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Complete set of commuting symmetry operators for the Klein-Gordon equation in generalized higher-dimensional Kerr-NUT-(A)dS spacetimes

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We consider the Klein-Gordon equation in generalized higher-dimensional Kerr-NUT-(A)dS spacetime without imposing any restrictions on the functional parameters characterizing the metric. We establish commutativity of the second-order operators constructed from the Killing tensors found in [J. High Energy Phys. 02 (2007) 004] and show that these operators, along with the first-order operators originating from the Killing vectors, form a complete set of commuting symmetry operators (i.e., integrals of motion) for the Klein-Gordon equation. Moreover, we demonstrate that the separated solutions of the Klein-Gordon equation obtained in [J. High Energy Phys. 02 (2007) 005] are joint eigenfunctions for all of these operators. We also present an explicit form of the zero mode for the Klein-Gordon equation with zero mass. In the semiclassical approximation we find that the separated solutions of the Hamilton-Jacobi equation for geodesic motion are also solutions for a set of Hamilton-Jacobi-type equations which correspond to the quadratic conserved quantities arising from the above Killing tensors.

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I. INTRODUCTION

Investigation of the properties of higher-dimensional black-hole spacetimes has recently attracted considerable attention, in particular, in connection with the string theory. The metrics describing black holes of increasing generality were found in [1–5]. The most general metric of this kind known so far corresponds to a higher-dimensional generally rotating (however neither charged nor accelerated) black hole with the NUT parameters and arbitrary cosmological constant. This metric was found by Chen, Lü, and Pope [6] in the form which generalizes Carter's four-dimensional Kerr-NUT-(anti-)de Sitter metric [7,8].

The spacetime with the metric from [6] has a lot of interesting properties. In D dimensions it possesses explicit and hidden symmetries encoded in the series of $n = [D/2]$ rank-two Killing tensors and $D - n$ Killing vectors. The former ones can be constructed from the so-called principal Killing-Yano tensor [9], and in fact the spacetime in question is the only one admitting a rank-two closed conformal Killing-Yano tensor with certain further properties [10]. The symmetries allow one to define a complete set of D quantities conserved along geodesics. These quantities are linear and quadratic in canonical momenta. Moreover, they are functionally independent and in involution [11,12] and thus their existence guarantees complete integrability of the geodesic motion.

The existence of such integrals of motion is intimately related to separability of the Hamilton-Jacobi and Klein-Gordon equations. In [13] it was shown that the presence of these integrals yields the so-called separability structure. The latter guarantees separability of the Hamilton-Jacobi equation and, for the Einstein spaces, also separability of the Klein-Gordon equation. Separability of the latter equation in the spacetime under study was explicitly demonstrated in [14].

In the present paper we discuss operator counterparts of the conserved quantities constructed from the Killing vectors and rank-two Killing tensors. Namely, we convert the integrals of motion into operators using the rule $\boldsymbol{p} \to -i\alpha\boldsymbol{\nabla}$ and employing the symmetric ordering of derivatives, and we demonstrate that all these operators commute. Since one of these operators is, up to an overall constant factor, the Klein-Gordon operator itself, we thus obtain symmetry operators for the Klein-Gordon equation in the sense of [15,16]. Moreover, we show that the separated solutions of the Klein-Gordon equation found in [14] are joint eigenfunctions of all symmetry operators with eigenvalues corresponding to the separation constants. As a byproduct, we obtain a zero mode solution (30) for the Klein-Gordon equation with zero mass.

We further demonstrate that semiclassical approximations of the eigenvalue equations yield a set of Hamilton-Jacobi-type equations. The latter can be solved using the separation of variables in the same fashion as in [14].

It is worth noticing that all these properties actually hold for a broader class of spacetimes than just the black-hole

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spacetimes of [6]. These properties depend on the algebraic structure of the metric (1) rather than on the explicit form of metric functions X_μ. For this reason in what follows we do not require our metric to satisfy the vacuum Einstein equation.

