Electromagnetic field on the background of the higher-dimensional black holes

plus

Two no-go theorems for generalizations to higher-dimensional Plebański–Demiański metric

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- **Metric of the Kerr–NUT–(A)dS spacetime**
- **Plebański–Demiański metric in $D = 4$**
- **No-go theorem for accelerating the Kerr–NUT–(A)dS metric**
- **Algebraically special test electromagnetic field**
- **No-go theorem for charging the Kerr–NUT–(A)dS metric**

based on the papers:

- Krtouš P.:
  Electromagnetic field on the background of higher-dimensional black holes
  arXiv:0707.0002 [hep-th]
- Kubizňák D., Krtouš P.:
  On conformal Killing–Yano tensors of Plebański–Demiański family of solutions
  arXiv:0706.0409 [gr-qc]

Metric of the Kerr–NUT–(A)dS spacetime

in even dimensions $D = 2n$

$$\boldsymbol{g} \;=\; \sum_{\mu=1}^{n} \left[\, \frac{U_\mu}{X_\mu}\,\boldsymbol{d}x_\mu^2 \;+\; \frac{X_\mu}{U_\mu}\Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\boldsymbol{d}\psi_k\Big)^2 \,\right]$$

$X_\mu = X_\mu(x_\mu)$	metric functions to be determined by the Einstein equations
x_μ	radial and latitudinal coordinates $(\mu = 1, \dots, n)$
ψ_k	time and longitudinal coordinates $(k = 0, \dots, n{-}1)$ – symmetries of the spacetime

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k,\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2)$$

- Myers R. C., Perry M. J.: *Black Holes in Higher Dimensional Space-Times*, Ann. Phys. 172, 304 (1986)
- Gibbons G. W., L H., Page D. N., Pope C. N.: *Rotating Black Holes in Higher Dimensions with a Cosmological Constant*, Phys. Rev. Lett. 93, 171102 (2004), arXiv:hep-th/0409155
- Chen W., L H., Pope C. N.: *General Kerr-NUT-AdS Metrics in All Dimensions*, Class. Quant. Grav. 23, 5323 (2006), arXiv:hep-th/0604125

Orthogonal form of the metric

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left(\frac{U_\mu}{X_\mu} \boldsymbol{\epsilon}^\mu \boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu} \boldsymbol{\epsilon}^{\hat{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}} \right) = \sum_{\mu=1}^{n} \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}} \right)$$

$$\boldsymbol{\epsilon}^\mu = \boldsymbol{d}x_\mu \qquad \boldsymbol{\epsilon}_\mu = \boldsymbol{\partial}_{x_\mu}$$

$$\boldsymbol{\epsilon}^{\hat{\mu}} = \sum_{k=0}^{n-1} A_\mu^{(k)} \boldsymbol{d}\psi_k \qquad \boldsymbol{\epsilon}_{\hat{\mu}} = \frac{1}{U_\mu} \sum_{k=0}^{m} (-x_\mu^2)^{n-1-k} \boldsymbol{\partial}_{\psi_k}$$

$$\boldsymbol{e}^\mu = \left(\tfrac{U_\mu}{X_\mu}\right)^{1/2} \boldsymbol{\epsilon}^\mu \qquad \boldsymbol{e}_\mu = \left(\tfrac{X_\mu}{U_\mu}\right)^{1/2} \boldsymbol{\epsilon}_\mu$$

$$\boldsymbol{e}^{\hat{\mu}} = \left(\tfrac{X_\mu}{U_\mu}\right)^{1/2} \boldsymbol{\epsilon}^{\hat{\mu}} \qquad \boldsymbol{e}_{\hat{\mu}} = \left(\tfrac{U_\mu}{X_\mu}\right)^{1/2} \boldsymbol{\epsilon}_{\hat{\mu}}$$

$$\mu = 1, \dots, n \qquad \hat{\mu} = \mu + n$$

Curvature

$$\mathbf{Ric} = -\sum_{\mu=1}^{n} r_\mu \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}\right) \qquad \text{Ricci tensor}$$

$$r_\mu = \frac{1}{2}\frac{X''_\mu}{U_\mu} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu}\frac{x_\nu X'_\nu - x_\mu X'_\mu}{x_\nu^2 - x_\mu^2} - \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \frac{1}{U_\nu}\frac{X_\nu - X_\mu}{x_\nu^2 - x_\mu^2}$$

$$\mathcal{R} = -\sum_{\nu=1}^{n} \frac{X''_\nu}{U_\nu} \qquad \text{scalar curvature}$$

