

Ring wormholes and time machines

Valeri P. Frolov^{1,*}, Pavel Krtouš^{2,†} and Andrei Zelnikov^{1,‡}

¹*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2E1*

²*Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic*



(Received 5 May 2023; accepted 4 June 2023; published 14 July 2023)

In the present paper we discuss properties of a model of a ring wormhole, recently proposed by Gibbons and Volkov [Phys. Rev. D **96**, 024053 (2017); J. Cosmol. Astropart. Phys. **05** (2017) 039; Phys. Lett. B **760**, 324 (2016)]. Such a wormhole connects two flat spacetimes which are glued through disks of the radius a bounded by the string with negative angle deficit -2π . The presence of the string's matter violating the null-energy condition makes the wormhole static and traversable. We study gravitational field of static sources in such a spacetime in the weak-field approximation. In particular, we discuss how a field of an oblate thin massive shell surrounding one of the wormhole's mouths is modified by its presence. We also obtain a solution of a similar problem when both mouths of the wormhole are located in the same space. This approximate solution is found for the case when the distance L between these mouths is much larger than the radius a of the ring. We demonstrate that the corresponding locally static gravitational field in such a multiply connected space is nonpotential. As a result of this, the proper time gap for the clock's synchronization linearly grows with time and closed timelike curves are formed. This process inevitably transforms such a traversable ring wormhole into a time machine. We estimate the timescale of this process.

DOI: [10.1103/PhysRevD.108.024034](https://doi.org/10.1103/PhysRevD.108.024034)

I. INTRODUCTION

Even in the absence of matter, the space topology in general relativity can be very nontrivial. A well-known example is the famous Einstein-Rosen bridge connecting two asymptotically flat spaces. Examples of solutions of the initial value problem describing wormholes in a three-dimensional space were constructed by Misner [1] and Brill and Lindquist [2]. Wormholes and their properties were described in detail by Wheeler in his famous book on geometrodynamics [3]. A generic property of vacuum wormhole solutions of the Einstein equations is that as a result of their time evolution they shrink and form a singularity so fast that one cannot use them to “travel in space.” Gannon [4] proved that any asymptotically flat spacetime with a nonsimply connected Cauchy surface has singular time evolution if it satisfies the weak energy condition (see also [5–7]).

A special class of wormholes, called traversable, attracted in the past a lot of attention. Their characteristic property is that a region with nontrivial topology is located inside a compact spatial domain and the particles and light can penetrate through a topological handle and return to the exterior region without meeting a singularity [8–11]. As it was demonstrated by Thorne and collaborators [9,12] a relative motion of the traversable wormhole's mouths can generate closed timelike curves, so that such a wormhole becomes a “time machine.” A similar effect of time-machine creation from a traversable wormhole can be achieved when the wormhole's mouths are not moving, but one of them is surrounded by some mass distribution [13]. In the presence of closed timelike curves a problem of self-consistency of the standard physics becomes highly nontrivial [12,14–16]. For example, infinite amplification of zero-point vacuum fluctuation during the creation of the time machine may result in (formally) infinite growth of the value of the renormalized vacuum stress-energy tensor, that indicates that its backreaction would become important [17,18]. Hawking formulated this problem as a chronology protection conjecture [19]. For general discussion of wormholes and a time-machine problem see, e.g., Refs. [20–22].

The simplest model of a traversable wormhole connecting two asymptotically flat spaces has a spherically symmetric geometry [9–11]. For such a wormhole there exists a two-sphere of the minimal area in the wormhole throat and the null-energy condition should be violated in

*vfrolov@ualberta.ca

†Pavel.Krtous@utf.mff.cuni.cz

‡zelnikov@ualberta.ca

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

its vicinity (see, e.g., [10]). Recently, an interesting model of a traversable wormhole was proposed by Gibbons and Volkov [23–25]. In this model two flat spacetimes are connected through a disk such that its boundary is a circular ring where there exists a conical curvature singularity with the angle deficit -2π . This model of a traversable wormhole is a particular implementation of a loop-based wormhole discussed by Visser [20]. One can identify this ring with a cosmic string with the corresponding negative energy distribution. Namely, the presence of this matter violates the null-energy condition and makes possible the existence of a static traversable wormhole. The geometry of the spacetime everywhere outside the ring (including the disk where two flat spacetime geometries are glued) is flat. The radius of the ring a can be arbitrary large and a particle propagating through the disk from one flat space to the other meets neither negative energy density regions nor a strong gravitational field. We refer to this type of geometries as the ring wormhole. A main purpose of this paper is to demonstrate how such a ring wormhole can be transformed into a time machine.

For this goal we discuss a gravitational field created by static sources in the presence of a ring wormhole. We consider two cases: (i) The ring wormhole connects two different flat spaces and (ii) such a wormhole connects two spatially separated regions of the same spacetime. In the latter case such a space is multiply connected. We assume that the gravitational field is weak and use the linearized gravity equations. We also assume that the source of the gravitational field is an oblate massive thin shell spheroid surrounding a mouth of the ring wormhole confocal to the string ring. An exact solution for the gravitational field of such a shell for the ring wormhole connecting two flat spaces is obtained.

For the second problem (the wormhole in a single space) an approximate solution is found for the case when the distance L between the mouths is much larger than the ring’s size a . We demonstrate that in the second case the locally static gravitational field is nonpotential and as a result of this closed timelike curves are created some time after the massive source surrounding one of the mouths is “switched on.” We also estimate the corresponding time required for this as a function of mass of the shell, its size and the distance between the mouths.

The paper is organized as follows. In the next section we remind the reader properties of ring wormholes. In Sec. III we explain the meaning of an approximation of a weak gravitational field in application to wormhole spacetime. Gravitational field in the presence of a ring wormhole connecting two flat spaces is discussed in Sec. IV. A similar problem for a string wormhole connecting two separated regions of a single space is considered in Sec. V. Closed timelike paths formation, i.e., appearance of a time machine, in the space of a wormhole with a nonpotential locally static gravitational field is discussed in Sec. VI.

In particular, it contains estimation of the time required for closed timelike curve formation. In Sec. VII we discuss the obtained results and their possible consequences. Two appendixes collect information concerning the static gravitational field in the weak-field approximation and general properties of locally static nonpotential gravitational fields.

In this paper we use sign conventions adopted in Ref. [26].

II. GEOMETRY OF A RING WORMHOLE

A. A ring wormhole connecting two asymptotically flat spaces

Let us discuss first a spatial configuration of the wormhole spacetimes. We begin with a case of a ring wormhole connecting two flat spaces. Such a wormhole can be obtained as follows. Consider two copies of a three-dimensional flat space. Denote by $X^i = (X_{\pm}, Y_{\pm}, Z_{\pm})$ standard Cartesian coordinates in these spaces. Let D_{\pm} be disks in R_{\pm} of radius a in the planes $Z_{\pm} = 0$ with the center at the origin of these spaces (see Fig. 1).

To join the spaces R_- and R_+ with a wormhole, we identify surfaces of the disks D_- and D_+ . This identification is done as follows. We denote by $D_{\pm}^>$ and $D_{\pm}^<$ the “right” and “left” faces of the disks. Namely, if the disk D_{\pm} is reached by a point with the positive value of Z_{\pm} , we say that it belongs to $D_{\pm}^>$. In the opposite case when a point approaches the disk with $Z_{\pm} < 0$ we say that it belongs to $D_{\pm}^<$. One identifies $D_{-}^<$ with $D_{+}^>$ and $D_{+}^>$ with $D_{-}^<$, without any rotation around the z axis.

This means that a free particle 1 moving in space R_+ , which meets the disk $D_{-}^<$, enters space R_- at $D_{+}^>$ with identical coordinates $X_- = X_+$, $Y_- = Y_+$, $Z_- = Z_+ = 0$ and continues its motion with increasing Z_- coordinate. Similarly if a particle 2 enters $D_{+}^>$, it appears at $D_{-}^<$ part of the disk D_- as illustrated in Fig. 1.

In what follows we denote \hat{R} the space obtained by unifying two spaces R_+ and R_- with identification described above. We call the disks D_- and D_+ in R_- and R_+ , respectively, the wormhole mouths and their identification D the wormhole throat.

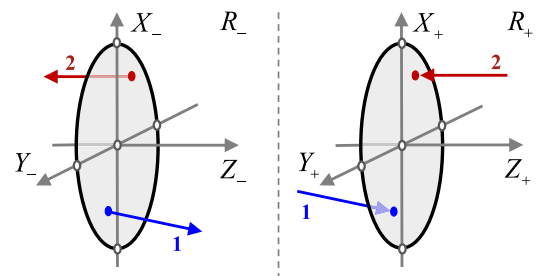


FIG. 1. A ring wormhole connecting two flat spaces R_+ and R_- .

The radius a of the disk has the dimension of length. It is convenient to use it as a natural scale and write the metric in a flat space in the form

$$dL^2 = \delta_{ij} dX^i dX^j = a^2 dl^2, \quad (2.1)$$

$$dl^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2. \quad (2.2)$$

In what follows we shall use the dimensionless metric dl^2 and dimensionless coordinates $x^j = X^j/a$. We denote by (z, ρ, φ) dimensionless cylindrical coordinates

$$\rho^2 = x^2 + y^2, \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad (2.3)$$

with the metric taking the form

$$dl^2 = d\rho^2 + dz^2 + \rho^2 d\varphi^2. \quad (2.4)$$

The ring equations are

$$\rho = 1, \quad z = 0. \quad (2.5)$$

It will be very useful to introduce also the oblate spheroidal coordinates $(\chi, \vartheta, \varphi)$ related to the cylindrical coordinates as follows:

$$z = \sinh \chi \cos \vartheta, \quad \rho = \cosh \chi \sin \vartheta. \quad (2.6)$$

A two-dimensional surface $\chi = \text{const}$ is an oblate spheroid, confocal with the ring, with $\cosh \chi$ being its larger semiaxis and $\sinh \chi$ being the smaller one. A surface $\vartheta = \text{const}$ is a hyperboloid confocal also with the ring. Let us note that the relations (2.6) can be written in the following complex form:

$$z + i\rho = \sinh(\chi + i\vartheta). \quad (2.7)$$

The flat metric in the oblate spheroidal coordinates is

$$dl^2 = (\sinh^2 \chi + \cos^2 \vartheta)(d\chi^2 + d\vartheta^2) + \cosh^2 \chi \sin^2 \vartheta d\varphi^2. \quad (2.8)$$

For a single flat space R , the oblate spheroidal coordinates have a twofold degeneracy: $(\chi, \vartheta, \varphi)$ and $(-\chi, \pi - \vartheta, \varphi)$ correspond to the same point. For this reason, it would be sufficient to consider the ranges of the oblate spheroidal coordinates $\chi \in (0, \infty)$, $\vartheta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$.

In the wormhole space \tilde{R} , we introduce separate copies of flat $(x_{\pm}, y_{\pm}, z_{\pm})$ and cylindrical $(z_{\pm}, \rho_{\pm}, \varphi_{\pm})$ coordinates in spaces R_- and R_+ . However, we take advantage of the degeneracy of the oblate spheroidal coordinates and define just single coordinates $(\chi, \vartheta, \varphi)$ covering the whole wormhole space, where $\chi \in (-\infty, \infty)$, simply assuming that $\chi > 0$ in R_+ and $\chi < 0$ in R_- .

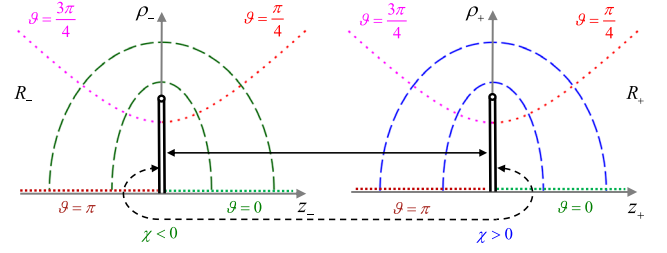


FIG. 2. Ring wormhole in space \tilde{R} with two asymptotic regions R_- and R_+ . Each of the domains R_{\pm} is covered by cylindrical coordinates $\rho_{\pm}, \varphi_{\pm}, z_{\pm}$. Directions related to angular coordinate φ_{\pm} are not shown. They can be obtained by rotations around axes $\rho_{\pm} = 0$. The whole space \tilde{R} is covered by one copy of the oblate spheroidal coordinates $\chi \in (-\infty, \infty)$, $\vartheta \in (0, \pi)$ and $\varphi \equiv \varphi_{\pm} \in (-\pi, \pi)$. Dashed lines corresponds to $\chi = \text{const}$, dotted lines to $\vartheta = \text{const}$. Double lines indicate the disks which are identified as indicated by arrows. They represent the mouths of the wormhole.

Figure 2 shows space \tilde{R} in (χ, ϑ) coordinates. To specify a point, one needs to add an azimuthal angle φ . The coordinates $(\chi, \vartheta, \varphi)$ cover the complete space and they are continuous at the disks representing the mouths of the ring wormhole. The left figure corresponds to R_- domain where $\chi < 0$, while the right one corresponds to R_+ where $\chi > 0$.

Let us note that it is possible to generalize the described ring wormhole space to the case when there exist not only one but several disks in space R_+ , each “connected” with its own version of flat space. We do not consider such multiwormhole configurations in the present paper.

B. A ring wormhole in a single space

It is easy to use a simple procedure to construct a space that contains both mouths of a single wormhole. One just places two disks representing the wormhole’s mouths into one flat space. Obviously, there is an ambiguity in the choice of positions and orientations of the disks. We consider the simplest case when the disks are orthogonal to a common axis and separated by a distance L .

If we introduce global dimensionless flat and cylindrical coordinates (x, y, z) and (z, ρ, φ) , the centers of two disks D_{\pm} of radius 1 are located along the z axis at $z = \pm \ell/2$, and the disks are orthogonal to the z axis. Here, we denoted by $\ell = L/a$ the dimensionless version of the disks’ distance.

To form the wormhole, we identify the disks D_- and D_+ in a similar manner as we described above: Each of the disks has two faces which we denoted by D_{\pm}^{\geq} and D_{\pm}^{\leq} . One identifies D_-^{\leq} with D_+^{\geq} and D_+^{\geq} with D_-^{\leq} , without any rotation around the z axis. This identification is illustrated in Fig. 3.

We call the resulting space R_{wh} . The two-face disks D_- and D_+ are called left and right mouths of the wormhole. Identified, they form the throat of the wormhole. Strictly speaking, the standard coordinates (x, y, z) and (ρ, φ, z) ,

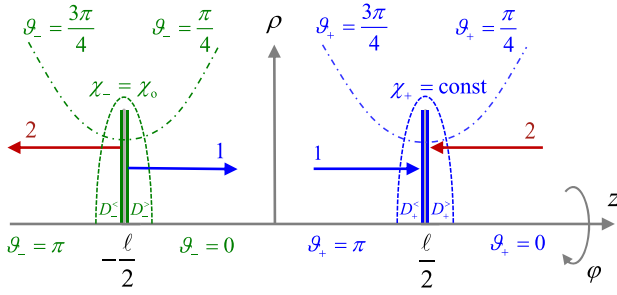


FIG. 3. Ring wormhole space R_{wh} with one asymptotic region. We have global cylindrical coordinates ρ, φ, z and two copies of the oblate spheroidal coordinates $\chi_{\pm}, \vartheta_{\pm}, \varphi_{\pm}$. The coordinate $\varphi \equiv \varphi_{\pm}$ is not shown. The disks D_{\pm} representing the wormhole's mouths are indicated by double lines located at $z = \pm \ell/2$. The dashed ($\chi_{\pm} = \text{const}$) and dash-dotted ($\vartheta_{\pm} = \text{const}$) lines correspond to the domains \bar{V}_{\pm} described in the text and Fig. 4. The coordinates in \bar{V}_{\pm} are related to the opposite coordinates in the leaked domains \hat{V}_{\mp} by $\chi_{+} = -\chi_{-}$ and $\vartheta_{+} = \pi - \vartheta_{-}$. Compare also Fig. 5. In the further discussion we put a massive shell at a constant value of $\chi_{-} = \chi_{0}$.

when understood as coordinate maps on R_{wh} , are well defined only outside the throat.

