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Geodesic motion and test fields in the background of higher-dimensional black holes[1]

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ABSTRACT
In a series of papers, we recently investigated properties of geodesic motion and test scalar fields in the background of generic rotating higher-dimensional black holes. In this contribution, we briefly discuss the integrability of geodesic motion, the construction of constants of motion, and the relation to the separability of the Hamilton–Jacobi equation. We also present a class of algebraically special test electromagnetic fields which generalize the electromagnetic field of a charged black hole in four dimensions. It will be, however, shown that in higher dimensions such fields cannot be easily modified in such a way that they would satisfy full Maxwell–Einstein equations.

Keywords: Black holes – higher dimensions – geodesic motion – integrability and separability – test fields

1 INTRODUCTION

Spacetimes of higher dimensions ($D > 4$) have become much studied as a result of their role in unification theories, such as the string/M theory. One important class of such spacetimes is a sequence of higher-dimensional black-hole metrics of greater and greater generality that have been discovered over the years.

The first such higher-dimensional black-hole spacetime was the metric for a nonrotating black hole in $D > 4$ (the generalization of the 1916 Schwarzschild solution), found in Tangherlini (1963). Next was the metric for a rotating black hole in higher dimensions (the generalization of the 1963 Kerr metric in four dimensions), discovered in Myers and Perry (1986) in the case of zero cosmological constant. Then in 1999 Hawking, Hunter and Taylor-Robinson (Hawking et al., 1999) found the general $D = 5$ version of the $D = 4$ rotating black hole with a cosmological constant (called also the Kerr–(anti-)de Sitter metric). In 2004 Gibbons, Lü, Page and Pope (Gibbons et al., 2004, 2005) discovered the general

[1] This contribution is a review of the results which have been obtained together with Don N. Page, Valeri Frolov, David Kubizňák, and Muraari Vasudevan last and this year and have been published in the papers (Page et al., 2007; Frolov et al., 2007; Krtouš et al., 2007a,b; Krtouš, 2007).

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Kerr–de Sitter metrics in all higher dimensions, and in 2006 Chen, Lü and Pope (Chen et al., 2006) put these into a simple form similar to that of Carter (1968a,b) and were able to add a NUT parameter (though not electric charge) to get the general Kerr–NUT–(a)dS metrics for all D. The properties of these metrics have been extensively studied in recent years. In the following, we give overview of some of these results.

One of the key spacetime properties is the nature of the corresponding geodesic motion. In our papers (Page et al., 2007; Krtouš et al., 2007b) we have found a full set of D conserved quantities for geodesic motion and demonstrated that this motion is completely integrable. The constants of motion have been constructed with help of the principal Killing–Yano tensor – an important geometrical structure that has been thoroughly investigated in (Krtouš et al., 2007a).

Closely related to the integrability of the geodesic motion is the separability of the Hamilton–Jacobi equation. It was proved, together with the separability of the Klein–Gordon equation (in Frolov et al., 2007).

Finally, we will discuss a test electromagnetic field specially aligned with the high-dimensional black hole background which was found in (Krtouš 2007; cf. also Chen and Lü 2007) and a no-go theorem for "charging" the rotating black hole in higher dimensions with the electromagnetic field of this type. Let us note that another no-go theorem for "accelerating" black holes in a way analogous to the four dimensional case has been presented in Kubizňák and Krtouš (2007).

In the following sections, we will revisit these topics in more detail. For simplicity, we will concentrate on the case of even dimensions. However, all discussed properties are valid also in odd dimensions – see the originals papers for corresponding expressions and modifications.

2 METRIC OF A GENERALLY ROTATING BLACK HOLE IN HIGHER DIMENSIONS

The metric of the general Kerr–NUT–(anti-)de Sitter spacetime in $D = 2n$ dimensions discovered by Chen et al. (2006) can be written

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu}\, \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)}\, \boldsymbol{d}\psi_k \right)^2 \right] . \tag{1}$$

Here, the coordinates x_μ ($\mu = 1, \dots, n$) correspond to (Wick rotated) radial and latitudinal directions, ψ_k ($k = 0, \dots, n-1$) to temporal and azimuthal directions. The metric functions U_μ, $A_\mu^{(k)}$, together with auxiliary functions $A^{(k)}$, are given by

$$U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \left(x_\nu^2 - x_\mu^2 \right) , \quad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k,\, \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 . \tag{2}$$

Each of the remaining metric functions X_μ is a function of a single variable x_μ and their exact form is given by the Einstein equations. However, most of the properties discussed below are independent of the exact form of the metric functions X_μ.

