Separabilita Diracovy rovnice ve vícedimenzionálních Kerr-NUT-(A)dS prostoročasech

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- Metrika obecně rotující černé díry ve více dimenzích
- Explicitní a skryté symetrie
- Integrabilita geodetického pohybu
- Skalární separabilita

- Cliffordův bundle and vnější algebra
- Diracovy spinory
- Diracův operátor
- Operátory komutující s Diracovým operátorem

- Killingovy–Yanovy závorky
- Úplná sada komutujících operátorů v BH prostoročasech
- Součinová struktura Diracova bundlu
- Společná tenzorová separabilita komutujících operátorů

Dimension and signature of the metric

Results will be presented in an even dimension $n = 2N$.

All results are known also in odd dimensions.
Expressions are usually modified by additional terms.
We do not show them here just for simplicity.

For convenience, we use formal euclidian signature $(+, +, \dots, +)$.

The metric can be in most cases understood as a Lorentzian metric with signature $(-, +, \dots, +)$ assuming that the coordinate x_N (and some metric parameters) takes imaginary values $x_N = i\,r$.

However, there are some differences for Dirac spinors!

Metric of the Kerr–NUT–(A)dS spacetime

in even dimensions $n = 2N$

$$\boldsymbol{g} \;=\; \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

$X_\mu = X_\mu(x_\mu)$	metric functions to be determined by the Einstein equations
x_μ	radial and latitudinal coordinates ($\mu = 1, \dots, N$)
ψ_k	temporal and longitudinal coordinates ($k = 0, \dots, N-1$) – symmetries of the spacetime

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k,\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad U_\mu = \prod_{\substack{\nu \\ \nu\neq\mu}} (x_\nu^2 - x_\mu^2)$$

- Myers R. C., Perry M. J.: *Black Holes in Higher Dimensional Space-Times*, Ann. Phys. 172, 304 (1986)
- Gibbons G. W., Lü H., Page D. N., Pope C. N.: *Rotating Black Holes in Higher Dimensions with a Cosmological Constant*, Phys. Rev. Lett. 93, 171102 (2004), arXiv:hep-th/0409155
- Chen W., Lü H., Pope C. N.: *General Kerr-NUT-AdS Metrics in All Dimensions*, Class. Quant. Grav. 23, 5323 (2006), arXiv:hep-th/0604125

Orthogonal form of the metric

$$\boldsymbol{g} \;=\; \sum_{\mu} \left(\boldsymbol{e}^{\mu}\boldsymbol{e}^{\mu} \;+\; \hat{\boldsymbol{e}}^{\mu}\hat{\boldsymbol{e}}^{\mu} \right)$$

1-forms

$$\boldsymbol{e}^{\mu} = \left(\frac{U_\mu}{X_\mu}\right)^{1/2} \boldsymbol{d}x_\mu$$

$$\hat{\boldsymbol{e}}^{\mu} = \left(\frac{X_\mu}{U_\mu}\right)^{1/2} \sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k$$

vectors

$$\boldsymbol{e}_{\mu} = \left(\frac{X_\mu}{U_\mu}\right)^{1/2} \boldsymbol{\partial}_{x_\mu}$$

$$\hat{\boldsymbol{e}}_{\mu} = \left(\frac{U_\mu}{X_\mu}\right)^{1/2} \sum_k \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}$$

$$\mu = 1, \dots, N$$

Curvature

$$\mathbf{Ric} = -\sum_{\mu} r_{\mu} \left(\boldsymbol{e}^{\mu}\boldsymbol{e}^{\mu} + \hat{\boldsymbol{e}}^{\mu}\hat{\boldsymbol{e}}^{\mu}\right) \qquad \text{Ricci tensor}$$

$$r_{\mu} = \frac{1}{2}\frac{X''_{\mu}}{U_{\mu}} + \sum_{\substack{\nu \\ \nu\neq\mu}} \frac{1}{U_{\nu}} \frac{x_{\nu}X'_{\nu} - x_{\mu}X'_{\mu}}{x_{\nu}^2 - x_{\mu}^2} - \sum_{\substack{\nu \\ \nu\neq\mu}} \frac{1}{U_{\nu}} \frac{X_{\nu} - X_{\mu}}{x_{\nu}^2 - x_{\mu}^2}$$

$$\mathcal{R} = -\sum_{\nu} \frac{X''_{\nu}}{U_{\nu}} \qquad \text{scalar curvature}$$

- Hamamoto N., Houri T., Oota T., Yasui Y.: *Kerr-NUT-de Sitter Curvature in All Dimensions*, J. Phys. A40, F177 (2007), arXiv:hep-th/0611285

Vacuum spacetimes with a cosmological constant

$$X_{\mu} = b_{\mu}\, x_{\mu} + \sum_{k} c_k \left(-x_{\mu}^2\right)^{N-1-k}$$

b_{μ} and c_k are related to mass, NUT parameters, angular momenta and cosmological constant

Explicit and hidden symmetries

Principal Killing–Yano tensor

- closed conformal KY 2-form
- splits the tangent space to N 2-planes spanned on $\boldsymbol{e}_\mu$ and $\hat{\boldsymbol{e}}_\mu$
- each 2-plane is associated with an eigenvalue function x_μ
- existence of $\boldsymbol{h}$ implies the discussed form of the metric

$$\boldsymbol{h} = \sum_\mu x_\mu \, \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu$$

- Kubizňák D., Frolov V. P.: *Hidden Symmetry of Higher Dimensional Kerr-NUT-AdS Spacetimes*, Class. Quant. Grav. 24, F1 (2007), arXiv:gr-qc/0610144
- Houri T., Oota T., Yasui Y.: *Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness*, Phys. Lett. B 656, 214 (2007), arXiv:0708.1368 [hep-th]
- Krtouš P., Frolov V.P., Kubizňák D.: *Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime*, Phys. Rev. D **78**, 064022 (2008), arXiv:0804.4705 [hep-th]
- Houri T., Oota T., Yasui Y.: *Closed conformal Killing-Yano tensor and uniqueness of generalized Kerr-NUT-de Sitter spacetime*, Class. Quant. Grav **26**, 045015 (2009) arXiv:0805.3877 [hep-th]
- Houri T., Oota T., Yasui Y.: *Generalized Kerr-NUT-de Sitter metrics in all dimensions*, Phys. Lett. B 666, 391 (2008), arXiv:0805.0838 [hep-th]

Killing–Yano tensors

antisymmetric form:

$$\nabla_{a_0} f_{a_1\dots a_p} \;=\; \text{antisymmetric part} \;+\; \text{divergence part} \;+\; \text{harmonic part}$$

conformal Killing–Yano tensor:

$$\nabla_{a_0} f_{a_1\dots a_p} \;=\; \nabla_{[a_0} f_{a_1\dots a_p]} \;+\; \tfrac{p}{D-p+1}\, g_{a_0[a_1} \nabla_{|e|} f^{e}{}_{a_2\dots a_p]} \;+\; 0$$