Clearly, the wormhole spacetime R_{wh} has a different topology than the original empty flat space. Its first homology group is nontrivial, since we have noncontractible loops spanned through the wormhole.

In R_{wh} , we introduce two sets of oblate spheroidal coordinates. One of them, $(\chi_{+}, \vartheta_{+}, \varphi_{+})$, is centered on the ring at $z = +\ell/2$, and the other one, $(\chi_{-}, \vartheta_{-}, \varphi_{-})$, is centered on the ring at $z = -\ell/2$.

For $\chi_{\pm} > 0$, we restrict the coordinates $(\chi_{+}, \vartheta_{+}, \varphi_{+})$ to the domain where $z > -\tilde{\delta}$ and similarly, we restrict the coordinates $(\chi_{-}, \vartheta_{-}, \varphi_{-})$ to the domain where $z < \tilde{\delta}$ and $\tilde{\delta}$ is assumed to be a small parameter.

We assume that these coordinates are valid not only for positive values of χ_{\pm} but are extended to some negative value $-\delta < \chi_{\pm}$, where δ is small. This provide us with a ‘‘leakage’’ of these coordinates through wormhole mouths. Namely, coordinates $(\chi_{+}, \vartheta_{+}, \varphi_{+})$ cover not only the space outside disks D_{+} but also some domain $-\delta < \chi_{+}$ in the vicinity of the disk D_{-} . Similarly, the coordinates $(\chi_{-}, \vartheta_{-}, \varphi_{-})$ cover the exterior of the disk D_{-} and a domain $-\delta < \chi_{-}$ in the vicinity of D_{+} .

We denote the corresponding domains of definitions of coordinates $(\chi_{\pm}, \vartheta_{\pm}, \varphi_{\pm})$ as V_{\pm} , respectively; see Fig. 4. These two domains intersect in a narrow strip \bar{V} where $-\tilde{\delta} < z < \tilde{\delta}$ and in a spheroidal neighborhood \hat{V} of the throat, given by $-\delta < \chi_{\pm} < \delta$; see Fig. 4. We assume δ and $\tilde{\delta}$ to be sufficiently small so that \bar{V} and \hat{V} do not overlap.

We denote the part of V_{+} with $\chi_{+} > 0$ as \bar{V}_{+} and the ‘‘leaked’’ part of V_{+} , i.e., the part with $-\delta < \chi_{+} \leq 0$, as \hat{V}_{+} . We define \bar{V}_{-} and \hat{V}_{-} analogously. The domains are related:

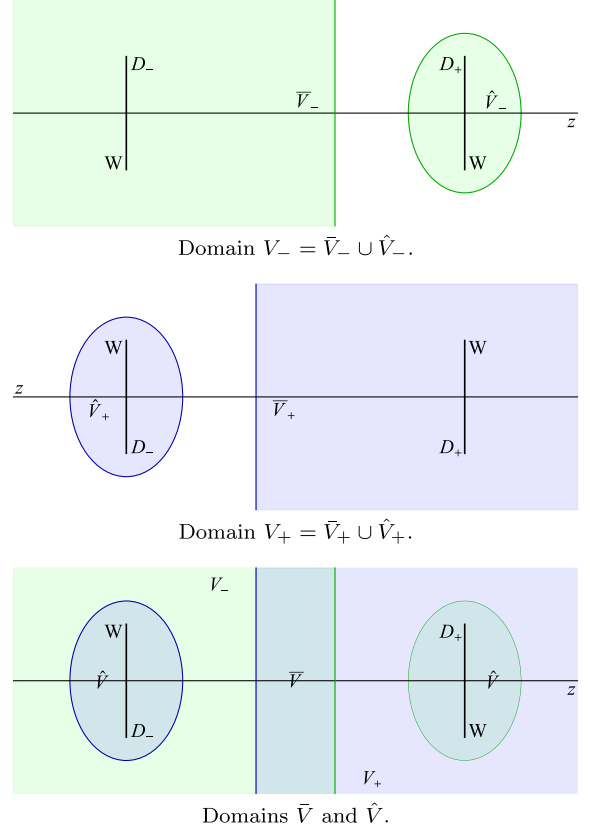


FIG. 4. Domains V_{-} (top) and V_{+} (middle) are covering together the whole space. They are composed by the ‘‘primary’’ parts \bar{V}_{\pm} and leaked parts \hat{V}_{\pm} . The bottom diagram shows two intersections \bar{V} and \hat{V} of these two domains. Horizontal direction corresponds to the z axis, vertical to the ρ direction. The mouths D_{\pm} of the wormhole W are represented by the thick vertical lines. The distance between mouths is ℓ .

$$\begin{aligned} V_{-} &= \bar{V}_{-} \cup \hat{V}_{-}, & V_{+} &= \bar{V}_{+} \cup \hat{V}_{+}, \\ \bar{V} &= \bar{V}_{-} \cap \bar{V}_{+}, & \hat{V} &= \hat{V}_{+} \cup \hat{V}_{-}. \end{aligned} \quad (2.9)$$

In domains \bar{V}_{+} and \bar{V}_{-} , the oblate spheroidal coordinates are related to the global cylindrical coordinates as

$$\begin{aligned} z &= \pm \frac{\ell}{2} + \sinh \chi_{\pm} \cos \vartheta_{\pm}, \\ \rho &= \cosh \chi_{\pm} \sin \vartheta_{\pm}, \\ \chi_{\pm} &> 0, & \vartheta_{\pm} &\in (0, \pi). \end{aligned} \quad (2.10)$$

These formulas also establish relations between coordinates $(\chi_{+}, \vartheta_{+})$ and $(\chi_{-}, \vartheta_{-})$ in \bar{V} :

$$\begin{aligned} \cosh \chi_{+} \sin \vartheta_{+} &= \cosh \chi_{-} \sin \vartheta_{-}, \\ -\ell/2 + \sinh \chi_{-} \cos \vartheta_{-} &= \ell/2 + \sinh \chi_{+} \cos \vartheta_{+}. \end{aligned} \quad (2.11)$$

In the leaked domains \hat{V}_{-} and \hat{V}_{+} , the oblate coordinates $(\chi_{+}, \vartheta_{+})$ and $(\chi_{-}, \vartheta_{-})$ are related to the global cylindrical coordinates (ρ, z) as

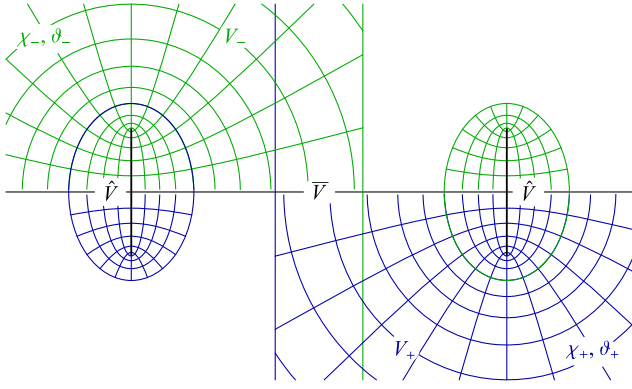


FIG. 5. Coordinate maps (χ_-, ϑ_-) in the domain V_- (top half) and (χ_+, ϑ_+) in V_+ (bottom half).

$$\begin{aligned} z &= \mp \frac{\ell}{2} + \sinh \chi_{\pm} \cos \vartheta_{\pm}, \\ \rho &= \cosh \chi_{\pm} \sin \vartheta_{\pm}, \\ \chi_{\pm} < 0, \quad \vartheta_{\pm} \in (0, \pi). \end{aligned} \quad (2.12)$$

As a consequence, we have in \hat{V}

$$\chi_+ = -\chi_-, \quad \vartheta_+ = \pi - \vartheta_-. \quad (2.13)$$

The relation of coordinates φ_{\pm} is trivial. We can consistently set $\varphi = \varphi_- = \varphi_+$ on all overlaps of the domains of definition.

C. Static wormhole spacetimes

Let us note that after fixing three-dimensional spatial geometry R , we have to fix also full spacetime geometry M . In this paper we will assume that the time direction is added to the given space in a trivial way, namely, that the resulting spacetime will be locally static with time slices locally equivalent to the chosen spatial geometry.

The simplest example is a globally static spacetime $M = R \times \mathbb{R}$, with a global dimensionless coordinate t independent of the spatial geometry, and with spacetime metric

$$ds^2 = -dt^2 + dl^2. \quad (2.14)$$

We can generalize this case in several ways. First, we can admit a redshift factor

$$ds^2 = -e^{2U} dt^2 + dl^2, \quad (2.15)$$

where U is a globally well-defined spatial function.

When the spatial manifold is topologically nontrivial, as the space R_{wh} discussed above, we can have more elaborate generalizations. First, we can identify the mouths of the wormhole at different times.

Namely, suppose we introduce time t synchronized with clocks at infinity, defined everywhere except at the

wormhole throat. It complements the global flat coordinates (x, y, z) or cylindrical coordinates (ρ, z, φ) .

Now, we do not identify the disks D_{\pm} at the same values of time t , but we identify D_- at $t = t_0$ with D_+ at $t = t_0 + \Delta t$, with Δt being a constant. We require that the spacetime geometry is still described by the metric

$$ds^2 = -dt^2 + dl^2. \quad (2.16)$$

It is straightforward to see that the metric is continuous through the wormhole, since the constant shift Δt disappears in the differential dt .

Consider a particle which enters the disk D_- at time $t = t_0$. It appears from the disk D_+ at time $t = t_0 + \Delta t$. We thus say that there exists a time gap Δt associated with the wormhole. Since the proper time at both sides of the identified disks must be the same (the geometry is continuous through the wormhole), the time gap must remain the same. This means that if a second particle is sent through D_- shortly after the first one at time $t = t_0 + \delta t$, it will appear at D_+ at time $t = t_0 + \delta t + \Delta t$.

However, if we put some matter around one of the mouths of the wormhole, then its gravitational field affects a proper time of static observers and the static spacetime metric needs further modification. In general, the time shift of coordinate t will not remain constant. Our spacetime becomes only locally static as we will describe in the following sections.

D. Nature of the ring

Before going further let us make a following remark. Gibbons and Volkov [23–25] demonstrated that in fact there exists a δ -like singularity of the curvature at the location of the ring. The corresponding distribution of matter violates the null-energy condition which makes possible the very existence of a static traversable wormhole. This distribution of matter can be identified with an infinitely thin cosmic string (strut) with a negative angle deficit -2π . In order to exclude the δ -like singularity at the ring one can smear the distribution of the matter of the corresponding cosmic string. This makes the spacetime smooth, but the equation of state of the corresponding string is not very realistic. If λ is the radius of the smeared string, we assume that the parameter $d = \lambda/a$ is small. This always can be achieved since the radius of the ring a can be arbitrary large. In what follows we always assume that, if necessary, the corresponding smearing is done.

III. WEAK GRAVITATIONAL FIELD

A. Gravitational potential

In the weak-field approximation, a solution of the Einstein equation for a static matter distribution can be written in the form

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dl^2, \quad (3.1)$$

where the gravitational potential U obeys the Poisson equation

$$\Delta U = 4\pi\mu. \quad (3.2)$$

Here μ is the mass density defined with respect to the flat metric dl^2 . The metric (3.1) is a perturbation of the flat background spacetime

$$ds_0^2 = -dt^2 + dl^2. \quad (3.3)$$

For details and further discussion of static metrics in the weak-field approximation see Appendix A.

We denote ξ^μ a timelike Killing vector and u^μ a normalized four-velocity of static observers moving along this Killing vector:

$$u^\alpha = e^{-U} \xi^\alpha. \quad (3.4)$$

Obviously,

$$u_\alpha = -e^U t_{,\alpha}. \quad (3.5)$$

The four-acceleration of these static observers,

$$w^\mu = u^\mu{}_{;\nu} u^\nu, \quad (3.6)$$

satisfies the relation $w_{[\alpha,\beta]} = 0$ and hence, at least locally, one has

$$w_\alpha = U_{,\alpha}. \quad (3.7)$$

We call w^α the acceleration field and $-U_{,\alpha}$ the gravitational field strength. Since w^α is orthogonal to the time direction u^α , it can be restricted to its spatial components. The field strength is a fictitious force that one assumes in the noninertial static frame to explain a tendency of free observers to move with respect to the frame. Nonmoving static observers have to “compensate” this force by a real force equal to w_j per unit mass. Of course, in the full spacetime description, the static observers move along nongeodesic trajectories with four-acceleration w^α caused by the real force.

In static spacetimes we can effectively “ignore” the time coordinate. The weak-field equation (3.2) can be understood as the equation on spatial section $t = \text{const}$, or, better, on a factorized three-dimensional space, points of which are orbits of the Killing vector ξ^α . We will see in a moment that the latter approach is more general.

B. Locally static spacetimes

The above description of the static spacetime, as well as of the weak field, was local. Let us make some remarks

about global aspects. We start with a general static spacetime without any approximation.

First, we observe that metric (3.1) is invariant under a scaling transformation

$$U = \hat{U} + \Delta U, \quad t = e^{-\Delta U} \hat{t}, \quad dl^2 = e^{2\Delta U} \hat{dl}^2, \quad (3.8)$$

parametrized by a constant¹ ΔU . This rescaling tells us that two potentials on the background spacetime which differ only by a constant are in some sense equivalent.

This opens a very important possibility for globally nontrivial spacetimes. Consider spacetime with a nontrivial first homology group, i.e., spacetime containing noncontractible loops. From the duality between homology and de Rham cohomology we know that in such a spacetime closed 1-forms do not have to be exact²:

$$\sigma_{[\mu,\nu]} = 0 \not\Rightarrow \sigma_\mu = \alpha_{,\mu}. \quad (3.9)$$

In particular, we can have spacetime with global static observers given by globally well-defined four-velocity u_α . It gives a globally well-defined four-acceleration w_α . Although it is closed,

$$w_{[\mu,\nu]} = 0, \quad (3.10)$$

it may be nonpotential. Such w_α corresponds to a nontrivial element of the first cohomology group.

Of course, locally, on a topologically trivial domain V , we can always find a potential U such that (3.7) holds. But in a general case such a potential does not have a global smooth extension to the whole spacetime.

If we have two such potentials U and \hat{U} on domains V and \hat{V} , respectively, on their intersection $V \cap \hat{V}$ the potentials may differ only by a constant:

$$w_\alpha = U_{,\alpha} = \hat{U}_{,\alpha} \Rightarrow \Delta U = U - \hat{U} = \text{const}. \quad (3.11)$$

We can cover the whole manifold by simply connected domains with associated with them potentials. Next, we may shift the potentials by a constant on some of these domains to fit potentials in the neighborhood domains. However, in topologically nontrivial spacetime it may not be possible to do it consistently everywhere. It may happen that we find two domains V and \hat{V} with potentials which are already continuously extended along some path in the manifold. However, these potentials do not fit; they differ on the intersections of these domains by a constant ΔU .