- Hamamoto N., Houri T., Oota T., Yasui Y.: *Kerr-NUT-de Sitter Curvature in All Dimensions*, J. Phys. A40, F177 (2007), arXiv:hep-th/0611285

Vacuum spacetimes with a cosmological constant

$$X_\mu = m_\mu\, x_\mu + \sum_{k=0}^{n-1} c_k \left(-x_\mu^2\right)^{n-1-k}$$

m_μ and c_k are related to mass, angular momenta, NUT parameters and cosmological constant

Principal Killing–Yano tensor

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu\, \boldsymbol{\epsilon}^{\mu} \wedge \boldsymbol{\epsilon}^{\hat{\mu}} = \sum_{\mu=1}^{n} x_\mu\, \boldsymbol{e}^{\mu} \wedge \boldsymbol{e}^{\hat{\mu}} \qquad \text{closed conformal KY tenzor}$$

$$\boldsymbol{f} = \sum_{\mu=1}^{n} x_\mu\, \underbrace{\boldsymbol{e}^{1} \wedge \cdots \wedge \boldsymbol{e}^{D}}_{\boldsymbol{e}^{\mu},\, \boldsymbol{e}^{\hat{\mu}} \text{ skipped}} \qquad \text{KY tenzor}$$

- Kubizňák D., Frolov V. P.: *Hidden Symmetry of Higher Dimensional Kerr-NUT-AdS Spacetimes*, Class. Quant. Grav. 24, F1 (2007), arXiv:gr-qc/0610144
- Krtouš P., Kubizňák D., Page D. N., Frolov V. P.: *Killing–Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions*, JHEP02(2007)004, arXiv:hep-th/0612029

Complete integrability

- Page D. N., Kubizňák D., Vasudevan M., Krtouš P.: *Complete Integrability of Geodesic Motion in General Kerr-NUT-AdS Spacetimes*, Phys. Rev. Lett. 98, 061102 (2007), arXiv:hep-th/0611083
- Krtouš P., Kubizňák D., Page D. N., Vasudevan M.: *Constants of Geodesic Motion in Higher-Dimensional Black-Hole Spacetimes*, arXiv:0707.0001 [hep-th]

Separability of the Hamilton–Jacobi and Klein–Gordon equations

- Frolov V. P., Krtouš P., Kubizňák D.: *Separability of Hamilton–Jacobi and Klein-Gordon Equations in General Kerr-NUT-AdS Spacetimes*, JHEP02(2007)005, arXiv:hep-th/0611245

Case $D = 4$

Redefinition of the coordinates and the metric functions:

$$\begin{array}{llll} \psi_0 = t & x_1 = ir & X_1 = Q & m_1 = 2im \\ \psi_1 = \sigma & x_2 = p & X_2 = P & m_2 = 2n \end{array}$$
$$c_0 = k \qquad c_1 = -\varepsilon \qquad c_2 = -\Lambda/3$$

Metric:

$$\boldsymbol{g} = -\frac{Q}{r^2+p^2}\big(\boldsymbol{d}t + p^2\boldsymbol{d}\sigma\big)^2 + \frac{r^2+p^2}{Q}\boldsymbol{d}r^2 + \frac{r^2+p^2}{P}\boldsymbol{d}p^2 + \frac{P}{r^2+p^2}\big(\boldsymbol{d}t - r^2\boldsymbol{d}\sigma\big)^2$$

$$Q \ = k - 2\,m\,r + \varepsilon\,r^2 - \frac{\Lambda}{3}\,r^4$$

$$P \ = k + 2\,n\,p - \varepsilon\,p^2 - \frac{\Lambda}{3}\,p^4$$

Kerr–NUT–(A)dS black hole solution in $D = 4$

- Carter B.: *Hamilton-Jacobi and Schrodinger Separable Solutions of Einstein's Equations*, Commun. Math. Phys. 10, 280 (1968)

