Hidden symmetries of Kerr-NUT-(A)dS geometry and structure of its Riemann tensor

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Outline

- **Kerr–NUT–(A)dS metric**
  a generally rotating black hole in higher dimensions
- **Explicit and hidden symmetries**
  objects describing symmetries
- **Principal CCKY form and Killing tower**
  building symmetries from CCKY form
- **Structure of Riemann tensor**
  alignment of curvature with CCKY form
- **Consequences for the geometry**
  integrability and commutativity
- **Related spaces**
  integrability without commutativity

Metric of the Kerr–NUT–(A)dS spacetime

(for simplicity only in even dimensions $D = 2N$)

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

x_μ	radial and latitudinal coordinates ($\mu = 1, \dots, N$)
ψ_k	temporal and longitudinal coordinates ($k = 0, \dots, N-1$)
$X_\mu = X_\mu(x_\mu)$	metric functions to be determined by the Einstein equations

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k,\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad U_\mu = \prod_{\substack{\nu \\ \nu\neq\mu}} (x_\nu^2 - x_\mu^2)$$

- Myers R. C., Perry M. J.: *Black Holes in Higher Dimensional Space-Times*, Ann.Phys. 172 (1986) 304
- Gibbons G. W., L H., Page D. N., Pope C. N.: *Rotating Black Holes in Higher Dimensions with a Cosmological Constant*, Phys.Rev.Lett. 93 (2004) 171102
- Chen W., L H., Pope C. N.: *General Kerr-NUT-AdS Metrics in All Dimensions*, Class.Quant.Grav. 23 (2006) 5323

Properties

- **Integrability of geodesic motion**
- **Separability of the Hamilton–Jacobi equations**
- **Commuting scalar symmetry operators**
- **Separability of the Klein–Gordon equations**
- **Commuting Dirac symmetry operators**
- **Separability of the Dirac equations**

Explicit and hidden symmetries

Explicit symmetries — N independent **Killing vectors**:

$$\boldsymbol{l}_{(j)} = \boldsymbol{\partial}_{\psi_j} \qquad j = 0, \ldots, N-1$$

Hidden symmetries — N independent **Killing tensors of rank 2**:

$$\boldsymbol{k}_{(j)} = \sum_{\mu} A_{\mu}^{(j)} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right] \qquad j = 0, \ldots, N-1$$

$$\boldsymbol{k}_{(0)} = \boldsymbol{g} \qquad \text{metric for } j = 0$$

Integrability of geodesic motion

Observables linear in momenta

$$L_{(j)} = \boldsymbol{l}^a_{(j)} \boldsymbol{p}_a$$

energy, angular momenta

Observables quadratic in momenta

$$K_{(j)} = \boldsymbol{k}^{ab}_{(j)} \boldsymbol{p}_a \boldsymbol{p}_b$$

Hamiltonian, Carter constants

Integrability — conserved quantities are in involution

$$\begin{aligned} \{L_{(i)}\,, L_{(j)}\} &= 0 \\ \{L_{(i)}\,, K_{(j)}\} &= 0 \\ \{K_{(i)}, K_{(j)}\} &= 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \big[\boldsymbol{l}_{(i)}\,, \boldsymbol{l}_{(j)}\big]_{\mathrm{NS}} &= 0 \\ \big[\boldsymbol{l}_{(i)}\,, \boldsymbol{k}_{(j)}\big]_{\mathrm{NS}} &= 0 \\ \big[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\big]_{\mathrm{NS}} &= 0 \end{aligned}$$

Nijenhuis–Schouten brackets: $\quad [\boldsymbol{a}, \boldsymbol{b}]_{\mathrm{NS}}^{n_1 \dots n_{r+s-1}} = s\, \boldsymbol{b}^{n(n_1 \dots} \boldsymbol{\nabla}_n\, \boldsymbol{a}^{\dots n_{r+s-1})} - r\, \boldsymbol{a}^{n(n_1 \dots} \boldsymbol{\nabla}_n\, \boldsymbol{b}^{\dots n_{r+s-1})}$