II. PRELIMINARIES

Consider the D-dimensional spacetime with the metric

$$\begin{aligned} \boldsymbol{g} &= \sum_{\mu=1}^{n}\Big[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu{}^2 + \frac{X_\mu}{U_\mu}\Big(\sum_{j=0}^{n-1}A_\mu^{(j)}\boldsymbol{d}\psi_j\Big)^2\Big] \\ &\quad + \varepsilon\frac{c}{A^{(n)}}\Big(\sum_{k=0}^{n}A^{(k)}\boldsymbol{d}\psi_k\Big)^2 \qquad (1)\\ &= \sum_{\mu=1}^{n}\Big(\frac{U_\mu}{X_\mu}\boldsymbol{\epsilon}^\mu\boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu}\boldsymbol{\epsilon}^{\hat\mu}\boldsymbol{\epsilon}^{\hat\mu}\Big) + \varepsilon\frac{c}{A^{(n)}}\boldsymbol{\epsilon}^{\hat 0}\boldsymbol{\epsilon}^{\hat 0}. \end{aligned}$$

Here $n = [D/2]$, $\varepsilon = D - 2n$, and c is an arbitrary constant; x_μ, $\mu = 1, \ldots, n$ correspond to radial and latitudinal directions while ψ_k, $k = 0, \ldots, n + \varepsilon - 1$ correspond to temporal and longitudinal directions. The radial coordinate and some other related quantities are actually rescaled by the imaginary unit i in order to bring the metric into a more symmetric and compact form, cf. e.g. [6]. The signature of the metric depends on the signs of the metric functions; for the physically relevant ranges of coordinates it is $(- + \ldots +)$.

We use Latin indices from the beginning of the alphabet to label the whole coordinate set: $\{x^a\} = \{x_\mu, \psi_k\}$, where $a = 1, \ldots, D$, $\mu = 1, \ldots, n$, and $k = 0, \ldots, n + \varepsilon - 1$. The non-normalized one-forms $\{\boldsymbol{\epsilon}_\mu, \boldsymbol{\epsilon}_{\hat\mu}\}$ that diagonalize the metric, and the dual vector frame $\{\boldsymbol{\epsilon}^\mu, \boldsymbol{\epsilon}^{\hat\mu}\}$ have the form

$$\begin{aligned} &\boldsymbol{\epsilon}^\mu = \boldsymbol{d}x_\mu, \qquad \boldsymbol{\epsilon}^{\hat\mu} = \sum_{j=0}^{n-1}A_\mu^{(j)}\boldsymbol{d}\psi_j, \\ &\boldsymbol{\epsilon}^{\hat 0} = \sum_{k=0}^{n}A^{(k)}\boldsymbol{d}\psi_k, \qquad \boldsymbol{\epsilon}_\mu = \boldsymbol{\partial}_{x_\mu}, \qquad (2)\\ &\boldsymbol{\epsilon}_{\hat\mu} = \sum_{k=0}^{n+\varepsilon-1}\frac{(-x_\mu^2)^{n-1-k}}{U_\mu}\boldsymbol{\partial}_{\psi_k}, \qquad \boldsymbol{\epsilon}_{\hat 0} = \frac{1}{A^{(n)}}\boldsymbol{\partial}_{\psi_n}. \end{aligned}$$

The quantities $\boldsymbol{\epsilon}^{\hat 0}$ and $\boldsymbol{\epsilon}_{\hat 0}$ are defined only for odd $D = 2n + 1$. By $\boldsymbol{\partial}_{x_\mu}$ and $\boldsymbol{\partial}_{\psi_k}$ we denote the coordinate vectors.

The functions U_μ, $A_\mu^{(k)}$, U, and $A^{(k)}$ used below are defined as follows:

$$\begin{aligned} &A_\mu^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k,\,\nu_i\neq\mu}}^{n} x_{\nu_1}^2\cdots x_{\nu_k}^2, \qquad A^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k}}^{n} x_{\nu_1}^2\cdots x_{\nu_k}^2, \\ &U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2), \qquad U = \prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{n}(x_\mu^2 - x_\nu^2) = \det A_\mu^{(j)}. \end{aligned} \qquad (3)$$