Killing–Yano tensor:

$$\nabla_{a_0} f_{a_1\dots a_p} \;=\; \nabla_{[a_0} f_{a_1\dots a_p]} \;+\; 0 \;+\; 0$$

closed conformal Killing–Yano tensor:

$$\nabla_{a_0} f_{a_1\dots a_p} \;=\; 0 \;+\; \tfrac{p}{D-p+1}\, g_{a_0[a_1} \nabla_{|e|} f^{e}{}_{a_2\dots a_p]} \;+\; 0$$

Hodge dual

$*$: antisymetric part $\leftrightarrow$ divergence part

$*$: Killing–Yano tensor $\leftrightarrow$ closed conformal Killing–Yano tensor

Tower of symmetries

Primary Killing vector

$$\boldsymbol{\xi} = \frac{1}{D-1}\,\boldsymbol{\nabla}\cdot\boldsymbol{h} = \boldsymbol{\partial}_{\psi_0}$$

Closed conformal KY tensors of rank $2j$

$$\boldsymbol{h}_{(j)} = \frac{1}{j!}\,\boldsymbol{h}^{\wedge j}$$

KY tensors of rank $2(N-j)$

$$\boldsymbol{f}_{(j)} = *\,\boldsymbol{h}_{(j)}$$

Killing tensors

$$\boldsymbol{k}_{(j)}{}^{a}{}_{b} \propto \boldsymbol{f}_{(j)}{}^{a n_1 n_2\dots}\;\boldsymbol{f}_{(j)\,b n_1 n_2\dots}$$

$$\boldsymbol{k}_{(j)} = \sum_{\mu} A^{(j)}_{\mu}\left(\boldsymbol{e}^{\mu}\boldsymbol{e}^{\mu} + \hat{\boldsymbol{e}}^{\mu}\hat{\boldsymbol{e}}^{\mu}\right)$$

$$\boldsymbol{k}_{(0)} = \boldsymbol{g}$$

Killing vectors

$$\boldsymbol{l}_{(j)}{}^{a} = \boldsymbol{k}_{(j)}{}^{a}{}_{n}\,\boldsymbol{\xi}^{n}$$

$$\boldsymbol{l}_{(j)} = \sum_{\mu} A^{(j)}_{\mu}\left(\frac{U_\mu}{X_\mu}\right)^{1/2}\boldsymbol{e}_{\mu} = \boldsymbol{\partial}_{\psi_j}$$

$$\boldsymbol{l}_{(0)} = \boldsymbol{\xi}$$

- Krtouš P., Kubizňák D., Page D. N., Frolov V. P.: *Killing–Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions*, JHEP02(2007)004, arXiv:hep-th/0612029

Integrability of geodesic motion

- n conserved quantities $L_{(j)}$ linear in momentum related to Killing vectors

$$L_{(j)} = \boldsymbol{l}_{(j)}{}^{a}\, \boldsymbol{p}_a = p_j \qquad\qquad j = 0, \dots, n-1$$

- n conserved quantities $K_{(j)}$ quadratic in momentum related to Killing tensors

$$K_{(j)} = \boldsymbol{k}_{(j)}{}^{ab}\, \boldsymbol{p}_a \boldsymbol{p}_b = \sum_{\mu} A_{\mu}^{(j)} \big(p_{\tilde{\mu}}^2 + p_{\hat{\mu}}^2\big) \qquad\qquad j = 0, \dots, n-1$$

where p_j, $p_{\tilde{\mu}}$ and $p_{\hat{\mu}}$ are components of $\boldsymbol{p}$

$$\boldsymbol{p} = \sum_{\mu} p_{\mu}\, \boldsymbol{d}x_{\mu} + \sum_{j} p_j\, \boldsymbol{d}\psi_j = \sum_{\mu} \big(p_{\tilde{\mu}}\, \boldsymbol{e}^{\mu} + p_{\hat{\mu}}\, \hat{\boldsymbol{e}}^{\mu}\big)$$

$L_{(k)}$ and $K_{(j)}$ are independent and in involution

$$\{L_{(k)}, L_{(l)}\} = \{L_{(k)}, K_{(j)}\} = \{K_{(i)}, K_{(j)}\} = 0$$

Geodesic motion is completely integrable

- Page D. N., Kubizňák D., Vasudevan M., Krtouš P.: *Complete Integrability of Geodesic Motion in General Kerr-NUT-AdS Spacetimes*, Phys. Rev. Lett. 98, 061102 (2007), arXiv:hep-th/0611083
- Krtouš P., Kubizňák D., Page D. N., Vasudevan M.: *Constants of Geodesic Motion in Higher-Dimensional Black-Hole Spacetimes*, Phys. Rev. D 76, 084034 (2007), arXiv:0706.0001 [hep-th]

Symmetry operators of the scalar Laplace operator

$$\mathcal{L}_{(k)} = -i\, \boldsymbol{l}_{(k)}{}^{a}\, \boldsymbol{\nabla}_a$$

$$\mathcal{K}_{(j)} = -\, \boldsymbol{\nabla}_a \left[\, \boldsymbol{k}_{(j)}{}^{ab}\ \boldsymbol{\nabla}_b \,\right]$$

operator version of conserved quantities $L_{(k)}$ and $K_{(j)}$

$$\boldsymbol{p} \quad \rightarrow \quad -i\boldsymbol{\nabla}$$

coordinate form:

$$\mathcal{L}_{(k)} = -i\, \frac{\partial}{\partial \psi_k}$$

$$\mathcal{K}_{(j)} = -\sum \mu\, \frac{A^j_\mu}{U_\mu} \left[\frac{\partial}{\partial x_\mu} \left[X_\mu \frac{\partial}{\partial x_\mu} \right] + \frac{1}{X_\mu} \left[\sum_k (-x_\mu^2)^{N-1-k} \frac{\partial}{\partial \psi_k} \right]^2 \right]$$

Properties of symmetry operators

- $\mathcal{L}_{(k)}$ first order and $\mathcal{K}_{(j)}$ second order differential operators
- symmetric operators (with respect to the natural $\mathbf{L}_2$ scalar product)
- $\mathcal{L}_{(k)}$ Lie derivatives along Killing vectors
- $\mathcal{K}_{(0)} = -\Box$ Laplace (d'Alembert) operator

- $\mathcal{L}_{(k)}$ and $\mathcal{K}_{(j)}$ commute with each other

$$[\mathcal{L}_{(k)}, \mathcal{L}_{(l)}] = 0 \qquad [\mathcal{L}_{(k)}, \mathcal{K}_{(j)}] = 0 \qquad [\mathcal{K}_{(i)}, \mathcal{K}_{(j)}] = 0$$

- system of common eigenfunctions

$$\mathcal{L}_{(k)}\phi = \Psi_k \, \phi \qquad \mathcal{K}_{(j)}\phi = \Xi_j \, \phi$$

- Sergyeyev A., Krtouš P.: *Complete Set of Commuting Symmetry Operators for Klein–Gordon Equation in Generalized Higher-Dimensional Kerr-NUT-(A)dS Spacetimes*, Phys. Rev. D 77, 044033 (2008), arXiv:0711.4623 [hep-th]

