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Weakly charged generalized Kerr–NUT–(A)dS spacetimes

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ABSTRACT

We find an explicit solution of the source free Maxwell equations in a generalized Kerr–NUT–(A)dS spacetime in all dimensions. This solution is obtained as a linear combination of the closed conformal Killing–Yano tensor h_{ab}, which is present in such a spacetime, and a derivative of the primary Killing vector, associated with h_{ab}. For the vanishing cosmological constant the obtained solution reduces to the Wald's electromagnetic field generated from the primary Killing vector.

Electromagnetic fields in the vicinity of (rotating) black holes in four dimensions have interesting astrophysical applications and have been investigated by many authors, see e.g. [1–8]. See also [9–15] for the studies in higher dimensions.

Let us first recall the Wald construction of a weak electromagnetic field in spacetimes with symmetry [1]. Consider a test electromagnetic field in a D-dimensional curved spacetime with metric g_{ab}. The corresponding source-free Maxwell equations are

$$F^{ab}{}_{;b} = 0, \qquad F_{[ab;c]} = 0\,. \tag{1}$$

The latter equation implies that the electromagnetic field strength can be found as an external derivative of the potential 1-form A_a, $F_{ab} = A_{b,a} - A_{a,b}$. Imposing the Lorenz gauge condition one can write the first Maxwell equation in the form

$$A_a{}^{;a} = 0, \qquad \Box A_a - R_a{}^c A_c = 0\,, \tag{2}$$

where $\Box A_a = \nabla_b \nabla^b A_a$.

Suppose now that the spacetime possesses a Killing vector ξ^a. Then the Killing equation $\xi_{(a;b)} = 0$ and its integrability conditions imply

$$\xi_a{}^{;a} = 0, \qquad \Box \xi_a + R_a{}^c \xi_c = 0\,. \tag{3}$$

The equations (2) and (3) are quite similar. The only difference is the sign of the curvature term. This means that any Killing vector in a vacuum spacetime serves as a vector potential for a test source-free Maxwell field. Therefore a special set of test source-free electromagnetic fields in the background of vacuum spacetimes can be generated by simply using the isometries of these spacetimes. In such a way one can generate a weakly charged Kerr black hole, or immerse this black hole in a 'uniform magnetic field' [1,16].

In this letter we show how to generalize the Wald approach to a wide class of spacetimes with a cosmological constant that admit a closed conformal Killing–Yano (CCKY) tensor h_{ab} of rank 2. Such spacetimes obey the Einstein equation

$$R_{ab} = \frac{2}{D-2} \Lambda g_{ab} \tag{4}$$

with the cosmological constant Λ, and admit a CCKY 2-form h_{ab} satisfying the equation

$$h_{ab;c} = g_{ca}\xi_b - g_{cb}\xi_a\,, \tag{5}$$

where

$$\xi_a = \frac{1}{D-1} h^b{}_{a;b}\,. \tag{6}$$

Since h_{ab} is closed, $h_{[ab,c]} = 0$, there exists a potential one form b_a, such that

$$h_{ab} = b_{b,a} - b_{a,b}\,. \tag{7}$$

Using the integrability condition for (5) one obtains

$$\xi_{(a;b)} = \frac{1}{2(D-2)} \left(h_{ac} R^c{}_b + h_{ab} R^c{}_a \right)\,. \tag{8}$$

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Thus, in an Einstein space, that is when (4) is satisfied, it follows that

$$\xi_{(a;b)} = 0\,, \tag{9}$$

and hence ξ^a is a Killing vector. To distinguish it from other Killing vectors which may be present in a given spacetime, we call it a *primary* Killing vector.

Let us now define the following 2-form constructed from h_{ab} and the primary Killing vector ξ^a:

$$\mathcal{F}_{ab} = 2q(\xi_{[b,a]} + \lambda h_{ab}), \qquad \lambda = \frac{2\Lambda}{(D-1)(D-2)}\,, \tag{10}$$

where the constant q parameterizes the field strength. This 2-form is closed and can be generated from a 1-form potential

$$\mathcal{A}_a = q(\xi_a + 2\lambda b_a)\,. \tag{11}$$

Moreover, using relations (3), (4), and (6) one can easily check that $\mathcal{F}_{ab}$ is also divergence free,

$$\mathcal{F}^{ab}{}_{;b} = 0\,. \tag{12}$$

In other words, in any Einstein space with the CCKY tensor h_{ab} there always exists a solution of the source-free Maxwell equations that can be written in the form (10). Since this electromagnetic field is generated from the primary Killing vector, we call it a *primary* electromagnetic field.