These functions satisfy the following important relations (see e.g. [14])

$$\begin{aligned} &\sum_{\mu=1}^{n}A_\mu^{(i)}\frac{(-x_\mu^2)^{n-1-j}}{U_\mu} = \delta_j^i, \qquad \sum_{j=0}^{n-1}A_\mu^{(j)}\frac{(-x_\nu^2)^{n-1-j}}{U_\nu} = \delta_\mu^\nu, \\ &\sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{x_\mu^2 U_\mu} = \frac{A^{(j)}}{A^{(n)}}, \qquad \sum_{j=0}^{n}A^{(j)}(-x_\mu^2)^{n-1-j} = 0, \end{aligned} \qquad (4)$$

for $i, j = 0, \ldots, n - 1$ and $\mu, \nu = 1, \ldots, n$. The determinant of the metric in the coordinates $\{x_\mu, \psi_k\}$ reads

$$\mathfrak{g} = \det g_{ab} = (cA^{(n)})^\varepsilon U^2. \qquad (5)$$

In the above definitions we did not specify the explicit form of the functions X_μ. In what follows we just assume that $X_\mu = X_\mu(x_\mu)$, $\mu = 1, \ldots, n$, i.e., each X_μ is a function of a single variable x_μ. In general, the metric (1) then does not satisfy the vacuum Einstein equations. We would recover the vacuum black-hole spacetime [6,17] by setting

$$X_\mu = b_\mu x_\mu^{1-\varepsilon} - \varepsilon\frac{c}{x_\mu^2} + \sum_{k=\varepsilon}^{n}c_k x_\mu^{2k}. \qquad (6)$$

The constants c, c_k, and b_μ are then related to the cosmological constant, angular momenta, mass, and NUT charges.

The spacetime with the metric (1) possesses explicit symmetries given by the Killing vectors

$$\boldsymbol{L}_{(k)} = \boldsymbol{\partial}_{\psi_k}, \qquad k = 0, \ldots, n + \varepsilon - 1, \qquad (7)$$

and hidden symmetries that can be generated using the principal Killing-Yano tensor [9,18]. Namely, it was shown in [9] that (1) admits a series of rank-two Killing tensors $\boldsymbol{K}_{(j)}$, $j = 0, \ldots, n - 1$ which are diagonal in the frame (2) and read

$$\begin{aligned} \boldsymbol{K}_{(j)} &= \sum_{\mu=1}^{n}A_\mu^{(j)}\Big(\frac{X_\mu}{U_\mu}\boldsymbol{\epsilon}_\mu\boldsymbol{\epsilon}_\mu + \frac{U_\mu}{X_\mu}\boldsymbol{\epsilon}_{\hat\mu}\boldsymbol{\epsilon}_{\hat\mu}\Big) \\ &\quad + \varepsilon A^{(j)}\frac{A^{(n)}}{c}\boldsymbol{\epsilon}_{\hat 0}\boldsymbol{\epsilon}_{\hat 0}. \end{aligned} \qquad (8)$$

In particular, for $j = 0$ we have $\boldsymbol{K}_{(0)} = \boldsymbol{g}^{-1}$, where $\boldsymbol{g}^{-1}$ is the contravariant metric. We use the Killing tensors with contravariant indices as they are more convenient for the construction of conserved quantities and symmetry operators used below.

Let $\mathcal{M}$ be our spacetime with the metric (1) and $\mathcal{T}^*\mathcal{M}$ be the corresponding cotangent bundle. The latter is naturally endowed with the canonical Poisson bracket. Let $\boldsymbol{p}$ be the one-form of canonical momenta, so that the components p_a of $\boldsymbol{p}$ are canonically conjugate to x^a, $a = 1, \ldots, D$. Then the above Killing vectors and tensors generate conserved quantities for the geodesic motion on $\mathcal{T}^*\mathcal{M}$. The conserved quantities in question are linear and quadratic in $\boldsymbol{p}$ and read

$$l_{(k)} = \boldsymbol{L}_{(k)} \cdot \boldsymbol{p}, \qquad k = 0, \ldots, n + \varepsilon - 1, \\ k_{(j)} = \boldsymbol{p} \cdot \boldsymbol{K}_{(j)} \cdot \boldsymbol{p}, \qquad j = 0, \ldots, n - 1. \tag{9}$$

Here and below $\cdot$ denotes contraction.

It was proved in [12] that these quantities are functionally independent and in involution with respect to the canonical Poisson bracket, and hence the geodesic motion on $\mathcal{T}^*\mathcal{M}$ is completely integrable.

