

Are “Superentropic” black holes superentropic?

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ABSTRACT: We study a critical limit in which asymptotically-AdS black holes develop maximal conical deficits and their horizons become non-compact. When applied to stationary rotating black holes this limit coincides with the “ultraspinning limit” and yields the *Superentropic* black holes whose entropy was derived recently and found to exceed the maximal possible bound imposed by the Reverse Isoperimetric Inequality [1, 2]. To gain more insight into this peculiar result, we study this limit in the context of accelerated AdS black holes that have unequal deficits along the polar axes, hence the maximal deficit need not appear on both poles simultaneously. Surprisingly, we find that in the presence of acceleration, the critical limit becomes smooth, and is obtained simply by taking various upper bounds in the parameter space that we elucidate. The *Critical* black holes thus obtained have many common features with *Superentropic* black holes, but are manifestly not superentropic. This raises a concern as to whether *Superentropic* black holes actually are superentropic.¹ We argue that this may not be so and that the original conclusion is likely attributed to the degeneracy of the resulting first law.

KEYWORDS: Black Holes, Classical Theories of Gravity

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¹We use the upper case *Superentropic* to indicate the specific black hole solution, and lower case superentropic to indicate the property that entropy violates the Reverse Isoperimetric Inequality.

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1 Introduction

Black hole thermodynamics represents a fascinating insight into the interaction of quantum physics with gravity. Without assigning an entropy to a black hole [3], we would have a violation of the second law of thermodynamics, widely considered to be one of the most fundamental laws of nature. Moreover, the discovery of black hole radiation by Hawking [4], consistent with the notion of black body radiation, presented definitive proof that black holes can indeed be assigned quantum properties. As the thermodynamics of black holes was extended and explored, a natural question was: what is the black hole equivalent of the pressure/volume term, PdV ? Early work [5, 6] proposed that the cosmological constant Λ could fulfil this role, however this was largely unexplored (though see [7, 8]) until the importance of anti-de Sitter (AdS) spacetime came to the fore in the context of the gauge-gravity duality in string theory. A crucial conceptual insight was that the ‘mass’, M , for the black hole should more properly be interpreted as enthalpy [9], the pressure with the (negative) cosmological constant, $P = -\Lambda/(8\pi)$, and the black hole volume with the corresponding conjugate quantity, $V = \partial M/\partial P$ [10–12], and with this, the subject enjoyed a renaissance, with many interesting critical phenomena and thermodynamic processes being explored, see [13] for a review.

Within the context of extended black hole thermodynamics there has been an interesting conjecture — the *Reverse Isoperimetric Inequality* [14], which is a statement about

the relation between the thermodynamic volume of the black hole and its entropy, or area. In mathematics, the *Isoperimetric Inequality* states that the surface area enclosing a given volume is minimised for a spherical surface, and indeed the area can be unboundedly large if a suitably deformed or wrinkly surface is chosen. From the physical perspective of black hole thermodynamics, however, this would be a disturbing inequality as, if true, the second law would imply that a black hole would want to be as deformed as possible to maximise its entropy, thus indicating a classical instability of black holes. However, in Cvetic et al. [14], it was demonstrated that in all (then) studied black hole solutions, the reverse of this inequality held, hence the *Reverse Isoperimetric Inequality Conjecture* (see also [15] for the de Sitter version of this conjecture).

Not long after a rather peculiar solution was investigated. In exploring possible black hole solutions in four-dimensional Fayet-Iliopoulos gauged supergravities, Gneccchi et al. [16] briefly presented a black hole with a novel horizon topology. The solution emerged as a certain limit of the Carter-Plebanski metric [17, 18] where the metric function governing the longitudinal angle develops a certain double root. That it can be interpreted as the *ultra-spinning* limit of the Kerr-AdS solution, where the rotation parameter a is taken to be critically large (equal to the AdS radius ℓ) was suggested in a letter by Klemm [1], and the corresponding limiting procedure was explicitly found in [2, 19, 20]. The result is a non-compact horizon of finite area, which is roughly spherical near its equator but becomes hyperbolic near the axis. The poles are removed from the spacetime and the horizon topology is that of a sphere with two punctures.

In a series of papers, Hennigar et al. [2, 19, 20] explored the thermodynamic implications of having such an extraordinary spacetime. These papers argued a distinct definition of thermodynamic variables from the standard Kerr-AdS variables, and intriguingly discovered that the black hole appeared to be *superentropic*. Specifically, the *reverse isoperimetric conjecture* [14, 15] was found to be violated by the ultra-spinning black hole, leading the authors to impose more stringent conditions under which the bound might be valid — the *Superentropic* black holes with non-compact horizons had to be excluded from the conjecture.

In this paper, we seek to determine the uniqueness of this latter discovery. A curious feature of the ultra spinning spacetime is that it is seemingly isolated from regularly-spinning black holes by any physical process. It is interesting therefore to ponder whether it truly is a special case, or whether this violation is present in further extensions of this solution. One way in which the set of black hole solutions can be extended beyond the usual generalisations to charged and/or rotating solutions is to consider acceleration. The solution that describes the accelerated black hole is known as the C-metric [21–27]. It is similar in form to Kerr-AdS, but has conical defect(s) along the polar axes that are different in magnitude, the differential deficit providing a nett force on the black hole, hence acceleration.

To probe superentropicity in this setting, we will re-visit the ultraspinning limit of Kerr-AdS black holes discussed above. We show that the same *Superentropic* black holes are obtained by running a conical deficit through the Kerr-AdS spacetime,¹ and making

¹In a physical picture, one might think about such a conical deficit as being caused by a cosmic string threading the black hole, though it is not entirely clear that this is a complete story for rotating black holes [28, 29].

this conical deficit maximal, equal to 2π . This provides an alternative to the ultraspinning limit that is also applicable to (not necessarily spinning) accelerated black holes. While the characteristic feature of the ultra-spinning black hole is the pair of maximal deficits at each pole, the accelerated solution has by default one deficit greater than the other, which means that we may only have one such maximal defect. Further, because the conical defects are present *a priori*, it is possible to maximise one simply by choosing a suitable values of the mass parameter, independent of whether the black hole is charged or rotating [30, 31]. The term “ultra-spinning” is therefore no longer appropriate to designate these special solutions, so we will use the term *critical* (for lack of an original word) to designate any black hole solution which exhibits a single (or a pair of) 2π -conical deficit(s).

The thermodynamic properties of black holes in AdS have been known for a while [7, 32–36], however the thermodynamics of accelerating black holes have only more recently been elucidated [37–44], and in the context of thermodynamics, we observe these accelerated critical solutions to behave differently to the original ultra-spinning black hole. For the ultra-spinning case, the thermodynamic quantities cannot simply be obtained by taking the $a \rightarrow \ell$ limit of the thermodynamic quantities of Kerr-AdS black holes, as these diverge in the limit $a \rightarrow \ell$. Instead, the thermodynamics of ultra-spinning black holes were constructed in [2, 19, 20] “afresh”, starting from the *Superentropic* metric and applying the standard procedures, such as the method of conformal completion [45]. In this way, a new set of consistent (and finite) thermodynamic quantities, that are evidently disconnected from those of Kerr-AdS black holes, were obtained and shown to satisfy the corresponding (degenerate) first law, and violate the reverse isoperimetric inequality. In contrast, here we find that when accelerated black holes are critical, their thermodynamic quantities can be obtained as a smooth (and finite) limit of the original thermodynamic quantities for the accelerated black holes [37, 38]. It then follows that the reverse isoperimetric conjecture, shown to be valid for the accelerated black holes [42], remains true also for the critical black holes.