It is useful to rewrite the metric in a diagonal form

$$\boldsymbol{g} = \sum_{\mu=1}^{n} (\boldsymbol{e}^{\mu}\boldsymbol{e}^{\mu} + \boldsymbol{e}^{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}}) \tag{3}$$

introducing an orthonormal frame of 1-forms $\{\boldsymbol{e}^{\mu}, \boldsymbol{e}^{\hat{\mu}}\}$ and the dual vector frame $\{\boldsymbol{e}_{\mu}, \boldsymbol{e}_{\hat{\mu}}\}$, with $\mu = 1, \ldots, n$ and $\hat{\mu} = \mu + n$:

$$\begin{aligned} \boldsymbol{e}^{\mu} &= \left(\frac{U_\mu}{X_\mu}\right)^{1/2} \boldsymbol{d}x_\mu\,, & \boldsymbol{e}_{\mu} &= \left(\frac{X_\mu}{U_\mu}\right)^{1/2} \boldsymbol{\partial}_{x_\mu}\,, \\ \boldsymbol{e}^{\hat{\mu}} &= \left(\frac{X_\mu}{U_\mu}\right)^{1/2} \sum_{k=0}^{n-1} A_\mu^{(k)}\, \boldsymbol{d}\psi_k\,, & \boldsymbol{e}_{\hat{\mu}} &= \left(\frac{1}{X_\mu U_\mu}\right)^{1/2} \sum_{k=0}^{n-1} \left(-x_\mu^2\right)^{n-1-k} \boldsymbol{\partial}_{\psi_k}\,. \end{aligned} \tag{4}$$

It was derived in Hamamoto et al. (2007) that the Ricci tensor is also diagonal in this frame

$$\boldsymbol{Ric} = -\sum_{\mu=1}^{n} r_\mu (\boldsymbol{e}^{\mu}\boldsymbol{e}^{\mu} + \boldsymbol{e}^{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}})\,, \tag{5}$$

with the component r_μ given by

$$r_\mu = \frac{1}{2}\frac{X_\mu''}{U_\mu} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu}\frac{x_\nu X_\nu' - x_\mu X_\mu'}{x_\nu^2 - x_\mu^2} - \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu}\frac{X_\nu - X_\mu}{x_\nu^2 - x_\mu^2}\,. \tag{6}$$

The scalar curvature then is

$$\mathcal{R} = -\sum_{\nu=1}^{n} \frac{X_\nu''}{U_\nu}\,. \tag{7}$$

Enforcing the vacuum Einstein equations we have to solve the conditions $r_\mu = 0$. It turns out that the general solution is

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n-1} c_k \left(-x_\mu^2\right)^{n-1-k}\,. \tag{8}$$

The constants b_μ and c_k are related to the mass, NUT parameters, angular momenta and cosmological constant (for details, see Gibbons et al., 2005; Chen et al., 2006).

3 PRINCIPAL KILLING–YANO TENSOR

Inspecting the metric, we immediately see that the metric has n Killing vectors $\boldsymbol{\partial}_{\psi_k}$. However, it also possesses hidden symmetries which can be demonstrated by the existence of the so-called *principal* Killing–Yano tensor $\boldsymbol{f}$

$$\boldsymbol{f} = \sum_{\mu=1}^{n} x_\mu \underbrace{\boldsymbol{e}^1 \wedge \cdots \wedge \boldsymbol{e}^D}_{\boldsymbol{e}^{\mu},\, \boldsymbol{e}^{\hat{\mu}} \text{ skipped}}\,. \tag{9}$$