Simultaneous separability of symmetry operators

eigenvalue equations

$$\mathcal{L}_{(k)}\phi \;=\; \Psi_k\,\phi \qquad\qquad \mathcal{K}_{(j)}\phi \;=\; \Xi_j\,\phi$$

can be solved by the separability ansatz

$$\phi = \prod_\mu R_\mu(x_\mu)\;\prod_k \exp\Big(\,i\Psi_k\psi_k\Big)$$

where each $R_\mu(x_\mu)$ satisfies an ordinary differential equation

$$\big(X_\mu R_\mu{}'\big)' + \bigg(\tilde{\Xi}_\mu - \frac{\tilde{\Psi}_\mu{}^2}{X_\mu}\bigg)R_\mu = 0$$

with

$$\tilde{\Psi}_\mu = \sum_k \Psi_k\,(-x_\mu^2)^{N-1-k} \qquad\qquad \tilde{\Xi}_\mu = \sum_k \Xi_k\,(-x_\mu^2)^{N-1-k}$$

$\Rightarrow$ **separability of the Klein–Gordon equation**

Separability of the Dirac equation

T. Oota and Y. Yasui (2008) showed the separability of the Dirac equation:

$$\Big[D + m \Big] \boldsymbol{\varphi} \; = \; 0$$

with spinor components $\psi^{\epsilon_1 \cdots \epsilon_N}$ indexed by N-tuple of signs in the form

$$\varphi^{\epsilon_1 \ldots \epsilon_N} = \phi_{\epsilon_1 \ldots \epsilon_N} \prod_k \exp\big(i \Psi_k \psi_k\big) \prod_\nu \chi_\nu^{\epsilon_\nu}(x_\mu)$$

where the prefactor $\phi_{\epsilon_1 \ldots \epsilon_N}$ depends on all coordinates x_ν

$$\phi_{\epsilon_1 \ldots \epsilon_N} = \prod_{\substack{\kappa,\lambda \\ \kappa<\lambda}} \Big(x_\kappa + \epsilon_\kappa \epsilon_\lambda x_\lambda \Big)^{-\frac{1}{2}}$$

and pairs of separability functions χ_ν^+, χ_ν^- satisfy coupled ordinary differential equations.

What kind of separability is it?

- Oota T., Yasui Y.: *Separability of Dirac equation in higher dimensional Kerr-NUT-de Sitter spacetime*, Phys. Lett. B 659, 688 (2008), arXiv:0711.0078 [hep-th]

Clifford bundle ...

Clifford bundle ClM

- Clifford objects form 2^n-dimensional space at each spacetime point
- abstract gamma matrices $\boldsymbol{\gamma}^a \in \mathbf{T}\otimes\mathbf{Cl}\, M$ satisfying

$$\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b + \boldsymbol{\gamma}^b\boldsymbol{\gamma}^a = 2\,\boldsymbol{g}^{ab}$$

- all Clifford object generated by abstract gamma matrices $\boldsymbol{\gamma}^a$

$$\boldsymbol{\omega} \;=\; \sum_p \frac{1}{p!}\,\boldsymbol{\omega}^{(p)}_{a_1\dots a_p}\,\boldsymbol{\gamma}^{a_1}\cdots\boldsymbol{\gamma}^{a_p}$$

Covariant derivative on ClM

- there exists a unique lift of the metric covariant derivative such that $\boldsymbol{\nabla}_n\boldsymbol{\gamma}^a = 0$
- for chosen orthonormal vielbein $\boldsymbol{e}^j \in \mathbf{T}^*M$ and the metric derivative $\boldsymbol{\nabla}_a$ reads

$$\boldsymbol{\nabla}_n\,\boldsymbol{\alpha}^{a\dots}_{b\dots} = \eth_n\boldsymbol{\alpha}^{a\dots}_{b\dots} + \boldsymbol{\omega}_n{}^a{}_k\,\boldsymbol{\alpha}^{k\dots}_{b\dots} + \dots - \boldsymbol{\omega}_n{}^k{}_b\,\boldsymbol{\alpha}^{a\dots}_{k\dots} - \dots + \left[\tfrac{1}{2}\boldsymbol{\omega}_n,\,\boldsymbol{\alpha}^{a\dots}_{b\dots}\right]$$

$\eth_a$	— the vielbein derivative annihilating $\boldsymbol{e}^j$ and $\boldsymbol{\gamma}^j$
$\boldsymbol{\omega}_n{}^a{}_b$	— spin coefficients of the metric connection, $\boldsymbol{\nabla}_n\,\boldsymbol{e}^a = -\boldsymbol{\omega}_n{}^a{}_k\,\boldsymbol{e}^k$
$\boldsymbol{\omega}_n$	— the lift of the spin coefficients, $\boldsymbol{\omega}_n = \frac{1}{2}\boldsymbol{\omega}_{nab}\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b$

. . . and Dirac spinors

Dirac bundle $\mathbf{D}M$

- Dirac spinors form 2^N-dimensional space at each spacetime point
- irreducible representation of $\mathbf{Cl}\,M$ on $\mathbf{D}M$
- the representation unique up to equivalence (and conjugation in some dimensions)
- the representation generated by gamma matrices $\boldsymbol{\gamma}_n{}^A{}_B$ understood as operators on $\mathbf{D}M$
- additional structures with details varying with the dimension
  e.g., (pseudo)scalar product and charge conjugations

Covariant derivative on $\mathbf{D}M$

- there exists a unique lift of the metric covariant derivative such that $\boldsymbol{\nabla}_n \boldsymbol{\gamma}^a = 0$
- the metric derivative $\boldsymbol{\nabla}_a$ acting on $\mathbf{D}M$ reads

$$\boldsymbol{\nabla}_n \boldsymbol{\psi}^A = \eth_n \boldsymbol{\psi}^A + \tfrac{1}{2} \boldsymbol{\omega}_n{}^A{}_K\, \boldsymbol{\psi}^K$$

$\boldsymbol{e}^j$, $\boldsymbol{\vartheta}^A$ — orthonormal frame in $\mathbf{T}^*M$ and associated frame in $\mathbf{D}M$ such that $\gamma_j{}^A{}_B$ are constants
$\eth_n$ — auxiliary derivative annihilating $\boldsymbol{e}^j$ and $\boldsymbol{\vartheta}^A$