¹Please, notice the difference between the character for Laplace operator, ΔU , and the letter Delta in the constant ΔU .

²Here, one should understand tensor indices as abstract indices, not coordinate ones. On a simple local coordinate map, any closed form is exact. But we speak here about tensor fields defined on the whole spacetime.

A maximal extension of the potential U can be specified by choosing a point p_0 of vanishing potential (typically in an asymptotical region) and paths γ_p from this point to each point p in the spacetime. The potential is given by

$$U = \int_{\gamma_p} w_\alpha dx^\alpha. \quad (3.12)$$

For a continuous change of the path γ_p the potential U changes smoothly. However, in a topologically nontrivial space some paths heading to two close points in the intersection $V \cap \hat{V}$ cannot be smoothly deformed to each other. Then, such a potential may not be continuous.

Impossibility to extend the potential to a single-valued global smooth function implies that also Killing vector ξ^α , given by (3.4), cannot be defined globally. It can be continuously extended to domains V and \hat{V} , similarly to potential U . But on the intersection $V \cap \hat{V}$, it will differ by rescaling

$$\xi^\alpha = e^{\Delta U} \hat{\xi}^\alpha. \quad (3.13)$$

The same applies to the time coordinate t related to the four-velocity by (3.5). On the intersection of domains V and \hat{V} we find

$$t = e^{-\Delta U} \hat{t}. \quad (3.14)$$

Spacetime with such a structure is called locally static [13], in contrast to the globally static spacetimes. In globally static spacetimes all quantities ds^2 , dl^2 , u^α , w^α , ξ^α , U , and t are defined globally. In locally static spacetimes, ξ^α , U , dl^2 , and t have only local meaning. See Appendix B for more details.

C. Locally static weak gravitational field

Our goal now is to build a locally static spacetime describing the gravitational field in the presence of a ring wormhole in the weak-field approximation.

We start with a globally static spacetime, in fact, with a flat spacetime which differs from the Minkowski spacetime just by a wormhole identification described in Sec. II. On this background we find a gravitational potential U which satisfies locally Poisson equation (3.2) with some physically reasonable sources. But we are interested in situations when the potential cannot be extended globally.

In such a case, the globally well-defined quantities are acceleration field w_α , four-velocity u^α , and, of course, the full perturbed metric ds^2 .

The weak-field equation (3.2) can be rewritten using globally defined quantity w^j . The potential relation (3.7) is superseded by (3.10). We get

$$\nabla_{[i} w_{j]} = 0, \quad (3.15)$$

$$\nabla_i w^i = 4\pi\mu. \quad (3.16)$$

These equation can be understood as the gravitostatic equations on the three-dimensional factorized space mentioned above.

Taking into account (3.1), we conclude that inertial time t and flat metric dl^2 , as well as the implicitly defined Cartesian coordinates x^i , depend on the choice of the potential. Their perturbed versions are not thus defined globally.

Similarly to the discussion above, if we find domains V and \hat{V} on which the potentials differ by a constant ΔU , expression (3.1) for the metric on these domains defines (t, x^j) and (\hat{t}, \hat{j}) which will be related by

$$\begin{aligned} U &= \hat{U} + \Delta U, \\ t &= e^{-\Delta U} \hat{t}, \quad x^j = e^{\Delta U} \hat{x}^j, \\ dl^2 &= e^{2\Delta U} \hat{dl}^2. \end{aligned} \quad (3.17)$$

Identification of the resulting spacetime with the original background with metric ds_0^2 thus cannot be done globally in the coordinates (t, x^j) .

However, the inertial coordinates are very simple and offer a straightforward local identification of the perturbed spacetime with the background. We want to take advantage of them. Therefore, we modify our perturbation procedure slightly.

Namely, we do not require that our perturbed metric ds^2 is a global perturbation of one unperturbed background ds_0^2 . To justify the validity of the weak-field equations, it is enough that the resulting spacetime is a perturbation of a background on local domains. The domains must cover the whole resulting spacetime. But the background can be assumed for each domain separately.

In this approach we can easily identify background spacetimes with the resulting perturbed spacetime using locally defined inertial coordinates. However, we have to face a problem of the inequality of these coordinates on overlaps of the domains.

In the specific cases of the wormhole spacetimes, we will be able to extend the potential and related quantities to the full spacetime, except the wormhole throat, using the procedure (3.12). However, similarly to the potential, the extended inertial coordinates will not be continuous through the wormhole and we will have to discuss how they are related. We will demonstrate this procedure in Sec. V.

Nonexistence of a global static time coordinate t is also a reason why we cannot use a global spatial section $t = \text{const}$. However, we can still introduce the three-dimensional factorized space of orbits of the Killing vector ξ^α and solve the weak-field equations (3.15) and (3.16) on it.

Finally, let us note that rescaling (3.17) can be used also for identification of locally inertial coordinates near an arbitrary point p . By a suitable choice of ΔU one can always put $\hat{U}(p) = 0$. For this gauge the coordinate \hat{t} at p coincides with the proper time at this point and locally Cartesian coordinates \hat{x}^j near p coincide with proper lengths along the axes.

D. Gravity field flux

Let S be a closed two-dimensional surface embedded in the spatial section $t = \text{const}$. We define a gravity field flux over this surface as

$$\Pi_S = \frac{1}{4\pi} \oint_S w^a n_a d^2 S. \quad (3.18)$$

Here n^a is an outward-pointing unit normal to S and $d^2 S$ is a two-dimensional surface area element on S .

Let S_1 and S_2 be two homological closed surfaces and V is a three-dimensional volume restricted by them:

$$\partial V = S_2 - S_1. \quad (3.19)$$

Then the Gauss theorem applied to volume V and the weak-field equation (3.16) implies that

$$\Pi_{S_2} - \Pi_{S_1} = \frac{1}{4\pi} \int \nabla_i w^i d^3 V = \int \mu d^3 V. \quad (3.20)$$

It has important consequences.

- (i) If there is no matter in the space between S_1 and S_2 , the fluxes through these surfaces coincide.

Applying (3.20) on a boundary ∂V of a compact volume V , we have to

- (ii) consider adiabatic change of the distribution of mass. It does not change the flux through a two-dimensional surface $S = \partial V$ enclosing a three-dimensional volume V until the matter crosses S .

In the case of space with a nontrivial topology or with several asymptotic regions, one can formulate a more general proposition.

- (iii) The flux through any closed two-dimensional surface S does not change under an adiabatic change of a mass distribution until the matter crosses the surface S .

This last proposition has a direct counterpart in a well-known result of the Maxwell electrodynamics, connecting the change of the flux over a closed surface with the current crossing it. The result can be easily proved by using the Stokes theorem. In the case of a weak gravitational field, the proposition does not follow just from properties of the Poisson equation. One would need to specify laws governing the dynamics. We will not discuss these laws here. Let us just mention that there exists a well-known analogy

between the weak gravitational field and the Maxwell equations; see, e.g., [27,28].

IV. GRAVITATIONAL FIELD NEAR THE WORMHOLE CONNECTING TWO ASYMPTOTIC DOMAINS

In this and the following sections, we work in a three-dimensional flat space with a nontrivial identification given by the wormhole. We will discuss some solutions of the field equation (3.2), or (3.15) and (3.16), respectively. As earlier, in Sec. II we use a radius a of the ring to define dimensional metric and coordinates. We will return to the spacetime description in Sec. VI.

A. Gravitational field of a massive shell

It is instructive to discuss first a gravitational field of a thick homogeneous ellipsoidal shell in a simple flat space which does not contain any wormhole. Let us consider a shell bounded by two similar ellipsoids filled by the matter of constant density. It will be sufficient for us to observe that according to the famous Newton theorem (cf. [29]) the gravitational force $-\nabla U$ inside the shell (inside the inner ellipsoid) vanishes.³ The potential in the cavity is thus constant $U = U_0$. Here, we normalize the potential U so that it decreases at the infinity outside the outer ellipsoid.

The field both outside and in the shell's interior (between both ellipsoids) can also be written explicitly [29,30]. However, these expressions are rather complicated. Instead of this, we will consider below a much simpler model in which the matter surrounding a throat is taken in the form of a thin spheroidal mass shell. This will allow us to obtain simple analytical expressions for the gravitational field.

Using the described field in the shell's cavity, it is easy to construct a solution of Eq. (3.2) in the space \tilde{R} where the ring wormhole connects two flat spaces. We assume that the disk associated with the ring wormhole in R_+ is inside the cavity. In this case the corresponding solution inside the shell and out of it remains the same as in a case without the ring wormhole, while in the cavity of the shell and everywhere in space R_- the potential U is constant equal to U_0 .

In a special case when all the axes of the ellipsoidal shell are equal, it takes the form of the sphere. For such a case, this conclusion remains valid even if the gravitational field of the shell is not weak. Namely, one can use a massive thin shell approach by Israel [31] to glue the Schwarzschild metric outside the shell with the inner flat solution. Imposing the ring wormhole inside the shell does not change the gravitational field around.

³Let us stress that Newton's theorem requires the bounding ellipsoids being *similar* and, therefore, they are not *confocal*. Thus, even when the ellipsoids are oblate spheroids, they cannot be specified just by two values of spheroidal coordinate χ .

Let us emphasize that the presence of a ring wormhole has nontrivial consequences. We can point out two aspects.

- (i) Even in the case of a ring wormhole connecting two flat spaces, the above described solution of the Poisson equation (3.2) with the described source is not unique.
- (ii) In the presence of a ring wormhole in a single space, the statement that the gravitating force vanishes in the cavity of the shell is not correct.

We will discuss these issues below.

B. Zero modes

The space \tilde{R} with the wormhole connecting two asymptotic regions has the first cohomology group trivial and thus, we do not have to face problems with nonexistence of a global potential. However, it has two asymptotic regions and it allows ambiguity in the potential, even for trivial sources. Let us illustrate this point.

Consider the function⁴

$$\mathcal{Z}(\chi) = -\operatorname{arccot}(\sinh \chi), \quad (4.1)$$

with $\operatorname{arccot} s$ taking values in the interval $(0, \pi)$. It has the following properties (see Fig. 6):

$$\mathcal{Z}'(\chi) = \frac{1}{\cosh \chi},$$

$$\mathcal{Z}(-\infty) = -\pi, \quad \mathcal{Z}(0) = -\frac{\pi}{2}, \quad \mathcal{Z}(+\infty) = 0. \quad (4.2)$$

Let m_∞ and C be two constants; then it is easy to check that

$$U_{\text{zm}}(\chi) = m_\infty \mathcal{Z}(\chi) + C \quad (4.3)$$

is a solution of the homogeneous Laplace equation

$$\Delta U_{\text{zm}} = 0. \quad (4.4)$$

We call such a solution a zero mode (see Fig. 7). It is convenient to use the standard ambiguity in the choice of the potential and to put $C = 0$; then

$$U_{\text{zm}}|_{\chi=+\infty} = 0, \quad U_{\text{zm}}|_{\chi=-\infty} = -m_\infty \pi. \quad (4.5)$$

In the absence of the ring wormhole, one should put $m_\infty = 0$ as well, in order to avoid a discontinuity in the first derivative at $\chi = 0$. For nonvanishing m_∞ , a jump in the field strength across the disk $\chi = 0$ would correspond to a

⁴The function \mathcal{Z} is related to Legendre function \mathbf{Q}_0 evaluated on the imaginary axis with a branch cut chosen such that it is analytical on the whole imaginary axis.

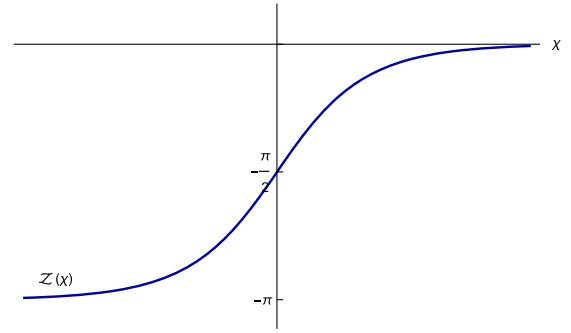


FIG. 6. Functions $\mathcal{Z}(\chi)$.

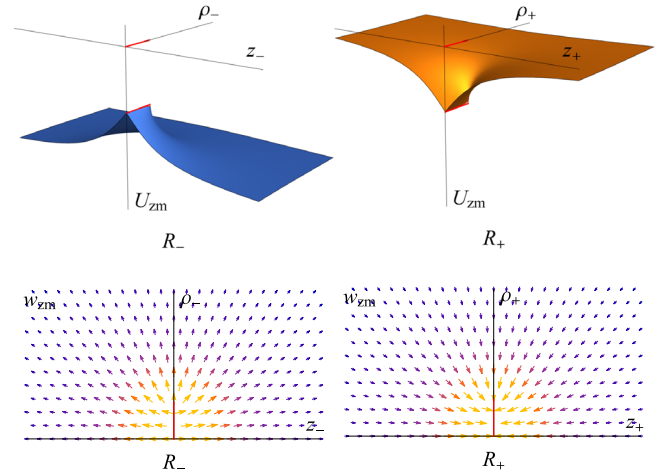


FIG. 7. Potential U_{zm} (top) and field strength w_{zm}^j (bottom) of the zero mode. The left diagram shows the field in space R_- , and the right one in R_+ . Thick segments on the ρ_\pm axes represent the mouths of the wormhole. Angular coordinate φ is ignored and can be recovered by a rotation around the axes $\rho_\pm = 0$.

mass distribution on the disk, and the potential (4.3) would not be a solution of source-free (4.4).

In the presence of the ring wormhole connecting spaces R_+ and R_- , the situation is quite different. Even if one imposes a condition that the potential vanishes at infinity of R_+ by putting $C = 0$, there exists a one-parameter family of homogeneous solutions parametrized by a constant m_∞ . The difference of potentials at the infinities of the spaces R_+ and R_- is

$$\Delta U = U_{\text{zm}}|_{\chi=+\infty} - U_{\text{zm}}|_{\chi=-\infty} = m_\infty \pi. \quad (4.6)$$

In the presence of a zero-mode potential, there exists a nonzero force acting on a unit-mass particle equal to

$$-\nabla U_{\text{zm}} = -\frac{m_\infty}{\cosh \chi} \nabla \chi. \quad (4.7)$$

For positive m_∞ this force is directed toward the decreasing χ . In R_+ this force describes an attraction of the particle by

the wormhole, while in R_- it is repulsive from the wormhole.

It is easy to calculate a flux Π_D through the wormhole associated with the zero mode. Consider a closed two-dimensional surface S surrounding the throat D of the wormhole, with normal oriented toward R_+ asymptotic. Since the zero mode has vanishing sources, the flux does not depend on exact shape and position of the surface; cf. argument (3.20). The calculation for spheroid $\chi = \text{const}$ gives

$$\Pi_S = m_\infty, \quad (4.8)$$

where we used

$$n^\alpha = \frac{1}{\sqrt{\sinh^2 \chi + \cos^2 \vartheta}} \delta_\chi^\alpha, \quad (4.9)$$

$$dS = \sqrt{\sinh^2 \chi + \cos^2 \vartheta} \cosh \chi \sin \vartheta d\vartheta d\varphi.$$

For surface D at $\chi = 0$, representing the identified disks, we obtain the flux Π_D through the wormhole throat.

