Complete Integrability of Geodesic Motion in General Higher-Dimensional Rotating Black-Hole Spacetimes

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We explicitly exhibit $n-1=[D/2]-1$ constants of motion for geodesics in the general D-dimensional Kerr-NUT-AdS rotating black hole spacetime, arising from contractions of even powers of the 2-form obtained by contracting the geodesic velocity with the dual of the contraction of the velocity with the $(D-2)$-dimensional Killing-Yano tensor. These constants of motion are functionally independent of each other and of the $D-n+1$ constants of motion that arise from the metric and the $D-n=[(D+1)/2]$ Killing vectors, making a total of D independent constants of motion in all dimensions D. The Poisson brackets of all pairs of these D constants are zero, so geodesic motion in these spacetimes is completely integrable.

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With motivations especially from string theory, many people have shown much recent interest in black hole metrics in higher dimensions. Nonrotating black hole metrics in higher dimensions were first given in 1963 by Tangherlini [1]. In 1986, Myers and Perry [2] generalized the 1963 Kerr metric [3] for a 4-dimensional rotating black hole to all higher dimensions D. In 1968, Carter [4] added a cosmological constant to get a 4-dimensional rotating Kerr-de Sitter metric. In 1998, Hawking, Hunter, and Taylor-Robinson [5] found the general 5-dimensional extension of this metric, and in 2004, Gibbons, Lü, Page, and Pope [6,7] discovered the general Kerr-de Sitter metrics in all higher dimensions. In 2006, Chen, Lü, and Pope [8] were able to add a NUT [9] parameter to get the general Kerr-NUT-AdS metrics in all dimensions, which they presented in an especially simple form, analogous to the Plebański-Demiański [10] 4-dimensional generalization of Carter's Kerr-de Sitter metric.

With $n=[D/2]$, $\varepsilon=D-2n$, $m=n-1+\varepsilon$, Latin indices running over 1 through $D=2n+\varepsilon$, and Greek indices running over 1 through n, these Kerr-NUT-AdS metrics [8] that solve the Einstein equation $R_{ab}=-(D-1)g^2 g_{ab}$ may, after suitable analytic continuations, be written in the orthonormal form (cf. [11])

$$ds^2 \equiv \mathbf{g} = \sum_{a=1}^{D}\sum_{b=1}^{D}\delta_{ab}\mathbf{e}^a\mathbf{e}^b = \sum_{\mu=1}^{n}(\mathbf{e}^\mu\mathbf{e}^\mu + \mathbf{E}^\mu\mathbf{E}^\mu) + \varepsilon\boldsymbol{\omega}\boldsymbol{\omega}, \tag{1}$$

where the orthonormal basis one-forms are

$$\mathbf{e}^\mu = Q_\mu^{-1/2}dx_\mu, \qquad \mathbf{e}^{n+\mu} = \mathbf{E}^\mu = Q_\mu^{1/2}\sum_{k=0}^{n-1}A_\mu^{(k)}d\psi_k,$$
$$\mathbf{e}^{2n+1} = \boldsymbol{\omega} = (-c/A^{(n)})^{1/2}\sum_{k=0}^{n}A^{(k)}d\psi_k, \tag{2}$$

and where

$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = {\prod_{\nu=1}^{m}}'(x_\nu^2 - x_\mu^2), \qquad c = \prod_{k=1}^{m}a_k^2,$$
$$X_\mu = (-1)^\varepsilon\frac{g^2x_\mu^2-1}{x_\mu^{2\varepsilon}}\prod_{k=1}^{m}(a_k^2-x_\mu^2) + 2M_\mu(-x_\mu)^{(1-\varepsilon)},$$
$$A_\mu^{(k)} = {\sum_{\nu_1<\dots<\nu_k}}' x_{\nu_1}^2\dots x_{\nu_k}^2, \qquad A^{(k)} = \sum_{\nu_1<\dots<\nu_k} x_{\nu_1}^2\dots x_{\nu_k}^2. \tag{3}$$

Primes on the sum and product symbols mean that the index $\nu=\mu$ is omitted. The a_k and M_μ are related to angular momentum, mass, and NUT parameters.