Recall that the geodesic motion in this context is generated by the Hamiltonian $H = \frac{1}{2}k_{(0)} = \frac{1}{2}\boldsymbol{p} \cdot \boldsymbol{g}^{-1} \cdot \boldsymbol{p}$, and the fact that $l_{(k)}$ and $k_{(j)}$ are conserved quantities (i.e., integrals) for the geodesic motion means that the $l_{(k)}$ and $k_{(j)}$ Poisson commute with H.

III. SYMMETRY OPERATORS AND SEPARABILITY FOR THE KLEIN-GORDON EQUATION

It is natural to study the operators obtained from the above conserved quantities using the heuristic rule $\boldsymbol{p} \to -i\alpha\boldsymbol{\nabla}$, where $\boldsymbol{\nabla}$ is the usual covariant derivative with respect to the metric $\boldsymbol{g}$. Upon fixing the symmetric operator ordering for the second-order operators we define

$$\mathcal{L}_{(k)} = -i\alpha \boldsymbol{L}_{(k)} \cdot \boldsymbol{\nabla}, \qquad \mathcal{K}_{(j)} = -\alpha^2 \boldsymbol{\nabla} \cdot [\boldsymbol{K}_{(j)} \cdot \boldsymbol{\nabla}]. \tag{10}$$

We employ here the convention that the square brackets do not prevent the action of the derivatives to the right. Here α is a constant giving a scale to be used in order to obtain a semiclassical (geometric-optical) approximation in the next section. Of course, in the quantum context α would be nothing but the Planck constant $\hbar$. Writing out the above operators in the coordinates $\{x_\mu, \psi_k\}$ we obtain

$$\mathcal{L}_{(k)} = -i\alpha \frac{\partial}{\partial \psi_k}, \qquad k = 0, \ldots, n + \varepsilon - 1, \tag{11}$$

$$\begin{aligned} \mathcal{K}_{(j)} = -\alpha^2 \sum_{\mu=1}^{n} \frac{A_\mu^{(j)}}{U_\mu} \bigg[& \frac{\partial}{\partial x_\mu}\bigg[X_\mu \frac{\partial}{\partial x_\mu}\bigg] + \varepsilon \frac{X_\mu}{x_\mu}\frac{\partial}{\partial x_\mu} \\ & + \frac{1}{X_\mu}\bigg[\sum_{k=0}^{n+\varepsilon-1} (-x_\mu^2)^{n-1-k} \frac{\partial}{\partial \psi_k}\bigg]^2\bigg] \\ & - \varepsilon \frac{\alpha^2 A^{(j)}}{cA^{(n)}}\bigg[\frac{\partial}{\partial \psi_n}\bigg]^2, \qquad j = 0, \ldots, n-1. \end{aligned} \tag{12}$$

Here we used the fact that $\mathcal{K}_{(j)}$ from (10) can be written as $\mathcal{K}_{(j)} = -\alpha^2 \mathfrak{g}^{-1/2}\partial_a[\mathfrak{g}^{1/2}K_{(j)}^{ab}\partial_b]$, Eq. (5), and a trivial identity $\frac{\partial}{\partial x_\mu}(A_\mu^{(j)} U/U_\mu) = 0$.

In [9] it was shown that the "quasiclassical limits" $l_{(k)}$ and $k_{(j)}$ of $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$ are in involution, i.e., they Poisson commute. However, the argument of [9] does not directly imply the commutativity of the corresponding "quantum" operators, i.e., of $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$.

Our goal is to show that the operators $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$ commute for all $k = 0, \ldots, n + \varepsilon - 1$ and $j = 0, \ldots, n - 1$.

The commutators of $\mathcal{L}_{(k)}$ among themselves and of $\mathcal{L}_{(k)}$ with $\mathcal{K}_{(j)}$ obviously vanish, i.e.,

$$[\mathcal{L}_{(k)}, \mathcal{L}_{(l)}] = 0, \qquad [\mathcal{L}_{(k)}, \mathcal{K}_{(j)}] = 0 \tag{13}$$

for all $k, l = 0, \ldots, n + \varepsilon - 1$ and $j = 0, \ldots, n - 1$, and we only have to prove that

$$[\mathcal{K}_{(i)}, \mathcal{K}_{(j)}] = 0 \tag{14}$$

for all $i, j = 0, \ldots, n - 1$.