Representation of $so(n)$ algebra

$so(n)$ **Lie algebra** (= small isometries)

encoded using antisymmetric 2-forms $\boldsymbol{M}_{kl}$

representations M on various spaces

$\mathbf{T}^1_0 M$	$\mathsf{M}\boldsymbol{a}^n = \boldsymbol{M}^n{}_k\, \boldsymbol{a}^k$
$\mathbf{T}^0_1 M$	$\mathsf{M}\boldsymbol{a}_n = -\boldsymbol{M}^k{}_n\, \boldsymbol{a}_k$
$\mathbf{\Lambda}\, M$	$\mathsf{M}\boldsymbol{\sigma} = -\boldsymbol{M} \underset{1}{\wedge} \boldsymbol{\sigma} = \left[\frac{1}{2}\boldsymbol{M},\, \boldsymbol{\sigma}\right]$
$\mathbf{Cl}\, M$	$\mathsf{M}\boldsymbol{\alpha} = \left[\frac{1}{2}\,\not{\boldsymbol{M}},\, \boldsymbol{\alpha}\right]$
$\mathbf{D} M$	$\mathsf{M}\boldsymbol{\psi} = \frac{1}{2}\,\not{\boldsymbol{M}}\, \boldsymbol{\psi}$

commutator of the metric derivative

$$\mathsf{R}_{ab} \equiv \boldsymbol{\nabla}_a \boldsymbol{\nabla}_b - \boldsymbol{\nabla}_b \boldsymbol{\nabla}_a$$

representation of $so(n)$ algebra generated by the Riemann tensor $\boldsymbol{R}_{ab\,kl}$

Equivalence of Clifford bundle $\mathbf{Cl}\,M$ and exterior algebra $\mathbf{\Lambda}\,M$

relation $\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b + \boldsymbol{\gamma}^b\boldsymbol{\gamma}^a = 2\,\boldsymbol{g}^{ab}$ allows eliminate symmetric products of $\boldsymbol{\gamma}$'s

$$\Downarrow$$

unique correspondence

$$\boldsymbol{\omega} \quad \leftrightarrow \quad \not{\boldsymbol{\omega}} \;=\; \sum_p \frac{1}{p!}\,\boldsymbol{\omega}^{(p)}_{a_1\dots a_p}\,\boldsymbol{\gamma}^{a_1}\cdots\boldsymbol{\gamma}^{a_p}$$

where coefficients $\boldsymbol{\omega}^{(p)}$ are homogeneous antisymmetric p-forms and

$$\boldsymbol{\omega} = \sum_p \boldsymbol{\omega}^{(p)}$$

is non-homogeneous form from exterior algebra $\mathbf{\Lambda}M$

Clifford object can be represented by non-homogeneous antisymmetric forms

the correspondence induces *Clifford multiplication* on antisymmetric forms

$$\boldsymbol{\omega}\circ\boldsymbol{\sigma} \quad \leftrightarrow \quad \not{\boldsymbol{\omega}}\,\not{\boldsymbol{\sigma}}$$

the infix '$\circ$' will be omitted

Clifford multiplication and related objects

Clifford objects $\leftrightarrow$ antisymmetric forms

Clifford multiplication of p-form $\boldsymbol{\alpha}$ and q-form $\boldsymbol{\beta}$

$$\boldsymbol{\alpha}\boldsymbol{\beta} = \sum_m \frac{(-1)^{m(p-m)+[m/2]}}{m!}\, \boldsymbol{\alpha} \underset{m}{\wedge} \boldsymbol{\beta}$$

contracted wedge (first contract j indices, then wedge)

$$(\boldsymbol{\alpha} \underset{j}{\wedge} \boldsymbol{\beta})_{a_1\dots b_1\dots} = \frac{(p+q-2j)!}{(p-j)!(q-j)!}\, \boldsymbol{\alpha}_{[a_1\dots}{}^{c_1\dots c_j}\, \boldsymbol{\beta}_{b_1\dots]c_1\dots c_j}$$

canonical form $\boldsymbol{e}^a$ – the equivalent of the gamma matrices in $\boldsymbol{\Lambda}M$

$$\boldsymbol{e}^a \quad \leftrightarrow \quad \not{\boldsymbol{e}}^a = \boldsymbol{\gamma}^a \qquad \text{i.e.} \quad \boldsymbol{e}^a_b = \boldsymbol{\delta}^a_b$$

canonical vectors $\boldsymbol{X}_a$ – the object dual to $\boldsymbol{e}^a$

$$\boldsymbol{X}_a \cdot \boldsymbol{e}^b = \boldsymbol{\delta}^b_a \qquad\qquad \boldsymbol{X}_a \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}_a$$

degree operator π and **parity operator** η

$$\pi\boldsymbol{\omega} = \sum_p p\, \boldsymbol{\omega}^{(p)} \qquad\qquad \eta\boldsymbol{\omega} = (-1)^\pi \boldsymbol{\omega}$$

Dirac operator

Dirac operator on ClM and DM

$$\mathcal{D} = \boldsymbol{\gamma}^a \, \boldsymbol{\nabla}_a \qquad \text{i.e.,} \qquad \mathcal{D}\boldsymbol{\psi}^A = \boldsymbol{\gamma}^{nA}{}_B \boldsymbol{\nabla}_n \, \boldsymbol{\psi}^B$$

Operators on $\boldsymbol{\Lambda}M$

$$\begin{aligned} \mathcal{D} &= \boldsymbol{\nabla}\circ &&= \boldsymbol{\nabla}_a \, \boldsymbol{e}^a \circ \\ \boldsymbol{d} &= \boldsymbol{\nabla}\wedge &&= \boldsymbol{\nabla}_a \, \boldsymbol{e}^a \wedge \\ -\boldsymbol{\delta} = \mathbf{div} &= \boldsymbol{\nabla}\cdot &&= \boldsymbol{\nabla}^a \, \boldsymbol{X}_a \cdot \end{aligned}$$

Dirac operator on $\boldsymbol{\Lambda}M$

$$\mathcal{D}\boldsymbol{\sigma} = \boldsymbol{\nabla} \wedge \boldsymbol{\sigma} + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{d}\boldsymbol{\sigma} - \boldsymbol{\delta}\boldsymbol{\sigma}$$

Laplace operator and $\mathcal{D}^2$

Laplace operators on $\boldsymbol{\Lambda}\, M$

Laplace-Beltrami operator	$\nabla^2 \equiv \boldsymbol{g}^{ab} \boldsymbol{\nabla}_a \boldsymbol{\nabla}_b$
Laplace-de Rham operator	$\triangle \equiv \boldsymbol{d}\,\mathbf{div} + \mathbf{div}\,\boldsymbol{d}$
Dirac square	$\mathcal{D}^2 \equiv \mathcal{D}\,\mathcal{D}$

Commutator

$$\mathsf{R}_{ab} \equiv \boldsymbol{\nabla}_a \boldsymbol{\nabla}_b - \boldsymbol{\nabla}_b \boldsymbol{\nabla}_a$$

$$\mathsf{R} \equiv \tfrac{1}{2}\, \boldsymbol{e}^a \wedge \boldsymbol{e}^b\, \mathsf{R}_{ab} \qquad \text{or} \qquad \mathsf{R} \equiv \tfrac{1}{2}\, \boldsymbol{\gamma}^a \boldsymbol{\gamma}^b\, \mathsf{R}_{ab}$$