Proof:

- Polynomial constants of motion
  Page D. N., Kubizňák D., Vasudevan M., Krtouš P.: *Complete Integrability of Geodesic Motion in General Kerr-NUT-AdS Spacetimes*, Phys.Rev.Lett. 98 (2007) 061102
- Quadratic constants of motion and their equivalence with polynomial constants
  Krtouš P., Kubizňák D., Page D. N., Vasudevan M.: *Constants of Geodesic Motion in Higher-Dimensional Black-Hole Spacetimes*, Phys.Rev.D 76 (2007) 084034

Principal CCKY form — origin of the symmetries

Principal closed conformal Killing-Yano form $\boldsymbol{h}$:

- 2-form
- non-degeneracy
- functionally independent eigenvalues
- closed conformal Killing–Yano (CCKY) form

$$\boldsymbol{\nabla}_a \boldsymbol{h}_{bc} = \boldsymbol{g}_{ab}\boldsymbol{\xi}_c - \boldsymbol{g}_{ac}\boldsymbol{\xi}_b \qquad \text{with} \qquad \boldsymbol{\xi}_a = \tfrac{1}{D-1}\boldsymbol{\nabla}^n \boldsymbol{h}_{na}$$

Kerr–NUT–(A)dS geometry:

$$\boldsymbol{h} = \sum_\mu x_\mu\, \boldsymbol{d}x_\mu \wedge \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k\Big) = \frac{1}{2}\sum_j \boldsymbol{d}A^{(j+1)} \wedge \boldsymbol{d}\psi_j$$

- Kubizňák D., Frolov V. P.: *Hidden Symmetry of Higher Dimensional Kerr-NUT-AdS Spacetimes*, Class.Quant.Grav. 24 (2007) F1

Geometry admitting principal CCKY form

very strong restriction on the geometry which guarantees:

- **existence of Killing tower of symmetries**
- **NS-commutativity of Killing vectors and Killing tensors**
- **integrability of geodesic motion**
- **commutativity of scalar operators**

without reference to explicit form of the metric!

it determines the form of the metric:

- **geometry is given by off-shell Kerr–NUT–(A)dS metric**

Symmetry objects

Killing tensor (KT)

$$\nabla^{(a}\boldsymbol{k}^{bc\dots)} = 0$$

$\boldsymbol{k}$ symmetric

Conformal Killing tensor (CKT)

$$\nabla^{(a}\boldsymbol{q}^{bc\dots)} = \boldsymbol{g}^{(ab}\,\boldsymbol{\sigma}^{c\dots)}$$

$\boldsymbol{q}$ symmetric

$\boldsymbol{\sigma}$ = combination of traces of $\nabla\boldsymbol{q}$

Killing-Yano (KY) form

$$\nabla_a \boldsymbol{f}_{bc\dots} = \nabla_{[a}\boldsymbol{f}_{bc\dots]}$$

$\boldsymbol{f}$ antisymmetric

Closed conformal Killing-Yano (CCKY) form

$$\nabla_a \boldsymbol{h}_{bc\dots} = p\,\boldsymbol{g}_{a[b}\,\boldsymbol{\xi}_{c\dots]}$$

$\boldsymbol{h}$ antisymmetric p-form

$$\boldsymbol{\xi}_{c\dots} = \tfrac{1}{D-p+1}\nabla^n \boldsymbol{h}_{nc\dots}$$

Killing tower

$\boldsymbol{h}$ — principal CCKY form

$\left(\boldsymbol{\xi} = \frac{1}{D-1}\boldsymbol{\nabla}\cdot\boldsymbol{h}\right)$

$\boldsymbol{h}^{(j)} = \frac{1}{j!}\boldsymbol{h}^{\wedge j}$ **CCKY $2j$-forms**