The discrepancy of the two results is astonishing. We attribute it to the two inter-related basic facts. First, a crucial step towards establishing thermodynamics of rotating black holes is to correctly identify a possible rotation at infinity Ω_∞ [7, 32]. This affects the conjugate quantity to the angular momentum, and in its turn also modifies the thermodynamic mass. In the case of ultra-spinning black holes it is very difficult to identify Ω_∞ as this formally diverges; a particular choice of $\Omega_\infty = 0$ was made in [2, 19, 20]. Second, due to the ultraspinning limit $a \rightarrow \ell$, the (would be) mass or enthalpy M_S is directly dependent on the (would be) angular momentum charge J_S , the two obeying the ‘chirality condition’ $J_S = M_S \ell$ that clearly interrelates M_S , J_S , and P [1]. The corresponding first law is thus degenerate and the thermodynamic quantities in it are no longer uniquely defined. To illustrate this point, in section 4 we construct a different set of thermodynamic quantities for ultraspinning black holes, that are also consistent and can be derived from the standard methods (with different Ω_∞), but do not violate the reverse isoperimetric inequality. Together with the results on the thermodynamics of the critical black holes this raises an interesting question: are the original *Superentropic* black holes truly superentropic?

In the next section we review the accelerating black hole geometry, focussing on the slowly accelerating black hole [24], discuss the corresponding admissible parameter space,

conical deficits, and thermodynamics. In section 3, we construct the critical black holes and confirm that they obey the reverse isoperimetric inequality. In section 4 we compare the obtained accelerating results to those of ultra-spinning black holes; a novel ‘derivation’ of the ultra-spinning thermodynamics is presented and subjected to critical comments. We summarise in section 5, and discuss the example of superentropic thermodynamics in the BTZ black hole.

2 Accelerated black holes

2.1 The generalized C-metric

The geometry of an accelerating black hole is given by the ‘‘C-metric’’ (so called because of a classification scheme of axisymmetric metrics [21]) that describes a local black hole type of horizon, distorted by conical deficits that provide the accelerating force acting on the black hole [22, 23]. In anti de Sitter (AdS) spacetime, where $\ell = \sqrt{|\Lambda|/3}$ is the AdS lengthscale, the metric can be written in the following form [46–48]:

$$\begin{aligned}
 ds^2 = & \frac{f(r)}{\Sigma H^2} \left[\frac{dt}{\alpha} - a \sin^2 \theta \frac{d\phi}{K} \right]^2 - \frac{\Sigma dr^2}{f(r)H^2} - \frac{\Sigma r^2}{g(\theta)H^2} d\theta^2 \\
 & - \frac{g(\theta) \sin^2 \theta}{\Sigma r^2 H^2} \left[\frac{adt}{\alpha} - (r^2 + a^2) \frac{d\phi}{K} \right]^2,
 \end{aligned} \tag{2.1}$$

where the metric functions are

$$\begin{aligned}
 f(r) &= (1 - A^2 r^2) \left[1 - \frac{2m}{r} + \frac{a^2 + e^2}{r^2} \right] + \frac{r^2 + a^2}{\ell^2}, \\
 g(\theta) &= 1 + 2mA \cos \theta + (\Xi - 1) \cos^2 \theta, \\
 \Sigma &= 1 + \frac{a^2}{r^2} \cos^2 \theta, \quad H = 1 + Ar \cos \theta, \\
 \Xi &= 1 + e^2 A^2 - \frac{a^2}{\ell^2} (1 - A^2 \ell^2),
 \end{aligned} \tag{2.2}$$

and the electromagnetic potential is given by

$$B = -\frac{e}{\Sigma r} \left[\frac{dt}{\alpha} - a \sin^2 \theta \frac{d\phi}{K} \right] + \Phi_t dt, \quad \Phi_t = \frac{er_+}{\alpha(a^2 + r_+^2)}. \tag{2.3}$$

The parameters a , e , m , and $A \geq 0$ are related to the angular momentum, charge, mass and acceleration of the black hole, respectively. It is worth commenting on a few aspects of this geometry before turning to the features we will be exploring in the next section.

Note the presence of the parameter K associated with the ϕ coordinate. In (2.1), the range of the angular parameter ϕ is taken to be 2π , thus the parameter K will encode in part the conical deficits along each axis. Next, note that the time coordinate has been rescaled by α . It might seem therefore that a new parameter has been introduced, however, because the time coordinate is non-compact, the rescaling by α represents a gauge degree of freedom: time is usually chosen relative to an asymptotic observer, which for the

accelerating black hole is not entirely straightforward to define. In [41], using holographic renormalization, this was found to be

$$\alpha = \frac{\sqrt{(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}}{1 + a^2A^2}. \tag{2.4}$$

The conformal factor, H , sets the location of the boundary at $r_{\text{bd}} = -1/A \cos \theta$, that lies “beyond infinity” for $\theta < \pi/2$. The coordinates in (2.1) therefore do not cover the full spacetime (though can easily be extended — see subsection 2.2), but are nonetheless useful coordinates as they intuitively extend the familiar Kerr metric to include acceleration. Finally, note that usually a uniformly accelerating observer has an acceleration horizon, however, if $A\ell < \mathcal{O}(1)$ (again see section 2.2), the function f is positive outside the black hole event horizon, suggesting that there is no acceleration horizon and the black hole is simply suspended in AdS at a finite displacement from the centre. This is known as a *slowly accelerating* black hole [24], and will be the focus of our study, although the actual bound on $A\ell$ is slightly modified to account for the lack of an acceleration horizon beyond $r = \infty$ as we describe below. We now turn to this, and other parametric restrictions before discussing the conical deficit structure and the critical limit.

2.2 Coordinate ranges and parametric restrictions

To explore what restrictions might apply to the parameters in this metric, we must translate the physical requirements for the slowly rotating black hole into statements about the functions $f(r)$ and $g(\theta)$ that then give constraints on the parameters in the metric. That we are dealing with a black hole means that we have a zero for $f(r)$ that corresponds to $2m$ in the limit that $\ell \rightarrow \infty$, $e, a, A \rightarrow 0$, and lies entirely inside the AdS bulk. That the black hole lacks an acceleration horizon means that there is no other relevant zero of f . Finally, that θ corresponds to the angular coordinate on the (deformed) 2-sphere requires that $g(\theta) \geq 0$ on $[0, \pi]$.

The constraint that there is a black hole horizon corresponds to the existence of an r_+ such that $f(r_+) = 0$, with $f'(r_+) \geq 0$, and that this horizon lies fully within the spacetime. The former requirement is relevant in the case of a charged or rotating black hole, and corresponds to the black hole being sub-extremal, or extremal if $f'(r_+) = 0$. The latter requirement translates to $Ar_+ < 1$; as otherwise it would be possible for $1/Ar_+ = -\cos \theta_+$ for some θ_+ , hence the event horizon would reach the boundary.

To explore these constraints, for convenience set the scale of the dimensionful parameters using the acceleration:

$$\tilde{r} = Ar, \quad \tilde{m} = Am, \quad \tilde{e} = Ae, \quad \tilde{a} = Aa, \quad \tilde{\ell} = A\ell. \tag{2.5}$$

We can now solve the extremality constraint $f(r_+) = f'(r_+) = 0$ leading to constraints on the mass and cosmological constant (i.e. ℓ) expressed in terms of the charge and angular momentum (or vice versa). These can conveniently be parametrised in terms of horizon radius:

$$\tilde{m} = \frac{(\tilde{r}_+^2 + \tilde{a}^2)^2 + \tilde{e}^2(\tilde{a}^2 - \tilde{r}_+^4 + 2\tilde{r}_+^2)}{\tilde{r}_+ (\tilde{a}^2(1 + \tilde{r}_+^2) + \tilde{r}_+^2(2 - \tilde{r}_+^2))}, \quad \tilde{\ell}^2 = \frac{\tilde{r}_+^2(\tilde{r}_+^2 - \tilde{a}^2\tilde{r}_+^2 - 3\tilde{r}_+^2 - \tilde{a}^2)}{(1 - \tilde{r}_+^2)^2(\tilde{r}_+^2 - \tilde{a}^2 - \tilde{e}^2)}. \tag{2.6}$$