The inverse Kerr-NUT-AdS metric has the form

$$\left(\frac{\partial}{\partial s}\right)^2 = \sum_{a=1}^{D}\sum_{b=1}^{D}\delta^{ab}\mathbf{e}_a\mathbf{e}_b = \sum_{\mu=1}^{n}(\mathbf{e}_\mu\mathbf{e}_\mu + \mathbf{E}_\mu\mathbf{E}_\mu) + \varepsilon\mathbf{E}\mathbf{E}, \tag{4}$$

where the orthonormal basis vectors are

$$\mathbf{e}_\mu = Q_\mu^{1/2}\frac{\partial}{\partial x_\mu},$$
$$\mathbf{e}_{n+\mu} = \mathbf{E}_\mu = Q_\mu^{-1/2}U_\mu^{-1}\sum_{k=0}^{m}(-1)^{n-1-k}x_\mu^{2(n-1-k)}\frac{\partial}{\partial\psi_k},$$
$$\mathbf{e}_{2n+1} = \mathbf{E} = (-cA^{(n)})^{-1/2}\frac{\partial}{\partial\psi_n}. \tag{5}$$

Kubizňák and Frolov [11] have shown that the Kerr-NUT-AdS metric possesses a $(D-2)$-rank Killing-Yano tensor

$$\mathbf{f} = *\mathbf{k}, \tag{6}$$

where the closed 2-form $\mathbf{k}$ can easily be shown to be

$$\mathbf{k} = \sum_{\mu=1}^{n} x_\mu \mathbf{e}^\mu \wedge \mathbf{E}^\mu. \quad (7)$$

In the general case, a Killing-Yano tensor [12] of rank p is a p-form $\mathbf{f}$ that satisfies the equations

$$f_{a_1 \ldots a_p} = f_{[a_1 \ldots a_p]}, \qquad f_{a_1 \ldots (a_p ; a_{p+1})} = 0. \quad (8)$$

Kubizňák and Frolov [11] then show how to use the conformal Killing tensor

$$\mathbf{Q} = Q_{ab}\mathbf{e}^a\mathbf{e}^b = k_{ac}k_b{}^c\mathbf{e}^a\mathbf{e}^b = \sum_{\mu=1}^{n} x_\mu^2(\mathbf{e}^\mu\mathbf{e}^\mu + \mathbf{E}^\mu\mathbf{E}^\mu) \quad (9)$$

to construct the 2nd-rank Killing tensor that can easily be shown to be

$$\mathbf{K} = \mathbf{Q} - \frac{1}{2}Q_c{}^c\mathbf{g} = -\sum_{\mu=1}^{n} A_\mu^{(1)}(\mathbf{e}^\mu\mathbf{e}^\mu + \mathbf{E}^\mu\mathbf{E}^\mu) - \varepsilon A^{(1)}\omega\omega. \quad (10)$$

A Killing tensor [13–15] of rank r is a totally symmetric tensor $\mathbf{K}$ that satisfies the equations

$$K_{a_1 \ldots a_r} = K_{(a_1 \ldots a_r)}, \qquad K_{(a_1 \ldots a_r ; a_{r+1})} = 0. \quad (11)$$

Geodesic motion gives conserved constants from contractions of one velocity $\mathbf{u} = u^a\mathbf{e}_a$ with each of the $D - n = n + \varepsilon = m + 1$ Killing vectors $\partial/\partial\psi_k$, from the contraction of two velocities with the metric, and from contractions of velocities with any Killing tensors present. With one 2nd-rank Killing tensor present that is independent of the metric (which is always a Killing tensor), one thus has $D + 2 - n$ constants of motion. For $n \leq 2$ or $D \leq 5$, this gives a full set of D constants to make the geodesic motion integrable. However, for $D > 5$, it was not previously known how to find a full set of D constants of geodesic motion for the general Kerr-NUT-AdS metrics [8], or even for the general Myers-Perry (MP) metrics [2] obtained by eliminating the NUT parameters and the cosmological constant. For earlier work on geodesic motion and Killing tensors in the MP, Kerr-(NUT)-AdS, and related metrics in higher dimensions, see [11,16–32].

The point of the present Letter is to show that one can obtain a full set of D independent constants in involution for geodesic motion in the general Kerr-NUT-AdS metrics, thereby making this motion completely integrable.