$\boldsymbol{f}^{(j)} = *\,\boldsymbol{h}^{(j)}$ **KY $(D-2j)$-forms**

$\boldsymbol{Q}^{ab}_{(j)} = \frac{1}{(2j-1)!}\,\boldsymbol{h}^{(j)a}{}_{mn\ldots}\,\boldsymbol{h}^{(j)bmn\ldots}$ **CKTs of rank 2**

$\boldsymbol{k}^{ab}_{(j)} = \frac{1}{(D-2j-1)!}\,\boldsymbol{f}^{(j)a}{}_{mn\ldots}\,\boldsymbol{f}^{(j)bmn\ldots}$ **KTs of rank 2**

$\boldsymbol{l}^{a}_{(j)} = \boldsymbol{k}^{an}_{(j)}\,\boldsymbol{\xi}_n$ **Killing vectors = KTs of rank 1**

How to prove that $\boldsymbol{k}_{(j)}$ and $\boldsymbol{l}_{(j)}$ are KTs and have vanishing NS-brackets?

- Krtouš P., Kubizňák D., Page D. N., Frolov V. P.: *Killing–Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions*, JHEP02(2007)004

Generating functional

Tensors depending on an auxiliary parameter β

$$\boldsymbol{q}(\beta) = \boldsymbol{g} - \beta^2\, \boldsymbol{h}^2$$

$$A(\beta) = \left(\frac{\mathrm{Det}\,\boldsymbol{q}(\beta)}{\mathrm{Det}\,\boldsymbol{g}}\right)^{\frac{1}{2}}$$

$$\boldsymbol{k}(\beta) = A(\beta)\, \boldsymbol{q}(\beta)^{-1}$$

$$\boldsymbol{l}(\beta) = \boldsymbol{k}(\beta)\cdot\boldsymbol{\xi}$$

$$\Rightarrow$$

$$A(\beta) = \sum_j A^{(j)}\beta^{2j}$$

$$\boldsymbol{l}(\beta) = \sum_j \boldsymbol{l}_{(j)}\beta^{2j}$$

$$\boldsymbol{k}(\beta) = \sum_j \boldsymbol{k}_{(j)}\beta^{2j}$$

Vanishing NS-brackets

$$\big[\,\boldsymbol{l}(\beta_1)\,,\,\boldsymbol{l}(\beta_2)\,\big]_{\mathrm{NS}} = 0$$

$$\big[\,\boldsymbol{l}(\beta_1)\,,\boldsymbol{k}(\beta_2)\big]_{\mathrm{NS}} = 0$$

$$\big[\boldsymbol{k}(\beta_1), \boldsymbol{k}(\beta_2)\big]_{\mathrm{NS}} = 0$$

Since $\boldsymbol{k}(0) = \boldsymbol{g}$, it also implies that $\boldsymbol{l}(\beta)$ and $\boldsymbol{k}(\beta)$ are KTs.

Canonical frame

Eigenvalue problem for principal CCKY form $\boldsymbol{h}$ with normalization given by metric $\boldsymbol{g}$

$\Downarrow$

Coordinates x_μ and canonical normalized frames

$$\boldsymbol{e}_\mu,\quad \hat{\boldsymbol{e}}_\mu \qquad \mu = 1,\dots,N \qquad \text{vectors frame}$$

$$\boldsymbol{e}^\mu,\quad \hat{\boldsymbol{e}}^\mu \qquad \mu = 1,\dots,N \qquad \text{dual frame of 1-forms}$$

such that

$$\boldsymbol{h} = \sum_\mu x_\mu\, \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu \qquad\qquad \boldsymbol{e}^\mu \propto \boldsymbol{d}x_\mu$$