Plebański–Demiański solution in $D = 4$

= charged accelerated Kerr–NUT–(A)dS black holes

$$\boldsymbol{g} = \Omega^2 \left[-\frac{Q}{r^2+p^2}\left(\boldsymbol{d}t + p^2 \boldsymbol{d}\sigma\right)^2 + \frac{r^2+p^2}{Q}\boldsymbol{d}r^2 + \frac{r^2+p^2}{P}\boldsymbol{d}p^2 + \frac{P}{r^2+p^2}\left(\boldsymbol{d}t - r^2 \boldsymbol{d}\sigma\right)^2 \right]$$

$$\begin{aligned} \Omega^{-1} &= 1 - \alpha\, p\, r \\ Q &= k + e^2 - 2\, m\, r + \varepsilon\, r^2 - 2\,\alpha\, n\, r^3 - \left(\Lambda/3 + \alpha^2 (k - g^2)\right) r^4 \\ P &= k - g^2 + 2\, n\, p - \varepsilon\, p^2 + 2\,\alpha\, m\, p^3 - \left(\Lambda/3 + \alpha^2 (k + e^2)\right) p^4 \end{aligned}$$

$$\boldsymbol{A} = -\frac{1}{r^2+p^2}\Big(e\, r\, \left(\boldsymbol{d}t + p^2\, \boldsymbol{d}\sigma\right) + g\, p\, \left(\boldsymbol{d}t - r^2\, \boldsymbol{d}\sigma\right)\Big)$$

r	radial coordinate	p	latitudinal coordinate
t	time coordinate	σ	longitudinal coordinate

$m,\, n,\, k,\, \varepsilon, \Lambda$	parameters related to mass, NUT charge, angular momentum, and cosmological constant
$e,\, g$	electric and magnetic charges
α	acceleration parameter

- Plebiański J. F., Demiński M.: *Rotating, Charged, and Uniformly Accelerating Mass in General Relativity*, Ann. Phys. (N.Y.) 98, 98 (1976)
- Podolský J., Griffiths J. B.: *Accelerating Kerr-Newman black holes in (anti-)de Sitter space-time*, Phys. Rev. D 73, 044018 (2006), arXiv:gr-qc/050601130

No-go theorem for accelerating the Kerr–NUT–(A)dS metric in higher dimensions

Metric ansatz:

$$\tilde{\boldsymbol{g}} = \Omega^2 \left[\sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{k=0}^{n-1} A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right] \right]$$

Unknown metric functions:

$$\Omega = \Omega(x_1, \dots, x_n) \qquad\qquad X_\mu = X_\mu(x_\mu)$$

Ricci tensor:

$$\tilde{\mathbf{Ric}} = \mathbf{Ric} + (D-2)\Omega \boldsymbol{\nabla}\boldsymbol{\nabla}\Omega^{-1} + \boldsymbol{g}\Big(\Omega \boldsymbol{\nabla}^2 \Omega^{-1} - (D-1)\Omega^2 \big(\boldsymbol{\nabla}\Omega^{-1}\big)^2 \Big)$$

Both **Ric** and $\boldsymbol{g}$ are diagonal in the frame $\boldsymbol{e}^a$!

Conformal factor

conditions on the off-diagonal terms of the Ricci tensor $\tilde{\mathbf{Ric}}$

$\Downarrow$

Even dimensions:

$$\Omega^{-1} \;=\; c + a\, x_1 \,\ldots\, x_n$$

– a generalization of the $D = 4$ case

Odd dimensions:

$$\Omega^{-1} \;=\; \text{constant}$$

– trivial solution

Metric functions for $D = 2n$

conditions on the diagonal terms $\Rightarrow$

(i) Solution dual to the unaccelerated metric

$$\Omega^{-1} = x_1 \ldots x_2 \qquad\qquad X_\mu = \bar{b}_\mu\, x_\mu^{2n-1} + \sum_{k=0}^{n} c_k\, x_\mu^{2k}$$

Duality

$$x_\mu = 1/\bar{x}_\mu \qquad\qquad \psi_j = \bar{\psi}_{n-1-j} \qquad\qquad X_\mu = \bar{x}_\mu^{-n+1} \bar{X}_\mu$$

transforms the rescaled metric $\tilde{\boldsymbol{g}}$ back to the unscaled metric $\boldsymbol{g}$ in 'barred' coordinates.