In order to explore the constraint from slow acceleration, note that outside the black hole horizon $f(r)$ is positive, but while r is a familiar coordinate for describing the properties of the black hole, it does not cover the full spacetime, instead $y = -1/Ar$, running from $-1/Ar_+$ on the horizon to $\cos\theta$ on the boundary proves to be a better coordinate. The region of spacetime beyond $r = \infty$ is now covered by positive values of y , and the lack of an acceleration horizon in this region corresponds to $F(y) > 0$, where

$$F(y) = \tilde{\ell}^2 y^2 f(-1/Ay) = 1 + \tilde{a}^2 y^4 - \tilde{\ell}^2 (1 - y^2) (1 + 2\tilde{m}y + (\Xi - 1)y^2). \quad (2.7)$$

F has a minimum on $[0, 1]$, so the borderline case as the acceleration horizon forms is $F(y_0) = F'(y_0) = 0$, giving

$$\tilde{m} = y_0 \frac{(1 + \tilde{a}^2 y_0^2)^2 - \tilde{e}^2 (1 - 2y_0^2 - \tilde{a}^2 y_0^4)}{1 - y_0^2 (3 + \tilde{a}^2 (1 + y_0^2))}, \quad \tilde{\ell}^2 = \frac{1 - 3y_0^2 - \tilde{a}^2 y_0^2 (1 + y_0^2)}{(1 - y_0^2)^2 (1 - y_0^2 (\tilde{a}^2 + \tilde{e}^2))}. \quad (2.8)$$

Finally, the constraint that $g(\theta) \geq 0$ on $[0, \pi]$, i.e.

$$1 + 2\tilde{m}x + (\Xi - 1)x^2 \geq 0 \quad \text{for } x \in [-1, 1] \quad (2.9)$$

translates to

$$\tilde{m} \leq \begin{cases} \Xi/2 & \Xi \leq 2 \\ \sqrt{\Xi - 1} & \Xi > 2. \end{cases} \quad (2.10)$$

However, the requirement that $Ar_+ < 1$ implies that the term in $f(r_+)$ inside square brackets is negative:

$$\tilde{r}_+^2 - 2\tilde{m}\tilde{r}_+ + \tilde{e}^2 + \tilde{a}^2 < 0. \quad (2.11)$$

Clearly this quadratic must have real roots, and this in turn requires that its discriminant be positive:

$$\tilde{m}^2 > \tilde{e}^2 + \tilde{a}^2 = \Xi - 1 + \frac{\tilde{a}^2}{\tilde{\ell}^2} \geq \Xi - 1, \quad (2.12)$$

in clear contradiction with (2.10) for $\Xi \geq 2$. Thus, the constraints arising from the angular coordinate require

$$\Xi < 2 \quad \text{and} \quad \tilde{m} \leq \Xi/2. \quad (2.13)$$

To sum up: the constraint from $g(\theta)$ gives an upper bound on \tilde{m} , the constraint from extremality gives a lower bound on \tilde{m} , and the constraint from slow acceleration gives an upper bound on $\tilde{\ell}$, that is \tilde{m} -dependent.

2.3 The conical defect

The presence of a conical deficit in the spacetime is parametrised (in part) by the parameter K . Whether or not there is acceleration, if $K \neq 1$, the metric will not be flat along at least one of the axes. To see this, expand the angular part of the metric in (2.1) near the poles by setting $\theta = \theta_{\pm} \pm \rho$ (with $\theta_+ = 0$ and $\theta_- = \pi$) near each axis:

$$ds^2 \sim \frac{1}{H^2} \frac{\Sigma r^2}{g(\theta_{\pm})} \left[d\rho^2 + \frac{g^2(\theta_{\pm})\rho^2}{K^2} d\phi^2 \right]. \quad (2.14)$$

The deficit on each axis is then read off as:

$$\delta_{\pm} = 2\pi \left[1 - \frac{g(\theta_{\pm})}{K} \right] = 2\pi \left[1 - \frac{\Xi \pm 2mA}{K} \right]. \quad (2.15)$$

If $A = 0$, both deficits are identical and can be interpreted as a cosmic string through the black hole [49, 50] of tension

$$\mu = \frac{\delta}{8\pi} = \frac{1}{4} \left[1 - \frac{\Xi}{K} \right]. \quad (2.16)$$

If A is nonzero however, then there is an asymmetry in the spacetime, with differing deficits at north and south poles:

$$\mu_{\pm} = \frac{1}{4} \left[1 - \frac{\Xi \pm 2\tilde{m}}{K} \right], \quad (2.17)$$

that produces a nett force on the black hole, hence acceleration.

It is now evident that if we choose K to obtain a particular value of the conical deficit on one axis, that choice of K has a global impact: A then regulates the distribution of tensions between the axes. It is also worth mentioning that although a negative deficit (otherwise known as an excess) is possible, it would be sourced by a negative energy object and hence in general associated with instabilities (though see [51–54]). We therefore restrict ourselves to positive energy sources, thus (taking $A > 0$ without loss of generality) $K \geq \Xi + 2\tilde{m}$. In most of the literature on accelerating black holes, the deficit along one axis (here, the north) is chosen to vanish, i.e. $K = \Xi + 2\tilde{m}$. However, we will not make this restriction here, unless stated explicitly.

2.4 Thermodynamics of accelerated black holes

The properties of slowly accelerating black holes have been studied in recent years and our understanding of their thermodynamics has greatly improved over time [37–40]. The full thermodynamics for the general accelerating black hole is given by the extended first law [38]:

$$\delta M = T\delta S + \Phi\delta Q + \Omega\delta J + V\delta P + \lambda_+\delta\mu_+ + \lambda_-\delta\mu_-, \quad (2.18)$$

where the enthalpy is

$$M = \frac{m(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}{K\Xi\alpha(1 + a^2A^2)}, \quad (2.19)$$

(with α defined in (2.4)) and the six thermodynamic charges S, Q, J, P, μ_{\pm} together with their corresponding potentials $T, \Phi, \Omega, V, \lambda_{\pm}$ are given in terms of the six black hole pa-

rameters A, a, m, e, ℓ, K as [41]

$$\begin{aligned}
 T &= \frac{f'_+ r_+^2}{4\pi\alpha(r_+^2 + a^2)}, \quad S = \frac{\pi(r_+^2 + a^2)}{K(1 - A^2 r_+^2)}, \\
 J &= \frac{ma}{K^2}, \quad \Omega = \Omega_H - \Omega_\infty = \left(\frac{Ka}{\alpha(r_+^2 + a^2)} \right) - \left(-\frac{aK(1 - A^2 \ell^2 \Xi)}{\ell^2 \Xi \alpha(1 + a^2 A^2)} \right), \\
 Q &= \frac{e}{K}, \quad \Phi = \Phi_t = \frac{er_+}{(r_+^2 + a^2)\alpha}, \\
 P &= \frac{3}{8\pi\ell^2}, \quad V = \frac{4\pi}{3K\alpha} \left[\frac{r_+(r_+^2 + a^2)}{(1 - A^2 r_+^2)^2} + \frac{m[a^2 + A^2 \ell^4 \Xi^2]}{(1 + a^2 A^2)\Xi} \right], \\
 \lambda_\pm &= \frac{-r_+}{\alpha(1 \pm Ar_+)} + \frac{m}{\alpha} \frac{[\Xi + \frac{a^2}{\ell^2}(2 - A^2 \ell^2 \Xi)]}{(1 + a^2 A^2)\Xi^2} \pm \frac{A\ell^2(\Xi + a^2/\ell^2)}{\alpha(1 + a^2 A^2)}.
 \end{aligned} \tag{2.20}$$

These charges also satisfy a Smarr relation [55]

$$M = 2(TS + \Omega J - PV) + \Phi Q. \tag{2.21}$$

A description of how the potentials were obtained, using both conformal and holographic techniques is given in Anabalon et al. [41].