$\Downarrow$

$$\boldsymbol{q}(\beta) = \boldsymbol{g} - \beta^2 \boldsymbol{h}^2 \qquad = \sum_\nu (1+\beta^2 x_\nu^2)\big(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu\big)$$

$$A(\beta) = \Big(\frac{\operatorname{Det}\boldsymbol{q}(\beta)}{\operatorname{Det}\boldsymbol{g}}\Big)^{\frac{1}{2}} \qquad = \prod_\nu (1+\beta^2 x_\nu^2) \qquad = \sum_j A^{(j)}\beta^{2j}$$

$$\boldsymbol{k}(\beta) = A(\beta)\ \boldsymbol{q}(\beta)^{-1} \qquad = \sum_\mu \Big(\prod_{\substack{\nu\\ \nu\neq\mu}} (1+\beta^2 x_\mu^2)\Big)\big(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu\big) \qquad = \sum_j \boldsymbol{k}_{(j)}\,\beta^{2j}$$

where

$$\boldsymbol{k}_{(j)} = \sum_\mu A_\mu^{(j)}\big(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu\big) \qquad A^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j\\ \nu_1<\dots<\nu_j}} x_{\nu_1}^2\dots x_{\nu_j}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j\neq\mu\\ \nu_1<\dots<\nu_j}} x_{\nu_1}^2\dots x_{\nu_j}^2$$

Generating functional for CCKY and KY forms

$$\boldsymbol{H}(\beta) = \hat{\exp}(\beta \boldsymbol{h}) \qquad\qquad \boldsymbol{F}(\beta) = *\,\boldsymbol{H}(\beta)$$

non-homogeneus forms which satisfy

CCKY form condition

$$\boldsymbol{\nabla_a}\,\boldsymbol{H}(\beta) = \boldsymbol{a}\wedge\boldsymbol{\Xi}(\beta) \qquad \text{where} \qquad \boldsymbol{\Xi} = \beta\,\boldsymbol{\xi}\wedge\boldsymbol{H} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{H} = (D-\pi)\,\boldsymbol{\Xi}$$

KY form condition

$$\boldsymbol{\nabla_a}\,\boldsymbol{F}(\beta) = \boldsymbol{a}\cdot\boldsymbol{\Phi}(\beta) \qquad \text{where} \qquad \boldsymbol{\Phi} = -\beta\,\boldsymbol{\xi}\cdot\boldsymbol{F} \qquad \boldsymbol{\nabla}\wedge\boldsymbol{F} = \pi\,\boldsymbol{\Phi}$$

they generate tower of CCKY and KY forms

$$\boldsymbol{H}(\beta) = \sum_j \boldsymbol{h}^{(j)}\beta^j \qquad\qquad \boldsymbol{F}(\beta) = \sum_j \boldsymbol{f}^{(j)}\beta^j$$

Proof of vanishing NS-brackets

$$\boldsymbol{\nabla}_a \boldsymbol{h}_{bc} = \boldsymbol{g}_{ab}\boldsymbol{\xi}_c - \boldsymbol{g}_{ac}\boldsymbol{\xi}_b \qquad \Leftarrow \quad \text{CCKY form}$$

$$\boldsymbol{\nabla}_c \boldsymbol{q}_{ab} = 2\beta^2\big(\boldsymbol{g}_{c(a}\boldsymbol{h}_{b)n} + \boldsymbol{h}_{c(a}\boldsymbol{g}_{b)n}\big)\boldsymbol{\xi}^n \qquad \Leftarrow \quad \boldsymbol{q} = \boldsymbol{g} - \beta^2\boldsymbol{h}^2$$

$$\boldsymbol{\nabla}_a A = 2\beta^2 \boldsymbol{h}_{am}\boldsymbol{k}^{mn}\boldsymbol{\xi}_n \qquad \Leftarrow \quad A = (\mathrm{Det}\,\boldsymbol{q}/\,\mathrm{Det}\,\boldsymbol{g})^{\frac{1}{2}}$$

$$\boldsymbol{\nabla}^c \boldsymbol{k}^{ab} = \frac{2\beta^2}{A}\big(\boldsymbol{k}^{ab} \quad \boldsymbol{k}^{cn}\boldsymbol{h}_n{}^m + \boldsymbol{h}^m{}_n\boldsymbol{k}^{n(a}\boldsymbol{k}^{b)c} - \boldsymbol{k}^{m(a}\boldsymbol{k}^{b)n}\boldsymbol{h}_n{}^c\big)\,\boldsymbol{\xi}_m \qquad \Leftarrow \quad \boldsymbol{k} = A\,\boldsymbol{q}^{-1}$$