Relation between operators

$$\mathcal{D}^2 = \triangle = \nabla^2 + \mathsf{R}$$

Weitzenböck identities

Laplace operator on Dirac bundle $\mathbf{D}M$

$$\mathsf{R}_{ab}\boldsymbol{\psi} = \tfrac{1}{2}\not{\boldsymbol{R}}_{ab}\,\boldsymbol{\psi} = \tfrac{1}{4}\boldsymbol{R}_{ab\,kl}\,\boldsymbol{\gamma}^k\boldsymbol{\gamma}^l\,\boldsymbol{\psi}$$

$$\mathsf{R}\boldsymbol{\psi} = \tfrac{1}{2}\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b\,\mathsf{R}_{ab}\boldsymbol{\psi} = \tfrac{1}{8}\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b\,\boldsymbol{R}_{ab\,kl}\,\boldsymbol{\gamma}^k\boldsymbol{\gamma}^l\,\boldsymbol{\psi} = -\tfrac{1}{4}\mathcal{R}\,\boldsymbol{\psi}$$

$$\Downarrow$$

$$\mathcal{D}^2 = \nabla^2 - \frac{1}{4}\mathcal{R}$$

Laplace operator on exterior algebra $\boldsymbol{\Lambda}\,M$

$$\mathsf{R}\,\boldsymbol{\sigma} = \tfrac{1}{2}\,\boldsymbol{e}^a\wedge\boldsymbol{e}^b\,\mathsf{R}_{ab}\,\boldsymbol{\sigma} = -\mathbf{Ric}_n\wedge\boldsymbol{\sigma}^n + \boldsymbol{R}_{mn}\wedge\boldsymbol{\sigma}^{mn}$$

$\boldsymbol{R}_{mn} = \tfrac{1}{2}\boldsymbol{R}_{mn\,kl}\,\boldsymbol{e}^k\wedge\boldsymbol{e}^l$ curvature 2-form

$\mathbf{Ric}_n = \mathbf{Ric}_{nk}\,\boldsymbol{e}^k$ Ricci 1-form

$\boldsymbol{\sigma}^n = \boldsymbol{X}^n\cdot\boldsymbol{\sigma}$ $\quad\boldsymbol{\sigma}^{mn} = \boldsymbol{X}^n\cdot\boldsymbol{\sigma}^m$

$$\Downarrow$$

$$\triangle\boldsymbol{\sigma} = \nabla^2\boldsymbol{\sigma} - \mathbf{Ric}_n\wedge\boldsymbol{\sigma}^n + \boldsymbol{R}_{mn}\wedge\boldsymbol{\sigma}^{mn}$$

Operators commuting with the Dirac operator

operators linear in $\boldsymbol{\nabla}_a$ commuting with $\mathcal{D}$

$$[\mathcal{D}, \mathcal{S}] = 0 \qquad \Leftrightarrow \qquad \mathcal{S} = \mathcal{L}_{\boldsymbol{\kappa}} + \mathcal{M}_{\boldsymbol{\alpha}}$$

$\boldsymbol{\kappa}$ odd KY form

$\boldsymbol{\alpha}$ even CCKY form

$$\mathcal{L}_{\boldsymbol{\kappa}} = \boldsymbol{X}^a \cdot \boldsymbol{\kappa}\ \boldsymbol{\nabla}_a \ + \ \frac{\pi - 1}{2\pi} (\boldsymbol{\nabla} \wedge \boldsymbol{\kappa})$$

$$\mathcal{M}_{\boldsymbol{\alpha}} = \boldsymbol{e}^a \wedge \boldsymbol{\alpha}\ \boldsymbol{\nabla}_a \ + \ \frac{n - \pi - 1}{2(n - \pi)} (\boldsymbol{\nabla} \cdot \boldsymbol{\alpha})$$

- Benn I. M., Charlton P.: *Dirac symmetry operators from conformal Killing-Yano tensors*, Class. Quantum Grav. 14, 1037 (1997), arXiv:gr-qc/9612011
- Benn I. M., Kress J., *First-order Dirac symmetry operators*, Class. Quantum Grav. 21, 427 (2004)
- Cariglia M., Krtouš P., Kubizňák D.: *Commuting symmetry operators of the Dirac equation, Killing–Yano and Schouten-Nijenhuis brackets*, Phys. Rev. D84, 024004 (2011), arXiv:1102.4501 [hep-th]

Killing–Yano brackets

first-order commutation

commutators of first-order symmetry operators $\mathcal{L}_{\boldsymbol{\kappa}}$ and $\mathcal{M}_{\boldsymbol{\alpha}}$

$$\left[\mathcal{L}_{\boldsymbol{\kappa}}, \mathcal{L}_{\boldsymbol{\lambda}}\right] \qquad \left[\mathcal{L}_{\boldsymbol{\mu}}, \mathcal{M}_{\boldsymbol{\omega}}\right] \qquad \left[\mathcal{M}_{\boldsymbol{\alpha}}, \mathcal{M}_{\boldsymbol{\beta}}\right]$$

are of the **first order** if the following **algebraic conditions** hold:

$$[\boldsymbol{X}^{(a} \cdot \boldsymbol{\kappa}, \boldsymbol{X}^{b)} \cdot \boldsymbol{\lambda}] = 0 \qquad [\boldsymbol{X}^{(a} \cdot \boldsymbol{\mu}, \boldsymbol{e}^{b)} \wedge \boldsymbol{\omega}] = 0 \qquad [\boldsymbol{e}^{(a} \wedge \boldsymbol{\alpha}, \boldsymbol{e}^{b)} \wedge \boldsymbol{\beta}] = 0$$

here [,] denotes Clifford commutator

Killing–Yano brackets

in such cases we can define brackets acting on **odd KY** and **even CCKY** forms

$$\begin{aligned} \left[\mathcal{L}_{\boldsymbol{\kappa}}, \mathcal{L}_{\boldsymbol{\lambda}}\right] &= \mathcal{L}_{[\boldsymbol{\kappa},\boldsymbol{\lambda}]_{\mathrm{KY}}} \\ \left[\mathcal{L}_{\boldsymbol{\mu}}, \mathcal{M}_{\boldsymbol{\omega}}\right] &= \mathcal{M}_{[\boldsymbol{\mu},\boldsymbol{\omega}]_{\mathrm{KY}}} \\ \left[\mathcal{M}_{\boldsymbol{\alpha}}, \mathcal{M}_{\boldsymbol{\beta}}\right] &= \mathcal{L}_{[\boldsymbol{\alpha},\boldsymbol{\beta}]_{\mathrm{KY}}} \end{aligned}$$

- Cariglia M., Krtouš P., Kubizňák D.: *Commuting symmetry operators of the Dirac equation, Killing–Yano and Schouten-Nijenhuis brackets*, Phys. Rev. D84, 024004 (2011), arXiv:1102.4501 [hep-th]