To this end we first observe that the operators $\mathcal{K}_{(j)}$ can be written as

$$\mathcal{K}_{(j)} = \sum_{\mu=1}^{n} \frac{A_\mu^{(j)}}{U_\mu} \tilde{\mathcal{K}}_{(\mu)}, \tag{15}$$

where

$$\begin{aligned} \tilde{\mathcal{K}}_{(\mu)} = -\alpha^2 \bigg[& \frac{\partial}{\partial x_\mu}\bigg[X_\mu \frac{\partial}{\partial x_\mu}\bigg] + \frac{\varepsilon X_\mu}{x_\mu}\frac{\partial}{\partial x_\mu} \\ & + \frac{\varepsilon}{cx_\mu^2}\bigg[\frac{\partial}{\partial \psi_n}\bigg]^2 + \frac{1}{X_\mu}\bigg[\sum_{k=0}^{n+\varepsilon-1} (-x_\mu^2)^{n-1-k} \frac{\partial}{\partial \psi_k}\bigg]^2\bigg]. \end{aligned} \tag{16}$$

The operators (16) enjoy a remarkable property: for any given μ the operator $\tilde{\mathcal{K}}_{(\mu)}$ involves only $\partial/\partial x_\mu$ and x_μ but does not involve $\partial/\partial x_\nu$ and x_ν for $\nu \neq \mu$. Therefore $\tilde{\mathcal{K}}_{(\mu)}$ commute, i.e., we have

$$[\tilde{\mathcal{K}}_{(\mu)}, \tilde{\mathcal{K}}_{(\nu)}] = 0 \tag{17}$$

for all $\mu, \nu = 1, \ldots, n$.

Using the identities (4) we find that

$$\tilde{\mathcal{K}}_{(\mu)} = \sum_{k=0}^{n-1} (-x_\mu^2)^{n-1-k} \mathcal{K}_{(k)}, \tag{18}$$

and hence the commutativity of $\tilde{\mathcal{K}}_{(\mu)}$ entails that of $\mathcal{K}_{(j)}$.

Indeed, consider (17) for $\mu \neq \nu$ and rewrite it as

$$\tilde{\mathcal{K}}_{(\mu)}\tilde{\mathcal{K}}_{(\nu)} = \tilde{\mathcal{K}}_{(\nu)}\tilde{\mathcal{K}}_{(\mu)}. \tag{19}$$

Using (18) for $\tilde{\mathcal{K}}_{(\nu)}$ on the left-hand side of (19) and for $\tilde{\mathcal{K}}_{(\mu)}$ on the right-hand side of (19) and employing a trivial identity $[\tilde{\mathcal{K}}_{(\mu)}, (-x_\nu^2)^{n-1-l}] = 0$ valid for $\mu \neq \nu$ yields

$$\sum_{l=0}^{n-1} (-x_\nu^2)^{n-1-l} \tilde{\mathcal{K}}_{(\mu)} \mathcal{K}_{(l)} = \sum_{k=0}^{n-1} (-x_\mu^2)^{n-1-k} \tilde{\mathcal{K}}_{(\nu)} \mathcal{K}_{(k)}. \tag{20}$$

Again using (18) we see that (20) boils down to

$$\sum_{k,l=0}^{n-1}(-x_\mu^2)^{n-1-k}(-x_\nu^2)^{n-1-l}[\mathcal{K}_{(k)},\mathcal{K}_{(l)}]=0, \tag{21}$$

whence by nonsingularity of the matrix $B_\mu^j=(-x_\mu^2)^{n-1-j}$ we readily obtain (14). It is important to stress that the above reasoning makes substantial use of the fact that the matrix B_μ^j is the Stäckel matrix, i.e., its μth column depends on x_μ alone, cf. e.g. [19].

Notice an important corollary of (14): since the Klein-Gordon equation $\Box\phi-\frac{m^2}{\alpha^2}\phi=0$ can be written as $\mathcal{K}_{(0)}\phi=-m^2\phi$, Eqs. (13) and (14) with $j=0$ imply that $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(i)}$ are *symmetry operators* for the Klein-Gordon equation, see e.g. [15,16] and references therein for the general theory of such operators. Let us mention that for the special case of the Kerr and Kerr-Newman metrics with $D=4$ this was established by Carter [20].

