \tilde{\mathbf{g}}_{b]d} + \bar{\mathbf{R}}_{ab}{}^c{}_d - 2\tilde{w}^2 \tilde{\lambda}^2 \tilde{\delta}_{[a}^c \tilde{\mathbf{g}}_{b]d}, \quad (\text{B24})$$

$$\mathbf{Ric}_{ab} = \tilde{\mathbf{Ric}}_{ab} - \frac{\bar{N}}{\tilde{w}} \tilde{\mathbf{H}}_{ab} + \bar{\mathbf{Ric}}_{ab} - \tilde{w}^2 \left(\frac{\tilde{\mathcal{H}}}{\tilde{w}} + (\bar{N} - 1) \tilde{\lambda}^2 \right) \tilde{\mathbf{g}}_{ab}. \quad (\text{B25})$$

Scalar curvature \mathcal{R} and the Kretschmann scalar \mathcal{K} are

$$\mathcal{R} = \tilde{\mathcal{R}} + \frac{\bar{\mathcal{R}}}{\tilde{w}^2} - \frac{2\bar{N}}{\tilde{w}} \tilde{\mathcal{H}} - \bar{N}(\bar{N} - 1) \tilde{\lambda}^2, \quad (\text{B26})$$

$$\mathcal{K} = \tilde{\mathcal{K}} + \frac{4\bar{N}}{\tilde{w}^2} \tilde{\mathbf{H}}^{ab} \tilde{\mathbf{H}}_{ab} + \frac{1}{\tilde{w}^4} \bar{\mathcal{K}} - 4 \frac{\tilde{\lambda}^2}{\tilde{w}^2} \bar{\mathcal{R}} + 2(\bar{N} - 1) \bar{N} \tilde{\lambda}^4. \quad (\text{B27})$$

Here, the Hessian tensor $\tilde{\mathbf{H}}$ and scalar $\tilde{\mathcal{H}}$ are defined as

$$\tilde{\mathbf{H}}_{ab} = \tilde{\nabla}_a \tilde{\nabla}_b \tilde{w}, \quad \tilde{\mathcal{H}} = \tilde{\mathbf{g}}^{ab} \tilde{\mathbf{H}}_{ab}, \quad (\text{B28})$$

and

$$\tilde{\lambda}^2 = \tilde{w}^{-2} \tilde{\mathbf{g}}^{ab} \mathbf{d}_a \tilde{w} \mathbf{d}_b \tilde{w}. \quad (\text{B29})$$

In particular, for the warp factor (3.13) the Hessian tensor and scalars $\tilde{\mathcal{H}}$, $\tilde{\lambda}^2$ can be expressed as

$$\begin{aligned} \tilde{\mathbf{H}} &= \sum_{\tilde{\mu}} \frac{\tilde{w}}{2\tilde{x}_{\tilde{\mu}}} \left(\frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \right) (\tilde{\mathbf{e}}^{\tilde{\mu}} \tilde{\mathbf{e}}^{\tilde{\mu}} + \hat{\mathbf{e}}^{\tilde{\mu}} \hat{\mathbf{e}}^{\tilde{\mu}}), \\ \tilde{\mathcal{H}} &= \sum_{\tilde{\mu}} \frac{\tilde{w} \tilde{X}'_{\tilde{\mu}}}{\tilde{x}_{\tilde{\mu}} \tilde{U}_{\tilde{\mu}}}, \quad \tilde{\lambda}^2 = \sum_{\tilde{\mu}} \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{x}_{\tilde{\mu}}^2 \tilde{U}_{\tilde{\mu}}}. \end{aligned} \quad (\text{B30})$$

B.3. Multiple-spin-zero limit: survival of the Killing tower

Let us now provide more details about the multiple-spin-zero limit of the Killing tower discussed briefly in section 3.4. We mentioned there that although the limit of the CCKY 2-form $\mathbf{h} \approx \tilde{\mathbf{h}}$ becomes degenerate, the full Killing tower of symmetries nevertheless

survives. In some sense this is caused by the emergence of a new 2-form $\tilde{\mathbf{h}}$ that is non-degenerate on manifold $\tilde{\mathcal{M}}$. Let us first briefly explain how this additional 2-form emerges.

