Hidden symmetries and test fields in rotating black hole spacetimes

part II

Separability

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with

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Contents

Off-shell Kerr–NUT–(A)dS geometry

geometry given by the existence of hidden symmetries encoded in the principal tensor

(non-degenerate closed conformal Killing–Yano tensor of rank 2)

Off-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit polynomials in coordinates x_μ

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1 \\ \nu_1<\dots<\nu_k}}^{N} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\dots<\nu_j \\ \nu_i\neq\mu}}^{N} x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

unspecified N metric functions of one variable

$$X_\mu = X_\mu(x_\mu)$$

On-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Einstein equations $\Rightarrow$

$$X_\mu \quad = \quad \lambda \prod_{\nu=1}^{N} (a_\nu^2 - x_\mu^2) - 2b_\mu \, x_\mu$$

Parameters:

λ	cosmological parameter related to the cosmological constant $\Lambda = (2N-1)(N-1)\lambda$
b_μ	mass and NUT parameters
a_μ	rotational parameters

freedom in scaling of coordinates $\Rightarrow$ one parameter can be fixed by a gauge condition

exact interpretation of parameters depends on coordinate ranges, signature, and gauge choices

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Wick rotation:

time: $\tau = \psi_0$ radial coordinate: $x_N = ir$ mass: $b_N = im$

Gauge condition:

$a_N^2 = -\frac{1}{\lambda}$ (suitable for limit $\lambda \to 0$)

New Killing coordinates:

$$t = \tau + \sum_{\bar{k}} \bar{\mathcal{A}}^{(\bar{k}+1)} \bar{\psi}_{\bar{k}} \qquad \frac{\phi_{\bar{\mu}}}{a_{\bar{\mu}}} = \lambda\tau - \sum_{\bar{k}} \big(\bar{\mathcal{A}}_{\bar{\mu}}^{(\bar{k})} - \lambda \bar{\mathcal{A}}_{\bar{\mu}}^{(\bar{k}+1)} \big) \bar{\psi}_{\bar{k}}$$

barred quantities refer to ranges of indices given by $N \to \bar{N} = N - 1$

(i.e., $\bar{\mu} = 1, \dots, \bar{N}$ and $\bar{j} = 0, \dots, \bar{N} - 1$, etc.)

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\left(\prod_{\bar{\nu}}\frac{1+\lambda x_{\bar{\nu}}^2}{1+\lambda a_{\bar{\nu}}^2}\,\boldsymbol{d}t - \sum_{\bar{\nu}}\frac{\bar{J}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1+\lambda a_{\bar{\nu}}^2)\bar{\mathcal{U}}_{\bar{\nu}}}\boldsymbol{d}\phi_{\bar{\nu}}\right)^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2$$

$$+\sum_{\bar{\mu}}\frac{(r^2{+}x_{\bar{\mu}}^2)}{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}\,\boldsymbol{d}x_{\bar{\mu}}^2 + \sum_{\bar{\mu}}\frac{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}{(r^2{+}x_{\bar{\mu}}^2)}\left(\frac{1{-}\lambda r^2}{1{+}\lambda x_{\bar{\mu}}^2}\prod_{\bar{\nu}}\frac{1{+}\lambda x_{\bar{\nu}}^2}{1{+}\lambda a_{\bar{\nu}}^2}\,\boldsymbol{d}t + \sum_{\bar{\nu}}\frac{(r^2{+}a_{\bar{\nu}}^2)\,\bar{J}_{\bar{\mu}}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1{+}\lambda a_{\bar{\nu}}^2)\,\bar{\mathcal{U}}_{\bar{\nu}}}\boldsymbol{d}\phi_{\bar{\nu}}\right)^2$$

$$\Delta_r = -X_N = \big(1{-}\lambda r^2\big)\prod_{\bar{\nu}}\big(r^2{+}a_{\bar{\nu}}^2\big) - 2mr \qquad \Sigma = U_N = \prod_{\bar{\nu}}(r^2 + x_{\bar{\nu}}^2)$$

$$\Delta_{\bar{\mu}} = -X_{\bar{\mu}} = \big(1{+}\lambda x_{\bar{\mu}}^2\big)\bar{\mathcal{J}}(x_{\bar{\mu}}^2) + 2b_{\bar{\mu}}x_{\bar{\mu}} \qquad \bar{U}_{\bar{\mu}} = \prod_{\substack{\bar{\nu}\\ \bar{\nu}\neq\bar{\mu}}}(x_{\bar{\nu}}^2 - x_{\bar{\mu}}^2)$$