$$\boldsymbol{l}^a = \boldsymbol{k}^{an}\boldsymbol{\xi}_n$$

$\Downarrow$ substituting into NS-brackets

$$\big[\boldsymbol{k}_1, \boldsymbol{k}_2\big]^{abc}_{\mathrm{NS}} = 3\big(\boldsymbol{k}_1^{n(a}\boldsymbol{\nabla}_n\boldsymbol{k}_2^{bc)} - \boldsymbol{k}_2^{n(a}\boldsymbol{\nabla}_n\boldsymbol{k}_1^{bc)} \qquad = 0$$

$$\big[\boldsymbol{k}_1, \boldsymbol{l}_2\big]^{ab}_{\mathrm{NS}} = 2\,\boldsymbol{k}_1^{n(a}\boldsymbol{\nabla}_n\boldsymbol{l}_2^{b)} - \boldsymbol{l}_2^n\boldsymbol{\nabla}_n\boldsymbol{k}_1^{ab)} \qquad = \big(\boldsymbol{k}_1^{am}\,(\boldsymbol{\nabla}_m\boldsymbol{\xi}_n)\,\boldsymbol{k}_2^{nb} + \boldsymbol{k}_2^{am}\,(\boldsymbol{\nabla}_n\boldsymbol{\xi}_m)\,\boldsymbol{k}_1^{nb}\big)$$

$$\big[\boldsymbol{l}_1, \boldsymbol{l}_2\big]^{a}_{\mathrm{NS}} = \boldsymbol{l}_1^n\boldsymbol{\nabla}_n\boldsymbol{l}_2^{a)} - \boldsymbol{l}_2^n\boldsymbol{\nabla}_n\boldsymbol{l}_1^{a)} \qquad = \big(\boldsymbol{k}_1^{am}\,(\boldsymbol{\nabla}_m\boldsymbol{\xi}_n)\,\boldsymbol{k}_2^{nb} - \boldsymbol{k}_2^{am}\,(\boldsymbol{\nabla}_m\boldsymbol{\xi}_n)\,\boldsymbol{k}_1^{nb}\big)\,\boldsymbol{\xi}_b$$

$\Uparrow$

$$\mathcal{S}(\boldsymbol{\nabla}\boldsymbol{\xi}) = 0 \qquad \text{and} \qquad \big[\boldsymbol{\nabla}\boldsymbol{\xi}, \boldsymbol{k}(\beta)\big] = 0 \quad \Leftarrow \quad \big[\boldsymbol{\nabla}\boldsymbol{\xi}, \boldsymbol{h}\big] = 0$$

where $\boldsymbol{k}_1 = \boldsymbol{k}(\beta_1)$, $\boldsymbol{k}_2 = \boldsymbol{k}(\beta_2)$, $\boldsymbol{l}_1 = \boldsymbol{l}(\beta_1)$, etc.

Integrability conditions for principal CCKY form

$$\boldsymbol{\nabla}_a \boldsymbol{h}_{bc} = \boldsymbol{g}_{ab}\boldsymbol{\xi}_c - \boldsymbol{g}_{ac}\boldsymbol{\xi}_b$$

$\Downarrow$ taking second derivative $2\boldsymbol{\nabla}_{[a}\boldsymbol{\nabla}_{b]}\boldsymbol{h}_{mn}$

(IC1) $\boldsymbol{R}^{ab}{}_{c[m}\,\boldsymbol{h}^{c}{}_{n]} = 2\delta^{[a}_{[m}\boldsymbol{\nabla}^{b]}\boldsymbol{\xi}_{n]}$