(ii) Trivial solution

$$\Omega^{-1} = 1 + a\, x_1 \ldots x_2 \qquad\qquad X_\mu = \sum_{k=0}^{n} c_k\, x_\mu^{2k}$$

Conditions for $D > 4$ are too restrictive – they enforce the 'trivial' metric functions X_μ.

Maximally symmetric spacetimes in 'rotating' 'accelerating' coordinates

- Kubizňák D., Krtouš P.: *On conformal Killing–Yano tensors of Plebański–Demiański family of solutions*, arXiv:0706.0409 [gr-qc]

Algebraically special test electromagnetic field

Assumption of an 'alignment' of the EM field

$$\boldsymbol{F} = \sum_{\mu=1}^{n} f_\mu \, \boldsymbol{\epsilon}^{\mu} \wedge \boldsymbol{\epsilon}^{\hat{\mu}} \qquad f_\mu = f_\mu(x_1, \dots, x_n)$$

Electromagnetic potential

$$\boldsymbol{F} = \boldsymbol{dA} \qquad \boldsymbol{dF} = 0$$

$$\Downarrow$$

$$\boldsymbol{A} = \sum_{\mu=1}^{n} \frac{g_\mu \, x_\mu}{U_\mu} \, \boldsymbol{\epsilon}^{\hat{\mu}} \qquad g_\mu = g_\mu(x_\mu)$$

$$f_\mu = \frac{g_\mu}{U_\mu} + \frac{x_\mu \, g'_\mu}{U_\mu} + 2 \, x_\mu \sum_{\nu=1}^{n} \frac{1}{U_\nu} \, \frac{x_\nu \, g_\nu - x_\mu \, g_\mu}{x_\nu^2 - x_\mu^2}$$

'Scalar potential'

$$f_\mu = \phi_{,\mu} \qquad \phi = \sum_{\nu=1}^{n} \frac{g_\nu \, x_\nu}{U_\nu} \qquad \phi_{,\mu\nu} = 2 \, \frac{x_\nu \, \phi_{,\mu} - x_\mu \, \phi_{,\nu}}{x_\mu^2 - x_\nu^2}$$

Electromagnetic source

$$\boldsymbol{J} = -\boldsymbol{\nabla}\cdot\boldsymbol{F} \qquad\qquad \boldsymbol{J} = \sum_{\mu=1}^{n} j_\mu\, \boldsymbol{\epsilon}_{\hat{\mu}}$$

$$j_\mu = \frac{1}{x_\mu}\left(\sum_{\nu=1}^{n}\frac{x_\nu^2\, g_\nu'}{U_\nu}\right)_{,\,\mu}$$

Source-free Maxwell equations

$$\boldsymbol{J} = 0 \qquad\qquad j_\mu = 0$$

$$\Downarrow$$

$$g_\mu \;=\; e_\mu + \frac{1}{x_\mu}\sum_{i=0}^{n-1} a_i\left(-x_\mu^2\right)^{n-1-i}$$

$$\boldsymbol{A} \;=\; \sum_{\mu=1}^{n}\frac{e_\mu\, x_\mu}{U_\mu}\,\boldsymbol{\epsilon}^{\hat{\mu}} + \sum_{i=0}^{n-1} a_i\,\boldsymbol{d}\psi_i \qquad\qquad \phi \;=\; a_0 + \sum_{\mu=1}^{n}\frac{e_\mu\, x_\mu}{U_\mu}$$

$$\boldsymbol{F} \;=\; \sum_{\mu=1}^{n} f_\mu\,\boldsymbol{\epsilon}^{\mu}\wedge\boldsymbol{\epsilon}^{\hat{\mu}} \qquad\qquad f_\mu \;=\; \phi_{,\mu} \;=\; \frac{e_\mu}{U_\mu} + 2\,x_\mu\sum_{\nu=1}^{n}\frac{1}{U_\nu}\,\frac{x_\nu\, e_\nu - x_\mu\, e_\mu}{x_\nu^2 - x_\mu^2}$$