We start from the CCKY 2-form \mathbf{h} . In the limit $\sigma \rightarrow 0$ its exterior power $\mathbf{h}^{\wedge \tilde{N}}$ becomes a rank $2\tilde{N}$ antisymmetric tensor that vanishes in the directions tangent to $\tilde{\mathcal{M}}$. It can be also understood as a tensor on the submanifold $\tilde{\mathcal{M}}$. On this submanifold we can define the Levi-Civita tensor $\tilde{\varepsilon}$ and, obviously, $\mathbf{h}^{\wedge \tilde{N}}$ must be proportional to it with some suitable prefactor,

$$\frac{1}{\tilde{N}!} \mathbf{h}^{\wedge \tilde{N}} \approx \frac{1}{\tilde{N}!} \tilde{\mathbf{h}}^{\wedge \tilde{N}} = \sqrt{\tilde{A}^{(\tilde{N})}} \tilde{\varepsilon}. \quad (\text{B31})$$

One also has

$$\frac{1}{(\tilde{N} + 1)!} \mathbf{h}^{\wedge (\tilde{N} + 1)} \approx \sqrt{\tilde{A}^{(\tilde{N})}} \tilde{\varepsilon} \wedge \mathbf{h} \approx \mathcal{O}(\sigma). \quad (\text{B32})$$

As we will show, if this object is rescaled by σ^{-1} , its limit has a structure of the wedge-product of tensors from $\tilde{\mathcal{M}}$ and $\bar{\mathcal{M}}$

$$\mathbf{h}^{\wedge (\tilde{N} + 1)} \propto \tilde{\varepsilon} \wedge \tilde{\mathbf{h}}, \quad (\text{B33})$$

see relations (3.23) for $\bar{k} = 1$. This determines a new 2-form $\tilde{\mathbf{h}}$ on \mathcal{M} which is non-degenerate on the unspinned manifold $\tilde{\mathcal{M}}$. In some sense this 2-form is responsible for the survival of the Killing tower.

More concretely, as a result of the limiting procedure of the original CCKY tensor \mathbf{h} one obtains a CCKY tensor $\tilde{\mathbf{h}}$ and a new rank two antisymmetric tensor $\tilde{\mathbf{h}}$. Although $\tilde{\mathbf{h}}$ is degenerate on the full manifold, it is non-degenerate on $\tilde{\mathcal{M}}$ where it plays a role similar to \mathbf{h} on \mathcal{M} . The other tensor, $\tilde{\mathbf{h}}$, is not a CCKY 2-form of the limiting metric (since we factorized out some nontrivial prefactors); however we will show that it can effectively play a role of a non-degenerate CCKY tensor on $\tilde{\mathcal{M}}$. Indeed, as discussed in the main text, the limiting geometry has a structure of the warped product (3.10) and $\tilde{\mathbf{h}}$ is the CCKY form of metric $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ is the CCKY form of metric $\bar{\mathbf{g}}$; both metrics belonging to the off-shell Kerr-NUT-(A)dS class. This guarantees the survival of the Killing tower.

With the above intuitive explanation in mind let us proceed to the actual limit of the Killing objects. The CCKY forms $\mathbf{h}^{(k)}$ defined in (2.36) can be written as

$$\mathbf{h}^{(k)} = \frac{1}{\sqrt{\mathcal{A}^{(k)}}} \sum_{\mu_1 < \dots < \mu_k} x_{\mu_1} \dots x_{\mu_k} \omega^{\mu_1} \wedge \dots \wedge \omega^{\mu_k}. \quad (\text{B34})$$

For $k = \tilde{k} \leq \tilde{N}$, the leading order contains just unscaled terms x_{μ_i} with $\mu_i = \tilde{N} + \tilde{\mu}_i > \tilde{N}$,

$$\mathbf{h}^{(\tilde{k})} = \frac{1}{\sqrt{\mathcal{A}^{(\tilde{k})}}} \sum_{\tilde{\mu}_1 < \dots < \tilde{\mu}_k} \tilde{x}_{\tilde{\mu}_1} \dots \tilde{x}_{\tilde{\mu}_k} \tilde{\omega}^{\tilde{\mu}_1} \wedge \dots \wedge \tilde{\omega}^{\tilde{\mu}_k}, \quad (\text{B35})$$

which together with (3.7) proves the first relation (3.23).