Relations (13) and (14) suggest that we may seek for the joint eigenfunctions ϕ of the operators $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$ with the respective eigenvalues Ψ_k and Ξ_j,

$$\mathcal{L}_{(k)}\phi=\Psi_k\phi, \tag{22}$$

$$\mathcal{K}_{(j)}\phi=\Xi_j\phi. \tag{23}$$

We will now show that these eigenfunctions can be found by the separation of variables, i.e., by assuming that they have the form (see Eq. (4.2) of [14])

$$\phi=\prod_{\mu=1}^{n}R_\mu(x_\mu)\prod_{k=0}^{n+\varepsilon-1}\exp\left(\frac{i}{\alpha}\Psi_k\psi_k\right), \tag{24}$$

with each function $R_\mu(x_\mu)$ depending on a single variable x_μ only.

The functions (24) clearly satisfy Eqs. (22). Using (18) we can combine Eqs. (23) into an equivalent set of equations

$$\left[\tilde{\mathcal{K}}_{(\mu)}-\sum_{j=0}^{n-1}\Xi_j(-x_\mu^2)^{n-1-j}\right]\phi=0. \tag{25}$$

Substituting (16) into (25) we find that (24) solves all of the Eqs. (25)—and therefore Eqs. (23) as well—provided the functions R_μ satisfy the following *ordinary* differential equations:

$$(X_\mu R_\mu')'+\varepsilon\frac{X_\mu}{x_\mu}R_\mu'+\frac{1}{\alpha^2}\left(\tilde{\Xi}_\mu-\frac{\tilde{\Psi}_\mu^2}{X_\mu}\right)R_\mu=0, \tag{26}$$

where

$$\begin{aligned}\tilde{\Psi}_\mu&=\sum_{k=0}^{n+\varepsilon-1}\Psi_k(-x_\mu^2)^{n-1-k},\\ \tilde{\Xi}_\mu&=\sum_{k=0}^{n+\varepsilon-1}\Xi_k(-x_\mu^2)^{n-1-k},\end{aligned} \tag{27}$$

and for odd $D=2n+1$ we set $\Xi_n=\Psi_n^2/c$ for convenience. Upon setting $\alpha=1$ in these equations we recover, of course, the separated ordinary differential equations (4.10) from [14].

Thus, we proved that the functions ϕ of the form (24) with R_μ satisfying (26) are eigenfunctions of the operators $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$. In particular, as $\mathcal{K}_{(0)}=-\alpha^2\Box$ (see above), we recover the result of [14] that ϕ of the form (24) satisfies the Klein-Gordon equation

$$\Box\phi-\frac{m^2}{\alpha^2}\phi=0, \tag{28}$$

where $m^2=-\Xi_0$.

As a final remark note that if $\Psi_k=0$, $k=0,\ldots,n+\varepsilon-1$, and $\Xi_j=0$, $j=0,\ldots,n-1$, then the general solution of (26) is easily found to be

$$R_\mu(x_\mu)=h_\mu+f_\mu\int\frac{dx_\mu}{X_\mu(x_\mu)x_\mu^\varepsilon}, \tag{29}$$

where h_μ and f_μ are arbitrary constants. Therefore,

$$\phi_0=\prod_{\mu=1}^{n}\left(h_\mu+f_\mu\int\frac{dx_\mu}{X_\mu(x_\mu)x_\mu^\varepsilon}\right) \tag{30}$$

is a zero mode, i.e., it satisfies

$$\mathcal{L}_{(j)}\phi_0=0,\qquad \mathcal{K}_{(k)}\phi_0=0 \tag{31}$$

for all $j=0,\ldots,n+\varepsilon-1$ and $k=0,\ldots,n-1$.