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\left(\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t - \sum_{\bar\nu}\frac{\bar{J}(a_{\bar\nu}^2)}{a_{\bar\nu}(1+\lambda a_{\bar\nu}^2)\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2$$

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Rotating black holes with NUTs and Λ

Off-shell Kerr–NUT–(A)dS geometry

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit function polynomial in coordinates x_μ (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{N} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1\\ \nu_1<\dots<\nu_j\\ \nu_i\neq\mu}}^{N} x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

unspecified N metric functions of one variable

$$X_\mu = X_\mu(x_\mu)$$

Killing tower and hidden symmetries

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Principal tensor

$$\boldsymbol{h} = \sum_\mu x_\mu \, \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu$$

Primary Killing vector

$$\boldsymbol{\xi} = \boldsymbol{\partial}_{\psi_0}$$

Hidden symmetries – Killing tensors

$$\boldsymbol{k}_{(j)} = \sum_\mu A_\mu^{(j)} \left(\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu \right)$$

Explicit symmetries – Killing vectors

$$\boldsymbol{l}_{(j)} = \boldsymbol{\partial}_{\psi_j}$$

$$\left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\text{NS}} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\text{NS}} = 0 \qquad \left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\text{NS}} = 0$$

Darboux frame

forms:

$$\boldsymbol{e}^\mu = \Big(\frac{U_\mu}{X_\mu}\Big)^{\frac{1}{2}} \boldsymbol{d}x_\mu \qquad \hat{\boldsymbol{e}}^\mu = \Big(\frac{X_\mu}{U_\mu}\Big)^{\frac{1}{2}} \sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j$$

vectors:

$$\boldsymbol{e}_\mu = \Big(\frac{X_\mu}{U_\mu}\Big)^{\frac{1}{2}} \boldsymbol{\partial}_{x_\mu} \qquad \hat{\boldsymbol{e}}_\mu = \Big(\frac{U_\mu}{X_\mu}\Big)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}$$

Integrability of the geodesic motion

Killing tower

hidden symmetries – Killing tensors of rank 2

$$\boldsymbol{k}_{(j)}$$

explicit symmetries – Killing vectors

$$\boldsymbol{l}_{(j)}$$

commutation in sense of Nijenhuis–Schouten brackets

$$\left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\rm NS} = 0$$

Conserved quantities

observables quadratic in momentum

$$K_j = \boldsymbol{p} \cdot \boldsymbol{k}_{(j)} \cdot \boldsymbol{p}$$

observables linear in momentum

$$L_j = \boldsymbol{l}_{(j)} \cdot \boldsymbol{p}$$

Hamiltonian

$$\boldsymbol{k}_{(0)} = \boldsymbol{g} \qquad \Rightarrow \qquad H = \frac{1}{2} K_0 = \frac{1}{2} \boldsymbol{p}^2$$

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Integrability

D observables K_j and L_j which are in involution and commute with the Hamiltonian

$$\big\{K_i,K_j\big\}=0 \qquad \big\{K_i,L_j\big\}=0 \qquad \big\{L_i,L_j\big\}=0$$

Integrability of the geodesic motion

Killing tower

hidden symmetries – Killing tensors of rank 2

$$\boldsymbol{k}_{(j)}$$

explicit symmetries – Killing vectors

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$$\big\{K_i, K_j\big\} = 0 \qquad \big\{K_i, L_j\big\} = 0 \qquad \big\{L_i, L_j\big\} = 0$$

Separability of Hamilton–Jacobi equation

Separability of field equations

The field equations on the off-shell Kerr–NUT–(A)dS background can be separated for:

- **Scalar field** [2007] VF, PK, DK, A. Seregyeyev
- **Dirac field** [2011] T. Oota, Y. Yasui, M. Cariglia, PK, DK
- **Vector field** [2017] O. Lunin, VF, PK, DK
  (including the electromagnetic case)