For $k = \tilde{N} + \bar{k} > \tilde{N}$ the leading term in (B34) must contain \bar{k} rescaled terms, since there are just \tilde{N} unscaled coordinates $x_{\tilde{N}+1}, \dots, x_N$. The unscaled terms can be factorized in front of the sum and with the help of (3.7), (3.20), and, in the second equality, with the help of (3.13), (2.17), and again (B34), we obtain

$$\begin{aligned}
\mathbf{h}^{(\tilde{N}+\bar{k})} &\approx \sigma^{-\bar{k}} \left(\frac{1}{\tilde{\mathcal{A}}^{(\tilde{N})} \tilde{\mathcal{A}}^{(\bar{k})}} \right)^{\frac{1}{2}} \sigma^{\bar{k}} \tilde{w}^{2\bar{k}} \tilde{x}_1 \dots \tilde{x}_{\tilde{N}} \tilde{\omega}^1 \wedge \dots \wedge \tilde{\omega}^{\tilde{N}} \\
&\quad \wedge \sum_{\tilde{\mu}_1 < \dots < \tilde{\mu}_{\bar{k}}} \tilde{x}_{\tilde{\mu}_1} \dots \tilde{x}_{\tilde{\mu}_{\bar{k}}} \tilde{\omega}^{\tilde{\mu}_1} \wedge \dots \wedge \tilde{\omega}^{\tilde{\mu}_{\bar{k}}} \\
&= \tilde{w}^{2\bar{k}+1} \tilde{\varepsilon} \wedge \tilde{\mathbf{h}}^{(\bar{k})},
\end{aligned} \tag{B36}$$

which concludes the proof of relations (3.23).

The limit of KY forms $\mathbf{f}^{(k)}$ can be obtained employing definition (2.37). The Hodge dual is given by the contraction with the Levi-Civita tensor ε which splits into unscaled and scaled parts (3.14). For $k = \bar{k} \leq \tilde{N}$ we use the first relation of (3.23) and obtain

$$\mathbf{f}^{(\bar{k})} = \mathbf{h}^{(\bar{k})} \bullet \varepsilon \approx \tilde{w}^{2\tilde{N}} \tilde{\mathbf{h}}^{(\bar{k})} \bullet (\tilde{\varepsilon} \wedge \bar{\varepsilon}). \tag{B37}$$

Since $\tilde{\mathbf{h}}^{(\bar{k})}$ is nontrivial only in the unscaled directions, we can write

$$\mathbf{f}^{(\bar{k})} \approx \tilde{w}^{2\tilde{N}} (\tilde{\mathbf{h}}^{(\bar{k})} \bar{\bullet} \tilde{\varepsilon}) \wedge \bar{\varepsilon} = \tilde{w}^{2\tilde{N}} \tilde{\mathbf{f}}^{(\bar{k})} \wedge \bar{\varepsilon}, \tag{B38}$$

which is the first relation (3.24).

For $k = \tilde{N} + \bar{k} > \tilde{N}$ we employ the second relation (3.23), to get

$$\begin{aligned}
\mathbf{f}^{(\tilde{N}+\bar{k})} &= \mathbf{h}^{(\tilde{N}+\bar{k})} \bullet \varepsilon \\
&\approx (\tilde{w}^{2\bar{k}+1} \tilde{\varepsilon} \wedge \tilde{\mathbf{h}}^{(\bar{k})}) \bullet (\tilde{w}^{2\tilde{N}} \tilde{\varepsilon} \wedge \bar{\varepsilon}) \\
&\approx \tilde{w}^{2\bar{k}+1} \tilde{w}^{2\tilde{N}} \tilde{w}^{-4\bar{k}} (\tilde{\varepsilon} \bar{\bullet} \tilde{\varepsilon}) (\tilde{\mathbf{h}}^{(\bar{k})} \bar{\bullet} \bar{\varepsilon}).
\end{aligned} \tag{B39}$$

The additional warp-factor $\tilde{w}^{-4\bar{k}}$ has appeared due to a conversion of \bullet into $\bar{\bullet}$, since these operations involve the inverse metric \mathbf{g}^{-1} , $\bar{\mathbf{g}}^{-1}$, respectively, used for raising \bar{k} indices before the contraction. Since $\tilde{\varepsilon} \bar{\bullet} \tilde{\varepsilon} = 1$, and recalling (2.37), we proved (3.24),

$$\mathbf{f}^{(\tilde{N}+\bar{k})} \approx \tilde{w}^{2(\tilde{N}-\bar{k})+1} \tilde{\mathbf{f}}^{(\bar{k})}. \tag{B40}$$