Despite the fact that the tensions μ_\pm are natural variables, and indeed correspond to physical objects (cosmic strings emerging from the event horizon [50, 56]), expressing the charges and potentials in terms of extensive variables [42] reveals that the thermodynamics is more naturally delineated into an overall and differential conical deficit, Δ and C respectively:

$$\begin{aligned}
 \Delta &= 1 - 2(\mu_+ + \mu_-) = \frac{\Xi}{K}, \\
 C &= \frac{\mu_- - \mu_+}{\Delta} = \frac{\tilde{m}}{\Delta K} = \frac{mA}{\Xi}.
 \end{aligned} \tag{2.22}$$

Since the tensions are bounded from below by the positivity of energy, and above by the maximum conical deficit of 2π , we have

$$0 \leq \mu_+ \leq \mu_- \leq 1/4, \tag{2.23}$$

which translates into bounds for C :

$$0 \leq C \leq \min \left\{ \frac{1}{2}, \frac{1 - \Delta}{2\Delta} \right\}. \tag{2.24}$$

The Christodoulou-like formula for the enthalpy then reads [42]

$$M^2 = \frac{\Delta S}{4\pi} \left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right)^2 + \left(1 + \frac{8PS}{3\Delta} \right) \left(\frac{4\pi^2 J^2}{\Delta^2 S^2} - \frac{3C^2 \Delta}{2PS} \right) \right], \tag{2.25}$$

while the other expressions are

$$\begin{aligned}
 V &= \frac{2S^2}{3\pi M} \left[1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} + \frac{2\pi^2 J^2}{(\Delta S)^2} + \frac{9C^2 \Delta^2}{32P^2 S^2} \right], \\
 T &= \frac{\Delta}{8\pi M} \left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right) \left(1 - \frac{\pi Q^2}{\Delta S} + \frac{8PS}{\Delta} \right) - \frac{4\pi^2 J^2}{(\Delta S)^2} - 4C^2 \right], \\
 \Omega &= \frac{\pi J}{SM\Delta} \left(1 + \frac{8PS}{3\Delta} \right), \\
 \Phi &= \frac{Q}{2M} \left(1 + \frac{\pi Q^2}{S\Delta} + \frac{8PS}{3\Delta} \right), \\
 \lambda_{\pm} &= \frac{S}{4\pi M} \left[\left(\frac{8PS}{3\Delta} + \frac{\pi Q^2}{\Delta S} \right)^2 + \frac{4\pi^2 J^2}{(\Delta S)^2} \left(1 + \frac{16PS}{3\Delta} \right) - (1 \mp 2C)^2 \pm \frac{3C\Delta}{2PS} \right],
 \end{aligned} \tag{2.26}$$

or considering the conjugates to Δ and C instead,

$$\begin{aligned}
 \lambda_{\Delta} &= -\frac{S}{8\pi M} \left[\left(\frac{8PS}{3\Delta} + \frac{\pi Q^2}{\Delta S} \right)^2 + \frac{4\pi^2 J^2}{(\Delta S)^2} \left(1 + \frac{16PS}{3\Delta} \right) - 1 + C^2 \left(4 - \frac{3\Delta}{PS} \right) \right], \\
 \lambda_C &= -\frac{\Delta CS}{\pi M} \left[1 + \frac{3\Delta}{4PS} \right].
 \end{aligned} \tag{2.27}$$

These expressions, (2.25), (2.26), and (2.27) are most useful for exploring the general thermodynamical properties of the black holes, however, we will refer to the parametric expressions (2.20) when discussing the ultraspinning black hole.

2.5 Reverse Isoperimetric Inequality

The fact that the thermodynamic quantities of the accelerated black holes obey the *Reverse Isoperimetric Inequality* [14] (roughly, a statement that black holes like to be round) has been shown in [42]. Let us repeat here the corresponding argument.

Squaring the formula (2.26) for V , we have

$$\left(\frac{3V}{4\pi} \right)^2 \left(\frac{\pi}{S} \right)^3 = \frac{S}{4\pi} \frac{1}{M^2} \left[1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} + \frac{2\pi^2 J^2}{(\Delta S)^2} + \frac{9C^2 \Delta^2}{32P^2 S^2} \right]^2. \tag{2.28}$$

Thence, upon using the Christodoulou formula (2.25) to eliminate M , this yields

$$\begin{aligned}
 \Delta \left(\frac{3V}{4\pi} \right)^2 \left(\frac{\pi}{S} \right)^3 &= \frac{\left[1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} + \frac{2\pi^2 J^2}{(\Delta S)^2} + \frac{9C^2 \Delta^2}{32P^2 S^2} \right]^2}{\left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right)^2 + \left(1 + \frac{8PS}{3\Delta} \right) \left(\frac{4\pi^2 J^2}{\Delta^2 S^2} - \frac{3C^2 \Delta}{2PS} \right) \right]} \\
 &\geq \frac{\left[1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} + \frac{2\pi^2 J^2}{(\Delta S)^2} \right]^2}{\left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right)^2 + 2 \left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right) \frac{2\pi^2 J^2}{\Delta^2 S^2} \right]} \geq 1.
 \end{aligned} \tag{2.29}$$

We have thus verified the refined Reverse Isoperimetric Inequality [42]

$$\left(\frac{V}{V_0} \right)^2 \geq \frac{1}{\Delta} \left(\frac{A}{A_0} \right)^3, \tag{2.30}$$

where V_0 and A_0 are the volume and area of a unit ball, $V_0 = \frac{4}{3}\pi$ and $A_0 = 4\pi$, and the inequality is saturated if and only if $C = 0 = J$.

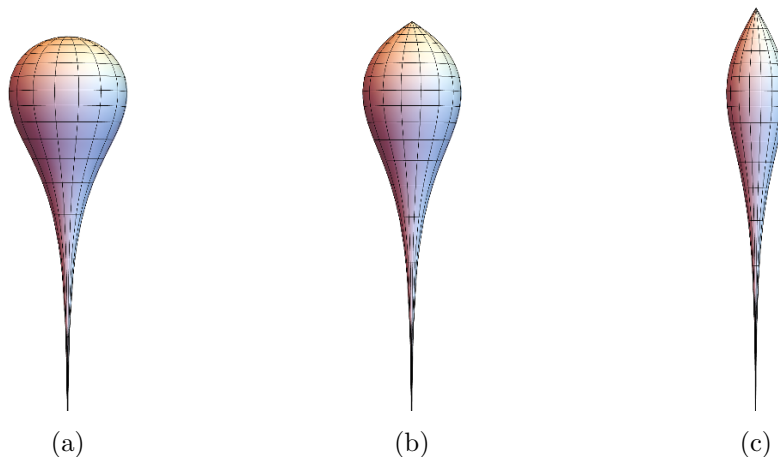


Figure 1. Horizon embedding of critical black holes. We display the horizon embeddings in \mathbb{R}^3 of the critical C-metrics for $m = 9\ell$, $A = 0.04\ell$ and (a) $K = 2\Xi$ (b) $K = 2.4\Xi$ (c) $K = 4\Xi$.

3 Critical black holes and their thermodynamics

3.1 Critical limit

Having discussed the slowly accelerating C-metric, and the parametric restrictions that this geometry requires, we now turn to the critical black holes we are interested in exploring.

The term *critical* is used to describe a geometry in which at least one of the tensions has its maximal value of $1/4$, i.e., where the deficit becomes 2π as in the ultra-spinning black hole. For the ultraspinning Kerr-AdS black hole, this corresponds to saturating an upper bound on rotation, however, in our accelerating black hole metric, the deficit along one axis can become 2π , even in the absence of rotation, e.g. for $mA = 1/2$ in the ‘black bottles’ of [30, 31]. We can therefore think of criticality as saturation of an upper bound for the mass parameter \tilde{m} ,

$$mA = \Xi/2. \tag{3.1}$$

With this choice, the south pole axis is effectively removed from the spacetime, while the north pole axis may still have a conical deficit, determined by the ratio of K/Ξ , see figure 1, where the embedding of the event horizon of a critical black hole is displayed for various such ratios.