A zero-mode field can be created in the following process. Let us start with empty space with vanishing potential. Consider a particle of mass m at the infinity R_+ and move it adiabatically to bring it close to the throat. The flux Π_D through the surface D at the throat is not changed in such a process until the mass crosses D . After crossing D the flux becomes $\Pi_D = m$ and it remains such if the mass stays in R_- . When the mass m is brought to infinity in the space R_- , one is left with a space without any matter but with a nonvanishing flux Π_D through the throat. It will be exactly the zero mode described above.

C. A field of a massive thin shell around the wormhole

Let us assume now that there exists a static mass distribution surrounding the wormhole. In particular, we consider a thin massive shell which has a form of an oblate spheroid S_0 in R_+ surrounding the wormhole and confocal to its ring. In what follows we assume that this massive shell is formed by massive particles that are brought together from R_+ infinity. During this process no zero-mode field would be created and they should be excluded in our solution.

In oblate spheroidal coordinates $(\chi, \vartheta, \varphi)$ the equation of shell is $\chi = \chi_0 > 0$. Let M be the mass of the massive shell and $m = M/a$ be its dimensionless variant. The mass is not distributed homogeneously over the shell surface and its surface density is

$$\sigma = \frac{m}{4\pi \sqrt{\sinh^2 \chi + \cos^2 \vartheta} \cosh \chi}. \quad (4.10)$$

The distributional volume density is⁵

$$\mu = \sigma \delta_{S_0} = \frac{m}{4\pi(\sinh^2 \chi + \cos^2 \vartheta) \cosh \chi} \delta(\chi - \chi_0). \quad (4.11)$$

Surprisingly, this mass distribution can be obtained [29] as a thin shell limit of the thick homogeneous spheroidal shell discussed in Sec. IV A. Nonhomogeneity of the surface mass density arises from the fact that the thick shell is formed by two similar oblate spheroids and its thickness varies with position on the spheroid. We will see that for the thin shell the potential U_{sh} depends only on the χ coordinate and hence it is constant on the shell.⁶

In fact, to obtain a solution of the Poisson equation for the thin shell we can use solutions $\mathcal{Z}(\chi)$ of the Laplace equation given by (4.1). The field U_{sh} analogous to that of the thick shell discussed in Sec. IV A is (see Fig. 8)

$$U_{\text{sh}}(\chi) = \begin{cases} m\mathcal{Z}(\chi) & \text{for } \chi > \chi_0, \\ U_0 = m\mathcal{Z}(\chi_0) & \text{for } \chi < \chi_0. \end{cases} \quad (4.12)$$

The potential inside the shell, as well as in R_- , is constant. Outside the shell, it is the same as the zero mode, and it vanishes at the infinity of R_+ . The jump in the field strength at the shell corresponds to the surface mass distribution (4.10).

Of course, we can superpose this solution with the zero-mode solution

$$U = U_{\text{sh}} + U_{\text{zm}}. \quad (4.13)$$

At the infinity of R_+ the potential U again vanishes, while at the infinity of R_- its value is

$$U|_{\chi=-\infty} = m\mathcal{Z}(\chi_0) - m_\infty \pi. \quad (4.14)$$

We thus found that a general potential (4.13) of the thin shell vanishing in the infinity of R^+ contains two constants, m and m_∞ . Their meaning can be understood using the

⁵Here, δ_S is a covariant δ function localized on a surface S ,

$$\int f \delta_S dV = \int_S f dS,$$

and $\delta(\chi - \chi_0)$ is its coordinate representation. They are related:

$$\delta_S dl = \delta(\chi - \chi_0) d\chi,$$

with the proper length dl orthogonal to S :

$$dl = \sqrt{\sinh^2 \chi + \cos^2 \vartheta} d\chi.$$

⁶This solution is similar to the well-known solution for a static electric potential of a charged conducting oblate spheroid which is discussed, e.g., in Ref. [30].

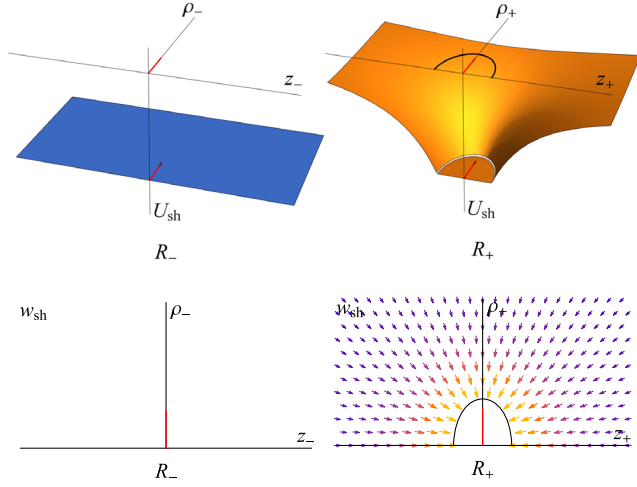


FIG. 8. Potential U_{sh} (top) and field strength w_{sh}^i (bottom) of a thin massive spheroidal shell located around the wormhole in R_+ . The left diagram shows the field in space R_- , and the right one in R_+ . Thick segments on the ρ_{\pm} axes represent mouths of the wormhole. The shell (depicted by the thick ellipse) is confocal with the wormhole mouth. Angular coordinate φ is ignored and can be recovered by a rotation around the axes $\rho_{\pm} = 0$.

gravity field flux. As we have seen [cf. (3.20)], the flux may be used to define the mass function $m_S = \Pi_S$ characterizing mass inside a surface S . For the equipotential surface $\chi = \text{const}$ the same calculation as in (4.8) gives

$$m_S(\chi) = m\Theta(\chi - \chi_0) + m_{\infty}. \quad (4.15)$$

It should be emphasized that both in R_+ and R_- the normal to the spheroid $\chi = \text{const}$ is chosen in the direction of growing χ .

In the exterior of the shell, the mass function is $m_S = m + m_{\infty}$. Inside the shell (including space R_- , where $\chi < 0$), the mass function is just $m_S = m_{\infty}$. It correctly jumps by the mass of the shell when crossing the shell. Since m_S does not change when we further decrease χ , we can interpret m_{∞} as a mass “far behind” the wormhole, located under arbitrary negative χ , i.e., located in the infinity of space R_- .

However, this mass had to be brought into the infinity of R_- through the wormhole from the infinity of R_+ . Because by fixing vanishing potential U at the infinity of R_+ , we have chosen that all matter forming the gravitational field has to be brought from this infinity. The mass m_{∞} at the infinity of R_- thus had to be moved from the infinity of R_+ , and the potential difference (4.6) corresponds to the work needed for that.

We interpreted the potential assuming that it vanishes at infinity of R_+ . We selected one of the asymptotics as the origin of all matter. We thus have broken the symmetry of the system. Of course, we can also allow the matter to

originate at the infinity of R_- . In such a case, we have to recover the constant C in (4.3).

Then, one can interpret the zero mode as the gravitational field trapped by the wormhole, which holds memory about a previous activity how the matter has been moved through the wormhole.

D. Wormhole in a homogeneous gravitational field

Let us consider a homogeneous gravitational field in a flat space. Its potential is

$$U = w_0 z. \quad (4.16)$$

The acceleration field has constant value w_0 and is directed along the z axis:

$$w_i = \nabla_i U = w_0 \nabla_i z. \quad (4.17)$$

Let us assume now that a ring wormhole leading to another space is immersed in this gravitational field. For simplicity, we assume the wormhole is in the origin $z = 0$, with its mouth orthogonal to the z axis. To find how such a wormhole modifies the field, one needs to solve the Laplace equation in \tilde{R} space which has an asymptotic (4.16) at R_+ infinity. The corresponding solution is

$$U_{\text{hom}} = -w_0 \mathcal{Z}_+(\chi) \cos \vartheta. \quad (4.18)$$

Here, we introduce functions⁷ $\mathcal{Z}_{\pm}(\chi)$ (see Fig. 9)

$$\mathcal{Z}_{\pm}(\chi) = \mp \frac{1}{2} \sinh \chi - \frac{1}{\pi} (1 + \sinh \chi \arctan \sinh \chi). \quad (4.19)$$

We require $\arctan s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Clearly

$$\mathcal{Z}_+(\chi) = \mathcal{Z}_-(-\chi) = -\sinh \chi + \mathcal{Z}_-(\chi). \quad (4.20)$$

Function $\mathcal{Z}_+(\chi)$ asymptotically behaves as

$$\mathcal{Z}_+|_{\chi \rightarrow -\infty} = 0, \quad \mathcal{Z}_+|_{\chi \rightarrow +\infty} \approx -\sinh \chi, \quad (4.21)$$

and function $\mathcal{Z}_-(\chi)$ as

$$\mathcal{Z}_-|_{\chi \rightarrow -\infty} \approx \sinh \chi, \quad \mathcal{Z}_-|_{\chi \rightarrow +\infty} = 0, \quad (4.22)$$

and

$$\mathcal{Z}_{\pm}'(0) = \mp \frac{1}{2}. \quad (4.23)$$

⁷Functions \mathcal{Z}_{\pm} are related to Legendre function \mathbf{Q}_1 evaluated on the imaginary axis, with properly chosen asymptotic, and with a branch cut chosen such that it is analytical on the whole imaginary axis.

To check the validity of the boundary condition of the field U_{hom} at $z \rightarrow \infty$ in R_+ , we use relation (4.20)

$$U_{\text{hom}} = w_0 \sinh \chi \cos \vartheta - w_0 \mathcal{Z}_-(\chi) \cos \vartheta. \quad (4.24)$$

In R_+ , where $z = \sinh \chi \cos \vartheta$, the first term correctly reproduces the asymptotic behavior (4.16), since the second term falls down at $\chi \rightarrow +\infty$.

One can interpret this result as follows. The ring wormhole changes the homogeneous gravitational field (4.16), which was originally introduced only in R_+ . When we consider it in the full wormhole space \tilde{R} , we call it truncated field U_t . In the presence of the wormhole, it will be modified by an induced field U_i . The contribution of the induced field in the region R_+ is described by term $-w_0 \mathcal{Z}_-(\chi) \cos \vartheta$, while in the region R_- it is given by field (4.18) penetrating through the wormhole, i.e., by $-w_0 \mathcal{Z}_+(\chi) \cos \vartheta$. Together,

$$U_{\text{hom}} = U_t + U_i, \quad (4.25)$$

where the truncated and the induced fields are, respectively,

$$U_t = w_0 \Theta(\chi) \sinh \chi \cos \vartheta, \quad (4.26)$$

$$U_i = -w_0 \mathcal{Z}_-(|\chi|) \cos \vartheta, \quad (4.27)$$

$\Theta(\chi)$ being the Heaviside step function.

These fields have a natural physical interpretation. The truncated potential U_t vanishes in R_- and describes the homogeneous field in R_+ . It is continuous everywhere, including the wormhole mouth $\chi = 0$, but it is not smooth here. Its field strength is discontinuous at $\chi = 0$. In fact, it is discontinuous at both faces $D^<$ and $D^>$ of the ring wormhole. The jump of the normal component of the field strength is $+w_0$ and $-w_0$ at $D^<$ and $D^>$, respectively, since the normal is pointing in a negative z direction at $D^<$ and in a positive z direction at $D^>$. Hence, U_t does not satisfy the homogeneous Laplace equation at the wormhole. The jump corresponds to a massive thin shell at the wormhole mouth $\chi = 0$ with a surface density $\mp \frac{w_0}{4\pi}$.

The potential induced by the wormhole, U_i , is also continuous everywhere and not smooth at the wormhole. Using properties of function \mathcal{Z}_+ one can find that it satisfies the Laplace equation everywhere except the wormhole throat $\chi = 0$. The jump of the normal component of the field strength at $\chi = 0$ is in this case $-w_0$ and $+w_0$ at $D^<$ and $D^>$, respectively. So, it also corresponds to a massive shell at the throat of the wormhole with a surface density $\pm \frac{w_0}{4\pi}$. Potential U_i has dipole character both in R_- and R_+ .

Clearly, if one adds both potentials U_t and U_i , then the sum (4.24) is a solution of the source-free Laplace equation everywhere, including the wormhole throat $\chi = 0$.

Let us note that the potential U_{hom} defined by (4.18) is finite and its value is proportional to w_0 (see Fig. 10).

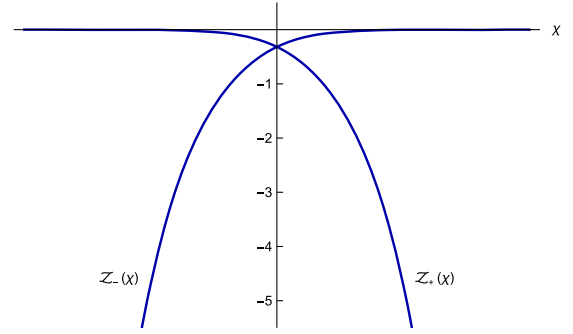


FIG. 9. Functions $\mathcal{Z}_-(\chi)$ and $\mathcal{Z}_+(\chi)$.

Hence, for small w_0 the field strength outside the ring is uniformly small. However, the corresponding field strength diverges at the ring. This divergency is a consequence of the assumption that the corresponding cosmic string located at the ring is infinitely thin. As it was discussed in Sec. II, the smearing of the distribution of the matter of the string makes the background spacetime of the wormhole regular. One can expect that the gradient of the potential \tilde{U} on such a smooth background would be regular in the vicinity of smeared string as well. The adopted weak-field approximation remains valid until a narrow “tube” surrounds the smeared string. The behavior of the gravitational field inside such a tube depends on details of the string’s matter distribution.

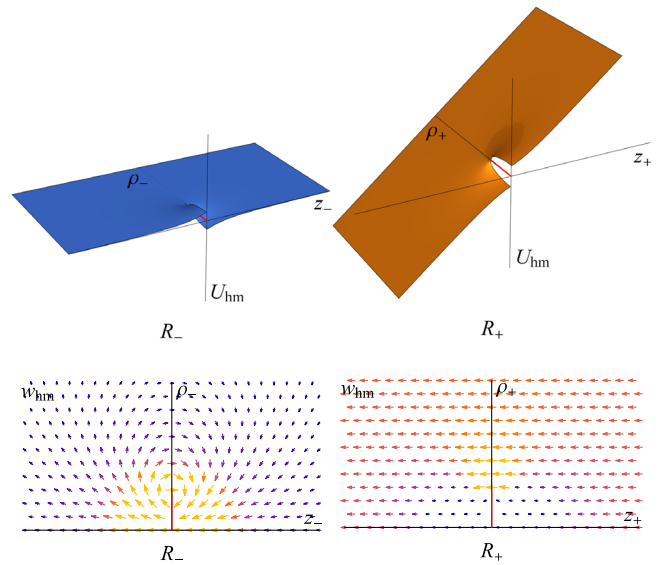


FIG. 10. Potential U_{hom} (top) and field strength w_{hom}^j (bottom) of an asymptotically homogeneous field in R_+ . The left diagram shows the induced field in space R_- , and the right one the modified homogeneous field in R_+ . Thick segments on the ρ_{\pm} axes represent mouths of the wormhole. Angular coordinate φ is ignored and can be recovered by a rotation around the axes $\rho_{\pm} = 0$.

V. GRAVITATIONAL FIELD NEAR THE WORMHOLE IN SPACETIME WITH ONE ASYMPTOTIC

In this section, we will study a wormhole connecting two places in one asymptotically Minkowski spacetime, and we will discuss a gravitational perturbation of such a space. We described the background wormhole spacetime R_{wh} in Sec. II.