Standard method of separation of variables

Equation for a sum of terms depending on different variables

$$\sum_{\mu=1}^{N} f_\mu(x_\mu) = S$$

implies

$$f_\mu(x_\mu) = S_\mu$$

where S_μ are constants satisfying $\displaystyle\sum_{\mu=1}^{N} S_\mu = S$

Solution is given by $N-1$ independent separation constants

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Solution is given by $N-1$ independent separation constants

U-separation of variables

Equation for a *composition* of terms depending on different variables

$$\sum_{\mu=1}^{N} \frac{1}{U_\mu} f_\mu(x_\mu) = C_0$$

implies

$$f_\mu(x_\mu) = \tilde{C}_\mu$$

where $\tilde{C}_\mu = \displaystyle\sum_{j=0}^{N-1} C_j (-x_\mu^2)^{N-1-j}$ are polynomials of degree $N-1$ with the same coefficients C_j

Solution is given by $N-1$ independent separation constants C_j

U-separation of variables

Equation for a *composition* of terms depending on different variables

$$\sum_{\mu=1}^{N} \frac{1}{U_\mu} f_\mu(x_\mu) = C_0 \qquad\qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

implies

$$f_\mu(x_\mu) = \tilde{C}_\mu \qquad\qquad \tilde{C}_\mu = \sum_{j=0}^{N-1} C_j (-x_\mu^2)^{N-1-j}$$

$\tilde{C}_\mu$ are polynomials in variable x_μ^2 of degree $N-1$ with the same coefficients C_j

Solution is given by $N-1$ independent separation constants C_j, $j = 1, \ldots, N-1$

Related to the fact that $A_\mu^{(j)}$ and $\frac{(-x_\mu^2)^{N-1-j}}{U_\mu}$ are inverse matrices

Separability of the scalar field equation

Massive scalar field equation

$$\Box\phi - m^2\phi = 0 \qquad\qquad \Box = \boldsymbol{\nabla}\cdot\boldsymbol{g}\cdot\boldsymbol{\nabla}$$

Separability of the scalar field equation

Scalar field equation as the eigenfunction equation

$$\mathcal{K}_0\,\phi = K_0\,\phi \qquad\qquad \mathcal{K}_0 = -\boldsymbol{\nabla}\cdot\boldsymbol{g}\cdot\boldsymbol{\nabla} \qquad K_0 = -m^2$$

Multiplicative separation ansatz

$$\phi = \prod_{\mu=1}^{N} R_\mu \prod_{k=0}^{N-1} \exp\bigl(iL_k\psi_k\bigr) \qquad\qquad R_\mu = R_\mu(x_\mu)$$

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Eigenfunction equation in coordinates

$$\mathcal{K}_0\,\phi = K_0\,\phi \qquad \Leftrightarrow \qquad \sum_\nu \frac{1}{U_\nu}\,\frac{1}{R_\nu}\biggl(\Bigl(X_\nu R'_\nu\Bigr)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu\biggr) = K_0$$

Separability of the scalar field equation

Scalar field equation as the eigenfunction equation

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Eigenfunction equation in coordinates

$$\mathcal{K}_0\,\phi = K_0\,\phi \qquad \Leftrightarrow \qquad \sum_\nu \frac{1}{U_\nu} \underbrace{\frac{1}{R_\nu}\bigg(\Big(X_\nu R_\nu'\Big)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu\bigg)}_{=\tilde{K}_\mu} = K_0$$

Separated ODEs

$$\Big(X_\nu R_\nu'\Big)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu - \tilde{K}_\nu R_\nu = 0$$

Separation constants as eigenvalues

Multiplicative separation ansatz

$$\phi = \prod_{\mu=1}^{N} R_\mu \prod_{k=0}^{N-1} \exp\big(iL_k\psi_k\big) \qquad\qquad R_\mu = R_\mu(\,x_\mu\,;\,K_j\,,L_j\,)$$

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Separated ODEs

$$\Bigl(X_\nu R'_\nu\Bigr)' - \frac{\tilde{L}_\nu^2}{X_\nu} R_\nu - \tilde{K}_\nu R_\nu = 0$$

Symmetry operators corresponding to the Killing tower

$$\mathcal{K}_j = -\boldsymbol{\nabla}_a\, \boldsymbol{k}_{(j)}^{ab}\, \boldsymbol{\nabla}_b \qquad\qquad \boldsymbol{\mathcal{L}}_j = -i\, \boldsymbol{l}_{(j)}^a \boldsymbol{\nabla}_a$$