Since in the process of taking the critical limit, only one parameter is eliminated, by imposing (3.1), we have a three-parameter family of critical accelerating black holes, parametrised by $\tilde{e} = eA$, $\tilde{a} = aA$, and $\tilde{\ell} = A\ell$, with the mass given by (3.1). Once again, these parameters are constrained by $g(\theta)$, and the slow-acceleration/extremal limits for the

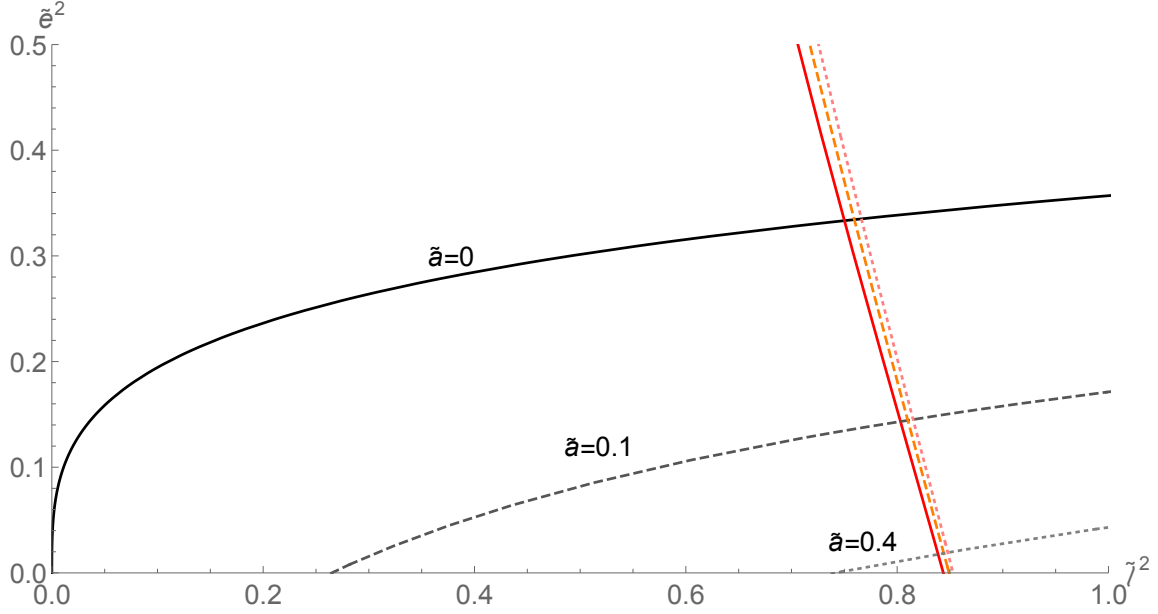


Figure 2. The allowed values of $\tilde{\ell}$ and \tilde{e} : the upper bound for \tilde{e} from extremality is shown in black/grey, and the upper bound for $\tilde{\ell}$ from the slow acceleration limit is shown in red/pink for sample values of \tilde{a} as labelled. The upper bound for \tilde{a} is $\tilde{a}^2 = 3 - 2\sqrt{2}$.

black hole:

$$\begin{aligned}
 \text{extremal limit} & \begin{cases} \tilde{\ell}_{\text{ext}}^2 = \frac{\tilde{a}^2 + 3\tilde{a}^2\tilde{r}_+ + 4\tilde{r}_+^3 + \tilde{r}_+^4 - \tilde{r}_+^5}{(1 - \tilde{r}_+)^3(1 + \tilde{r}_+)^2}, \\ \tilde{e}_{\text{ext}}^2 = \frac{-\tilde{a}^4 - 3\tilde{a}^4\tilde{r}_+ + 2\tilde{a}^2\tilde{r}_+^2 - 2\tilde{a}^2\tilde{r}_+^3 + 3\tilde{r}_+^4 + \tilde{r}_+^5}{\tilde{a}^2 + 3\tilde{a}^2\tilde{r}_+ + 4\tilde{r}_+^3 + \tilde{r}_+^4 - \tilde{r}_+^5}, \end{cases} \\
 \text{slow acc. limit} & \begin{cases} \tilde{\ell}_{\text{acc}}^2 = \frac{1 + y_+ - 4y_+^2 - 3\tilde{a}^2y_+^4 + \tilde{a}^2y_+^5}{(1 - y_+)^2(1 + y_+)^3}, \\ \tilde{e}_{\text{acc}}^2 = \frac{-1 + 3y_+ + 2\tilde{a}^2y_+^2 + 2\tilde{a}^2y_+^3 + 3\tilde{a}^4y_+^4 - \tilde{a}^4y_+^5}{1 + y_+ - 4y_+^2 - 3\tilde{a}^2y_+^4 + \tilde{a}^2y_+^5}. \end{cases}
 \end{aligned} \tag{3.2}$$

See figure 2 for a plot of parameter space. Note, the constraint from $g(\theta)$ is automatically (marginally) satisfied due to the choice of \tilde{m} .

3.2 Thermodynamics and absence of superentropy

The above constructed critical black holes ($\mu_- = 1/4$) were simply obtained by setting $2mA = \Xi$, that is,

$$C = \frac{1}{2}, \quad \Delta = \frac{1}{2} - 2\mu_+. \tag{3.3}$$

The associated allowed range for Δ is $\Delta \in [0, 1/2]$, with the lower (upper) bound corresponding to the upper (lower) value that μ_+ can take; thus $\mu_+ = 0 \iff \Delta = 1/2$, and $\mu_+ \rightarrow 1/4 \iff \Delta \rightarrow 0$.

The criticality condition (3.3) is hard to impose at the parametric level, (2.20), but very simple for the expressions (2.25), (2.26), and (2.27). The limit is smooth and simply

yields

$$\begin{aligned}
 M^2 &= \frac{\Delta S}{4\pi} \left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right)^2 + \left(1 + \frac{8PS}{3\Delta} \right) \left(\frac{4\pi^2 J^2}{\Delta^2 S^2} - \frac{3\Delta}{8PS} \right) \right], \\
 V &= \frac{2S^2}{3\pi M} \left[1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} + \frac{2\pi^2 J^2}{(\Delta S)^2} + \frac{9\Delta^2}{128P^2 S^2} \right], \\
 T &= \frac{\Delta}{8\pi M} \left[\left(1 + \frac{\pi Q^2}{\Delta S} + \frac{8PS}{3\Delta} \right) \left(1 - \frac{\pi Q^2}{\Delta S} + \frac{8PS}{\Delta} \right) - \frac{4\pi^2 J^2}{(\Delta S)^2} - 1 \right], \\
 \Omega &= \frac{\pi J}{SM\Delta} \left(1 + \frac{8PS}{3\Delta} \right), \\
 \Phi &= \frac{Q}{2M} \left(1 + \frac{\pi Q^2}{S\Delta} + \frac{8PS}{3\Delta} \right), \\
 \lambda_\Delta &= -\frac{S}{8\pi M} \left[\left(\frac{8PS}{3\Delta} + \frac{\pi Q^2}{\Delta S} \right)^2 + \frac{4\pi^2 J^2}{(\Delta S)^2} \left(1 + \frac{16PS}{3\Delta} \right) - \frac{3\Delta}{4PS} \right].
 \end{aligned} \tag{3.4}$$

These quantities obey the full cohomogeneity first law,

$$\delta M = T\delta S + \Phi\delta Q + \Omega\delta J + V\delta P + \lambda_\Delta\delta\Delta, \tag{3.5}$$

together with the corresponding Smarr relation (2.21).

Of course, the proof of the reverse isoperimetric inequality (2.30) presented above for accelerated black holes goes through exactly the same way for their critical subfamily and this is despite the fact that the horizon of critical black holes is non-compact (as is the horizon of ultraspinning black holes). Note also that since $C = 1/2$ the inequality can no longer be saturated and is a strict inequality.

Also, note that by taking another limit, $\Delta \rightarrow 0$, one formally obtains a critical black hole with maximal conical deficits on both poles, that in fact is the superspinning black hole. However, it is obvious from the expressions (3.4) above, that the thermodynamic quantities such as mass M , and angular velocity Ω diverge in this limit. Therefore, either one accepts that the thermodynamics is ill-defined in this limit, or one looks for new (renormalised) thermodynamic parameters.