Now we consider the gravitational potentials U of the massive thin shell located around the wormhole mouth D_- . In oblate spheroidal coordinates $(\chi_-, \vartheta_-, \varphi_-)$ constructed around the mouth D_- , the shell is given by $\chi_- = \chi_0 > 0$. It is highly nontrivial to find the full, consistent solution of the Poisson equation in the wormhole spacetime R_{wh} . Namely, one needs to obtain a solution of the Laplace equation with specially chosen boundary conditions. Besides natural conditions of decreasing of the gravitational potential U at the infinity and jump conditions for it on the surface of the shell one needs to impose additionally a condition of the regularity and absence of matter at the disks which are identified in such a space. The latter condition means that the field's strength ∇U should be continuous on the disks. This problem belongs to a class of so-called mixed boundary value problems. A general discussion of such problems and references to the papers where they have been studied can be found in Ref. [32].

We will solve this problem only approximately, for the wormhole mouths sufficiently distant, $\ell \gg 1$.⁸

A. Large-distance approximation—Zeroth order

Let us start with the zeroth-order approximation. We return to the first-order corrections below. In the zeroth order, we assume that the field of the shell near the second mouth D_+ is negligible, and we can thus ignore any interaction with the wormhole. The potential is thus given by field (4.12) positioned around D_- , which is achieved by employing $(\chi_-, \vartheta_-, \varphi_-)$ coordinates:

$$U = \begin{cases} m\mathcal{Z}(\chi_-) & \text{for } \chi_- > \chi_0, \\ U_0 = m\mathcal{Z}(\chi_0) & \text{for } 0 < \chi_- < \chi_0. \end{cases} \quad (5.1)$$

However, even in the zeroth approximation, we have to extend the potential through the wormhole. We perform that separately in the domain V_- around D_- and in V_+ around D_+ (see Fig. 11). We thus obtain two potentials, U_- and U_+ , each of which satisfies the Laplace equation. On

⁸We must distinguish two approximate schemes employed here. First, we use the perturbative expansion of the gravitational interaction, where we consider only the first-order approximation in which the gravity description reduces to the Newtonian potential satisfying the Poisson equation. Second, for the particular wormhole spacetime, we look for the gravitational potential of the massive shell using the expansion for large mouths' distance ℓ . Here, we discuss the later expansion.

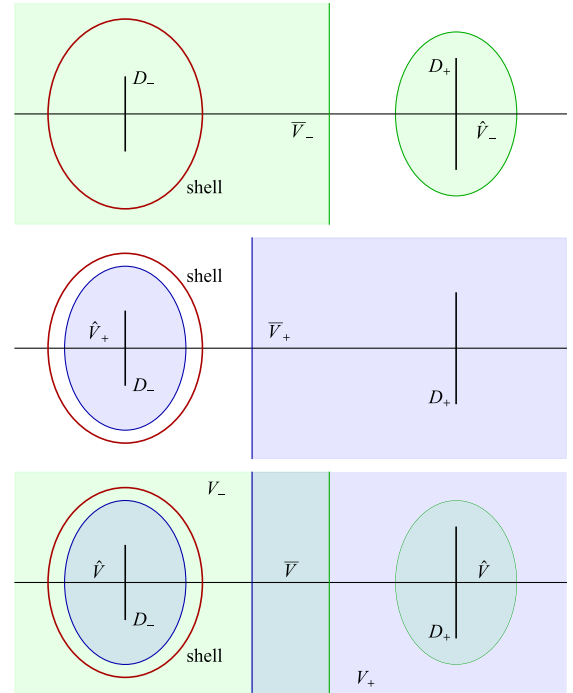


FIG. 11. Coordinate domains around the wormhole in the presence of a gravitational field of a thin massive shell near the left wormhole mouth. Domains V_- (top) and V_+ (middle) are covering together the whole space. The division into main part \bar{V}_\pm and leaked part \hat{V}_\pm is indicated. The bottom diagram shows two intersections \bar{V} and \hat{V} of these two domains. The wormhole mouths D_\pm are represented by the thick vertical lines. The real size of both mouths is the same. However, since the gravitational field of the shell affects the geometry, the global Cartesian coordinates used for mapping to the diagram do not represent the geometry properly. The coordinate size of both mouths is thus different, as can be observed in the diagram. The distance between mouths should be large—diagrams do not reflect this feature accurately.

the intersection \bar{V} , both potentials agree. In the zeroth-order approximation, the potential U_- is constant inside the shell around the mouth D_- , and it thus leaks as constant U_0 “behind” the wormhole, into the domain \hat{V}_- . On the other hand, the potential U_+ is negligible around D_+ in the zeroth order, and thus nothing leaks behind the wormhole into the domain \hat{V}_+ . See Fig. 12.

Clearly, the potentials differ around the mouths as

$$U_+ = U_- + \Delta U, \quad (5.2)$$

and in this order of approximation we obtain

$$\Delta U = -U_0 = -m\mathcal{Z}(\chi_0) \quad \text{in } \hat{V}. \quad (5.3)$$

It indicates that the global potential is not well defined and we deal with a locally static spacetime. The globally well-defined quantity is the geometry.

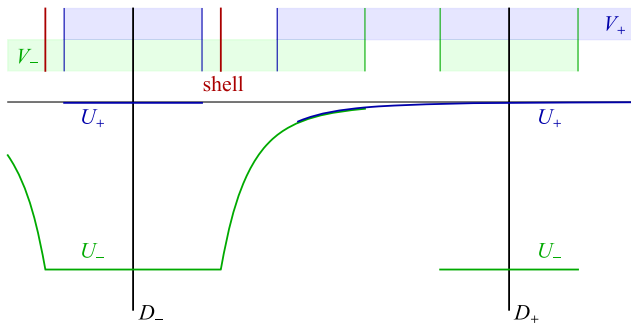


FIG. 12. Diagram shows the zeroth-order approximation of potentials U_{\pm} drawn along the axis z . The potentials U_{-} and U_{+} are continuous on the domains V_{-} and V_{+} , respectively. They coincide on the intersection domain \bar{V} but differ by a constant ΔU on the intersection domain \hat{V} . The falloff of the potential is exaggerated to compensate a small distance between mouths. The top part of the diagram indicates the domains V_{-} and V_{+} and the position of the wormhole and of the massive shell.

The perturbed metrics in the domains V_{-} and V_{+} are

$$ds_{\pm}^2 = -e^{2U_{\pm}} dt_{\pm}^2 + e^{-2U_{\pm}} dl_{\pm}^2. \quad (5.4)$$

We want that these two metrics describe one globally well-defined geometry. Therefore, they have to coincide as tensors on both intersections \bar{V} and \hat{V} . Their identification will be simple on the intersection \bar{V} , since the potentials U_{-} and U_{+} coincide here. We can express the both metrics just in global cylindrical coordinates (ρ, z, φ) on \bar{V} ; cf. (2.3).

However, the potentials differ on other intersection \hat{V} . Therefore, we have to adjust the coordinate identification in this domain to match both metrics. By identifying the proper time of the static observers, we get

$$e^{U_{-}} dt_{-} = e^{U_{+}} dt_{+}. \quad (5.5)$$

Identifying spatial distance, we find

$$e^{-U_{-}} dl_{-} = e^{-U_{+}} dl_{+}. \quad (5.6)$$

Here, the background proper lengths dl_{-} and dl_{+} should be expressed using (2.8) in appropriate spheroidal coordinates $(\chi_{-}, \vartheta_{-}, \varphi_{-})$ and $(\chi_{+}, \vartheta_{+}, \varphi_{+})$, respectively. Relation (5.6) then gives the identification of coordinates $(\chi_{\pm}, \vartheta_{\pm}, \varphi_{\pm})$ in the domain \hat{V} , which guarantees the matching geometry. The resulting relation is, however, rather complicated.

The identification of coordinates is simpler if expressed in cylindrical coordinates. For that, however, we have to introduce gravitationally modified cylindrical coordinates leaked behind the wormhole. Let us consider first coordinates $(\chi_{-}, \vartheta_{-}, \varphi_{-})$ around the wormhole mouth D_{-} . In the domain \bar{V}_{-} (above the mouth), they are related to the original global cylindrical coordinates (ρ, z, φ) as (2.10). However, behind the mouth, in the domain \hat{V}_{-} , they are

related by analogous relations (2.12) to new leaked cylindrical coordinates $(\rho_{-}, z_{-}, \varphi_{-})$. Of course, the flat background metric dl_{-}^2 is given on both sides of the wormhole throat by (2.3):

$$\begin{aligned} dl_{-}^2|_{\bar{V}_{-}} &= d\rho^2 + dz^2 + \rho^2 d\varphi^2, \\ dl_{-}^2|_{\hat{V}_{-}} &= d\rho_{-}^2 + dz_{-}^2 + \rho_{-}^2 d\varphi_{-}^2. \end{aligned} \quad (5.7)$$

Similarly, dl_{+}^2 is related by (2.10) to (ρ, z, φ) in \bar{V}_{+} and by (2.12) to new leaked coordinates $(\rho_{+}, z_{+}, \varphi_{+})$ in domain \hat{V}_{+} , and

$$\begin{aligned} dl_{+}^2|_{\bar{V}_{+}} &= d\rho^2 + dz^2 + \rho^2 d\varphi^2, \\ dl_{+}^2|_{\hat{V}_{+}} &= d\rho_{+}^2 + dz_{+}^2 + \rho_{+}^2 d\varphi_{+}^2. \end{aligned} \quad (5.8)$$

Comparing (5.7) and (5.8) using relation (5.6), we find

$$e^{-U_{-}} d\rho_{-} = e^{-U_{+}} d\rho, \quad e^{-U_{-}} dz_{-} = e^{-U_{+}} dz \quad (5.9)$$

in domain \hat{V}_{-} and

$$e^{-U_{-}} d\rho = e^{-U_{+}} d\rho_{+}, \quad e^{-U_{-}} dz = e^{-U_{+}} dz_{+} \quad (5.10)$$

in \hat{V}_{+} .

Assuming that the mouths of the wormhole are identified at times $t_{-} = t_{0-}$ and $t_{+} = t_{0+}$, the differential relation (5.5) gives

$$(t_{+} - t_{0+}) = e^{-\Delta U} (t_{-} - t_{0-}), \quad (5.11)$$

valid in \hat{V} .

For spatial coordinates, we get that in domain \hat{V}_{+} the leaked cylindrical coordinates $(\rho_{+}, z_{+}, \varphi_{+})$ are related to the global cylindrical coordinates (ρ, z, φ) as

$$\rho_{+} = e^{\Delta U} \rho, \quad z_{+} + \frac{\ell}{2} = e^{\Delta U} \left(z + \frac{\ell}{2} \right). \quad (5.12)$$

Similarly, in domain \hat{V}_{-} , the leaked cylindrical coordinates $(\rho_{-}, z_{-}, \varphi_{-})$ are related to the global cylindrical coordinates (ρ, z, φ) as

$$\rho_{-} = e^{\Delta U} \rho, \quad z_{-} - \frac{\ell}{2} = e^{\Delta U} \left(z - \frac{\ell}{2} \right). \quad (5.13)$$

See Fig. 13.

We can reformulate these relations in the rule, how the global cylindrical coordinates defined on both side of the wormhole throat should be identified through the throat. We observe that $t_{+} = t$ in \bar{V}_{+} and $t_{-} = t$ in \bar{V}_{-} . Relation (5.11) thus gives the identification of time coordinate $t = t_{0-} + \Delta t_{*}$ in \bar{V}_{-} with $t = t_{0+} + e^{-\Delta U} \Delta t_{*}$ in \bar{V}_{+} for any Δt_{*} . Similarly, we observe that the coordinates (ρ, φ)

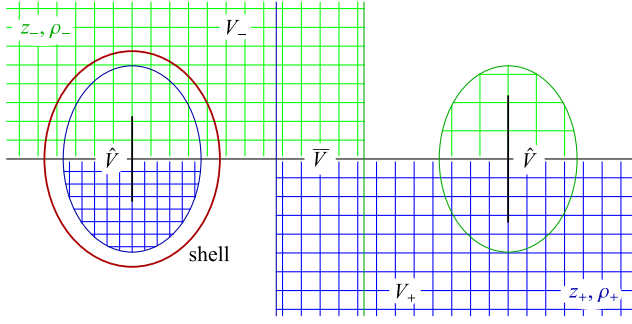


FIG. 13. Finite wormhole with distant mouths. The diagram shows coordinates x_-^j and x_+^j defined on the domains U_- and U_+ , respectively. Since the potential modifies the geometry, the coordinates continuously extended through the wormhole does not match the original coordinates.

approaching the mouth D_- from \bar{V}_- are matched by leaked coordinates (ρ_-, φ_-) approaching D_+ from \hat{V}_- . The result for the time coordinate and relations (5.13) thus give the identification

$$\begin{aligned} [t, \rho, z, \varphi]|_{\bar{V}_-} &= \left[t_{0-} + \Delta t_*, \rho_*, -\frac{\ell}{2} \pm 0, \varphi_* \right] \\ [t, \rho, z, \varphi]|_{\hat{V}_+} &= \left[t_{0+} + e^{-\Delta U} \Delta t_*, e^{\Delta \phi} \rho_*, \frac{\ell}{2} \mp 0, \varphi_* \right] \end{aligned} \quad (5.14)$$

for any Δt_* , ρ_* and φ_* . The term ± 0 distinguishes the faces of the mouths.

It means that the identification of points through the wormhole throat changes under the gravitational perturbation. It causes a complication with the size of the wormhole. To keep the proper size of the wormhole mouths a the same from both sides, one has to choose a different coordinate radius for each mouth, a_- , and a_+ , such that

$$a = e^{-U_-} a_- = e^{-U_+} a_+, \quad \text{i.e., } a_+ = e^{\Delta U} a_-. \quad (5.15)$$

Notice that the change in the coordinate size of the wormhole mouths is of the first order in the gravity perturbations and it produces only next-order perturbations to the gravitational potential. Thus, we can continue to use the potential calculated on the original wormhole background.

Let us summarize. We have perturbed the background with metric (3.3) and found the perturbed metrics (5.4). Since the potentials U_- and U_+ cannot be extended to a global function on the background spacetime, the perturbation has to be performed separately on domains \bar{V}_- and \bar{V}_+ . In these domains we can use global cylindrical coordinates (t, ρ, z, φ) to identify the background spacetime with the perturbed spacetimes.

The perturbed metrics ds_-^2 and ds_+^2 can be easily extended to domain V_- and V_+ through the wormhole throat, using, for example, spheroidal coordinates

$(t_-, \chi_-, z_-, \varphi_-)$ and $(t_+, \chi_+, z_+, \varphi_+)$, respectively. These metrics match trivially on the intersection \bar{V} , as can be seen in global coordinates (t, ρ, z, φ) , since the potentials are equal here.

However, since the potentials differ on the other intersection \hat{V} , the metrics cannot be matched without a proper identifications of the coordinates. Because the potentials U_- and U_+ differ only by a constant [cf. (5.2)], the metrics ds_-^2 and ds_+^2 on \bar{V} differ only by a rescaling of time and spatial directions. It can be demonstrated using cylindrical coordinates $(t_\pm, \rho_\pm, z_\pm, \varphi_\pm)$ introduced behind the wormhole. Namely, the global cylindrical coordinates are related to leaked cylindrical coordinates by a rescaling (5.12) and (5.13).

However, this shows that we need to perform perturbation of the background in two domains independently, and only after that can we glue together the global geometry representing a weak gravitational field. As the result, the spacetime is not globally static but only locally static spacetime.