$\Downarrow$ Bianchi identities, contractions

(IC2) $(D{-}2)\boldsymbol{\nabla}_a\boldsymbol{\xi}_b = -\mathbf{Ric}_{an}\,\boldsymbol{h}^{n}{}_{b} + \frac{1}{2}\boldsymbol{h}_{mn}\boldsymbol{R}^{mn}{}_{ab}$

$\Downarrow$ substituting (IC2) to (IC1)

(IC3) $(D{-}2)\boldsymbol{R}^{ab}{}_{n[c}\,\boldsymbol{h}^{n}{}_{d]} - \boldsymbol{h}_{mn}\boldsymbol{R}^{mn[a}{}_{[c}\,\delta^{b]}_{d]} - 2\mathbf{Ric}^{[a}{}_{n}\,\delta^{b]}_{[c}\,\boldsymbol{h}^{n}{}_{d]} = 0$

$\mathcal{S}(\boldsymbol{\nabla\xi}) = \big[\boldsymbol{h}, \mathbf{Ric}\big]$ $\Leftarrow$ symmetrization of (IC2)

$(D{-}2)\big[\boldsymbol{\nabla\xi}, \boldsymbol{h}\big] = \big[\boldsymbol{h}, \mathbf{Ric}\big]\cdot\boldsymbol{h} + \frac{1}{2}\big[\mathbf{Rh}, \boldsymbol{h}\big]$ $\Leftarrow$ $\big[(\mathrm{IC2}), \boldsymbol{h}\big]$

Alignment of Riemann tensor with principal CCKY form

Taking various (anti)symmetrizations and contractions of (IC3) with $\boldsymbol{h}$ one can prove:

$$\Big[\,\boldsymbol{h}\,,\,\mathbf{Rh}^{(p)}\Big] = 0 \qquad \text{where} \quad \mathbf{Rh}^{(p)a}{}_b = \boldsymbol{h}^p{}_{mn}\,\boldsymbol{R}^{mn\,a}{}_b$$

$$\mathbf{Rh} = \mathbf{Rh}^{(1)}$$

$$\Big[\,\boldsymbol{h}\,,\,\mathbf{Rich}^{(2p)}\Big] = 0 \qquad \text{where} \quad \mathbf{Rich}^{(2p)}_{ab} = \boldsymbol{h}^p{}_{mn}\,\boldsymbol{R}^m{}_a{}^n{}_b$$

$$\mathbf{Rich}^{(0)} = \mathbf{Ric}$$

Any contraction of $\boldsymbol{R}$ with any power of $\boldsymbol{h}$ commutes with $\boldsymbol{h}$

Integrability of geodesic motion

Any contraction of $\boldsymbol{R}$ with any power of $\boldsymbol{h}$ commutes with $\boldsymbol{h}$

$\Rightarrow$
$$\mathcal{S}(\boldsymbol{\nabla\xi}) = \big[\boldsymbol{h}, \mathbf{Ric}\big] = 0$$
$$\big[\boldsymbol{\nabla\xi}, \boldsymbol{h}\big] = \tfrac{1}{D-2}\Big(\big[\boldsymbol{h}, \mathbf{Ric}\big]\cdot\boldsymbol{h} + \frac{1}{2}\big[\mathbf{Rh}, \boldsymbol{h}\big]\Big) = 0$$

$\Rightarrow$ **Vanishing NS-brackets**

$$\big[\boldsymbol{k}_1, \boldsymbol{k}_2\big]_{\text{NS}} = 0 \qquad \big[\boldsymbol{k}_1, \boldsymbol{l}_2\big]_{\text{NS}} = 0 \qquad \big[\boldsymbol{l}_1, \boldsymbol{l}_2\big]_{\text{NS}} = 0$$
$$\big[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\big]_{\text{NS}} = 0 \quad \big[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\text{NS}} = 0 \quad \big[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\text{NS}} = 0$$