To find the limit of the Killing tensors (2.42), it is useful to write them with indices in the lower position,

$$\mathbf{k}^{(j)} = \sum_{\mu} \frac{A_{\mu}^{(j)}}{\mathcal{A}^{(j)}} \left[\frac{U_{\mu}}{X_{\mu}} \mathbf{d}x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_k A_{\mu}^{(k)} \mathbf{d}\psi_k \right)^2 \right]. \tag{B41}$$

Employing relations (3.7) and (3.9), the sum splits into sums over scaled and unscaled coordinates. For $j = \tilde{j} \leq \tilde{N}$ both parts contribute to the leading order:

$$\begin{aligned}
\mathbf{k}^{(\tilde{j})} &\approx \frac{1}{\tilde{\mathcal{A}}^{(\tilde{j})}} \sum_{\tilde{\mu}} \tilde{A}_{\tilde{\mu}}^{(\tilde{j})} \left[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \mathbf{d}\tilde{x}_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \left(\sum_{\tilde{k}} \tilde{A}_{\tilde{\mu}}^{(\tilde{k})} \mathbf{d}\tilde{\psi}_{\tilde{k}} \right)^2 \right] \\
&\quad + \frac{\tilde{A}^{(\tilde{j})}}{\tilde{\mathcal{A}}^{(\tilde{j})}} \sum_{\tilde{\mu}} \left[\frac{1}{\sigma^2} \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} \frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \sigma^2 \mathbf{d}\tilde{x}_{\tilde{\mu}}^2 + \sigma^2 \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \frac{1}{\sigma^2} \left(\frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} \sum_{\tilde{k}} \tilde{A}_{\tilde{\mu}}^{(\tilde{k})} \mathbf{d}\tilde{\psi}_{\tilde{k}} \right)^2 \right] \\
&= \tilde{\mathbf{k}}^{(\tilde{j})} + \frac{\tilde{A}^{(\tilde{j})}}{\tilde{\mathcal{A}}^{(\tilde{j})}} \tilde{w}^2 \bar{\mathbf{g}}.
\end{aligned} \tag{B42}$$

For $j = \tilde{N} + \bar{j} > \tilde{N}$ the sum over unscaled coordinates disappears in the leading order:

$$\begin{aligned}
 \mathbf{k}^{(\tilde{N}+\bar{j})} &\approx \sigma^{-2\bar{j}} \frac{1}{\tilde{\mathcal{A}}^{(\tilde{N})} \tilde{\mathcal{A}}^{(\bar{j})}} \sigma^{2(\bar{j}+1)} \tilde{A}^{(\bar{j}+1)} \sum_{\bar{\mu}} \tilde{A}_{\bar{\mu}}^{(\tilde{N}-1)} \left[\frac{\tilde{U}_{\bar{\mu}}}{\tilde{X}_{\bar{\mu}}} d\tilde{x}_{\bar{\mu}}^2 + \frac{\tilde{X}_{\bar{\mu}}}{\tilde{U}_{\bar{\mu}}} \left(\sum_{\bar{k}} \tilde{A}_{\bar{\mu}}^{(\bar{k})} d\tilde{\psi}_{\bar{k}} \right)^2 \right] \\
 &+ \sigma^{-2\bar{j}} \frac{1}{\tilde{\mathcal{A}}^{(\tilde{N})} \tilde{\mathcal{A}}^{(\bar{j})}} \sigma^{2\bar{j}} \tilde{A}^{(\tilde{N})} \sum_{\bar{\mu}} \tilde{A}_{\bar{\mu}}^{(\bar{j})} \left[\frac{1}{\sigma^2} \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} \frac{\tilde{U}_{\bar{\mu}}}{\tilde{X}_{\bar{\mu}}} \sigma^2 d\tilde{x}_{\bar{\mu}}^2 \right. \\
 &\left. + \sigma^2 \frac{\tilde{\mathcal{A}}^{(\tilde{N})}}{\tilde{A}^{(\tilde{N})}} \frac{\tilde{X}_{\bar{\mu}}}{\tilde{U}_{\bar{\mu}}} \frac{1}{\sigma^2} \left(\frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}} \sum_{\bar{k}} \tilde{A}_{\bar{\mu}}^{(\bar{k})} d\tilde{\psi}_{\bar{k}} \right)^2 \right] \\
 &\approx \tilde{w}^4 \mathbf{k}^{(\bar{j})}.
 \end{aligned} \tag{B43}$$

Here, the functions $\tilde{X}_{\bar{\mu}}$ and $\tilde{X}_{\bar{\mu}}$ are given by relations (3.11) and (3.12). Raising indices with the limiting metric (3.10) we obtain relations (3.25).