Commutativity of the symmetry operators

$$\bigl[\mathcal{K}_k, \mathcal{K}_l\bigr] = 0 \qquad \bigl[\mathcal{K}_k, \boldsymbol{\mathcal{L}}_l\bigr] = 0 \qquad \bigl[\boldsymbol{\mathcal{L}}_k, \boldsymbol{\mathcal{L}}_l\bigr] = 0$$

Separation constants as eigenvalues

Multiplicative separation ansatz

$$\phi = \prod_{\mu=1}^{N} R_\mu \prod_{k=0}^{N-1} \exp\big(iL_k\psi_k\big) \qquad\qquad R_\mu = R_\mu(\, x_\mu\, ;\, K_j\, , L_j\,)$$

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Common eigenvalue problem with the eigenvalues given by the separation constants

$$\mathcal{K}_j\phi = K_j\phi \qquad\qquad \boldsymbol{\mathcal{L}}_j\phi = L_j\phi$$

Separability of the vector field equation

Proca field equation

$$\boldsymbol{\nabla}\cdot\mathbf{F} = m^2\,\mathbf{A} \qquad\qquad \mathbf{F} = \boldsymbol{d}\mathbf{A}$$

Vacuum Maxwell equations: $\boldsymbol{m^2 = 0}$

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 0 \qquad\qquad \mathbf{F} = \boldsymbol{d}\mathbf{A}$$

Lorenz condition

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0 \qquad \Leftarrow \quad \text{Proca equation}$$

Lunin's ansatz

Covariant form of the ansatz

$$\mathsf{A} = \boldsymbol{B} \cdot \boldsymbol{\nabla} Z$$

Polarization tensor $\boldsymbol{B}$

$$\boldsymbol{B} = (\boldsymbol{g} - \beta \boldsymbol{h})^{-1}$$

$\boldsymbol{B} \equiv \boldsymbol{B}(\beta)$ depends on an auxiliary parameter β

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Multiplicative separation ansatz

$$Z = \prod_{\mu=1}^{N} R_\mu \prod_{k=0}^{N-1} \exp\big(i L_k \psi_k\big)$$

$$R_\mu = R_\mu(x_\mu)$$

Vector field equations

Proca equation follows from the Lorenz condition and:

$$\Big[\Box + 2\beta\,\boldsymbol{\xi}\cdot\boldsymbol{B}\cdot\boldsymbol{\nabla}\Big]Z = m^2\,Z$$

$$\Updownarrow$$

$$\sum_\nu \frac{1}{U_\nu}\,\frac{1}{R_\nu}\bigg((1{+}\beta^2x_\nu^2)\Big(\frac{X_\nu}{1{+}\beta^2x_\nu^2}R'_\nu\Big)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu + i\beta\frac{1{-}\beta^2x_\nu^2}{1{+}\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu\bigg) = C_0 \equiv m^2$$

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Proca equation follows from the Lorenz condition and:

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$$\sum_\nu \frac{1}{U_\nu}\underbrace{\frac{1}{R_\nu}\left((1{+}\beta^2x_\nu^2)\left(\frac{X_\nu}{1{+}\beta^2x_\nu^2}R_\nu'\right)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu + i\beta\frac{1{-}\beta^2x_\nu^2}{1{+}\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu\right)}_{=\tilde{C}_\nu} = C_0 \equiv m^2$$

Separated ODEs:

$$(1{+}\beta^2x_\nu^2)\left(\frac{X_\nu}{1{+}\beta^2x_\nu^2}R_\nu'\right)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu + i\beta\frac{1{-}\beta^2x_\nu^2}{1{+}\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu - \tilde{C}_\nu R_\nu = 0$$

$$\tilde{C}_\nu = \sum_{j=0}^{N-1} C_j(-x_\nu^2)^{N-1-j}$$

separation constants C_j, $j = 0, \ldots, N-1$

Vector field equations

Lorenz condition:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}\cdot\boldsymbol{\nabla}Z = 0$$

$$\Updownarrow$$

$$\sum_\nu \frac{1}{U_\nu}\,\frac{1}{1+\beta^2x_\nu^2}\frac{1}{R_\nu}\left((1+\beta^2x_\nu^2)\Big(\frac{X_\nu}{1+\beta^2x_\nu^2}R_\nu'\Big)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu + i\beta\frac{1-\beta^2x_\nu^2}{1+\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu\right) = 0$$