Its Hodge dual gives the second-rank closed conformal Killing–Yano tensor

$$h = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}} . \tag{10}$$

The conformal Killing–Yano tensor (CKYT) was first proposed by Kashiwada (1968) and Tachibana (1969) as a generalization of the Killing–Yano tensors (Yano, 1952). Since then both these tensors found wide applications in physics related to hidden (super)symmetries, conserved quantities, symmetry operators, or separation of variables. Let us recall that CKYT of a general rank r is an antisymmetric r-form f the covariant derivative of which can be split into an antisymmetric part and a divergence part

$$\nabla f = \mathcal{A}\,\nabla f + \mathcal{T}\,\nabla f . \tag{11}$$

Here $\mathcal{A}$ is the standard anti-symmetrization and $\mathcal{T}$ is the projection onto the "trace" part of the tensor of rank $r+1$ which is antisymmetric in the last r indices,

$$\mathcal{T}\,A_{aa_1\dots a_r} = \frac{r}{D-r+1} g_{a[a_1} A^{e}{}_{|e|a_2\dots a_r]} . \tag{12}$$

The divergence part $\mathcal{T}\,\nabla f$ thus depends only on the divergence $\nabla_e f^{e}{}_{ab\dots}$. The operations $\mathcal{A}$ and $\mathcal{T}$ satisfy $\mathcal{A}^2 = \mathcal{A}$, $\mathcal{T}^2 = \mathcal{T}$, and $\mathcal{T}\,\mathcal{A} = \mathcal{A}\,\mathcal{T} = 0$. The condition (11) implies that ∇f does not have a harmonic part (given by the complement of the $\mathcal{A}$ and $\mathcal{T}$ projectors), i.e., f does not have a part for which both $d f$ and $\nabla \cdot f$ vanishes. A CKYT transforms into a CKYT under the Hodge duality. The antisymmetric part $\mathcal{A}\,\nabla f$ transforms into the divergence part $\mathcal{T}\,\nabla{*}f$ and vice versa.

A Killing–Yano tensor f is such a CKYT for which the divergence part is missing, i.e., $\nabla f = \mathcal{A}\,\nabla f$. The dual of a Killing–Yano tensor is a closed CKYT, i.e., an r-form obeying $\nabla f = \mathcal{T}\,\nabla f$.

In our case, the principal CKYT h is the crucial geometrical structure which allows us to construct additional conserved quantities for geodesic motion and which is closely related to the separability of the Hamilton–Jacobi equation.

4 INTEGRABILITY OF GEODESIC MOTION

Let us now investigate geodesic motion in the spacetime given by the metric (1) with unspecified metric functions X_μ. For such a motion, the non-normalized velocity plays the role of momentum p. Its norm

$$w = p \cdot p \tag{13}$$

is conserved along the motion. Having n Killing vectors ∂_{ψ_k}, we can construct n conserved quantities linear in momentum

$$L_j = \partial_{\psi_j} \cdot p , \qquad j = 0, \dots, n-1 . \tag{14}$$

The remaining $n-1$ independent constants of motion can be constructed starting from the generating function written in terms of the Killing–Yano tensor and momentum

$$W(\beta) = \det\left(\boldsymbol{I} - \sqrt{\beta}\,\boldsymbol{h}\cdot\boldsymbol{P}\right) . \tag{15}$$

Here, $\boldsymbol{P}$ is a projector on the directions orthogonal to the momentum $\boldsymbol{p}$. It was shown in Krtouš et al. (2007a) that $W(\beta)$ is conserved for any value of β. The independent constants of motion can be extracted as coefficients in the β-expansion

$$W(\beta) = \frac{1}{w}\sum_j C_j\beta^j , \tag{16}$$

leading to $C_0 = w$ and

$$C_j = \sum_{\mu=1}^{n} A_\mu^{(j)}\left(\bar{p}_\mu^2 + \bar{p}_{\hat{\mu}}^2\right) , \qquad j = 0,\dots,n-1 , \tag{17}$$

where $\bar{p}_\mu$, $\bar{p}_{\hat{\mu}}$ are components of momentum in the frame $\boldsymbol{e}^\mu$, $\boldsymbol{e}^{\hat{\mu}}$,

$$\boldsymbol{p} = \sum_{\mu=1}^{n}\left(\bar{p}_\mu\boldsymbol{e}^\mu + \bar{p}_{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}}\right) . \tag{18}$$

We have shown in Page et al. (2007); Krtouš et al. (2007b) that the constants L_j and C_j are not only independent, but that they are also in involution

$$\{L_k, L_l\} = \{L_k, C_l\} = \{C_k, C_l\} = 0 . \tag{19}$$

These are sufficient conditions for the motion to be completely integrable (see, e.g., Arnol'd, 1989).