4 Comparison to ultraspinning black holes

4.1 The superentropic argument

Let us now compare the critical limit to the ultraspinning limit of the Kerr-AdS spacetime. The charged Kerr-AdS black holes are obtained by setting $A = 0$ in the metric (2.1). For simplicity, and without loss of generality for the purposes of this discussion, we will also take the uncharged limit $e = 0$, so that the metric becomes

$$ds^2 = \frac{f(r)}{\Sigma} \left[dt - a \sin^2 \theta \frac{d\phi}{K} \right]^2 - \frac{\Sigma dr^2}{f(r)} - \frac{\Sigma r^2 d\theta^2}{g(\theta)} - \frac{g(\theta) \sin^2 \theta}{\Sigma r^2} \left[a dt - (r^2 + a^2) \frac{d\phi}{K} \right]^2, \tag{4.1}$$

where we now have

$$\begin{aligned}
 f(r) &= \left[1 - \frac{2m}{r} + \frac{a^2}{r^2} \right] + \frac{r^2 + a^2}{\ell^2}, & g(\theta) &= 1 + (\Xi - 1) \cos^2 \theta, \\
 \Sigma &= 1 + \frac{a^2}{r^2} \cos^2 \theta, & \Xi &= 1 - \frac{a^2}{\ell^2}.
 \end{aligned}
 \tag{4.2}$$

In the previous literature, one sets $K \equiv \Xi$, so that there is no conical deficit in the spacetime. The thermodynamics of these black holes was worked out definitively in [7, 32], the key insight being that the boundary has a non-zero angular velocity,

$$\Omega_\infty = \lim_{r \rightarrow \infty} -\frac{g_{t\phi}}{g_{\phi\phi}} = -\frac{aK}{\ell^2 \Xi},
 \tag{4.3}$$

implying that the total angular velocity ought to be re-normalised, $\Omega = \Omega_H - \Omega_\infty$. Further, a computation of the mass of the spacetime, using an appropriately normalised Killing vector, $\partial_t - \Omega_\infty \partial_\phi$, yielded $M = m/\Xi^2$ for the enthalpy. These results are entirely consistent with (2.19), (2.20), once one sets $K = \Xi$. Crucially, when considering a varying Λ , the inclusion of these normalisations for enthalpy and angular velocity leads to an enthalpy dependent correction term in the thermodynamic volume:

$$V = V_0 + V_1 = \frac{4\pi r_+(r_+^2 + a^2)}{3K} + \frac{4\pi M a^2}{3}.
 \tag{4.4}$$

The ultra-spinning limit is obtained by taking the limit in which $a \rightarrow \ell$ ($\Xi \rightarrow 0$), but because of the identification of K with Ξ , this results in an apparently singular metric. This was resolved in [1, 2] by defining a new angular coordinate, $\psi = \phi/\Xi$, so that ψ formally becomes noncompact in the ultra-spinning limit. This new angular coordinate is then given a finite range, $\Delta\psi = \mu_S$. Since $g(\theta) \rightarrow \sin^2\theta$ the limit yields the *Superentropic* black hole

$$\begin{aligned}
 ds^2 &= \frac{f(r)}{\Sigma} \left[dt - \ell \sin^2 \theta d\psi \right]^2 - \frac{\Sigma dr^2}{f(r)} - \frac{\Sigma r^2 d\theta^2}{\sin^2 \theta} - \frac{\sin^4 \theta}{\Sigma r^2} \left[\ell dt - (r^2 + \ell^2) d\psi \right]^2, \\
 f(r) &= \frac{\ell^2}{r^2} \left(1 + \frac{r^2}{\ell^2} \right)^2 - \frac{2m}{r}, & \Sigma &= 1 + \frac{\ell^2}{r^2} \cos^2 \theta,
 \end{aligned}
 \tag{4.5}$$

which was assigned the following thermodynamic parameters [1, 2]:

$$\begin{aligned}
 M_S &= \frac{\mu_S m}{2\pi}, & S_S &= \frac{\mu_S}{2} (r_+^2 + \ell^2), & T_S &= \frac{f'(r_+) r_+^2}{4\pi (r_+^2 + \ell^2)}, & J_S &= M_S \ell, \\
 \Omega_S &= \frac{\ell}{r_+^2 + \ell^2}, & V_S &= \frac{2\mu_S r_+}{3} (r_+^2 + \ell^2), & \lambda_S &= \frac{m (\ell^2 - r_+^2)}{4\pi (r_+^2 + \ell^2)},
 \end{aligned}
 \tag{4.6}$$

where the subscript S is used to denote these specific ‘superentropic’ definitions, and we have relabelled the thermodynamic length parameter, denoted K in [2] as λ_S , dual to a variation of the parameter μ_S ,

$$\delta M_S = T_S \delta S_S + \Omega_S \delta J_S + V_S \delta P + \lambda_S \delta \mu_S.
 \tag{4.7}$$

The first law is obviously degenerate as only 3 parameters, $\{r_+, \ell, \mu_S\}$, can be varied independently; the mass M_S and angular momentum J_S charges obey the ‘chirality condition’ $J_S = M_S \ell$.

Note that Ω_S is simply the angular velocity, defined by $\omega = -g_{t\psi}/g_{\psi\psi}$, evaluated on the horizon, $\Omega_S = \omega(r_+)$, while the corresponding quantity at infinity diverges. That is, in [2] the authors set formally $\Omega_\infty = 0$, there is no renormalisation of angular velocity, nor of the timelike Killing vector, and in consequence, there is no adjustment of the enthalpy ‘ M ’, nor a correction to the thermodynamic volume. As a result, the volume is simply the geometric volume, thus the standard mathematical *Isoperimetric* inequality applies, and the entropy is now minimised by the contained volume. This fascinating result has caused some puzzlement, as the thermodynamic parameters (4.6) are not obtained as an “ $a \rightarrow \ell$ ” limit of the conventional parameters, (2.20), nor does it seem possible to obtain one of these black holes by some sort of continuous process. In addition, the idea that the entropy can be unbounded for a fixed volume suggests that superentropic black holes should be somehow unstable, a notion explored (in a different context) by Johnson [57], see also [58, 59]. Thus the *Superentropic* black hole is worthy of further study.

One of the problems of the thermodynamic parameters of [2] is that setting $a \equiv \ell$ means that the angular momentum and thermodynamic pressure are no longer independent variables. In other words, the first law no longer has full cohomogeneity. Further, the discrete alteration of the periodicity of the angular coordinate is equivalent to a sudden shift of the conical deficit from 0 to 2π , as one is setting $K = \Xi$ for the sub-rotating black holes (giving $\mu = 0$) but for $a = \ell$, the periodicity of the original ϕ coordinate, set to $\mu\Xi$ by Hennigar et al. [2], now vanishes. However, since we have a set of thermodynamic variables that include potential variations in the conical deficit, we can now examine this superentropic ultra-spinning limit afresh, and try to understand what lies behind this phenomenon.

4.2 Kerr-AdS with conical deficits

Let us return to the general metric (4.1), retaining the parameter K , and re-examine the thermodynamics of the ultraspinning Kerr in the light of allowing for conical deficits. First, note that in the limit $a \rightarrow \ell$, $g(\theta) \rightarrow \sin^2 \theta$, thus defining a new angular coordinate

$$\Theta = \log \tan \frac{\theta}{2}, \tag{4.8}$$

the angular part of our metric in terms of Θ and ϕ is manifestly non-compact. The parameter K however now has no apparent physical meaning, as the deficit along the axis, defined by the tension (2.16) becomes maximal:

$$\delta = 8\pi\mu = 2\pi \left[1 - \frac{\Xi}{K} \right] \rightarrow 2\pi. \tag{4.9}$$

However, guided by the discussion in [2], define

$$\mu_S = \frac{2\pi}{K} = \frac{2\pi}{\Xi} (1 - 4\mu), \tag{4.10}$$

that will play the role of a “spectator tension”.