Briefly, the demonstration uses the fact that when the velocity is contracted with the Killing-Yano tensor of rank $D - 2$, this gives a $(D - 3)$-form that is covariantly constant along each geodesic. The dual of this $(D - 3)$-form gives a 3-form that is also covariantly constant along each geodesic, as is the 2-form contraction of this 3-form with the velocity. This 2-form has at least $n - 1$ nonzero complex-conjugate (pure imaginary) pairs of eigenvalues that give $n - 1$ constants of motion, which we can show [33] are independent of each other and of the $D - n + 1$ constants of motion obtainable from the Killing vectors and the metric, with all of the Poisson brackets between them vanishing. Therefore, we have D independent constants of motion in involution for geodesics in the general Kerr-NUT-AdS metrics, making the geodesics completely integrable (see, e.g., [34,35]).

Let us write the resulting 2-form as

$$\mathbf{F} = \mathbf{u} \cdot \{*[\mathbf{u} \cdot (*\mathbf{k})]\} = \frac{1}{2}F_{ab}\mathbf{e}^a \wedge \mathbf{e}^b, \quad (12)$$

with components

$$F_{ab} = (k_{ab}u_c + k_{bc}u_a + k_{ca}u_b)u^c. \quad (13)$$

Then since $\mathbf{F}$ is covariantly constant along geodesics, $u^cF_{ab;c} = 0$, the eigenvalues of $\mathbf{F}$ are constants of motion. In particular, the traces of even powers of the matrix form of $\mathbf{F}$ are constants. (The traces of odd powers are zero because of the antisymmetry of $\mathbf{F}$.)

Now let us give a formula for the new constants of motions C_j that are proportional to traces of the even powers of the matrix form of $\mathbf{F}$ and evaluate them explicitly for the 2nd, 4th, 6th, and 8th powers. For convenience, let us use matrix notation, in which F is the antisymmetric matrix with orthonormal components $F^a{}_b$, K is the antisymmetric matrix with components $k^a{}_b$ (not to be confused with a Killing tensor), $Q \equiv -K^2$ is the symmetric matrix with components $Q^a{}_b = -k^a{}_c k^c{}_b$, W is the symmetric matrix with components $u^a u_b$, $w \equiv \mathrm{Tr}(W) = u^c u_c$, $P \equiv I - W/w$ is the projection onto the hyperplane orthogonal to the velocity, and $S \equiv -PKPKP$. These matrices have the properties that $P^2 = P$ and $WK^{2j+1}W = 0$ for all nonnegative integers j. Then the component Eq. (13) becomes the matrix equation

$$F = wK - KW - WK = wPKP, \quad (14)$$

whose negative square is the symmetric matrix

$$-F^2 = w^2S = w^2PQP + wKWK. \quad (15)$$

One can now prove [33] that for all j,

$$\mathrm{Tr}(Q^j) + \mathrm{Tr}(S^j) = 2\mathrm{Tr}[(QP)^j]. \quad (16)$$

Therefore, we get the constants of motion

$$\begin{aligned} C_j &\equiv w^{-j}\mathrm{Tr}[(-F^2)^j] = w^j\mathrm{Tr}(S^j) \\ &= 2\mathrm{Tr}[(wQ - QW)^j] - w^j\mathrm{Tr}(Q^j) \\ &= w^j\mathrm{Tr}(Q^j) - 2jw^{j-1}\mathrm{Tr}(Q^jW) \\ &\quad + \sum_{c=2}^{j} \sum_{l_1 \leq \ldots \leq l_c, \sum_i l_i = j} (-1)^c N^j_{l_1 \ldots l_j} w^{j-c} \prod_{i=1}^{c} \mathrm{Tr}(Q^{l_i}W), \end{aligned} \quad (17)$$

where in the last expression the coefficients $N^j_{l_1 \ldots l_j}$ are some positive combinatoric factors [33]. We have used the fact that terms $\mathrm{Tr}(Q^jWQ^kW\cdots)$ factorize into $(\mathrm{Tr}Q^jW)\times(\mathrm{Tr}Q^kW)\cdots$. Notice that all traces in the last line contain strictly lower powers of Q than Q^j.