Explicit form of Killing–Yano brackets

KY brackets can be written in terms of potentials and copotentials

$$[\,\boldsymbol{\kappa},\,\boldsymbol{\lambda}\,]_{\mathrm{KY}} = -\frac{1}{\pi}\,\delta\sum_{k=0}\frac{(-1)^k}{(2k{+}1)!}\,(\pi\boldsymbol{\kappa})\underset{2k}{\wedge}(\pi\boldsymbol{\lambda})$$

$$[\,\boldsymbol{\mu},\,\boldsymbol{\omega}\,]_{\mathrm{KY}} = \frac{1}{n{-}\pi}\,d\sum_{k=0}\frac{(-1)^k}{(2k{+}1)!}\Big((\pi{-}2k)^{-1}\pi\,\boldsymbol{\mu}\Big)\underset{2k+1}{\wedge}\Big((n{-}\pi)\boldsymbol{\omega}\Big)$$

$$[\,\boldsymbol{\alpha},\,\boldsymbol{\beta}\,]_{\mathrm{KY}} = \frac{1}{\pi}\,\delta\sum_{k=0}\frac{(-1)^k}{(2k{+}1)!}\,\frac{1}{n{-}\pi{-}2k}\Big((n{-}\pi)\boldsymbol{\alpha}\Big)\underset{2k+1}{\wedge}\Big((n{-}\pi)\boldsymbol{\beta}\Big)$$

for **homogeneous** odd KY forms $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and even CCKY forms $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\omega}$ of degrees p, q, r and a, b, c it gives

$$[\,\boldsymbol{\kappa},\,\boldsymbol{\lambda}\,]_{\mathrm{KY}} \propto \delta\big(\,\boldsymbol{\kappa}\wedge\boldsymbol{\lambda}\,\big)$$

$$[\,\boldsymbol{\mu},\,\boldsymbol{\omega}\,]_{\mathrm{KY}} \propto d\big(\,\boldsymbol{\mu}\underset{r}{\wedge}\boldsymbol{\omega}\,\big)$$

$$[\,\boldsymbol{\alpha},\,\boldsymbol{\beta}\,]_{\mathrm{KY}} \propto \begin{cases} \delta\big(\,\boldsymbol{\alpha}\underset{p+q-n}{\wedge}\boldsymbol{\beta}\,\big) & n \text{ odd} \\ 0 & n \text{ even, } a{+}b\le n \\ \boldsymbol{\alpha}\underset{a+b-n-1}{\wedge}\big(\frac{1}{n-b+1}\delta\boldsymbol{\beta}\big) - \big(\frac{1}{n-a+1}\delta\boldsymbol{\alpha}\big)\underset{a+b-n-1}{\wedge}\boldsymbol{\beta} & n \text{ even, } a{+}b> n \end{cases}$$

First-order commutation conditions

algebraic conditions that commutator are of the first order

$$[\,\boldsymbol{X}^{(a}\cdot\boldsymbol{\kappa},\ \boldsymbol{X}^{b)}\cdot\boldsymbol{\lambda}\,]=0 \qquad [\,\boldsymbol{X}^{(a}\cdot\boldsymbol{\mu},\ \boldsymbol{e}^{b)}\wedge\boldsymbol{\omega}\,]=0 \qquad [\,\boldsymbol{e}^{(a}\wedge\boldsymbol{\alpha},\ \boldsymbol{e}^{b)}\wedge\boldsymbol{\beta}\,]=0$$

These are rather strong restrictions!

necessary contracted conditions

$$[\,\boldsymbol{X}_a\cdot\boldsymbol{\kappa},\ \boldsymbol{X}^a\cdot\boldsymbol{\lambda}\,]=0 \qquad [\,\boldsymbol{X}_a\cdot\boldsymbol{\mu},\ \boldsymbol{e}^a\wedge\boldsymbol{\omega}\,]=0 \qquad [\,\boldsymbol{e}_a\wedge\boldsymbol{\alpha},\ \boldsymbol{e}^a\wedge\boldsymbol{\beta}\,]=0$$

are equivalent to

non-homogeneous forms

$$\sum_{k=1}\frac{(-1)^k}{(2k-1)!}\,\boldsymbol{\kappa}\underset{2k}{\wedge}\boldsymbol{\lambda}=0$$

$$\sum_{k=0}\frac{(-1)^k}{(2k+1)!}\big((\pi-2k-1)\boldsymbol{\mu}\big)\underset{2k+1}{\wedge}\boldsymbol{\omega}=0$$

$$\sum_{k=0}\frac{(-1)^k}{(2k+1)!}(n-\pi-2k-1)\big(\boldsymbol{\alpha}\underset{2k+1}{\wedge}\boldsymbol{\beta}\big)=0$$

homogeneous forms

$$\boldsymbol{\kappa}\underset{2k}{\wedge}\boldsymbol{\lambda}=0 \qquad k=1,\,2,\,\ldots$$

$$\boldsymbol{\mu}\underset{2k+1}{\wedge}\boldsymbol{\omega}=0 \qquad k=0,\,1,\,\ldots$$

$$\boldsymbol{\alpha}\underset{2k+1}{\wedge}\boldsymbol{\beta}=0 \qquad k=0,\,1,\,\ldots$$

Physical significance of the Killing–Yano brackets

- There exists two important examples when the algebraic conditions are satisfied
  - explicit symmetry (one of the forms is Killing 1-form)
  - hidden symmetries of Kerr-NUT-(A)dS spacetimes

- For homogeneous odd KY forms equivalent to Schouten-Nijenhuis brackets

  $$[\boldsymbol{\kappa}, \boldsymbol{\lambda}]_{\mathrm{KY}} \propto [\boldsymbol{\kappa}, \boldsymbol{\lambda}]_{\mathrm{SN}}$$

- *Can be the Killing–Yano brackets non-trivial?*

- *If yes, do the Killing–Yano brackets form a closed algebra?*

Explicit symmetry – Lie derivative

one of the argument is KY 1-form $\boldsymbol{\lambda}$ equivalent to Killing vector $\boldsymbol{l} = \boldsymbol{\lambda}^\sharp$

algebraic conditions are automatically satisfied

$$[\,\boldsymbol{X}^{(a} \cdot \boldsymbol{\lambda},\ \boldsymbol{X}^{b)} \cdot \boldsymbol{\kappa}\,] = 0 \qquad\qquad [\,\boldsymbol{X}^{(a} \cdot \boldsymbol{\lambda},\ \boldsymbol{e}^{b)} \wedge \boldsymbol{\omega}\,] = 0$$

commutators of operators and KY brackets reduces to Lie derivative along $\boldsymbol{l}$

$$[\,\mathcal{L}_{\boldsymbol{\lambda}},\ \mathcal{L}_{\boldsymbol{\kappa}}\,] = \mathcal{L}_{\pounds_{\boldsymbol{l}}\,\boldsymbol{\kappa}} \qquad\qquad [\,\mathcal{L}_{\boldsymbol{\lambda}},\ \mathcal{M}_{\boldsymbol{\omega}}\,] = \mathcal{M}_{\pounds_{\boldsymbol{l}}\,\boldsymbol{\omega}}$$

$$[\,\boldsymbol{\lambda},\ \boldsymbol{\kappa}\,]_{\mathrm{KY}} = \pounds_{\boldsymbol{l}}\,\boldsymbol{\kappa} \qquad\qquad [\,\boldsymbol{\lambda},\ \boldsymbol{\omega}\,]_{\mathrm{KY}} = \pounds_{\boldsymbol{l}}\,\boldsymbol{\omega}$$