We now re-derive the thermodynamics of the superspinning black hole by taking a continuous limit of the generic, fully cohomogeneous, variables given in (2.20) by approaching the limit $a \rightarrow \ell$ from a more continuous perspective, taking a family of black holes with $a/\ell = \sqrt{1 - \Xi}$ fixed, then taking the limit $\Xi \rightarrow 0$.

Imposing the constraint that Ξ is a constant means that J and P are no longer independent thermodynamic variables ($\delta a = a\delta\ell/\ell$), thus keeping a first law with variations of both angular momentum and pressure is a bit disingenuous, and such a first law no longer has full cohomogeneity. Instead, the variation of the angular momentum yields contributions to the pressure variation, as well as terms that contribute to the enthalpy.

$$\delta J = a\delta\left(\frac{m}{K^2}\right) + \frac{m}{K^2}\delta a = a\delta\left(\frac{m}{K^2}\right) - \frac{4\pi\ell^2}{3}\frac{ma}{K^2}\delta P. \quad (4.11)$$

Therefore, in deriving the superentropic variables (4.6), Hennigar et al. have effectively used this equivalence between the variation of J and P to “re-organise” terms in the thermodynamic potentials, and (roughly) the angular momentum subtraction at infinity term cancels off the compensating thermodynamic volume term to yield an uncorrected V , hence a standard Isoperimetric inequality.

To track through the play-off between the various terms, start with the first law

$$\delta M = T\delta S + \Omega\delta J + V\delta P + 2\lambda\delta\mu, \quad (4.12)$$

with the thermodynamic variables pertinent to the discussion being:

$$\begin{aligned} M &= \frac{m}{K\Xi}, \quad J = \frac{ma}{K^2}, \quad \Omega = \Omega_H - \Omega_\infty = \frac{Ka}{(r_+^2 + a^2)} + \frac{aK}{\ell^2\Xi}, \\ V &= V_0 + V_1 = \frac{4\pi r_+(r_+^2 + a^2)}{3K} + \frac{4\pi ma^2}{3K\Xi}, \quad \lambda = -r_+ + \frac{m}{\Xi^2}\left(1 + \frac{a^2}{\ell^2}\right). \end{aligned} \quad (4.13)$$

Now, using the fact that Ξ is a constant, and noting the definition of μ_S above, (4.10), we see that

$$\delta M = \frac{1}{\Xi}\delta\left(\frac{m}{K}\right) = \frac{1}{\Xi}\delta\left(\frac{\mu_S m}{2\pi}\right) = \frac{1}{\Xi}\delta M_S, \quad (4.14)$$

where M_S is the superentropic variable in (4.6), but now defined at *finite* Ξ . We can also relate the variation of tension to the rescaled ‘spectator’ tension μ_S :

$$\delta\mu = -\frac{\Xi}{8\pi}\delta\mu_S. \quad (4.15)$$

Finally, looking at the angular momentum, and using (4.10), (4.11), (4.14) we see that $J = aM_S/K$, and we can write the variation of J in two useful alternate forms:

$$\begin{aligned} \delta J &= \frac{a}{K}\delta M_S + \frac{am}{K}\frac{\delta\mu_S}{2\pi} - \frac{4\pi\ell^2}{3}\frac{ma}{K^2}\delta P \\ &= \frac{1}{K}\delta(aM_S) + aM_S\frac{\delta\mu_S}{2\pi}, \end{aligned} \quad (4.16)$$

thus using the first expression in $\Omega_\infty\delta J$, and the second in $\Omega_H\delta J$, we see that

$$\Omega\delta J = \frac{a^2}{\ell^2\Xi}\delta M_S + \frac{a}{r_+^2 + a^2}\delta(aM_S) + \left[\frac{am}{r_+^2 + a^2} + \frac{a^2m}{\ell^2\Xi}\right]\frac{\delta\mu_S}{2\pi} - \frac{4\pi}{3}\frac{ma^2}{K\Xi}\delta P. \quad (4.17)$$

Note, the last piece in the above expression is in fact $-V_1\delta P$, so we now see how the compensating term in the thermodynamic volume that maintains the reverse isoperimetric inequality is cancelled.

Putting together all these pieces, we see our first law becomes

$$\begin{aligned} \delta M_S &= T \delta S + \frac{a}{r_+^2 + a^2} \delta(aM_S) + V_0 \delta P + \left[\frac{am}{r_+^2 + a^2} + \frac{a^2 m}{\ell^2 \Xi} - \frac{\lambda \Xi}{2} \right] \frac{\delta \mu_S}{2\pi} \\ &= T \delta S + \frac{a}{r_+^2 + a^2} \delta(aM_S) + \frac{2\mu_S r_+}{3} (r_+^2 + a^2) \delta P + \left[r_+ \Xi - m \frac{r_+^2 - a^2}{r_+^2 + a^2} \right] \frac{\delta \mu_S}{4\pi}. \end{aligned} \quad (4.18)$$

We see a clear parallel with the thermodynamic variables of (4.6), indeed, defining a new angular momentum charge, potential, and thermodynamic length

$$J_S = aM_S, \quad \Omega_S = \frac{a}{r_+^2 + a^2}, \quad \lambda_S = \frac{1}{4\pi} \left(r_+ \Xi - m \frac{r_+^2 - a^2}{r_+^2 + a^2} \right), \quad (4.19)$$

which are identical to those in (4.6) when $a = \ell$, we do indeed confirm the consistency of the Hennigar et al. first law, (4.7), but now for *finite* Ξ . We stress that although consistent for any finite Ξ , such a first law is not the correct one — it uses the wrong thermodynamic quantities, e.g., Ω_S lacks the contribution from rotation at infinity. Note also that after the limit $a \rightarrow \ell$, the spectator tension should really not be varied (the parameter K is no longer physical) and the term $\lambda_S \delta \mu_S$ in (4.7) should be omitted.

However, studying the system at finite Ξ reveals something interesting that was missed in [2]. Conventionally, consistency of thermodynamics has been used as the principle upon which to define the thermodynamic parameters, and provided there is full cohomogeneity this seems to be correct. However, once there is *not* full cohomogeneity, one must be more disciplined in deriving a consistent first law, and allow for general variations in the definition of variables. Note that

$$\delta J_S = \delta(aM_S) = a\delta M_S - \frac{4\pi}{3} a\ell^2 M_S \delta P. \quad (4.20)$$

Thus, if we return to our angular velocity subtraction, and notionally define²

$$\Omega_\infty = -\frac{\alpha}{a}, \quad (4.21)$$

then

$$(\Omega_S - \Omega_\infty) \delta J_S = \Omega_S \delta J_S + \alpha \delta M_S - \frac{4\pi}{3K} m a^2 \frac{\alpha}{1 - \Xi}, \quad (4.22)$$

that is, our first law (4.7) is also consistent (for any fixed a/ℓ) for the one-parameter family of variables

$$M_S^\alpha = (1 + \alpha)M_S, \quad V_S^\alpha = \frac{2\mu_S r_+}{3} (r_+^2 + a^2) + \frac{4\pi}{3K} \frac{\alpha m a^2}{1 - \Xi}, \quad \Omega_S^\alpha = \frac{a}{r_+^2 + \ell^2} + \frac{\alpha}{a}. \quad (4.23)$$

Thus there is a one parameter freedom that can be thought of as the ‘choice of angular velocity at infinity’. The actual computation yields infinite Ω_∞ so one can think of the

²This form may be motivated by the expression (4.3), by identifying $\alpha = K(1 - \Xi)/\Xi$, which is a ‘constant’ once the $a \rightarrow \ell$ limit is taken and K becomes a redundant parameter.

above value as some sort of renormalization. One can easily check that for $\alpha < 1/2$ the reverse isoperimetric inequality is violated, whereas it becomes satisfied for $\alpha > 1/2$.