To write the constants of motion in tensor notation, it is convenient to define the scalars $Q^{(j)}$ that are the traces of the jth power of the matrix Q,

$$Q^{(j)} \equiv \mathrm{Tr}(Q^j) = 2\sum_{\mu=1}^{n} x_\mu^{2j}, \qquad (18)$$

and the symmetric covariant tensor components $Q^{(j)}_{ab}$ that form the tensor $\mathbf{Q}^{(j)}$ corresponding to the jth power of the matrix Q,

$$\mathbf{Q}^{(j)} = Q^{(j)}_{ab}\mathbf{e}^a\mathbf{e}^b = \sum_{\mu=1}^{n} x_\mu^{2j}(\mathbf{e}^\mu\mathbf{e}^\mu + \mathbf{E}^\mu\mathbf{E}^\mu). \qquad (19)$$

For example, $Q^{(1)} = Q_c{}^c$, $Q^{(1)}_{ab} = Q_{ab}$, $Q^{(2)} = Q_c{}^dQ_d{}^c$, $Q^{(2)}_{ab} = Q_a{}^cQ_{cb}$, $Q^{(3)} = Q_c{}^dQ_d{}^eQ_e{}^c$, and $Q^{(3)}_{ab} = Q_a{}^cQ_c{}^dQ_{db}$.

Then one can easily see that the C_j's have the form

$$C_j = K_{a_1\ldots a_{2j}}u^{a_1}\ldots u^{a_{2j}} \qquad (20)$$

for some symmetric tensors $K_{a_1\ldots a_{2j}}$ formed from combinations of the metric g_{ab}, $Q^{(j)}$, and the $Q^{(i)}_{ab}$'s for $i \le j$. It can be shown [14] that these are Killing tensors in the sense of Eq. (11).

In particular, we get

$$\begin{aligned} C_1 &= w\mathrm{Tr}(Q) - 2\mathrm{Tr}(QW) = (Q^{(1)}g_{ab} - 2Q^{(1)}_{ab})u^au^b \\ &= -2K_{ab}u^au^b, \end{aligned} \qquad (21)$$

the constant from the previously-known 2nd-rank Killing tensor given in Eq. (10), and

$$\begin{aligned} C_2 &= w^2\mathrm{Tr}(Q^2) - 4w\mathrm{Tr}(Q^2W) + 2[\mathrm{Tr}(QW)]^2 \\ &= (Q^{(2)}g_{ab}g_{cd} - 4Q^{(2)}_{ab}g_{cd} + 2Q^{(1)}_{ab}Q^{(1)}_{cd})u^au^bu^cu^d \\ &= K_{abcd}u^au^bu^cu^d, \end{aligned} \qquad (22)$$

where the new 4th-rank Killing tensor has components

$$K_{abcd} = Q^{(2)}g_{(ab}g_{cd)} - 4Q^{(2)}_{(ab}g_{cd)} + 2Q_{(ab}Q_{cd)} \qquad (23)$$

and gives the Dth constant of motion for $D = 6$ and $D = 7$.

Continuing in a similar fashion to get the Dth constant of motion for $D = 8$ and $D = 9$,

$$\begin{aligned} C_3 &= w^3\mathrm{Tr}(Q^3) - 6w^2\mathrm{Tr}(Q^3W) + 6w\mathrm{Tr}(Q^2W)\mathrm{Tr}(QW) \\ &\quad - 2[\mathrm{Tr}(QW)]^3 \\ &= (Q^{(3)}g_{ab}g_{cd}g_{ef} - 6Q^{(3)}_{ab}g_{cd}g_{ef} + 6Q^{(2)}_{ab}Q^{(1)}_{cd}g_{ef} \\ &\quad - 2Q^{(1)}_{ab}Q^{(1)}_{cd}Q^{(1)}_{ef})u^au^bu^cu^du^eu^f \\ &= K_{abcdef}u^au^bu^cu^du^eu^f, \end{aligned} \qquad (24)$$

where the new 6th-rank Killing tensor is

$$\begin{aligned} K_{abcdef} &= Q^{(3)}g_{(ab}g_{cd}g_{ef)} - 6Q^{(3)}_{(ab}g_{cd}g_{ef)} \\ &\quad + 6Q^{(2)}_{(ab}Q_{cd}g_{ef)} - 2Q_{(ab}Q_{cd}Q_{ef)}. \end{aligned} \qquad (25)$$