B. Large-distance approximation—First order

Let us now improve our approximation of the shell field in the wormhole spacetime. We now include an interaction of the field with the wormhole in the first order of $\frac{1}{\ell}$. We start with some geometrical preliminaries.

The oblate spheroidal coordinates $(\chi_-, \vartheta_-, \varphi_-)$ and $(\chi_+, \vartheta_+, \varphi_+)$ introduced around the mouths D_- and D_+ are related by (2.11). We evaluate it at the mouth D_+ where $\chi_+ = 0$:

$$\sinh \chi_- \cos \vartheta_- = \ell, \quad \cosh \chi_- \sin \vartheta_- = \sin \vartheta_+. \quad (5.16)$$

Solving these equations one finds

$$\sin \vartheta_+ = \sqrt{\frac{\ell^2}{\cos^2 \vartheta_-} + 1} \sin \vartheta_- \geq \ell \sin \vartheta_-. \quad (5.17)$$

The value of $\sin \vartheta_+$ is restricted by 1; thus, the angle ϑ_- evaluated at D_+ is small, $\vartheta_-|_{D_+} \ll 1/\ell$. Substituting this result back to (5.16) we obtain that up to the quantity of order $O(\ell^{-2})$ the value of $\sinh \chi_-$ on the mouth D_+ is constant and equal to ℓ .

From the discussion of the first-order approximation, we expect that the potential U of the shell around the wormhole mouth D_- is not globally defined. As we discussed in Sec. III, we can define the potential U using integration (3.12) along a path from infinity not crossing the wormhole throat. It will not be continuous at the throat of the wormhole, but it is smooth everywhere else. And although the potential is not continuous through the wormhole, the field strength $-\nabla U$ is globally defined and smooth. We want to find an approximation of this potential.

In the zeroth approximation of the potential we assumed that the field (5.1) is negligible at the mouth D_+ . Now, we take into account that it is not negligible here. We denote this potential as truncated, U_{sht} . Equipotentials of U_{sht} are surfaces of $\chi_- = \text{const}$. Therefore, it follows from the discussion above that the field (5.1) can be treated as homogeneous near the mouth D_+ in the next order of approximation.

We can thus use our discussion from Sec. IV D. Here we learned that the truncated homogeneous field around the mouth of the wormhole has to be compensated by the field (4.27) adjusted to the mouth at $z = \frac{\ell}{2}$.

This field is nonvanishing on both sides of the wormhole; it thus changes the field not only in domain \bar{V}_+ “above” the mouth D_+ , but also in domain \hat{V}_+ “below” the mouth. Above the mouth D_+ we use expression (4.27) employing spheroidal coordinates $(\chi_+, \vartheta_+, \varphi_+)$. Below the mouth, we transfer the induced field into spheroidal coordinates $(\chi_-, \vartheta_-, \varphi_-)$, using relations⁹ (2.11). In these domains, values of relevant spheroidal coordinate χ_{\pm} are positive, and both sets of coordinates can be understood as functions of the global coordinates (ρ, z, φ) ; cf. (2.10).

We can thus write the induced field U_i outside the wormhole throat as

$$U_{\text{shi}} = w_0(\mathcal{Z}_-(\chi_-) \cos \vartheta_- - \mathcal{Z}_-(\chi_+) \cos \vartheta_+). \quad (5.18)$$

The corrected field of the shell thus reads (see Fig. 14)

$$U = U_{\text{sht}} + U_{\text{shi}} \quad (5.19)$$

with U_{sht} given by (5.1) and U_{shi} given by (5.18).

The constant w_0 in the induced field has to be fixed by the condition that the field strength should be continuous through the wormhole. Namely, we require

$$\begin{aligned} \left. \frac{dU}{dz} \right|_{D_+^<} &= \left. \frac{dU}{dz} \right|_{D_+^>}, \\ \left. \frac{dU}{dz} \right|_{D_-^>} &= \left. \frac{dU}{dz} \right|_{D_-^<}. \end{aligned} \quad (5.20)$$

In our approximation, we assume that the field is homogeneous and, thus, it is sufficient to check these conditions just along the z axis. Relation (2.12) gives

$$z \mp \ell/2 = \sinh \chi_{\pm} \cos \vartheta_{\pm}, \quad (5.21)$$

where, along the axis, values of $\cos \vartheta_{\pm}$ are just -1 or $+1$ depending on the position with respect to the wormhole mouths. Hence

⁹We do not employ here the corrections of the coordinate matching discussed after (5.6) in the previous section [which we also expressed using cylindrical coordinates in (5.14)] since it would produce corrections of higher order.

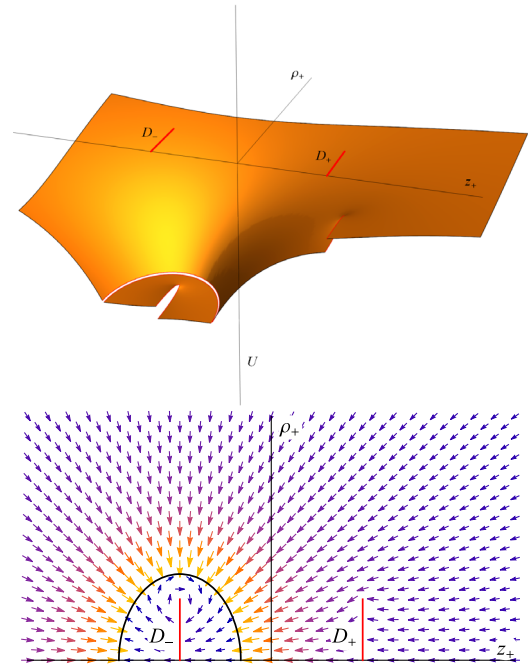


FIG. 14. Potential U [top] and field strength [bottom] of the thin massive shell localized around the wormhole mouth D_- . The potential depicted here is smooth outside the wormhole throat, but it is discontinuous at the throat. The zeroth-order value U_{sht} of the potential corresponds to the shell in space without a wormhole. The correction given by induced field U_{shi} makes the field strength smooth through the wormhole throat.

$$\frac{d\chi_{\pm}}{dz} = \frac{\cos \vartheta_{\pm}}{\cosh \chi_{\pm}}. \quad (5.22)$$

Using these relations, we can calculate z derivatives of the potential U at the mouths of the wormhole. Actually, it turns out that it is the same on both faces $D^<$ and $D^>$ of the mouth. Using relations (5.1), (5.18), (4.2) and (4.19) one finds

$$\begin{aligned} \left. \frac{dU}{dz} \right|_{D_+} &= \frac{m}{\ell^2 + 1} - \frac{w_0}{2}(1 + V_{\ell}), \\ \left. \frac{dU}{dz} \right|_{D_-} &= \frac{w_0}{2}(1 + V_{\ell}). \end{aligned} \quad (5.23)$$

Here V_{ℓ} is coming from the derivative of $\mathcal{Z}'_-(\ell)$, i.e., from the contribution of the induced field at distant wormhole mouth

$$V_{\ell} = \frac{2}{\pi} \left(\arctan \ell - \frac{\pi}{2} + \frac{\ell}{\ell^2 + 1} \right) = -\frac{4}{3\pi\ell^3} + \mathcal{O}(\ell^{-5}). \quad (5.24)$$

Now, it is easy to check that the continuity conditions (5.20) are satisfied provided

$$w_o = \frac{m}{(\ell^2 + 1)(1 + V_\ell)} = \frac{m}{\ell^2} + \mathcal{O}(\ell^{-4}). \quad (5.25)$$

We see that the contribution of the induced field is rather small.

The potential (5.19) is shown in Fig. 14. As we emphasized, it is not globally smooth; it has discontinuity across the wormhole throat.

Similarly as we did in the zeroth-order approximation, it can be extended as a solution of the homogeneous Laplace equation through each wormhole mouth, obtaining thus two potentials U_- and U_+ defined in domains V_- and V_+ , respectively. However, in this case we have to adjust the induced field U_{shi} to guarantee also the continuity of the potential. It can be achieved by adding a suitable constant to the extended field behind the the mouth. We obtain

$$U_+ = \begin{cases} U_{\text{sht}} + U_{\text{shi}} & \text{in } \bar{V}_+, \\ U_{\text{sht}}|_{z=\frac{\ell}{2}} + U_{\text{shi}} & \text{in } \hat{V}_+ \end{cases} \quad (5.26)$$

and

$$U_- = \begin{cases} U_{\text{sht}} + U_{\text{shi}} & \text{in } \bar{V}_-, \\ U_o + U_{\text{shi}} & \text{in } \hat{V}_-. \end{cases} \quad (5.27)$$

Here $U_o = U_{\text{sht}}|_{z=-\frac{\ell}{2}} = m\mathcal{Z}(\chi_o)$ is the value of the truncated potential inside the shell leaking behind the mouth D_- —the effect which we encountered already in the zeroth-order approximation..

The constant

$$U_{\text{sht}}|_{z=\frac{\ell}{2}} = -m \operatorname{arccot} \ell = -\frac{m}{\ell} + \mathcal{O}(\ell^{-3}) \quad (5.28)$$

leaking through the mouth D_+ into \hat{V}_+ appears only in the first order of approximation. Comparing with (5.25), we observe that it is actually more significant than the induced field U_{shi} itself.

Both potentials U_- and U_+ along the z axis are depicted in Fig. 15.

We can estimate the difference between both potentials. Analogically to (5.2) we obtain the correction to (5.3):

$$\begin{aligned} \Delta U &= U_{\text{sht}}|_{z=\frac{\ell}{2}} - U_o = -U_o - \frac{m}{\ell} + \mathcal{O}(\ell^{-2}) \\ &= m \left(\operatorname{arccot} \sinh \chi_o - \frac{1}{\ell} + \mathcal{O}(\ell^{-2}) \right). \end{aligned} \quad (5.29)$$

Since the jump ΔU plays a role in the identifications of the geometry through the wormhole (5.14), namely, in the identification of the time coordinate (5.11), it is worth to investigate it more from a different point of view.

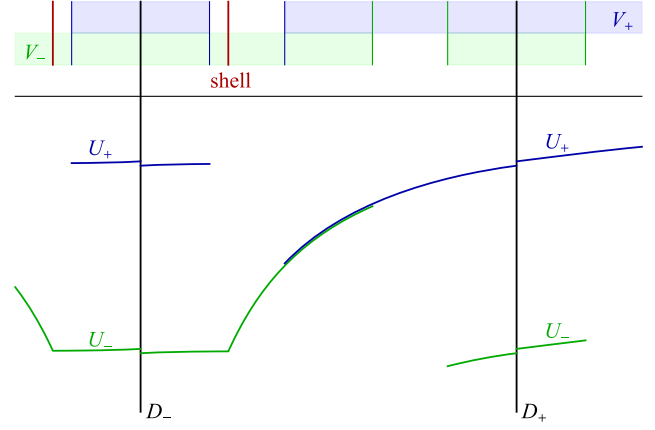


FIG. 15. The diagram shows the first-order approximation of potentials U_{\pm} drawn along the axis z . The potentials U_- and U_+ are continuous on the domains V_- and V_+ , respectively. The potentials include the induced field correcting the zeroth-order contribution on the wormhole throat. Both potentials coincide on the intersection domain \bar{V} but differ by a constant ΔU on the intersection domain \hat{V} . The top part of the diagram indicates the domains V_- and V_+ and the position of the wormhole and of the massive shell.

C. Invariant I_C calculation

In a multiply connected space with a nonpotential locally static gravitational field one can define the following integral:

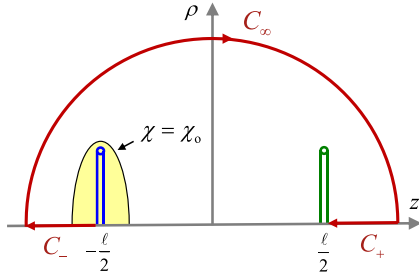
$$I_C = \oint_C w_i dx^i. \quad (5.30)$$

Here C is a closed contour, and w_i is a three-dimensional covector of the acceleration. This integral is a topological invariant in the following sense. The integral (5.30) is the same for any two closed paths, which can be obtained one from another by a continuous transformation.

Clearly, for a globally potential field $w_i = U_{,i}$, this integral vanishes. For a locally potential field which is not potential globally, it estimates the nonpotentiality of the field w_i . If evaluated along a path between different points which belongs to a domain, where the potentiality holds, it gives a potential difference. Therefore, we can conclude that for a closed path it estimates a potential difference which is an obstacle for a global definition of the potential:

$$\Delta U = I_C. \quad (5.31)$$

Let us calculate the invariant I_C for situation discussed in this section: the wormhole space R_{wh} with one of its mouths surrounded by a massive thin shell. We choose a closed C which passes through the wormhole. Since its value does not depend on a special choice of such a path we use for C the contour shown in Fig. 16. It consists of three parts C_- , C_+ and C_∞ . We choose C_- and C_+ to be intervals along


 FIG. 16. Contour C .

z - axes and C_∞ to be a circle $R \equiv \sqrt{\rho^2 + z^2} = \text{const}$ connecting the end points of these intervals. In the limit $R \rightarrow \infty$ the contribution of the part C_∞ to I_C vanishes.

For the other parts of the contour C one has

$$\begin{aligned} I_- &= \int_0^\infty w_{\chi_-} d\chi_- = U|_{D_-}^{\chi_- = \infty} \\ &= -U_{\text{sht}}|_{\chi_- = \chi_0} - U_{\text{shi}}|_{D_-}, \\ I_+ &= -\int_0^\infty w_{\chi_+} d\chi_+ = -U|_{D_+}^{\chi_+ = \infty} \\ &= U_{\text{sht}}|_{D_+} + U_{\text{shi}}|_{D_+}. \end{aligned} \quad (5.32)$$

The induced potential U_{shi} (5.18) is a symmetric functions with respect to reflection $\chi_- \rightleftharpoons \chi_+$, $\vartheta_- \rightleftharpoons \pi - \vartheta_+$ and, hence, satisfies the condition

$$U_{\text{shi}}|_{D_-} = U_{\text{shi}}|_{D_+}; \quad (5.33)$$

therefore, it cancels in the sum of I_- and I_+ . Then we obtain

$$I_C = -m[\mathcal{Z}(\chi_0) + \text{arccot} \ell]. \quad (5.34)$$

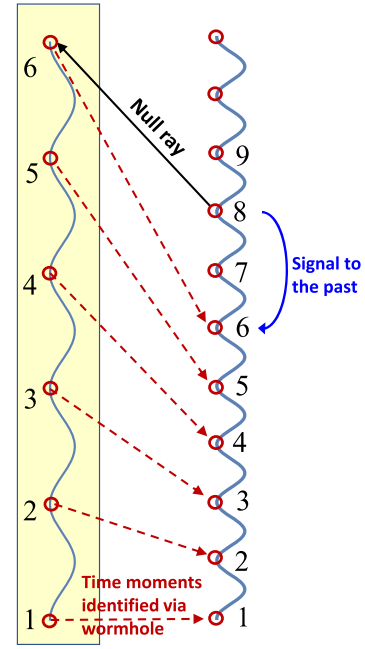
For $\ell \gg 1$ we finally get

$$I_C = m \left[\text{arccot} \sinh \chi_0 - \frac{1}{\ell} \right] + \mathcal{O}(\ell^{-2}). \quad (5.35)$$

One can check that when $\sinh \chi_0 \ll \ell$ the integral I_C is positive. The main contribution comes from the first term. The work along the closed path passing through the wormhole does not vanish and the gravitational field is nonpotential.