Note that similar to what was done in [2], the thermodynamic quantities (4.23) (in the limit $a \rightarrow \ell$) can be directly obtained from the *Superentropic* metric (4.5) by using the conformal method [45]. Namely, denoting $Q(\xi)$ a conformal charge corresponding to the Killing vector ξ , we find

$$Q(\partial_t) = M_S, \quad Q(\partial_\phi) = J_S, \tag{4.24}$$

and therefore

$$M_S^\alpha = Q(\partial_t - \Omega_\infty \partial_\phi) = (1 + \alpha)M_S. \tag{4.25}$$

To summarize, simply demanding consistency does not lead to a unique thermodynamics for the constrained Kerr-AdS black hole, for which $a = \sqrt{1 - \Xi}\ell$. The procedure is ambiguous in that there is (at least) a 1-parameter family of consistent thermodynamic quantities some of which do satisfy the isoperimetric inequality and some of which do not. The origin of this ambiguity is the degeneracy of the first law, which is no longer of full cohomogeneity and thence no longer fixes the thermodynamic quantities uniquely. We demonstrated this explicitly by working down from the full expressions (2.20), taking $a = \sqrt{1 - \Xi}\ell$, and defining a set of consistent parameters that can be seen to reduce to the Hennigar et al. expressions in the limit that both Ξ and α tend to zero.

5 Conclusions

In this paper we have discussed the general parameter space for slowly accelerating black holes and defined a *critical limit* in which at least one of the conical deficits becomes maximal at $\delta = 2\pi$. We then discussed thermodynamics of these critical black holes, in particular focussing on the Reverse Isoperimetric Inequality, reviewing a proof of the inequality and confirming that it holds in the critical limit, which is now smoothly connected to non-critical black holes. This is manifestly distinct from the argument for *Superentropic Black Holes* [2] therefore we have revisited this particular solution and critically examined the arguments in the literature.

We presented two possible alternate continuous ways of taking the superspinning limit, one by fixing a/ℓ and allowing it to tend to unity; the second taking an accelerating black hole, fixing one deficit to its maximal value of 2π and allowing the other deficit to approach 2π . In each case, the fully co-homogeneous thermodynamic parameters M , V and Ω diverge in the superspinning limit, thus in order to have finite charges a renormalisation prescription is required. Using the degeneracy of the thermodynamic variables that results from fixing a/ℓ , we showed how the first law can be reorganised, with a redefinition of the thermodynamic charges that results in a one parameter family. This new degree of freedom in turn raises doubts on the correctness of the superentropic thermodynamics and gives an alternate argument in favour of non-superentropic thermodynamics.

It remains to be shown whether similar doubts could arise also for other *Superentropic* black holes, for example the recently studied charged BTZ black holes [57–60] have an

apparently similar issue. The charged BTZ black hole is a three dimensional electrically charged solution of the Einstein-Maxwell equations [61]

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\varphi^2, \quad F = \frac{Q}{r} dt \wedge dr, \quad (5.1)$$

where Q is the black hole charge, $f(r)$ satisfies

$$f' = \frac{2r}{\ell^2} - \frac{8\pi G Q^2}{r}, \quad (5.2)$$

and ℓ is the AdS radius, defined by $\Lambda = -1/\ell^2$. In [61], the Newton's constant is fixed by setting $16\pi G = 1$, however, we will temporarily retain this dimensionful parameter, writing $8\pi G = L_p = M_p^{-1}$ the 3D Planck length, in order to emphasise the source of superentropy.

Integrating (5.2) yields the potential

$$f(r) = \frac{r^2}{\ell^2} - L_p Q^2 \log\left(\frac{r}{r_0}\right) - 2m, \quad (5.3)$$

where m is an integration constant we identify as the mass parameter, and r_0 is a dimensional integration parameter inserted to render the argument of log dimensionless. This is in part the reason for maintaining the dimensionful parameter G , if we set $8\pi G = 1$, or $1/2$ as in the original paper [61], then r becomes dimensionless, and we need not add any scale inside the logarithm, however, in keeping with convention, we introduce the scale r_0 as a second integration constant.

The extended thermodynamics of this black hole has been studied in [60, 62], and is shown to crucially depend on the choice of the integration parameter r_0 . The conventional choice in the literature is to set $r_0 = \ell$, however, in extended thermodynamics this has an important consequence: ℓ is related to the thermodynamic pressure, so varying P has the consequence of varying the integration constant. Imposing this value of r_0 leads to the thermodynamic variables

$$\begin{aligned} M &= \frac{m}{4} = \frac{r_+^2}{8\ell^2} - \frac{Q^2}{16} \log\left(\frac{r_+}{\ell}\right), & T &= \frac{r_+}{2\pi\ell^2} - \frac{Q^2}{8\pi r_+}, & S &= \frac{\pi r_+}{2}, \\ V &= \pi r_+^2 - \frac{1}{4} Q^2 \pi \ell^2, & \Phi &= -\frac{1}{8} Q \log\left(\frac{r_+}{\ell}\right), & P &= \frac{1}{8\pi\ell^2}, \end{aligned} \quad (5.4)$$

that obey the standard 1st law and Smarr relations:

$$\delta M = T\delta S + \Phi\delta Q + V\delta P, \quad TS = 2PV, \quad (5.5)$$

where we have set $L_p = 1/2$ to align with the literature [61].

Note the non-geometric correction to the black hole volume V , originating from the aforementioned variation of the integration constant. This is precisely the term that implies the violation of the reverse isoperimetric inequality. Although preferred in [60], this option is questionable from various perspectives. First, the potential Φ in (5.4) is that of the electrostatic potential evaluated on the horizon, however this is not gauge invariant; usually

one takes the potential difference between the horizon and infinity as a gauge invariant thermodynamic potential, however this is problematic in 3D as the potential at infinity obviously diverges. Secondly, the introduction of r_0 can be viewed as part of a renormalisation procedure, and indeed is discussed as such in [61]. If a cutoff is introduced, then one would expect that this cutoff would remain fixed as one is varying *physical* parameters in a thermodynamic process. This perspective leads to an alternative formulation of thermodynamics, where we identify r_0 in (5.1) as ‘enclosing’ the BTZ black hole in a circle of radius r_0 as in [61, 63]. Upon this, the potential at ‘infinity’ ($r = r_0$) vanishes and we obtain the following thermodynamic quantities [60]:

$$\begin{aligned}
 M &= \frac{m}{4} = \frac{r_+^2}{8\ell^2} - \frac{Q^2}{16} \log\left(\frac{r_+}{r_0}\right), & T &= \frac{r_+}{2\pi\ell^2} - \frac{Q^2}{8\pi r_+}, & S &= \frac{\pi r_+}{2}, \\
 \Phi &= -\frac{1}{8}Q \log\left(\frac{r_+}{r_0}\right), & V &= \pi r_+^2, & P &= \frac{1}{8\pi\ell^2},
 \end{aligned}
 \tag{5.6}$$

together with the circumference $C = 2\pi r_0$ and a dual thermodynamic potential $\mu_C = Q^2/16C$ if one wishes to vary the physical enclosure around the black hole. With these the first law (5.5) remains satisfied, but the Smarr relation now picks up a $C\mu_C$ term due to the scaling properties of r_0 . The thermodynamics (5.6) is obviously non-superentropic.

We therefore suspect that the traditional thermodynamics of this somewhat pathological solution is most likely not the correct one. Finally, in an interesting recent twist, the rotating (uncharged) BTZ black hole, which as a traditional Einstein solution is not superentropic, can be reimagined as a solution of the gravitational Chern-Simons action [58, 64]. The thermodynamic parameters become “exotic” (with mass and angular momentum charges reversed and the entropy no longer given by the horizon area). This it seems may also violate the reverse isoperimetric inequality, however since the *Reverse Isoperimetric Inequality* conjecture of [14] was originally put forward for Einstein gravity, this can not be regarded as a counter-example. What is abundantly clear however is that the existence and origin of superentropy most certainly deserves further investigation.

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