B.4. A successive application of the limit

In this appendix, we provide some details about the limiting procedure performed in section 4. Namely, we will repeatedly apply the multiple-spin-zero limit to the Kerr-NUT-(A) dS metric and its Killing tower objects.

Let us first list some useful relations for the intermediate metrics introduced in (4.1) and (4.2). Metric $\mathring{\mathbf{g}}$ and its Levi-Civita tensor can be split into a warped product as follows

$$\begin{aligned}
 \mathring{\mathbf{g}} &= \alpha \mathring{\mathbf{g}} + \alpha_W^2 \mathring{\mathbf{g}}, \\
 \mathring{\varepsilon} &= \alpha \mathring{\varepsilon} \wedge \alpha_W^{2(N-\alpha)} \mathring{\varepsilon},
 \end{aligned} \tag{B44}$$

where, for arbitrary α , the warp factor reads

$$\alpha_W = \xi_N \cdots \xi_{N-\alpha-1}. \tag{B45}$$

Intermediate metrics (4.1) and (4.2) satisfy recurrence relations

$$\begin{aligned}
 \alpha+1 \mathring{\mathbf{g}} &= \mathbf{q}_{\alpha+1} + \xi_{\alpha+1}^2 \alpha \mathring{\mathbf{g}}, & \alpha+1 \mathring{\varepsilon} &= \epsilon_{\alpha+1} \wedge \xi_{\alpha+1}^{2\alpha} \alpha \mathring{\varepsilon}, \\
 \alpha+1 \mathring{\mathbf{g}} &= \alpha \mathring{\mathbf{g}} + \alpha_W^2 \mathbf{q}_{N-\alpha}, & \alpha+1 \mathring{\varepsilon} &= \alpha \mathring{\varepsilon} \wedge \alpha_W^2 \epsilon_{N-\alpha}.
 \end{aligned} \tag{B46}$$

We want to take repeatedly the limit $\tilde{N} = N - 1$, $\tilde{N} = 1$. To obtain the limit of the metric, it is enough to look at one ‘inductive’ step. We claim that after $N - \alpha$ limiting steps, the metric is

$$\mathbf{g} \approx \mathring{\mathbf{g}} + \alpha_W^2 \alpha \mathring{\mathbf{g}}. \tag{B47}$$

Let us assume it for a particular α . Performing the $\tilde{N} = 1$ limit (3.10) of the metric $\alpha \mathring{\mathbf{g}}$ we get

$$\mathbf{g} \approx \mathring{\mathbf{g}} + \alpha_W^2 \left(\alpha \mathring{\mathbf{g}} + \frac{\tilde{x}_1^2}{\tilde{a}_1^2} \alpha \mathring{\mathbf{g}} \right). \tag{B48}$$

Introducing the rescaled coordinate $\xi_\alpha = \frac{\tilde{x}_1}{\tilde{a}_1} = \frac{x_\alpha}{a_\alpha}$, the 2-metric $\alpha \mathring{\mathbf{g}}$ can be rewritten as the atomic metric \mathbf{q}_α defined in (4.3). Relabeling back $\tilde{x}_{\bar{\mu}} \rightarrow x_{\bar{\mu}}$, $\tilde{a}_{\bar{\mu}} \rightarrow a_{\bar{\mu}}$ we get $\alpha \mathring{\mathbf{g}} \rightarrow \alpha-1 \mathbf{g}$. (Note that the relabeling of coordinates does not affect the definition of the rescaled coordinate ξ_μ in the following limiting steps, as $\xi_\mu = \frac{\tilde{x}_\mu}{\tilde{a}_\mu} = \frac{x_\mu}{a_\mu}$.) Taking into account relations (B46) and definition (B45), we prove the inductive step

$$\begin{aligned} \mathbf{g} &\approx (N-\alpha)\dot{\mathbf{g}} + \alpha w^2 \mathbf{q}_\alpha + \alpha w^2 \xi_\alpha^2 \alpha^{-1}\mathbf{g} \\ &= N-\alpha+1\dot{\mathbf{g}} + \alpha^{-1}w^2 \alpha^{-1}\mathbf{g}, \end{aligned} \quad (\text{B49})$$

and thus the relation (B47). A straightforward check of the last step in the limit shows that after N steps we obtain the desired relation $\mathbf{g} \approx N\dot{\mathbf{g}} = \dot{\mathbf{g}}$.