VI. TIME-MACHINE FORMATION

Let us show that in the spacetime with a ring wormhole, where one of its mouths is surrounded by matter, closed timelike curves are created. Assume that D_- mouth of the wormhole is inside the massive shell while D_+ mouth is located far away outside the shell (see Fig. 17) where its gravitational field is small. Because these mouths are



identified through the wormhole the proper times at D_- and D_+ are the same. Hence, the coordinate times t_- at D_- and t_+ at D_+ are related as (5.11). In this setup $\Delta U = I_C > 0$. It means that for the same proper time interval the coordinate time at the disk D_- which is inside the massive shell spans more than that at D_+ . Let us assume that clocks at the disks are synchronized at the moment $t = 0$. If a particle enters D_- at time t_1 , it appears from D_+ at slightly different time t_2 . Denote the time gap $\Delta t = t_2 - t_1$. Then in such a process the time gap is negative:

$$\Delta t = -[e^{\ell c} - 1]t_2 < 0. \quad (6.1)$$

Let us consider the following experiment. Suppose a null ray emitted at some time t_0 at the point with coordinate $z = 0$ propagates along the z axis to the left, enters the disk D_- at time t_1 and after it appears from D_+ at the time t_2 returns to the point of its emission at the time t_3 . Let us find how the corresponding time t_3 (as measured by an observer at infinity) depends on t_0 . For the metric (3.1) the equation of the null ray moving along the z axis is

$$\frac{dz}{dt} = -e^{2U(z)}, \quad (6.2)$$

where $U(z)$ is a value of the potential U on the z axis. A minus sign in the right-hand side of this equation indicates that at both stages (motion to D_- and motion from D_+) the ray moves in the direction opposite to the direction of the z axis. Using (6.2) one finds that the time of the arrival of the ray to D_- is

$$t_1 = t_0 + \int_{-\ell/2}^0 e^{-2U} dz. \quad (6.3)$$

Similarly, time t_3 of arrival to the original point $z = 0$ after the ray appears from D_- at time t_2 is

$$t_3 = t_2 + \int_0^{\ell/2} e^{-2U} dz. \quad (6.4)$$

Using Eqs. (6.1), (6.3), and (6.4) one finds

$$t_3 - t_0 = -[e^{I_C} - 1]t_2 + B, \quad (6.5)$$

$$B = \int_{-\ell/2}^{\ell/2} e^{-2U} dz > 0.$$

For a chosen direction of null ray motion $I_C > 0$. Hence, $t_3 - t_0$ is a decreasing function of the time t_2 at D_+ which for large ℓ is close to the time of an observer at infinity. This equation shows that t_3 becomes lesser than t_0 when

$$t_2 \geq T, \quad T = \frac{B}{e^{I_C} - 1}. \quad (6.6)$$

Because $B > 0$ and $I_C > 0$ this means that for sufficiently late time t_2 the null ray arrives at the initial point $z = 0$ earlier than it was emitted and a closed time curve is formed.

In the weak-field approximation adopted in this paper I_C is small and $B \approx \ell$. Then formation of the time machine happens after time

$$T \approx \frac{\ell}{I_C}. \quad (6.7)$$

Let $r = \sinh \chi_0$ be a characteristic size of the massive thin shell surrounding the D_- mouth of the ring wormhole. Then the characteristic time of the transformation of the wormhole into the time machine is

$$T \approx r\ell/m. \quad (6.8)$$

Restoring the dimensions one can write this expression in the form

$$T \approx \frac{RLc}{GM}. \quad (6.9)$$

Here T is the characteristic time of closed timelike curves formation in the initially traversable ring wormhole surrounded by a massive thin shell of radius R and mass M . L is the distance between mouths.

Let us emphasize that the main conclusion concerning closed timelike curve formation in the wormhole spacetime is a rather robust property. Such curves arise as a result of the asymmetry of the mass distribution surrounding its two mouths.

For more details concerning nonpotential gravitational field and time-machine formation in such fields see Refs. [13,20–22].

VII. DISCUSSION

In the present paper we study the gravitational field in a spacetime with a traversable ring wormhole. We considered two types of such wormholes: (i) a wormhole connecting two flat spaces and (ii) a wormhole connecting two distant domains in the same space. We focused on study solutions of the gravity equations in the weak-field approximation in the presence of matter in these spaces. For the wormhole connecting two flat spaces we demonstrated that there exist zero-mode solutions describing the gravitational field trapped by the wormhole. Such a field obeys the homogeneous Laplace equation. One can relate them to a proper choice of the matter distribution located at the infinity either of R_+ or/and R_- spaces. We also obtained a solution for the gravitational field of the massive thin oblate spheroidal shell confocal to the disk representing the ring wormhole. A main important feature of such a solution is that one cannot impose a condition that the corresponding gravitational potential U vanishes at both infinities in R_{\pm} . There always exists a nonvanishing difference of these potentials which is proportional to the mass m of the shell and depending on its size. We also found an exact solution for the case when a ring wormhole is immersed in the gravitational field which is homogeneous at R_+ infinity. Such a solution contains the dipole-type component which describes the modification of the homogeneous field by the wormhole.

In Sec. V we study the gravitational field of a massive thin shell surrounding one of the mouths of the ring wormhole connecting two distant domains in a single space. We solved the gravity equation in the approximation when the distance L between the mouths is much larger than the ring radius a . A main property of this solution is that the gravitational potential U becomes a multivalued function. This happens because the space is multiconnected. While the strength of the gravitational force \vec{w} is well defined and unique, a solution of the equation $\nabla U = -\vec{w}$ does not possess this property. One can integrate this equation and find the function U along any chosen curve connecting two points. But the value of U may be different for two different paths which cannot be transformed one to the other by means of a continuous transformation. As a result the

gravitational field is nonpotential. In such a case there exists a growing time gap for clocks synchronized along specially chosen noncontractible closed paths. This implies that a closed timelike curve formation occurs in such a space. This is a mechanism which transforms an original traversable wormhole without a time gap into a time machine.

Let us emphasize that above described properties of the ring wormholes are similar to well-known properties of the “standard” wormholes with a spherical topology of their throat. A main difference is that for the ring wormhole an observer passing through it moves in a flat (or practically flat spacetime), while in the case of standard wormholes he or she should pass a domain filled with the matter violating the null-energy condition. In this paper we focused on the classical aspects of the ring wormhole model. It is well known that for the standard wormholes the quantum effects are important and may dramatically change their property in the regime of the time-machine formation. Namely, it was shown [17,18] that at the moment close to the time when closed timelike curves are formed, the renormalized quantum average of the stress-energy tensor of a quantum field infinitely grows. Hawking formulated a chronology protection conjecture [19], according to which the back-reaction of quantum effects should always forbid formation of closed timelike curves. There exist quite a lot of papers where different aspects of the role of quantum effects in wormholes and time machines are discussed (see, e.g., [33–40]). It would be interesting to study quantum effects in ring wormholes and time machines. These models are rather simple from the mathematical point of view and they may provide one with simple analytical tools to study such a quantum mechanical problem.

ACKNOWLEDGMENTS

V.P.F. and A.Z. thank the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for their financial support. P.K. was supported by Czech Science Foundation Grant No. GA ĀR 22-14791S.

APPENDIX A: STATIC SPACETIME

1. Metric

Let us consider a static spacetime. It is locally described by metric

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = -e^{2U} dt^2 + dq^2, \\ dq^2 &= q_{ij} dx^i dx^j, \\ \partial_t U &= 0, \quad \partial_t q_{ij} = 0. \end{aligned} \quad (\text{A1})$$

We denote $\xi^\mu = \delta_t^\mu$ a timelike Killing vector and u^μ a four-velocity of static observers moving along this Killing vector:

$$u^\alpha = e^{-U} \xi^\alpha, \quad u^\mu u^\nu g_{\mu\nu} = -1. \quad (\text{A2})$$

Obviously,

$$u_\alpha = -e^U t_{,\alpha}. \quad (\text{A3})$$

The four-acceleration of these static observers,

$$w^\mu = u^\mu{}_{;\nu} u^\nu, \quad (\text{A4})$$

is orthogonal to u^α . Using the orthogonality, the Killing equation $\xi_{(\mu;\nu)} = 0$, and staticity $u^\alpha U_{,\alpha} = 0$, one finds that U is a local potential of w_μ :

$$w_\alpha = U_{,\alpha}. \quad (\text{A5})$$

We call U the gravitational potential, w^α the acceleration field, and $-U_{,\alpha}$ the gravitational field strength. Since w^α is orthogonal to the time direction u^α , it can be restricted to spatial components. The field strength is a fictitious force that one assumes in the static frame to explain a tendency of free observers to move with respect to the frame. Nonmoving static observers have to compensate this force by a real force equal to w_j per unit mass; cf. (3.7). Of course, in the spacetime description, the static observers move along nongeodesic trajectories with four-acceleration w^α caused by the real force.

2. Weak static gravitational field

Now we want to study a weak gravitational field on a flat background. We assume that the resulting spacetime is static, as described above, and we want to formulate the field equation in an approximation of a weak field. Since the metric (A1) is invariant under time reflection $t \rightarrow -t$, the extrinsic curvature of a surface $t = \text{const}$ vanishes, and Gauss-Codazzi equations imply that

$$G_t^i = -\frac{1}{2} \mathcal{R}. \quad (\text{A6})$$

Here, $G_{\mu\nu}$ is Einstein tensor of the spacetime metric ds^2 and \mathcal{R} is a scalar curvature of the spatial metric dq^2 [see, e.g., (B.2.6) in Ref. [22]].

We choose a stress-energy tensor for a static distribution of matter in the form

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu. \quad (\text{A7})$$

Here ε is the mass density. In the metric (A1) one has

$$dm = \varepsilon \sqrt{q} d^3x. \quad (\text{A8})$$

Using Einstein equations and relation (A6) one gets

$$\mathcal{R} = 16\pi\varepsilon. \quad (\text{A9})$$

Let us consider a special case when the spatial geometry is conformally flat with a factor given by e^{-2U} :

$$dq^2 = e^{-2U} dl^2, \quad (\text{A10})$$

$dl^2 = \delta_{ij} dx^i dx^j$ being a flat spatial metric. Then one has

$$\mathcal{R} = 2e^{2U}[2\Delta U - (\nabla U)^2]. \quad (\text{A11})$$

Here, ∇ , Δ and $(\dots)^2$ are the covariant derivative, the Laplace operator, and the square with respect to the flat spatial metric dl^2 , respectively.

We say that the gravitational field is weak if

$$|\nabla U \nabla U| \ll |\nabla \nabla U|. \quad (\text{A12})$$

In such a case the second term in the square brackets of (A11) can be omitted and one has $\mathcal{R} = 4e^{2U} \Delta U$. Equation (A9) implies that in the weak-field approximation the mass density ε is of the same first order as \mathcal{R} .

The second term in the brackets can be neglected in our approximation, but the prefactor is not negligible. From (A9) we obtain the equation for potential U

$$\Delta U = 4\pi e^{-2U} \varepsilon. \quad (\text{A13})$$

In fact, in the full theory, for a static distribution of the matter the stress-energy tensor should be slightly modified by adding a contribution responsible for repulsive forces keeping the matter at rest. Indeed, for the stress-energy tensor (A7) one has

$$T^{\mu\nu}{}_{;\nu} = \varepsilon w^\mu. \quad (\text{A14})$$

This means that a static distribution of matter is possible only when some repulsive forces are present that compensate the gravity attraction. This can be achieved by including the pressure p and writing $T_{\mu\nu}$ in the form

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (\text{A15})$$

For this tensor

$$T^{\mu\nu}{}_{;\nu} = (\varepsilon + p)w^\mu + p^{;\mu}. \quad (\text{A16})$$

In the weak-field approximation, both ε and w^α are small. We say that each of this quantities is of the first order in the weak-field approximation. To satisfy the conservation law $T^{\mu\nu}{}_{;\nu} = 0$, the pressure should be of the second order. Such pressure does not contribute to the Einstein equations in the leading first-order approximation.

Let us discuss now the other Einstein equations [besides (A11)] in the same leading-order approximation. It is easy to check that all time-spatial components of the Einstein tensor G_{ij} vanish identically, and the spatial components G_{ij} are of the second order in ∇U [see, e.g., (B.2.6) in Ref. [22]]. This means that in the leading order of the weak-field approximation the spatial components of the Einstein equations are also satisfied.

When working in the flat space, it is natural to use a mass density in (A8) normalized on the flat volume element $dV = d^3x$. The relation to ε follows from

$$dm = \mu dV. \quad (\text{A17})$$

Thanks to rescaling (A10) we obtain

$$\mu = e^{-3U} \varepsilon. \quad (\text{A18})$$

Then, Eq. (A13) for the potential takes the form

$$\Delta U = 4\pi e^U \mu. \quad (\text{A19})$$

Let us note that metric (A1), (A10) and the field equation (A19) are invariant under the following scaling transformation:

$$\begin{aligned} U &= \hat{U} + \Delta U, \\ t &= e^{-\Delta U} \hat{t}, \quad dl^2 = e^{2\Delta U} d\hat{l}^2, \\ \mu &= e^{-3\Delta U} \hat{\mu}. \end{aligned} \quad (\text{A20})$$

3. Small value of potential U case

Let us emphasize that in the above discussion we assume that the potential U is slowly changing in space, but we do not assume that its value is small.

To simplify the presentation we make an additional assumption that the potential U is uniformly small, that is, its value satisfies the relation

$$|U| \ll 1 \quad (\text{A21})$$

everywhere in space.

In this approximation Eq. (A19) takes the form

$$\Delta U = 4\pi\mu. \quad (\text{A22})$$

In our perturbative approach it is the same as ε in the leading order:

$$\mu = \varepsilon, \quad (\text{A23})$$

but for U that are not small they differ by the factor e^{-3U} ; cf. (A18).

APPENDIX B: LOCALLY STATIC SPACETIMES

In this appendix we demonstrate how to describe a locally static spacetime without introducing initially a static time coordinate t and a gravitational potential U as we did in Eq. (A1) in the previous appendix.

Let us consider a spacetime M with metric g and assume that it admits two vector fields, a future directed unit vector u and a vector w which obey the equations

$$\begin{aligned} u_\mu u^\mu &= -1, \\ u_{\mu;\nu} &= -w_\mu u_\nu, \\ w_{[\mu;\nu]} &= 0. \end{aligned} \quad (\text{B1})$$

We call such a spacetime locally static. The relations (B1) imply that

$$w^\mu = u^\nu u^\mu{}_{;\nu}, \quad w_\mu u^\mu = 0. \quad (\text{B2})$$

Denote by γ integral lines of u

$$\frac{dx^\mu}{d\tau} = u^\mu. \quad (\text{B3})$$

Normalization condition $u^2 = -1$ implies that τ is a proper time along a worldline γ .