$\Rightarrow$ **Observables $K_{(j)}$ and $L_{(j)}$ are in involution**

Scalar field operators

$$\boldsymbol{p} \qquad \longrightarrow \qquad -\, i\boldsymbol{\nabla}$$

$$L_{(j)} = \boldsymbol{l}^a_{(j)}\, \boldsymbol{p}_a \qquad \longrightarrow \qquad \mathcal{L}_{(j)} = -\frac{i}{2}\Big[\boldsymbol{l}^a_{(j)}\boldsymbol{\nabla}_a + \boldsymbol{\nabla}_a\, \boldsymbol{l}^a_{(j)}\Big]$$

$$K_{(j)} = \boldsymbol{k}^{ab}_{(j)}\, \boldsymbol{p}_a\, \boldsymbol{p}_b \qquad \longrightarrow \qquad \mathcal{K}_{(j)} = -\boldsymbol{\nabla}_a\, \boldsymbol{k}^{ab}_{(j)}\boldsymbol{\nabla}_b$$

Operators $\mathcal{K}_{(i)}$, $\mathcal{L}_{(j)}$ depend on a choice of $\boldsymbol{\nabla}$

Commutativity of operators

$$\left[\mathcal{L}_{(i)}, \mathcal{L}_{(j)}\right] = 0 \qquad \left[\mathcal{K}_{(i)}, \mathcal{L}_{(j)}\right] = 0 \qquad \left[\mathcal{K}_{(i)}, \mathcal{K}_{(j)}\right] = 0$$

$$\Updownarrow$$

integrability conditions:

$$\left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\mathrm{NS}} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\mathrm{NS}} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\mathrm{NS}} = 0$$

$$+$$

anomalous conditions:

$$\boldsymbol{\nabla}_a \boldsymbol{k}^{ab}_{(i)} \boldsymbol{\nabla}_b \big(\boldsymbol{\nabla}_c \boldsymbol{l}^c_{(j)}\big) = 0 \qquad \boldsymbol{\nabla}_n \boldsymbol{m}^{na} = 0$$

where $$\boldsymbol{m}^{ab} = \frac{2}{3}\Big(\boldsymbol{k}^{c[a}_{(i)} \boldsymbol{\nabla}_d \boldsymbol{\nabla}_c \boldsymbol{k}^{b]d}_{(j)} - \boldsymbol{k}^{c[a}_{(j)} \boldsymbol{\nabla}_d \boldsymbol{\nabla}_c \boldsymbol{k}^{b]d}_{(i)}\Big) - \frac{2}{3}\big(\boldsymbol{\nabla}_d \boldsymbol{k}^{c[a}_{(i)}\big)\big(\boldsymbol{\nabla}_c \boldsymbol{k}^{b]d}_{(j)}\big) - 2\boldsymbol{k}^{c[a}_{(i)} \,\mathbf{Ric}_{cd}\, \boldsymbol{k}^{b]d}_{(j)}$$

- Kolář I., Krtouš P.: *Weak electromagnetic field admitting integrability in Kerr-NUT-(A)dS spacetimes*, Phys.Rev.D 91 (2015) ??????, arXiv:1504.00524

Wave operator and its symmetry operators

$\boldsymbol{\nabla}$ is Levi-Civita covariant derivative of the metric $\boldsymbol{g}$

$\mathcal{K}_{(0)} = \Box$ wave operator

commutativity is satisfied

(thanks to a special structure of Ricci tensor for derivative $\boldsymbol{\nabla}$)

$$\Downarrow$$

Operators $\mathcal{K}_{(i)}$, $\mathcal{L}_{(j)}$ are mutually commuting symmetry operators of $\Box$

- Sergyeyev A., Krtouš P.: *Complete Set of Commuting Symmetry Operators for Klein–Gordon Equation in Generalized Higher-Dimensional Kerr-NUT-(A)dS Spacetimes*, Phys.Rev.D 77 (2008) 044033