For the multiple single-spin-zero limit of the CCKY forms we indicate just a few first steps which elucidate the resulting expressions. The relation (3.23) for $\tilde{N} = 1$, together with the same redefinitions as for the metric, allows us to write the first k limits as follows

$$\begin{aligned} \mathbf{h}^{(k)} &\approx \xi_N^{2(k-1)+1} \epsilon_N \wedge^{N-1} \mathbf{h}^{(k-1)} \\ &\approx \xi_N^{2(k-2)+1} \xi_{N-1}^{2(k-2)+1} \epsilon_N \wedge \xi_N^2 \epsilon_{N-1} \wedge^{N-2} \mathbf{h}^{(k-2)} \\ &\dots \\ &\approx \xi_N \dots \xi_{N-k+1} \epsilon_N \wedge \xi_N^2 \epsilon_{N-1} \wedge \xi_N^2 \xi_{N-1}^2 \epsilon_{N-2} \wedge \dots \wedge^{N-k} \mathbf{h}^{(0)} \\ &= k_W k_{\dot{\mathbf{E}}}. \end{aligned} \quad (\text{B50})$$

The remaining limits do not affect the resulting object any more. To obtain the limit of KY forms $\mathbf{f}^{(k)}$ we employ the relation (3.24) for the first k limits and the relation (3.14) for the remaining limits:

$$\begin{aligned} \mathbf{f}^{(k)} &\approx \xi_N^{2(N-k)+1} \epsilon_N \wedge^{N-1} \mathbf{f}^{(k-1)} \\ &\dots \\ &\approx (\xi_N \dots \xi_{N+1-k})^{2(N-k)+1} \epsilon_N \wedge^{N-k} \mathbf{f}^{(0)} = k_W^{2(N-k)+1} \epsilon_N \wedge^{N-k} \mathbf{f}^{(0)} \\ &\approx k_W^{2(N-k)+1} \epsilon_{N-k} \wedge \xi_{N-k}^{2(N-k-1)} \epsilon_{N-k-1} \wedge \dots \wedge \xi_{N-k}^2 \epsilon_1 \\ &\dots \\ &\approx k_W^{2(N-k)+1} \epsilon_{N-k} \wedge \xi_{N-k}^2 \epsilon_{N-k-1} \wedge \dots \wedge \xi_{N-k}^2 \dots \xi_2^2 \epsilon_1 \\ &= k_W^{2(N-k)+1} \epsilon_{N-k} \wedge \dots \wedge \xi_{N-k}^2 \epsilon_1. \end{aligned} \quad (\text{B51})$$

The first j limits in the multiple single-spin-zero limit of the Killing tensor $\mathbf{k}_{(j)}$ is given by the second relation of (3.25), the remaining limits act on the lower-dimensional Kerr-NUT-(A)dS metric in a way analogous to (4.5),

$$\mathbf{k}_{(j)} \approx N-j \mathbf{k}_{(j-1)} \approx \dots \approx N-j \mathbf{k}_{(0)} = N-j \mathbf{g}^{-1} \approx N-j \dot{\mathbf{g}}^{-1}, \quad (\text{B52})$$

which completes this appendix.

References

- [1] Plebański J P and Demiański M 1976 Rotating, charged, and uniformly accelerating mass in general relativity *Ann. Phys., NY* **98** 98–127
- [2] Carter B 1968 Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations *Commun. Math. Phys.* **10** 280–310
- [3] Plebański J P 1975 A class of solutions of Einstein-Maxwell equations *Ann. Phys., NY* **90** 196–255
- [4] Kerr R P 1963 Gravitational field of a spinning mass as an example of algebraically special metrics *Phys. Rev. Lett.* **11** 237–8
- [5] Taub A H 1951 Empty space-times admitting a three parameter group of motions *Ann. Math.* **53** 472–90
- [6] Newman E, Tamburino L and Unti T 1963 Empty-space generalization of the Schwarzschild metric *J. Math. Phys.* **4** 915–23
- [7] Misner C M 1963 The flatter regions of Newman, Unti, and Tamburino's generalized Schwarzschild space *J. Math. Phys.* **4** 924–37
- [8] Clément G, Gaițsov D and Guenouche M 2015 Rehabilitating space-times with NUTs *Phys. Lett. B* **750** 591–4