Algebraically special test electromagnetic field

$$\boldsymbol{A} = \sum_{\mu=1}^{n} \frac{e_\mu\, x_\mu}{U_\mu}\, \boldsymbol{\epsilon}^{\hat{\mu}} = \sum_{\mu=1}^{n} \frac{e_\mu x_\mu}{U_\mu} \sum_{i=0}^{n-1} A_\mu^{(j)}\, \boldsymbol{d}\psi_i$$

$$\boldsymbol{F} = \sum_{\mu=1}^{n} \left(\frac{e_\mu}{U_\mu} + 2\, x_\mu \sum_{\nu=1}^{n} \frac{1}{U_\nu}\, \frac{x_\nu\, e_\nu - x_\mu\, e_\mu}{x_\nu^2 - x_\mu^2} \right) \boldsymbol{\epsilon}^{\mu} \wedge \boldsymbol{\epsilon}^{\hat{\mu}}$$

- solves the Maxwell equations on the background with *general metric functions* X_μ
- adjusted to the frame preferred by the principal Killing–Yano tensor
- depends on n electric/magnetic charges e_μ ($\mu = 1, \ldots, n$)
- in $D = 4$ it has the same form as the electromagnetic field of Plebański–Demiański spacetime

Plebański–Demiański solution without acceleration in $D = 4$

Electromagnetic field:

$$\boldsymbol{A} = \frac{e_1\, x_1}{U_1}\left(A_1^{(0)}\boldsymbol{d}\psi_0 + A_1^{(1)}\boldsymbol{d}\psi_1\right) + \frac{e_2\, x_2}{U_2}\left(A_2^{(0)}\boldsymbol{d}\psi_0 + A_2^{(1)}\boldsymbol{d}\psi_1\right)$$

$$= \frac{e_1\, x_1}{U_1}\,\boldsymbol{\epsilon}^{\hat{1}} + \frac{e_2\, x_2}{U_2}\,\boldsymbol{\epsilon}^{\hat{2}} \qquad \leftarrow \text{the form derived in high dimensions}$$

Metric:

$$\boldsymbol{g} = \frac{U_1}{X_1}\boldsymbol{d}x_1^2 + \frac{U_2}{X_2}\boldsymbol{d}x_2^2 + \frac{X_1}{U_1}\left(A_1^{(0)}\boldsymbol{d}\psi_0 + A_1^{(1)}\boldsymbol{d}\psi_1\right)^2 + \frac{X_2}{U_2}\left(A_2^{(0)}\boldsymbol{d}\psi_0 + A_2^{(1)}\boldsymbol{d}\psi_1\right)^2$$

$$X_1 = c_0 + c_1\, x_1^2 + c_2\, x_1^4 + 2\, m_1\, x_1 - \boldsymbol{e_1^2}$$

$$X_2 = c_0 + c_1\, x_2^2 + c_2\, x_2^4 + 2\, m_2\, x_2 - \boldsymbol{e_2^2}$$

With a modification of the metric functions X_μ by the charges $\boldsymbol{e_\mu^2}$

the electromagnetic field couples to gravity!

No-go theorem for charging the Kerr–NUT–(A)dS metric in higher dimensions

Energy-momentum tensor:

$$8\pi\, \boldsymbol{T}_{\mathrm{EM}} = \sum_{\mu=1}^{n} \left(2 f_\mu^2 - f^2\right) \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}\right)$$

$$8\pi\, T_{\mathrm{EM}} = 2\,(2-n)\; f^2 \qquad\qquad f^2 = \sum_{\nu=1}^{n} f_\nu^2$$

Einstein equations:

$$\mathbf{Ric} - \frac{1}{2}\mathcal{R}\boldsymbol{g} + \Lambda \boldsymbol{g} = 8\pi\, \boldsymbol{T}_{\mathrm{EM}}$$

The equation for scalar curvature:

$$\mathcal{R} = \frac{2n}{n-1}\,\Lambda + 2\,\frac{n-2}{n-1}\; f^2 \qquad\qquad \mathcal{R} = -\sum_{\nu=1}^{n} \frac{X_\nu''}{U_\nu}$$

Cannot be satisfied for generic D !

(checked with *Mathematica* for small D's larger than 4)