Lie derivative on the Clifford bundle

$$\pounds_{\boldsymbol{l}}\,\boldsymbol{\sigma} \;=\; \nabla_{\boldsymbol{l}}\,\boldsymbol{\sigma} + \frac{1}{4}\,[\,\boldsymbol{d\lambda},\ \boldsymbol{\sigma}\,]$$

well defined only along a Killing vector $\boldsymbol{l}$

Complete set of commuting operators in BH spacetimes

explicit and hidden symmetries

- Killing vectors $\boldsymbol{l}_{(k)}$ corresponding to KY 1-forms $\boldsymbol{\lambda}_{(k)}$
- closed conformal KY $(2j)$-forms $\boldsymbol{h}_{(j)}$

Killing–Yano brackets

algebraic conditions are satisfied (namely, the non-trivial $[\,\boldsymbol{e}^{(a} \wedge \boldsymbol{h}_{(i)},\, \boldsymbol{e}^{b)} \wedge \boldsymbol{h}_{(j)}\,] = 0$)

Killing–Yano brackets are vanishing

$$[\,\boldsymbol{\lambda}_{(k)},\, \boldsymbol{\lambda}_{(l)}\,]_{\mathrm{KY}} = 0 \qquad [\,\boldsymbol{\lambda}_{(k)},\, \boldsymbol{h}_{(j)}\,]_{\mathrm{KY}} = 0 \qquad [\,\boldsymbol{h}_{(i)},\, \boldsymbol{h}_{(j)}\,]_{\mathrm{KY}} = 0$$

set of commuting operators

operators corresponding to the explicit and hidden symmetries

$$\mathcal{L}_{(k)} = \mathcal{L}_{\boldsymbol{\lambda}_{(k)}} \qquad \mathcal{M}_{(j)} = \mathcal{M}_{\boldsymbol{h}_{(j)}}$$

mutually commute

$$[\,\mathcal{L}_{(k)},\, \mathcal{L}_{(l)}\,] = 0 \qquad [\,\mathcal{L}_{(k)},\, \mathcal{M}_{(j)}\,] = 0 \qquad [\,\mathcal{M}_{(i)},\, \mathcal{M}_{(j)}\,] = 0$$

Commuting symmetry operators of the Dirac operator

Dirac operator

$$\mathcal{D} = \mathcal{M}_{(0)}$$

symmetry operators

$$\mathcal{L}_{(k)} = \pounds_{\boldsymbol{l}_{(k)}} = \boldsymbol{\nabla}_{\boldsymbol{l}_{(k)}} + \frac{1}{4}\, \boldsymbol{d\lambda}_{(k)}$$

$$\mathcal{M}_{(j)} = \boldsymbol{e}^a \wedge \boldsymbol{h}_{(j)} \;\boldsymbol{\nabla}_a + \frac{n-2j}{2(n-2j+1)} \,(\boldsymbol{\nabla} \cdot \boldsymbol{h}_{(j)})$$

mutual commutation

$$[\,\mathcal{L}_{(k)},\, \mathcal{L}_{(l)}\,] = 0 \qquad [\,\mathcal{L}_{(k)},\, \mathcal{M}_{(j)}\,] = 0 \qquad [\,\mathcal{M}_{(i)},\, \mathcal{M}_{(j)}\,] = 0$$

- *Do exist common spinorial eigenfunctions?* – **Yes!**
- *Do they separate in variables?* – **Yes!**

- Cariglia M., Krtouš P., Kubizňák D.: *Dirac Equation in Kerr-NUT-(A)dS Spacetimes: Intrinsic Characterization of Separability in All Dimensions*, Phys. Rev. D84, 024008 (2011), arXiv:1104.4123 [hep-th]

Product structure of the Dirac bundle

Dirac bundle

$\mathbf{D}_x M$ – 2^N-dimensional space with the representation of the Clifford bundle

product structure

the special vielbein $\{\boldsymbol{e}^\mu,\, \hat{\boldsymbol{e}}^\mu\}$ allows identify the tensor product structure

$$\mathbf{D}M = \mathbf{S}^N M$$

i.e., $\mathbf{D}_x M$ is the tensor product of N copies of 2-dimensional space $\mathbf{S}_x M$

spinor frame

spinorial frame $\boldsymbol{\vartheta}_{\epsilon_1 \dots \epsilon_N}$ labeled by N-tuple of signs, $\epsilon_j = \pm$

$$\boldsymbol{\vartheta}_{\epsilon_1 \dots \epsilon_N} = \boldsymbol{\vartheta}_{\epsilon_1} \otimes \cdots \otimes \boldsymbol{\vartheta}_{\epsilon_N}$$

where $\{\boldsymbol{\vartheta}_+, \boldsymbol{\vartheta}_-\}$ is a frame in two-dimensional space $\mathbf{S}_x M$

spinorial components

spinorial components $\psi^{\epsilon_1 \dots \epsilon_N}$ of a spinor $\boldsymbol{\psi} \in \mathbf{D}M$

$$\boldsymbol{\psi} = \psi^{\epsilon_1 \dots \epsilon_N}\, \boldsymbol{\vartheta}_{\epsilon_1 \dots \epsilon_N}$$

Representation of the gamma matrices

identity and Pauli operators $\boldsymbol{I}$, $\boldsymbol{\iota}$, $\boldsymbol{\sigma}$, $\hat{\boldsymbol{\sigma}}$ on $\mathbf{S}M$ (in frame $\boldsymbol{\vartheta}_\pm$)

$$I^\epsilon{}_\varsigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \iota^\epsilon{}_\varsigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^\epsilon{}_\varsigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}^\epsilon{}_\varsigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

extension on $\mathbf{D}M$

$$\boldsymbol{\alpha}_{\langle\mu\rangle} = \boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I}$$

with $\boldsymbol{\alpha}$ on the μ-th place in the tensor product

$$\boldsymbol{\alpha}_{\langle\mu_1\dots\mu_j\rangle} = \boldsymbol{\alpha}_{\langle\mu_1\rangle} \otimes \cdots \otimes \boldsymbol{\alpha}_{\langle\mu_j\rangle}$$

gamma matrices with respect to vielbein $\{\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu\}$

$$\boldsymbol{\gamma}^\mu = \boldsymbol{\iota}_{\langle 1\dots\mu-1\rangle} \boldsymbol{\sigma}_{\langle\mu\rangle} \qquad \boldsymbol{\gamma}^{\hat{\mu}} = \boldsymbol{\iota}_{\langle 1\dots\mu-1\rangle} \hat{\boldsymbol{\sigma}}_{\langle\mu\rangle}$$

Explicit form of the operators

covariant derivative on Dirac spinors

$$\boldsymbol{\nabla}_a = \eth_a + \frac{1}{4}\,\omega_{abc}\,\boldsymbol{\gamma}^b\boldsymbol{\gamma}^c$$