Vector field equations

Lorenz condition:

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$$\Updownarrow$$

$$\sum_\nu \frac{1}{U_\nu}\,\frac{1}{1+\beta^2x_\nu^2}\underbrace{\frac{1}{R_\nu}\left((1+\beta^2x_\nu^2)\left(\frac{X_\nu}{1+\beta^2x_\nu^2}R'_\nu\right)'-\frac{\tilde{L}_\nu^2}{X_\nu}R_\nu+i\beta\frac{1-\beta^2x_\nu^2}{1+\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu\right)}_{=\tilde{C}_\nu}=0$$

Vector field equations

Lorenz condition:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}\cdot\boldsymbol{\nabla}Z = 0$$

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$$\underbrace{\sum_\nu \frac{1}{U_\nu}\,\frac{1}{1+\beta^2x_\nu^2}\underbrace{\frac{1}{R_\nu}\left((1+\beta^2x_\nu^2)\left(\frac{X_\nu}{1+\beta^2x_\nu^2}R'_\nu\right)' - \frac{\tilde{L}_\nu^2}{X_\nu}R_\nu + i\beta\frac{1-\beta^2x_\nu^2}{1+\beta^2x_\nu^2}\beta^{2(1-N)}LR_\nu\right)}_{=\tilde{C}_\nu}}_{\propto C(\beta^2)} = 0$$

Condition on β:

$$C(\beta^2) \equiv \sum_{j=0}^{N-1} C_j\beta^{2j} = 0$$

β^2 must be a root of the polynomial with coefficients given by the separation constants

Polarizations (just guessing)

Condition on β:

$$C(\beta^2) \equiv \sum_{j=0}^{N-1} C_j \beta^{2j} = 0$$

β^2 must be a root of the polynomial with coefficients given by the separation constants

Polarization modes

choice of the root $\quad\Leftrightarrow\quad$ choice of the polarization

Polarizations (just guessing)

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$$C(\beta^2) \equiv \sum_{j=0}^{N-1} C_j \beta^{2j} = 0$$

β^2 must be a root of the polynomial with coefficients given by the separation constants

Polarization modes

choice of the root $\Leftrightarrow$ choice of the polarization

- $N-1$ roots
- $N-1$ complex polarizations
- $2N-2 = D-2$ real polarizations
- 1 polarization missing!

Quasi-normal modes of Proca field in 4 dimensions

[with J. E. Santos]

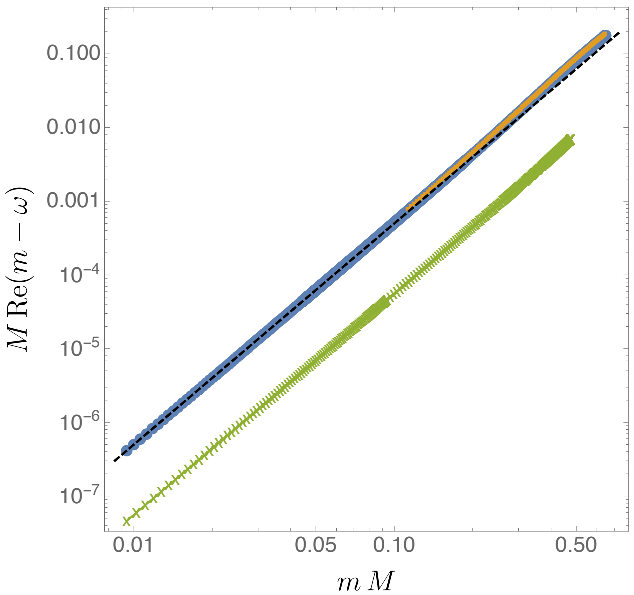
- Kerr solution in 4 dimensions
- massive vector field
- numerical search for quasi-normal modes