IV. HAMILTON-JACOBI EQUATION

Upon taking the solution ϕ of the Klein-Gordon equation in the form

$$\phi=A\exp\left(\frac{i}{\alpha}S\right) \tag{32}$$

we find that in the semiclassical (or geometric-optical, depending on the application in question) approximation the function S satisfies the Hamilton-Jacobi equation

$$\boldsymbol{d}S\cdot\boldsymbol{g}^{-1}\cdot\boldsymbol{d}S+m^2=0, \tag{33}$$

where $m^2=-\Xi_0$. Recall that passing to the semiclassical approximation in our case amounts to plugging the ansatz (32) into Eq. (28) multiplied by α^2 and taking the limit $\alpha\to 0$.

The same approximation leads to the Hamilton-Jacobi-type equations for each of the "wave" equations (23), namely

$$\boldsymbol{d}S\cdot\boldsymbol{K}_{(k)}\cdot\boldsymbol{d}S=\Xi_k. \tag{34}$$

In a similar fashion, the quasiclassical limit of Eqs. (22) yields

$$\boldsymbol{L}_{(k)}\cdot\boldsymbol{d}S=\Psi_k. \tag{35}$$

After the above discussion of the Klein-Gordon equation it is not surprising that all these conditions are satisfied by the separated solution found in [14].

Indeed, the additive separation of variables yields [14] the following ansatz for S:

$$S = \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{n+\varepsilon-1} \Psi_k \psi_k, \tag{36}$$

which automatically guarantees that the conditions (35) are satisfied.

The Hamilton-Jacobi-type equations (34) lead to the first-order differential equations

$$\sum_{\mu=1}^{n} \frac{A_\mu^{(j)}}{U_\mu}\left(X_\mu (S'_\mu)^2 + \frac{\tilde{\Psi}_\mu^2}{X_\mu}\right) + \varepsilon \frac{A^{(j)}}{cA^{(n)}} \Psi_n^2 = \Xi_j. \tag{37}$$

Taking linear combinations of these equations with the coefficients given again by the entries of the matrix $B_\mu^j = (-x_\mu^2)^{n-1-j}$ yields an equivalent set of ordinary differential equations for S_μ's,

$$(S'_\mu)^2 = \frac{\tilde{\Xi}_\mu}{X_\mu} - \frac{\tilde{\Psi}_\mu^2}{X_\mu^2}, \tag{38}$$

where $\tilde{\Psi}_\mu$ and $\tilde{\Xi}_\mu$ are given by (27) and we again set $\Xi_n = \Psi_n^2/c$ for odd $D = 2n + 1$. Equations (38) are precisely the separated first-order ordinary differential equations (3.7) of [14].

By direct inspection we also find that Eqs. (38) yield the semiclassical approximation of the separability conditions (26) for the Klein-Gordon equation, with the functions R_μ related to S_μ as $R_\mu = \exp(\frac{i}{\alpha} S_\mu)$.

Finally, upon identifying the momentum vector $\boldsymbol{p} = \boldsymbol{d}S$ in (34) and (35) we recover the original conserved quantities (9) in terms of the separation constants:

$$l_{(j)} = \Psi_j, \qquad k_{(j)} = \Xi_j. \tag{39}$$

V. CONCLUSIONS AND DISCUSSION

In the present paper we have established that the quantities $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$, see (10)–(12), form a complete set of commuting symmetry operators for the Klein-Gordon equation in the spacetime with the metric (1). The symmetry operators $\mathcal{L}_{(k)}$ are associated with the Killing vectors, and the operators $\mathcal{K}_{(j)}$ with the Killing tensors constructed from the Killing-Yano tensor [9]. We proved the commutativity of the symmetry operators in question for a general class of metrics (1) with each $X_\mu = X_\mu(x_\mu)$ being an arbitrary function of a single variable x_μ, $\mu = 1, \ldots, n$; this class includes higher-dimensional Kerr-NUT-AdS metrics [6] as special cases. We have further shown that the separated solutions (24) of the Klein-Gordon equation found in [14] provide, cf. e.g. [15,19], joint eigenfunctions of $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$, see (22) and (23). We have also analyzed the quasiclassical limit of the above results and compared this limit with the results of [14]. Finally, we found an explicit form of the zero mode (30) for the zero-mass Klein-Gordon equation.

In our opinion, it would be interesting to address similar issues for the Dirac equation in the spacetime with the metric (1), i.e., to find the symmetry operators whose joint eigenfunctions are the separated solutions of the Dirac equation found in [21].

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