5 SEPARABILITY OF THE HAMILTON–JACOBI AND KLEIN–GORDON EQUATIONS

Both the complete integrability and the existence of the Killing–Yano tensor are closely related to the separability of the Hamilton–Jacobi equation (see, e.g., Arnol'd, 1989; Floyd, 1973; Penrose, 1973; Benenti and Francaviglia, 1979, 1980).

The separability of the Hamilton–Jacobi equation for geodesic motion

$$\frac{\partial S}{\partial\tau} + \boldsymbol{d}S\cdot\boldsymbol{g}\cdot\boldsymbol{d}S = 0 \tag{20}$$

can be demonstrated assuming

$$S = -\tau w + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{i=0}^{n-1} L_i\psi_i ,$$

with $S_\mu(x_\mu)$ being functions of a single variable only. Substituting into (20), we obtain an ordinary differential equation for S_μ (Frolov et al., 2007)

$$S_\mu'^2 = \frac{1}{X_\mu}\sum_{i=0}^{n-1} C_i \left(-x_\mu^2\right)^{n-1-i} - \frac{1}{X_\mu^2}\left[\sum_{i=0}^{n-1} L_i \left(-x_\mu^2\right)^{n-1-i}\right]^2 , \tag{21}$$

which can be solved by quadratures.

Identifying the gradient $\boldsymbol{d}S$ with the momentum $\boldsymbol{d}S = \boldsymbol{p}$, we find that the separability constants w, L_j, and C_j are exactly those defined in the previous section in (13), (14), and (17), L_j being linear in momentum and C_j quadratic.

Similarly, it was also demonstrated in Frolov et al. (2007), that the massive Klein–Gordon equation for a scalar field

$$\left[\Box - m^2\right] \Phi = 0 \tag{22}$$

can be solved by the separability ansatz

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{m} \exp\left(\mathrm{i}\Psi_k \psi_k\right) . \tag{23}$$

It leads to differential equations for R_μ

$$\left(X_\mu R_\mu'\right)' - \left[\frac{1}{X_\mu}\left(\sum_{k=0}^{n-1} \Psi_k (-x_\mu^2)^{n-1-k}\right)^2 + \sum_{k=0}^{n-1} \Xi_k \left(-x_\mu^2\right)^{n-1-k}\right] R_\mu = 0 , \tag{24}$$

with Ψ_j and Ξ_k arbitrary separation constants.

6 ALGEBRAICALLY SPECIAL TEST ELECTROMAGNETIC FIELD

Following Krtouš (2007), we will discuss now a special kind of test electromagnetic fields on the background given by the metric (1). We are looking for a field that would share the explicit symmetry of the metric (it would be independent of ψ_j) and that would be aligned with the hidden symmetry of the spacetime, namely, its Maxwell tensor $\boldsymbol{F}$ would have the same eigenspaces as the principal conformal Killing–Yano tensor $\boldsymbol{h}$. We thus require

$$\boldsymbol{F} = \sum_{\mu=1}^{n} f_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat\mu} , \quad f_\mu = f_\mu(x_1, \ldots, x_n) . \tag{25}$$

The Maxwell tensor is generated by the vector potential, $\boldsymbol{F} = \boldsymbol{d}\boldsymbol{A}$. As a consequence of the assumption (25), we find that the vector potential can be written as

$$\boldsymbol{A} = \sum_{\mu=1}^{n} g_\mu \left(\frac{x_\mu}{U_\mu}\right)^{1/2} \boldsymbol{e}^{\hat\mu} , \tag{26}$$

where g_μ are functions of a *single variable* only, $g_\mu = g_\mu(x_\mu)$. Evaluating the Maxwell tensor, we get the components f_μ:

$$f_\mu = \frac{g_\mu}{U_\mu} + \frac{x_\mu g'_\mu}{U_\mu} + 2x_\mu \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu} \frac{x_\nu g_\nu - x_\mu g_\mu}{x_\nu^2 - x_\mu^2} \,. \tag{27}$$