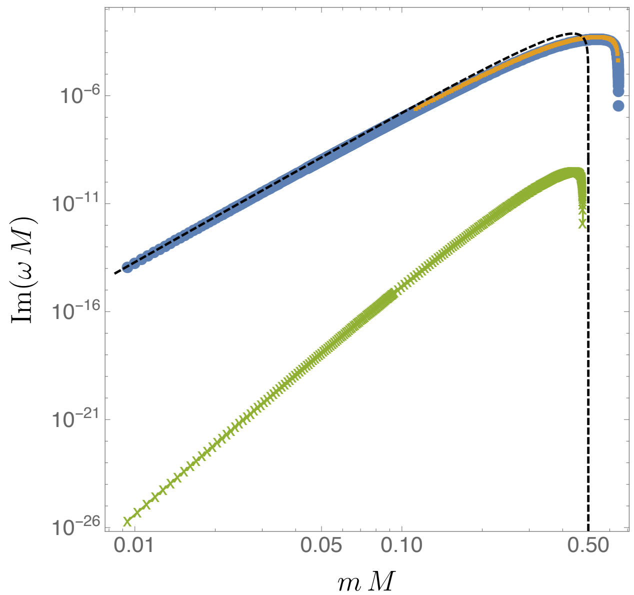
Quasi-normal modes of Proca field in 4 dimensions

[with J. E. Santos]

- Kerr solution in 4 dimensions
- massive vector field
- numerical search for quasi-normal modes

- separability allows much more effective calculations
- agreement with previous results based on numeric solution of PDEs and analytic aproximations
- possibility to extend a "computable" range of parameters
- recovering 2 polarizations (out of 3)





Quasi-normal modes of Proca field in 4 dimensions

Sam R. Dolan: *Instability of the Proca field on Kerr spacetime*, arXiv 1806.01604 (yesterday)

All three polarizations recovered!

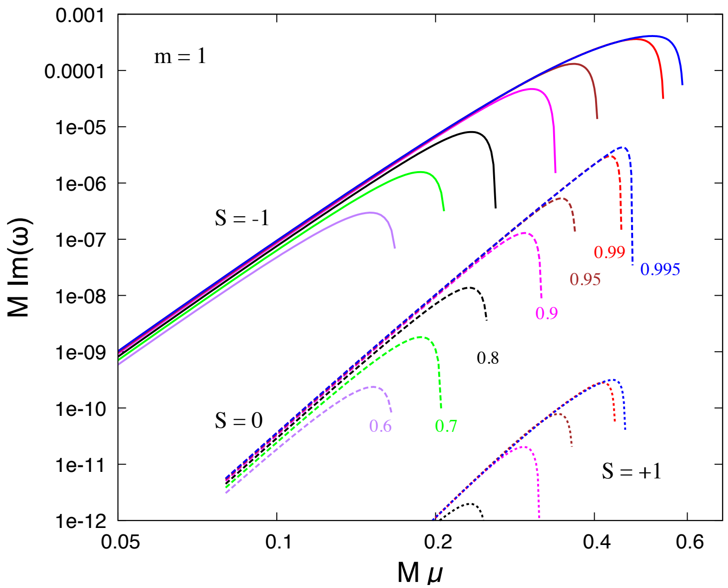


FIG. 1. Growth rate of the fundamental ($\hat{n} = 0$) corotating dipole ($m = 1$) modes of the Proca field, for the three polarizations $S = -1$ [solid], $S = 0$ [dashed] and $S = +1$ [dotted], and for BH spins of $a/M \in \{0.6, 0.7, 0.8, 0.9, 0.95, 0.99, 0.995\}$. The vertical axis shows the growth rate $\tau^{-1} = (GM/c^3)\text{Im}(\omega)$ on a logarithmic scale, and the horizontal axis shows $M\mu$.

Quasi-normal modes of Proca field in 4 dimensions

So, how is it with those polarizations?

- New Yorker Cartoon Poster Print
by Sam Gross at the Condé Nast Collection

Summary

Hidden symmetries encoded in the principal tensor determine geometry

- Off-shell Kerr–NUT–(A)dS spacetimes
  - includes generally rotating BH in higher dimensions
  - includes charged BH in 4 dimensions
  - includes conformally rescaled Plebański–Demiański metric

These spacetimes have nice properties:

- Integrability of the geodesic motion
- Separability of standard field equations
  - scalar field
  - Dirac field
  - vector field (including EM)

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    suitable for calculation of quasi-normal modes and study of an instability of the Proca field

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Scalar field

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