To finish the explicit expressions for all D constants of motion (not counting the constants from the metric and Killing vectors) up through $D = 11$, the highest dimension generally considered in superstring/M theory, we calculate

$$\begin{aligned} C_4 &= w^4\mathrm{Tr}(Q^4) - 8w^3\mathrm{Tr}(Q^4W) + 8w^2\mathrm{Tr}(Q^3W)\mathrm{Tr}(QW) \\ &\quad + 4w^2[\mathrm{Tr}(Q^2W)]^2 - 8w\mathrm{Tr}(Q^2W)[\mathrm{Tr}(QW)]^2 \\ &\quad + 2[\mathrm{Tr}(QW)]^4 \\ &= (Q^{(4)}g_{ab}g_{cd}g_{ef}g_{gh} - 8Q^{(4)}_{ab}g_{cd}g_{ef}g_{gh} \\ &\quad + 8Q^{(3)}_{ab}Q^{(1)}_{cd}g_{ef}g_{gh} + 4Q^{(2)}_{ab}Q^{(2)}_{cd}g_{ef}g_{gh} \\ &\quad - 8Q^{(2)}_{ab}Q^{(1)}_{cd}Q^{(1)}_{ef}g_{gh} + 2Q^{(1)}_{ab}Q^{(1)}_{cd}Q^{(1)}_{ef}Q^{(1)}_{gh}) \\ &\quad \times u^au^bu^cu^du^eu^fu^gu^h \\ &= K_{abcdefgh}u^au^bu^cu^du^eu^fu^gu^h, \end{aligned} \qquad (26)$$

with the corresponding 8th-rank Killing tensor being

$$\begin{aligned} K_{abcdefgh} &= Q^{(4)}g_{(ab}g_{cd}g_{ef}g_{gh)} - 8Q^{(4)}_{(ab}g_{cd}g_{ef}g_{gh)} \\ &\quad + 8Q^{(3)}_{(ab}Q_{cd}g_{ef}g_{gh)} + 4Q^{(2)}_{(ab}Q^{(2)}_{cd}g_{ef}g_{gh)} \\ &\quad - 8Q^{(2)}_{(ab}Q_{cd}Q_{ef}g_{gh)} + 2Q_{(ab}Q_{cd}Q_{ef}Q_{gh)}. \end{aligned} \qquad (27)$$

For a spacetime with $D = 2n + \varepsilon$ dimensions, we get $D - n = n + \varepsilon$ constants of motion from the $D - n$ Killing vectors and one constant of motion, $w = \mathbf{u}\cdot\mathbf{u}$, from the metric Killing tensor. Therefore, we need C_j up through $j = n - 1$ to give the remainder of the D constants of motion.

We can explicitly show [33] that all of these D constants of motion are independent of each other, as functions of the velocity components u^a, by calculating the Jacobian of this transformation. When all of the first n velocity components are nonzero, and at a generic point of the manifold where none of the $(x_\mu)^2$'s coincide (which would actually give a coordinate singularity if any did coincide), we find that the Jacobian is nonzero. Key to the proof is the fact that in the constant C_j, the coefficient of $\mathrm{Tr}(Q^jW)$, given explicitly in Eq. (17), is nonzero, as well as the fact that the n pairs of eigenvalues of the matrix Q, namely, the $(x_\mu)^2$'s, are all different when none of the $(x_\mu)^2$'s coincide. Therefore, the $\mathrm{Tr}(Q^jW)$ term for each higher j up to $n - 1$ gives a function of the velocity components that is independent of any of the terms with lower j. We can also prove [33] that the Poisson bracket between any pair of these D constant vanishes, so these constants are in involution, a sufficient condition for the integrable motion to be completely integrable (see, e.g., [34,35]).

In summary, we have shown that geodesic motion is completely integrable for all Kerr-NUT-AdS metrics [8] in all dimensions and with arbitrary rotation and NUT parameters. However, this has not enabled us (at least yet) to separate the Hamilton-Jacobi, Dirac, and Klein-Gordon equations.

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Note added.—For further recent work on this subject, including separation of the Hamilton-Jacobi and Klein-Gordon equations, see [36,37].

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