Alternatively, we could apply directly the first Maxwell equation $\boldsymbol{dF} = 0$ to find that f_μ are generated by an auxiliary potential ϕ,

$$f_\mu = \phi_{,\mu} \,, \tag{28}$$

which satisfies the equation

$$\phi_{,\mu\nu} = 2\frac{x_\nu \phi_{,\mu} - x_\mu \phi_{,\nu}}{x_\mu^2 - x_\nu^2} \quad \text{for} \quad \mu \neq \nu \,. \tag{29}$$

The field (26) is generated by the potential

$$\phi = \sum_{\nu=1}^{n} \frac{g_\nu x_\nu}{U_\nu} \,. \tag{30}$$

Calculating the source $\boldsymbol{J}$ of the electromagnetic field using the second Maxwell equation $\boldsymbol{J} = -\nabla \cdot \boldsymbol{F}$, we obtain

$$\boldsymbol{J} = \sum_{\mu=1}^{n} j_\mu \left(\frac{x_\mu}{U_\mu}\right)^{1/2} \boldsymbol{e}_{\hat{\mu}} \,, \tag{31}$$

with

$$j_\mu = -\frac{1}{x_\mu} \frac{\partial}{\partial x_\mu} \left(\phi - x_\mu^2 \sum_{\nu=1}^{n} x_\nu^{-1} \phi_{,\nu}\right) \,. \tag{32}$$

Substituting (30), we finally obtain

$$j_\mu = \frac{1}{x_\mu} \frac{\partial}{\partial x_\mu} \left(\sum_{\nu=1}^{n} \frac{x_\nu^2 g'_\nu}{U_\nu}\right) \,. \tag{33}$$

We are interested in source-free electromagnetic fields, so we require $\boldsymbol{J} = 0$. Using the special form of the sum in the square brackets in (33) we find that g'_μ are given by a single polynomial of order $(n-1)$ in variable x_μ^2. Integrating once more, we find

$$g_\mu x_\mu = e_\mu x_\mu + \sum_{k=0}^{n-1} a_k \left(-x_\mu^2\right)^{n-1-k} \,. \tag{34}$$

Substituting into the vector potential (26) or the scalar potential (30), we find that the terms containing the constants a_k are gauge-trivial and they can be ignored.

We have thus found that an algebraically special electromagnetic field (i.e., a field of the form (25)) satisfies Maxwell equations on the background described by the metric (1) if and only if it is generated by the vector potential

$$\boldsymbol{A} = \sum_{\mu=1}^{n} e_\mu \left(\frac{x_\mu}{U_\mu}\right)^{1/2} \boldsymbol{e}^{\hat{\mu}} . \tag{35}$$

The components f_μ of the Maxwell tensor are easily determined by (28) from the auxiliary potential

$$\phi = \sum_{\mu=1}^{n} \frac{e_\mu x_\mu}{U_\mu} , \tag{36}$$

and they read

$$f_\mu = \frac{e_\mu}{U_\mu} + 2x_\mu \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu} \frac{x_\nu e_\nu - x_\mu e_\mu}{x_\nu^2 - x_\mu^2} . \tag{37}$$

Here, e_μ are constants that can be related to the electric and magnetic charges of the field using the Gauss and Stokes theorems.

If we set all charges except for one, say e_ν, to zero, the Maxwell tensor $\boldsymbol{F}$ corresponds to the harmonic form $\boldsymbol{G}^{(\nu)}_{(2)}$ recently found and verified for particular cases in Chen and Lü (2007).

The surprising property of our field is that it satisfies Maxwell equations independently of the specific form of metric functions X_μ. Moreover, the stress-energy tensor corresponding to the field (25) has a form consistent with the structure of the Ricci (and Einstein) tensor (5). These facts open up a possibility that we could solve the full Einstein–Maxwell equations: modifying the metric functions X_μ, we could construct a spacetime in which the stress-energy tensor $\boldsymbol{T}$ would be a source for the Einstein equations, and the electromagnetic field would still satisfy Maxwell equations.