- [9] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact solutions of Einstein's field equations* (Cambridge: Cambridge University Press)
- [10] Griffiths J B and Podolský J 2006 A new look at the Plebański-Demiański family of solutions *Int. J. Mod. Phys. D* **15** 335–70
- [11] Griffiths J B and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge: Cambridge University Press)
- [12] Hubeny V E, Marolf D and Rangamani M 2010 Black funnels and droplets from the AdS C-metrics *Class. Quant. Grav.* **27** 025001
- [13] Gnechchi A, Hristov K, Klemm D, Toldo C and Vaughan O 2014 Rotating black holes in 4d gauged supergravity *J. High Energy Phys.* [JHEP01\(2014\)127](#)
- [14] Nozawa M and Hourī T 2015 Killing-Yano tensor and supersymmetry of the self-dual Plebański-Demiański solution, in preparation [arXiv:1510.07470](#) [hep-th]
- [15] Lü H, Mei J and Pope C N 2009 New black holes in five dimensions *Nucl. Phys. B* **806** 436–55
- [16] Chen W, Lü H and Pope C N 2006 General Kerr-NUT-AdS metrics in all dimensions *Class. Quantum Grav.* **23** 5323–40
- [17] Myers R C and Perry M J 1986 Black holes in higher dimensional space-times *Ann. Phys., NY* **172** 304–47
- [18] Gibbons G W, Lü H, Don N P and Pope C N 2004 Rotating black holes in higher dimensions with a cosmological constant *Phys. Rev. Lett.* **93** 171102
- [19] Gibbons G W, Lü H, Don N P and Pope C N 2005 The general Kerr-de Sitter metrics in all dimensions *J. Geom. Phys.* **53** 49–73
- [20] Mann R B and Stelea C 2004 Nuttier (A)dS black holes in higher dimensions *Class. Quant. Grav.* **21** 2937–62
- [21] Mann R B and Stelea C 2006 New multiply nutty spacetimes *Phys. Lett. B* **634** 448–55
- [22] Clarkson R and Mann R B 2006 Soliton solutions to the Einstein equations in five dimensions *Phys. Rev. Lett.* **96** 051104
- [23] Chen W, Lü H and Pope C N 2007 Kerr-de Sitter black holes with NUT charges *Nucl. Phys. B* **762** 38–54
- [24] Hourī T, Oota T and Yasui Y 2007 Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness *Phys. Lett. B* **656** 214–6
- [25] Krtouš P, Frolov V P and Kubizňák D 2008 Hidden symmetries of higher dimensional black holes and uniqueness of the Kerr-NUT-(A)dS spacetime *Phys. Rev. D* **78** 064022
- [26] Frolov V P and Kubizňák D 2007 'Hidden' symmetries of higher dimensional rotating black holes *Phys. Rev. Lett.* **98** 11101
- [27] Kubizňák D and Frolov V P 2007 Hidden symmetry of higher dimensional Kerr-NUT-AdS spacetimes *Class. Quantum Grav.* **24** F1–6
- [28] Frolov V P, Krtouš P and Kubizňák D 2008 Separability of Hamilton-Jacobi and Klein-Gordon equations in general Kerr-NUT-AdS spacetimes *J. High Energy Phys.* [JHEP0702\(2007\)005](#)
- [29] Sergyeyev A and Krtouš P 2008 Complete set of commuting symmetry operators for the Klein-Gordon equation in generalized higher-dimensional Kerr-NUT-(A)dS spacetimes *Phys. Rev. D* **77** 044033
- [30] Kolář I and Krtouš P 2016 Spacetimes with a separable Klein-Gordon equation in higher dimensions *Phys. Rev. D* **93** 024053
- [31] Oota T and Yasui Y 2008 Separability of Dirac equation in higher dimensional Kerr-NUT-de Sitter spacetime *Phys. Lett. B* **659** 688–93
- [32] Oota T and Yasui Y 2010 Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime *Int. J. Mod. Phys. A* **25** 3055–94
- [33] Cariglia M, Krtouš P and Kubizňák D 2011a Commuting symmetry operators of the Dirac equation, Killing-Yano and Schouten-Nijenhuis brackets *Phys. Rev. D* **84** 024004
- [34] Cariglia M, Krtouš P and Kubizňák D 2011b Dirac equation in Kerr-NUT-(A)dS spacetimes: intrinsic characterization of separability in all dimensions *Phys. Rev. D* **84** 024008
- [35] Page D N, Kubizňák D, Vasudevan M and Krtouš P 2007 Complete integrability of geodesic motion in general higher-dimensional rotating black hole spacetimes *Phys. Rev. Lett.* **98** 061102
- [36] Krtouš P, Kubizňák D, Page D N and Frolov V P 2007a Killing-Yano tensors, rank-2 Killing tensors, and conserved quantities in higher dimensions *J. High Energy Phys.* [JHEP0702\(2007\)004](#)
- [37] Hourī T, Oota T and Yasui Y 2008a Closed conformal Killing-Yano tensor and geodesic integrability *J. Phys. A* **41** 025204