$\eth_a$ is the vielbein derivative annihilating $\boldsymbol{e}^\mu$, $\hat{\boldsymbol{e}}^\mu$ and $\boldsymbol{\vartheta}_{\epsilon_1\dots\epsilon_N}$

${\omega_a}^b{}_c$ are spin coefficients of the metric connection, $\boldsymbol{\nabla}_a\,\boldsymbol{e}^b = -{\omega_a}^b{}_c\,\boldsymbol{e}^c$

operators corresponding to explicit symmetries

$$\mathcal{L}_{(k)} = \frac{\eth}{\partial\psi_k}$$

operators corresponding to hidden symmetries

$$\mathcal{M}_{(j)} = i^j\sum_\mu \sqrt{Q_\mu}\boldsymbol{B}_\mu^{(j)}\left[\frac{\eth}{\partial x_\mu}+\frac{X'_\mu}{4X_\mu}+\frac{1}{2}\sum_{\substack{\nu\\ \nu\neq\mu}}\frac{1}{x_\mu-\boldsymbol{\iota}_{\langle\mu\nu\rangle}x_\nu}-\frac{i\boldsymbol{\iota}_{\langle\mu\rangle}}{X_\mu}\sum_k(-x_\mu^2)^{N-1-k}\frac{\eth}{\partial\psi_k}\right]\boldsymbol{\gamma}^\mu$$

where $\boldsymbol{B}_\mu^{(k)} = \sum\limits_{\substack{\nu_1,\dots,\nu_k\\ \nu_1<\dots<\nu_k,\,\nu_i\neq\mu}} \boldsymbol{\iota}_{\langle\nu_1\rangle}x_{\nu_1}\cdots\;\boldsymbol{\iota}_{\langle\nu_k\rangle}x_{\nu_k}$

Tensorial R-separability

eigenvalue equations

$$\mathcal{L}_{(k)}\,\boldsymbol{\varphi} \;=\; \Psi_k\,\boldsymbol{\varphi} \qquad\qquad \mathcal{M}_{(j)}\,\boldsymbol{\varphi} \;=\; \Upsilon_j\,\boldsymbol{\varphi}$$

can be solved by the R-separability ansatz

$$\boldsymbol{\varphi} \;=\; \boldsymbol{R}\prod_k \exp\Big(i\Psi_k\psi_k\Big)\;\bigotimes_\nu \boldsymbol{\chi}_\nu(x_\nu)$$

where the prefactor $\boldsymbol{R}$ is

$$\boldsymbol{R} = \prod_{\substack{\kappa,\lambda\\ \kappa<\lambda}}\Big(x_\kappa + \boldsymbol{\iota}_{\langle\kappa\lambda\rangle}x_\lambda\Big)^{-\frac{1}{2}}$$

and each two-dimensional spinor $\boldsymbol{\chi}_\nu(x_\nu)$ satisfies an ordinary differential equation

$$\left[\Big[\frac{\eth}{\partial x_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu}\boldsymbol{\iota}_{\langle\nu\rangle}\Big]\,\boldsymbol{\sigma}_{\langle\nu\rangle} - \big(-\boldsymbol{\iota}_{\langle\nu\rangle}\big)^{N-\nu}\frac{\tilde{\Upsilon}_\nu}{\sqrt{|X_\nu|}}\right]\boldsymbol{\chi}_\nu = 0$$

with

$$\tilde{\Psi}_\mu = \sum_k \Psi_k\,(-x_\mu^2)^{N-1-k} \qquad\qquad \tilde{\Upsilon}_\mu = \sum_k \Upsilon_k\,(-x_\mu^2)^{N-1-k}$$

Separability in terms of spinorial components

components with respect to the frame $\boldsymbol{\vartheta}_{\epsilon_1\dots\epsilon_N}$, $\quad \epsilon_j = \pm$

$$\varphi^{\epsilon_1\dots\epsilon_N} = \phi_{\epsilon_1\dots\epsilon_N} \prod_k \exp\big(i\Psi_k\psi_k\big) \prod_\nu \chi_\nu^{\epsilon_\nu}(x_\mu)$$

- the matrix prefactor $\boldsymbol{R}$ is diagonal with respect to $\boldsymbol{\vartheta}_{\epsilon_1\dots\epsilon_N}$ with $\phi_{\epsilon_1\dots\epsilon_N}$ on the diagonal
- $\chi_\nu^\pm$ are two components of two-dimensional spinor $\boldsymbol{\chi}_\nu$

equations for χ_ν^+ and χ_ν^-

$$\left[\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu}\right]\chi_\nu^+ - \frac{\tilde{\Upsilon}_\nu}{\sqrt{|X_\nu|}}\chi_\nu^- = 0$$

$$\left[\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} - \frac{\tilde{\Psi}_\nu}{X_\nu}\right]\chi_\nu^- - (-1)^{N-\nu}\frac{\tilde{\Upsilon}_\nu}{\sqrt{|X_\nu|}}\chi_\nu^+ = 0$$

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Tensorial 'plain' separability

alternative gamma matrices

$$\bar{\boldsymbol{\gamma}}^a = \boldsymbol{R}^{-1}\,\boldsymbol{\gamma}^a\,\boldsymbol{R}$$

eigenvalue equations

$$\bar{\mathcal{L}}_{(k)}\,\bar{\boldsymbol{\varphi}} = \Psi_k\,\bar{\boldsymbol{\varphi}} \qquad \bar{\mathcal{M}}_{(j)}\,\bar{\boldsymbol{\varphi}} = \Upsilon_j\,\bar{\boldsymbol{\varphi}}$$

$$\bar{\mathcal{L}}_{(k)} = \boldsymbol{R}^{-1}\,\mathcal{L}_{(k)}\,\boldsymbol{R} = \mathcal{L}_{(k)} \qquad \bar{\mathcal{M}}_{(j)} = \boldsymbol{R}^{-1}\,\bar{\mathcal{M}}_{(j)}\,\boldsymbol{R} \qquad \bar{\boldsymbol{\varphi}} = \boldsymbol{R}^{-1}\,\boldsymbol{\varphi}$$

plain separability

$$\bar{\boldsymbol{\varphi}} = \prod_k \exp\Big(i\Psi_k\psi_k\Big) \bigotimes_\nu \boldsymbol{\chi}_\nu(x_\nu)$$

open question

The similarity transformation generated by $\boldsymbol{R}$ is spacetime dependent!

How is it with a 'locality' of the gamma matrices $\bar{\boldsymbol{\gamma}}^a$?

Shrnutí

Kerr-NUT-(A)dS prostoročasy:

- Integrabilita geodetického pohybu

- Úplná sada skalárních operátorů komutujících s vlnovým operátorem
- Vzájemná komutace operátorů a společné vlastní funkce
- Separabilita všech vlastních funkcí

- Úplná sada spinorových operátorů komutujících s Diracovým operátorem
- Vzájemná komutace operátorů a společné spinorové vlastní funkce
- Tenzorová separabilita všech spinorových vlastních funkcí

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