7 NO-GO THEOREM FOR CHARGING THE KERR–NUT–(A)DS METRIC

Indeed, this goal can be achieved in the physical dimension $D = 4$. In this case metric (1) with metric functions X_μ given by (8) corresponds to the uncharged black-hole solution in the form found by Carter (1968a,b) and elaborated by Plebański and Demiański (1976). However, if we modify the metric functions by adding constant terms $-e_1^2$ and $-e_2^2$,

$$\begin{aligned} X_1 &= c_0 + c_1 x_1^2 + c_2 x_1^4 + 2b_1 x_1 - e_1^2 , \\ X_2 &= c_0 + c_1 x_2^2 + c_2 x_2^4 + 2b_2 x_2 - e_2^2 , \end{aligned} \tag{38}$$

the metric (1) together with the electromagnetic field (35) solve the full Einstein–Maxwell equations – it corresponds to the Carter's charged black-hole solution.

In a generic dimension, we first evaluate the stress-energy tensor $\boldsymbol{T}$ of the electromagnetic field (25):

$$8\pi \boldsymbol{T} = \sum_{\mu=1}^{n} \left(2f_\mu^2 - f^2\right)\left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}\right) . \tag{39}$$

Its trace is

$$8\pi T = 2(2-n) f^2 , \tag{40}$$

where the function f^2 is defined as

$$f^2 = \sum_{\nu=1}^{n} f_\nu^2 . \tag{41}$$

We explicitly see that the trace of the stress-energy is non-vanishing for $D \neq 4$ which is related to the fact that the electromagnetic field is not conformally invariant in a general dimension.

Now we would like to solve the Einstein equations $\boldsymbol{Ric} - \frac{1}{2}\mathcal{R}\boldsymbol{g} + \Lambda \boldsymbol{g} = 8\pi \boldsymbol{T}$. The trace gives the condition

$$\mathcal{R} = 2\frac{D}{D-2}\Lambda + 2\frac{D-4}{D-2} f^2 . \tag{42}$$

However, the scalar curvature has the form (7) and it immediately follows that

$$\frac{\partial^{2n-2}}{\partial x_\mu^{2n-2}} (U_\mu \mathcal{R}) = -X_\mu^{[2n]} , \tag{43}$$

which is a function of x_μ only. Applying this to the right-hand-side of (42), we obtain the condition:

$$\frac{\partial^{2n-2}}{\partial x_\mu^{2n-2}} \left(U_\mu f^2\right) \quad \text{must be a function of } x_\mu \text{ only.} \tag{44}$$

This condition does not hold for the electromagnetic field given by (37), at least for the lowest non-trivial values of n. It seems that the main problem is that $\mathcal{R}$ behaves as $\sum h_\mu / U_\mu$ while f^2 as a square of such sums.

We can thus conclude that in a generic even dimension the electromagnetic field of the form (25), (37) cannot couple to the metric given by (1).

8 SUMMARY

In this contribution, we have reviewed some properties of the general higher-dimensional rotating black-hole spacetimes given by the metric (1). We have discussed the complete integrability of geodesic motion and explicitly found the full set of constants of motion. We have seen that the "nontrivial" constants are generated using the principal conformal Killing–Yano tensor and that they are quadratic in momenta and thus correspond to rank-2 Killing tensors (Krtouš et al., 2007a).

The complete integrability of the geodesic motion is related to the issue of separability of the Hamilton–Jacobi equation, which has been reviewed next. It was demonstrated that the separability constants for the Hamilton–Jacobi equation are the same as those constructed directly for geodesic motion.

Finally, we have presented an algebraically special test electromagnetic field. It depends on $n = D/2$ constants e_μ related to the global electric and magnetic charges. It generalizes the field known on the background of the Carter's black-hole solution in $D = 4$ dimensions. In this case, the metric functions can be modified in such a way that the field and the metric solve the full Einstein–Maxwell equations. Unfortunately, an analogous modification is not possible in a generic dimension.

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