- [38] Krtouš P, Kubizňák D, Page D N and Vasudevan M 2007b Constants of geodesic motion in higher-dimensional black-hole spacetimes *Phys. Rev. D* **76** 084034
- [39] Krtouš P 2015 Direct link between principal Killing-Yano tensor and hidden symmetries, in preparation
- [40] Frolov V P and Kubizňák D 2008 Higher-dimensional black holes: hidden symmetries and separation of variables *Class. Quant. Grav.* **25** 154005
- [41] Frolov V P 2014 Black holes, hidden symmetry and complete integrability: Brief Review *Proc., Relativity and Gravitation: 100 Years after Einstein in Prague* ed J Bičák and T Ledvinka (Cham: Springer) pp 261–73
- [42] Tangherlini F R 1963 Schwarzschild field in n dimensions and the dimensionality of space problem *Nuovo Cim.* **27** 636–51
- [43] Geroch R 1969 Limits of spacetimes *Commun. Math. Phys.* **13** 180–93
- [44] Houri T, Oota T and Yasui Y 2008b Generalized Kerr-NUT-de Sitter metrics in all dimensions *Phys. Lett. B* **666** 391–4
- [45] Houri T, Oota T and Yasui Y 2009 Closed conformal Killing-Yano tensor and uniqueness of generalized Kerr-NUT-de Sitter spacetime *Class. Quant. Grav.* **26** 045015
- [46] Krtouš P, Kubizňák D and Kolář I 2016 Killing-Yano forms and Killing tensors on a warped space *Phys. Rev. D* **93** 024057
- [47] Hamamoto N, Houri T, Oota T and Yasui Y 2007 Kerr-NUT-de Sitter curvature in all dimensions *J. Phys. A* **40** F177–84
- [48] Hawking S W 1977 Gravitational instantons *Phys. Lett. A* **60** 81–3
- [49] Page D N 1978a A compact rotating gravitational instanton *Phys. Lett. B* **79** 235–8
- [50] Page D N 1978b Taub—Nut Instanton With an Horizon *Phys. Lett. B* **78** 249
- [51] Gibbons G W and Hawking S W 1979 Classification of gravitational instanton symmetries *Commun. Math. Phys.* **66** 291–310
- [52] Eguchi T, Gilkey P B and Hanson A J 1980 Gravitation, gauge theories and differential geometry *Phys. Rep.* **66** 213–393
- [53] Hunter C J 1998 Action of instantons with a nut charge *Phys. Rev. D* **59** 024009
- [54] Mann R B 1999 Misner string entropy *Phys. Rev. D* **60** 104047
- [55] Chamblin A, Emparan R, Johnson C V and Myers R C 1999 Large N phases, gravitational instantons and the nuts and bolts of AdS holography *Phys. Rev. D* **59** 064010
- [56] Frolov V P 2008 Hidden symmetries of higher-dimensional black hole spacetimes *Prog. Theor. Phys. Suppl.* **172** 210–9
- [57] Mann R B, Shiekh A and Tarasov L 1990 Classical and quantum properties of two-dimensional black holes *Nucl. Phys. B* **341** 134–54
- [58] Frassino A M, Mann R B and Mureika J R 2015 Lower-dimensional black hole chemistry *Phys. Rev. D* **92** 124069
- [59] Hawking S W, Hunter C J and Taylor-Robinson M M 1999 Rotation and the AdS/CFT correspondence *Phys. Rev. D* **59** 064005
- [60] Kubizňák D 2009 On the supersymmetric limit of Kerr-NUT-AdS metrics *Phys. Lett. B* **675** 110–5
- [61] Hennigar R A, Kubizňák D, Mann R B and Musoke N 2015 Ultraspinning limits and super-entropic black holes *J. High Energy Phys.* **JHEP06(2015)096**
- [62] Frolov V P and Krtouš P 2011 Charged particle in higher dimensional weakly charged rotating black hole spacetime *Phys. Rev. D* **83** 024016
- [63] Dobarro F and Unal B 2005 Curvature of multiply warped products *J. Geom. Phys.* **55** 75