An important special case occurs when h_{ab} is non-degenerate. Let us denote $D = 2n + \varepsilon$ the number of spacetime dimensions, with $\varepsilon = 0$ in even dimensions and $\varepsilon = 1$ in odd dimensions. Then the non-degeneracy condition means that the tensor $Q^a{}_b = h^{ac} h_{bc}$ has n independent nonvanishing eigenvalues x_μ^2 ($\mu = 1, \ldots, n$). We call such a CCKY tensor a *principal tensor*. The most general solution of the Einstein equations (4) which possesses the principal tensor is the so-called *Kerr–NUT–(A)dS metric* [17–21]. This solution was found in [22]. It contains the following set of 'free' parameters: mass M, $(n-1+\varepsilon)$ rotation parameters, a_i, and $(n-1)$ NUT parameters, N_μ. For $M = N_\mu = 0$ the Kerr–NUT–(A)dS metric becomes the metric of an (anti-)de Sitter spacetime, and when also $\Lambda = 0$ it is a flat metric.

The Kerr–NUT–(A)dS spacetime has a number of remarkable properties. In particular, besides the primary Killing vector ξ^a, it has $n-1+\varepsilon$ additional commuting Killing vectors. A set of n equations $x_\mu = \text{const}$ defines a $(n-1+\varepsilon)$-dimensional submanifold, such that all Killing vectors are tangent to it. If one introduces coordinates ψ_i on this submanifold by conditions that the Killing vectors take the form ∂_{ψ_i}, then (x_μ, ψ_i) can be used as spacetime coordinates, which are called *canonical*. A remarkable property of the Kerr–NUT–(A)dS metric is that when written in canonical coordinates the tensor components of h_{ab} do not depend on the mass and NUT parameters. That is, they are the same as the corresponding components of this tensor in the (A)dS or flat spacetime. Thus, the operation (10) of 'upgrading' the Killing vector ansatz for an electromagnetic field can be interpreted as a subtraction from $2\xi_{[b,a]}$ a similar quantity, calculated for the corresponding (anti-)de Sitter background metric. Namely this prescription was used in [11,13] for obtaining the weakly charged Kerr–(A)dS black holes in all dimensions.

In order to obtain the components of the field $\mathcal{F}_{ab}$ in the canonical coordinates we introduce a non-normalized frame of 1-forms [15]

$$\epsilon^\mu = dx_\mu\,, \quad \epsilon^{\hat\mu} = \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k\,, \quad \epsilon^{\hat 0} = \sum_{k=0}^{n} A^{(k)} d\psi_k\,. \tag{13}$$

The last 1-form $\epsilon^{\hat 0}$ is present only in odd dimensions. The functions $A^{(k)}$ and $A_\mu^{(k)}$ are symmetric polynomials in x_μ^2,

$$A^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k}}^{n} x_{\nu_1}^2 \ldots x_{\nu_k}^2\,, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1,\ldots,\nu_j=1\\ \nu_1<\ldots<\nu_j\\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \ldots x_{\nu_j}^2\,.$$

In this frame h_{ab} takes the form [23]

$$h = \sum_{\mu=1}^{n} x_\mu \epsilon^\mu \wedge \epsilon^{\hat\mu}\,. \tag{14}$$

It is possible to show that the strength tensor $\mathcal{F}_{ab}$ has a similar form

$$\mathcal{F} = \sum_{\mu=1}^{n} f_\mu \epsilon^\mu \wedge \epsilon^{\hat\mu}\,, \tag{15}$$

where f_μ are functions of all coordinates x_ν. Hence this field is a special case of a wide class of solutions of the Maxwell equations which are 'aligned with' the CCKY tensor h_{ab}. This class was described and studied in details in [15]. Using the results of this paper one obtains

$$f_\mu = \phi_{,\mu}\,, \qquad \phi = -2q \sum_{\mu=1}^{n} \frac{N_\mu x_\mu^{1-\varepsilon}}{U_\mu}\,. \tag{16}$$

Here, N_μ are NUT and mass parameters[1] of the Kerr–NUT–(A)dS solution and functions U_μ are just polynomials in x_μ^2

$$U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2)\,. \tag{17}$$

Up to pure gauge terms, the corresponding vector potential (11) reads

$$\mathcal{A} = -2q \sum_{\mu=1}^{n} \frac{N_\mu x_\mu^{1-\varepsilon}}{U_\mu} \epsilon^{\hat\mu}\,. \tag{18}$$

In four dimensions it is possible to 'backreact' this electromagnetic field on the geometry, by simply modifying the metric functions, to obtain a fully charged Kerr–NUT–(A)dS spacetime [24,25]. In higher dimensions, though, this is no longer possible within the realms of pure Einstein–Maxwell theory, see however [26], and additional fields have to be introduced, e.g. [27].

If one omits the restriction that the CCKY tensor h_{ab} is non-degenerate, the class of the metrics which possess such a tensor and obey the Einstein equations (4) becomes much larger. These solutions, called *generalized Kerr–NUT–(A)dS* metrics, were described in [21,28–30]. They describe a huge family of geometries, ranging from the Kähler metrics, Einstein–Sasaki geometries, generalized Taub–NUT metrics, or rotating black holes with some equal and some vanishing rotation parameters. As evident from the discussion, our construction of the solution (10) for the test electromagnetic field works in these metrics as well.

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[1] Here, we set $N_n = iM$. It corresponds to the convention, that the radial coordinate is $x_n = ir$. See [17–21].

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