Separability

common eigenfunctions

$$\mathcal{L}_{(k)}\phi \;=\; \Psi_k\,\phi \qquad\qquad \mathcal{K}_{(j)}\phi \;=\; \Xi_j\,\phi$$

separability ansatz

$$\phi = \prod_{\mu} R_\mu(x_\mu)\;\prod_k \exp\Big(\,i\Psi_k\psi_k\Big)$$

with $R_\mu(x_\mu)$ satisfying ODEs

$$\big(X_\mu R_\mu{}'\big)' + \Bigg(\tilde{\Xi}_\mu - \frac{\tilde{\Psi}_\mu{}^2}{X_\mu}\Bigg)R_\mu = 0$$

where

$$\tilde{\Psi}_\mu = \sum_k \Psi_k\,(-x_\mu^2)^{N-1-k} \qquad\qquad \tilde{\Xi}_\mu = \sum_k \Xi_k\,(-x_\mu^2)^{N-1-k}$$

- Frolov V. P., Krtouš P., Kubizňák D.: *Separability of Hamilton–Jacobi and Klein-Gordon Equations in General Kerr-NUT-AdS Spacetimes*, JHEP02(2007)005
- Sergyeyev A., Krtouš P.: *Complete Set of Commuting Symmetry Operators for Klein–Gordon Equation in Generalized Higher-Dimensional Kerr-NUT-(A)dS Spacetimes*, Phys.Rev.D 77 (2008) 044033

Related geometries

— "cousins" to Kerr–NUT–(A)dS

2nd-rank Killing tensors $\boldsymbol{k}_{(i)}$ can play a role of an inverse metric!

"cousin" geometry given by the metric

$$ {}^{(i)}\boldsymbol{g} \quad = \quad \boldsymbol{k}_{(i)}^{-1} $$

defines covariant derivative and curvature

$$ {}^{(i)}\boldsymbol{g} \quad \rightarrow \quad {}^{(i)}\boldsymbol{\nabla} \quad \rightarrow \quad {}^{(i)}\boldsymbol{R}\,, \quad {}^{(i)}\mathbf{Ric} $$

Integrability and commutativity for related geometries

Commutativity of classical observables

$$\big[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\big]_{\text{NS}} = 0 \qquad \big[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\text{NS}} = 0 \qquad \big[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\text{NS}} = 0$$

NS-brackets do not depend on the choice of the geometry!

$$\Downarrow$$

- **Complete integrability of geodesic motion for all geometries $^{(i)}\boldsymbol{g}$**
- **$\boldsymbol{l}_{(j)}$ and $\boldsymbol{k}_{(j)}$ are Killing vectors and tensors for all geometries $^{(i)}\boldsymbol{g}$**

(Non)commutativity of scalar operators

Operators defined using covariant derivative $^{(i)}\boldsymbol{\nabla}$

Anomalous conditions are not satisfied for geometries $^{(i)}\boldsymbol{g}$, $i > 0$

$$\Downarrow$$

- **Scalar operators do not commute in spaces $^{(i)}\boldsymbol{g}$, $i > 0$**

Obstructions for $^{(i)}\boldsymbol{g}$, $i > 0$:

- $\boldsymbol{h}$ is not the CCKY form
- curvature $^{(i)}\boldsymbol{R}$ is not aligned with $\boldsymbol{h}$, namely, $[\boldsymbol{k}_{(j)}, {}^{(i)}\mathbf{Ric}] \neq 0$

Uniqueness of geometries admitting principal CCKY form

Geometry admits the principal CCKY form

$$\Downarrow$$

The metric can be written in the off-shell Kerr–NUT–(A)dS form

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{k=0}^{n-1} A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

- Houri T., Oota T., Yasui Y.: *Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness*, Phys.Lett.B 656 (2007) 214
- Krtouš P., Frolov V.P., Kubizňák D.: *Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime*, Phys.Rev.D 78 (2008) 064022

Summary

Principal CCKY form:

- **defines Killing tower of explicit and hidden symmetries**
- **implies alignment of the curvature tensors**
- **guarantees complete integrability of geodesic motion**
- **guarantees commutativity of scalar operators**
- **determines the geometry**

- The existence of a tower of Killing vectors and tensors *without* the principal CCKY form does not guarantee commutativity of scalar operators