The last of the relations in (B1) implies that the 1-form $w_\mu dx^\mu$ is closed. Let p_0 be a point in M and Ω_{p_0} be a simply connected region which contains this point. Then there exists such a function U in Ω_{p_0} that

$$w_\mu = U_{,\mu}. \quad (\text{B4})$$

Let p be a point in Ω_{p_0} and $C_{p_0 p}$ be a path in Ω_{p_0} connecting points p_0 and p . Then

$$U(p) = U(p_0) + \int_{C_{p_0 p}} w_\mu dx^\mu. \quad (\text{B5})$$

The function U called the gravitational potential is defined in Ω_{p_0} up to a constant $U_0 \equiv U(p_0)$. Let us denote

$$z^\mu = e^{\epsilon U} u^\mu, \quad \epsilon = \pm 1. \quad (\text{B6})$$

Using relations (B1) one gets

$$z_{\mu;\nu} = e^{\epsilon U} (-u_\nu w_\mu + \epsilon u_\mu w_\nu). \quad (\text{B7})$$

For $\epsilon = -1$ one has $z_{[\mu;\nu]} = 0$, while for $\epsilon = 1$ one has $z_{(\mu;\nu)} = 0$. We denote

$$\xi^\mu = e^U u^\mu, \quad \eta^\mu = -e^{-U} u^\mu. \quad (\text{B8})$$

Thus one has

$$\xi_{(\mu;\nu)} = 0, \quad \eta_{[\mu;\nu]} = 0, \quad \xi^2 = -e^{2U}. \quad (\text{B9})$$

The first of these equations shows that ξ is a Killing vector, while the second equation implies that there exists such a scalar function t in Ω_{p_0} that

$$\eta_\mu = t_{,\mu}. \quad (\text{B10})$$

It is also easy to check that

$$\xi^\mu \partial_\mu U = 0. \quad (\text{B11})$$

Consider displacement dx^μ along a worldline γ . Then one has

$$\begin{aligned} dt &= \eta_\mu dx^\mu = -e^{-U} u_\mu dx^\mu \\ &= -e^{-U} u_\mu u^\mu d\tau = e^{-U} d\tau. \end{aligned} \quad (\text{B12})$$

One also has

$$\frac{dx^\mu}{dt} = e^U \frac{dx^\mu}{d\tau} = e^U u^\mu = \xi^\mu. \quad (\text{B13})$$

Hence

$$\xi^\mu \partial_\mu = \partial_t, \quad (\text{B14})$$

and t is the Killing time parameter. Let us emphasize that the gravitational potential U in Ω_{p_0} is defined up to a constant. A shift $U \rightarrow U + c$ results in the rescaling both the Killing vector and Killing time

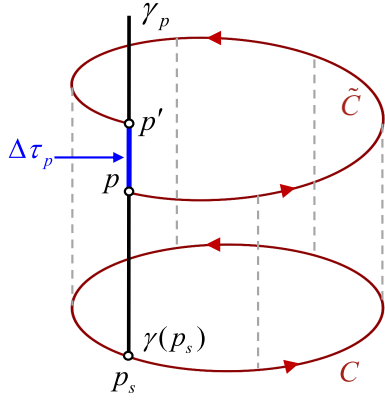
$$\xi^\mu \rightarrow e^{-c} \xi^\mu, \quad t \rightarrow e^c t. \quad (\text{B15})$$

The integral lines γ provide a foliation of the region Ω_{p_0} . Each of the Killing trajectories can be specified by giving three numbers y^i , $i = 1, 2, 3$, and a point p on a given trajectory can be specified by giving its Killing time value t . This defines coordinates $x^\mu = (t, x^i)$ in Ω_{p_0} . In these coordinates the metric takes the form

$$ds^2 = -e^{-2U} dt^2 + h_{ij} dy^i dy^j, \quad \partial_t U = \partial_t h_{ij} = 0. \quad (\text{B16})$$

In what follows we assume that ξ^2 does not vanish and remains negative everywhere in M . This property excludes a case of black holes. But in the spacetime considered in the present paper this property is valid.

Following Ref. [41] we denote by S a collection of all trajectories of ξ in M . There exists a natural map Ψ from M to S which is defined as follows: For any point in M there exists a Killing trajectory passing through it which determines a point in S (see Fig. 18). Three-dimensional space S has the induced metric

FIG. 18. Projection of M onto the space of Killing trajectories S .

$$dh^2 = h_{ij}dy^i dy^j. \quad (\text{B17})$$

Since $\mathcal{L}_\xi w = 0$ and $\xi \cdot w = 0$, the 4-vector of acceleration w has a natural projection \vec{w} on S . In (t, y^i) coordinates one has $w = (0, \vec{w})$. Let us consider a path $C_{p_0 p}^S$ connecting two points p_{S0} and p_S in S . We define a 3D potential $U(p_S)$ as follows:

$$U(p_S) = U_0 + \int_{C_{p_0 p}^S} w_i dy^i, \quad (\text{B18})$$

where U_0 is a constant which is chosen as a value of the potential at p_{S0} . The quantity $U(p_S) - U_0$ can be interpreted as a work done by the gravitational field on a particle of unit mass for its motion along the path C^S .

Consider a closed path C^S which starts at some point p_0 and returns to it again and denote

$$I_{C^S} = \oint_{C^S} w_i dy^i. \quad (\text{B19})$$

Since 3D 1-form w is closed, the Stokes' theorem implies that I_{C^S} depends only on the cohomology class of the path C^S . If S is a simply connected manifold, the invariant I_{C^S} vanishes. If S is not simply connected, there may exist paths for which $I_{C^S} \neq 0$. For such a closed path work done by the gravitational field does not vanish. We call such a field nonpotential. For this field the potential U is a multivalued function.

A path C^S connecting p_S and p'_S in S can be lifted to M as follows. Consider a Killing trajectory corresponding to p_S and choose a point p on it. If in local coordinates y^i the equation of C^S is $y^i = y^i(\lambda)$, then the equation of the path S in (t, y^i) coordinates in M is $x^\mu = (t_p, y^i(\lambda))$, where t_p is the time coordinate of point p . We require that the path C has a property that its tangent vector is orthogonal to u^μ . For a given initial point p and chosen path C^S in S this property uniquely defines C . It is easy to see that

$$\int_{C^S} w_i dy^i = \int_C w_\mu dx^\mu. \quad (\text{B20})$$

Let C^S be a closed path in multiply connected S passing through a point p_S and \tilde{C} be its lift to M passing through p . The end point p' of \tilde{C} lies on the same Killing trajectory as the initial point p but in a general case it does not coincide with it. Denote by ΔC a path from p' to p along their common Killing trajectory. The path $C = \tilde{C} + \Delta C$ is closed. Since w is orthogonal to a Killing trajectory, $\int_{\Delta C} w_\mu dx^\mu = 0$, and one has

$$I_C = \oint_{C^S} w_i dy^i = \oint_C w_\mu dx^\mu. \quad (\text{B21})$$

The value of this integral does not depend on a choice of the initial point p on the path, and relation (B21) is valid not only for a chosen lift of the path C^S but for any closed path C in M such that its projection on S coincides with C^S .

Let us consider a closed path C^S with a starting point p_S for which $I_C = 0$ and choose two close points p and \tilde{p} on the Killing trajectory determined by p_S (see Fig. 19). They have the same coordinates y^i . Let us denote by dt time difference between p and \tilde{p} and assume that dt is small. By construction, dt remains the same for the paths C and \tilde{C} with the initial points \tilde{p} and p , respectively. In particular this is true for the end points p' and \tilde{p}' of these curves. Since $I_C = 0$, the value of the potential U defined along the paths by Eq. (B18) at the end point coincides with its value at the initial point, and one concludes that the initial proper time interval $d\tau$ between p and \tilde{p} is the same as the final proper time interval $d\tau'$ between p' and \tilde{p}' . We call the proper time interval $\Delta\tau_p$ between the initial point p of C and its final point p' the proper time gap (see Fig. 18). The above discussion implies that the proper time gap is constant for closed paths for which the invariant I_C vanishes.

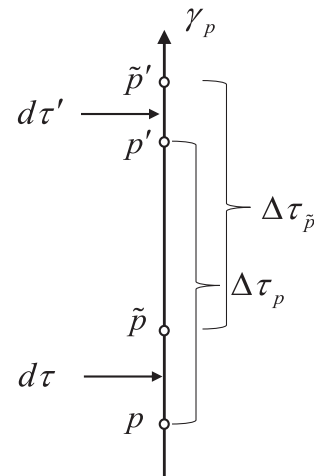


FIG. 19. Time gaps.

If a locally static spacetime M is simply connected so that $I_C = 0$ for any closed path in it, then

- (i) the gravitational potential is globally uniquely defined, up to a constant;
- (ii) the gravitational field is potential;
- (iii) the Killing vector field is globally uniquely defined after choice of its norm in some point.

In other words, this spacetime is (globally) static.

Let us now repeat the above consideration, but do not assume that the invariant I_C for a chosen closed path vanishes. Once again the Killing time interval dt for two close lifts of C^S with starting points at p and \tilde{p} remains constant along the path. Denote by $d\tau$ proper time interval between initial points p and \tilde{p} and by $d\tau'$ proper time interval between final points p' and \tilde{p}' (see Fig. 19). Then using (B12) one gets

$$d\tau = e^{U(p)} dt, \quad d\tau' = e^{U(p')} dt. \quad (\text{B22})$$

Thus

$$d\tau' = e^{U(p')-U(p)} d\tau = \exp(I_C) d\tau. \quad (\text{B23})$$

Denote by $\Delta\tau_p$ the proper time gap for the point p and by $\Delta\tau_{\tilde{p}}$ the proper time gap for \tilde{p} . Then one has (see Fig. 19)

$$d\Delta\tau \equiv \Delta\tau_{\tilde{p}} - \Delta\tau_p = d\tau' - d\tau = (\exp(I_C) - 1) d\tau. \quad (\text{B24})$$

Hence, the proper time gap $d\Delta\tau$ linearly grows with time τ so that

$$\frac{d\Delta\tau}{d\tau} = \exp(I_C) - 1. \quad (\text{B25})$$

For more details see Refs. [13,21,22].

-
- [1] C. W. Misner, Wormhole initial conditions, *Phys. Rev.* **118**, 1110 (1960).
 - [2] D. R. Brill and R. W. Lindquist, Interaction energy in geometrostatics, *Phys. Rev.* **131**, 471 (1963).
 - [3] *Geometrodynamics*, edited by J. A. Wheeler (Academic Press, New York, 1962).
 - [4] D. Gannon, Singularities in nonsimply connected space-times, *J. Math. Phys. (N.Y.)* **16**, 2364 (1975).
 - [5] C. W. Lee, A restriction on the topology of Cauchy surfaces in general relativity, *Commun. Math. Phys.* **51**, 157 (1976).
 - [6] G. J. Galloway, Minimal surfaces, spatial topology and singularities in space-time, *J. Phys. A* **16**, 1435 (1983).
 - [7] J. L. Friedman, K. Schleich, and D. M. Witt, Topological Censorship, *Phys. Rev. Lett.* **71**, 1486 (1993).
 - [8] M. S. Morris and K. S. Thorne, Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity, *Am. J. Phys.* **56**, 395 (1988).
 - [9] M. S. Morris, K. S. Thorne, and U. Yurtsever, Wormholes, Time Machines, and the Weak Energy Condition, *Phys. Rev. Lett.* **61**, 1446 (1988).
 - [10] M. Visser, Traversable wormholes: Some simple examples, *Phys. Rev. D* **39**, 3182 (1989).
 - [11] D. Hochberg and M. Visser, Geometric structure of the generic static traversable wormhole throat, *Phys. Rev. D* **56**, 4745 (1997).
 - [12] J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, K. S. Thorne, and U. Yurtsever, Cauchy problem in space-times with closed timelike curves, *Phys. Rev. D* **42**, 1915 (1990).
 - [13] V. P. Frolov and I. D. Novikov, Physical effects in wormholes and time machine, *Phys. Rev. D* **42**, 1057 (1990).
 - [14] F. Echeverria, G. Klinkhammer, and K. S. Thorne, Billiard balls in wormhole space-times with closed timelike curves: Classical theory, *Phys. Rev. D* **44**, 1077 (1991).
 - [15] A. Carlini, V. P. Frolov, M. B. Mensky, I. D. Novikov, and H. H. Soleng, Time machines: The principle of selfconsistency as a consequence of the principle of minimal action, *Int. J. Mod. Phys. D* **04**, 557 (1995); **05**, 99(E) (1996).
 - [16] J. Dolanský and P. Krtouš, Billiard in the space with a time machine, *Phys. Rev. D* **82**, 124056 (2010).
 - [17] S. W. Kim and K. S. Thorne, Do vacuum fluctuations prevent the creation of closed timelike curves?, *Phys. Rev. D* **43**, 3929 (1991).
 - [18] V. P. Frolov, Vacuum polarization in a locally static multiply connected space-time and a time machine problem, *Phys. Rev. D* **43**, 3878 (1991).
 - [19] S. W. Hawking, The chronology protection conjecture, *Phys. Rev. D* **46**, 603 (1992).
 - [20] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (AIP Press, New York City, 1995).
 - [21] *Black Hole Physics: Basic Concepts and New Developments*, edited by V. P. Frolov and I. D. Novikov (Kluwer Academic Publishers, Dordrecht, 1998).
 - [22] V. P. Frolov and A. Zelnikov, *Introduction to Black Hole Physics* (Oxford University Press, New York, 2011).
 - [23] G. W. Gibbons and M. S. Volkov, Zero mass limit of Kerr spacetime is a wormhole, *Phys. Rev. D* **96**, 024053 (2017).
 - [24] G. W. Gibbons and M. S. Volkov, Weyl metrics and wormholes, *J. Cosmol. Astropart. Phys.* **05** (2017) 039.
 - [25] G. W. Gibbons and M. S. Volkov, Ring wormholes via duality rotations, *Phys. Lett. B* **760**, 324 (2016).
 - [26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).

- [27] B. Mashhoon, Gravitoelectromagnetism: A brief review, [arXiv:gr-qc/0311030](https://arxiv.org/abs/gr-qc/0311030).
- [28] S. J. Clark and R. W. Tucker, Gauge symmetry and gravitoelectromagnetism, *Classical Quantum Gravity* **17**, 4125 (2000).
- [29] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press, New Haven, CT, 1969).
- [30] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1984).
- [31] W. Israel, Singular hypersurfaces and thin shells in general relativity, *Nuovo Cimento B* **44**, 1 (1966); **48**, 463(E) (1967).
- [32] I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (North-Holland, Amsterdam, 1966).
- [33] M. Visser, Hawking's chronology protection conjecture: Singularity structure of the quantum stress energy tensor, *Nucl. Phys.* **B416**, 895 (1994).
- [34] H. D. Politzer, Simple quantum systems in space-times with closed timelike curves, *Phys. Rev. D* **46**, 4470 (1992).
- [35] M. Visser, From wormhole to time machine: Comments on Hawking's chronology protection conjecture, *Phys. Rev. D* **47**, 554 (1993).
- [36] S. V. Krasnikov, On the quantum stability of the time machine, *Phys. Rev. D* **54**, 7322 (1996).
- [37] B. S. Kay, M. J. Radzikowski, and R. M. Wald, Quantum field theory on space-times with a compactly generated Cauchy horizon, *Commun. Math. Phys.* **183**, 533 (1997).
- [38] P. F. Gonzalez-Diaz, Ring holes and closed timelike curves, *Phys. Rev. D* **54**, 6122 (1996).
- [39] M. Visser, The Quantum physics of chronology protection, in *Proceedings of the Workshop on Conference on the Future of Theoretical Physics and Cosmology in Honor of Steven Hawking's 60th Birthday* (Cambridge University Press, Cambridge, 2003), pp. 161–176.
- [40] R. Emparan and M. Tomašević, Quantum backreaction on chronology horizons, *J. High Energy Phys.* **02** (2022) 182.
- [41] R. P. Geroch, A method for generating solutions of Einstein's equations, *J. Math. Phys. (N.Y.)* **12**, 918 (1971).