Hyperbolicity of the BSSN system

BSSN variables

The 3-metric γ_{ij} is decomposed into the tensor $\bar{\gamma}_{ij}$ and the scalar ϕ according to

$$\gamma_{ij} \equiv e^{4\phi}\bar{\gamma}_{ij} \tag{1}$$

so that $\det \bar{\gamma}_{ij} = 1$. Similarly

$$K_{ij} = e^{4\phi}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{2}$$

and since $K \equiv \gamma^{ij}K_{ij}$, we have tr $\bar{A}_{ij} = 0$. New quantities

$$\bar{\Gamma}^i = \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} \tag{3}$$

are introduced as independent variables.

These details are only partially important for the problem of hyperbolicity, because evolution equations determine all quantities. But these definitions are needed to determine background solution with respect to which we then consider linear perturbations around fixed background geometry and show that these perturbations evolve as waves without exponential or polynomial growth in time. Thus we have

Problem 5a

Find constant background values of $\alpha, \psi, K, \bar{\gamma}_{ij}, \bar{A}_{ij}$ and $\bar{\Gamma}^i$ for Minkowski spacetime (i.e. in Minkowski-Cartesian coordinates with $\beta^i \equiv 0$).

BSSN evolution equations

The Einstein equations with Bona-Massó generalized harmonic slicing condition (introducing constant parameter f) in these variables read

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^i\partial_i\phi + \frac{1}{6}\partial_i\beta^i\,, \tag{4}$$

$$\partial_t\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k\,, \tag{5}$$

$$\partial_t K = -\gamma^{ij}D_jD_i\alpha + \alpha(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2) + 4\pi\alpha(\rho + S) + \beta^i\partial_i K\,, \tag{6}$$

$$\begin{aligned}\partial_t\bar{A}_{ij} =& e^{-4\phi}(-(D_iD_j\alpha) + \alpha(R_{ij} - 8\pi S_{ij}))^{TF} + \alpha(K\bar{A}_{ij} - 2\bar{A}_{il}\bar{A}^l_j) \\ &+ \beta^k\partial_k\bar{A}_{ij} + \bar{A}_{ik}\partial_j\beta^k + \bar{A}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{A}_{ij}\partial_k\beta^k\,,\end{aligned} \tag{7}$$

$$\partial_t\bar{\Gamma}^i = 2\left(\bar{\Gamma}^i_{jk}\bar{A}^{jk} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\bar{A}^{ij}\partial_j\phi\right)\,, \tag{8}$$

$$\partial_t\alpha = -f\alpha K + \beta^i\partial_i\alpha\,, \tag{9}$$

$(...)^{\mathrm{TF}}$ means trace-free part of the tensor expression in brackets, e.g. $(W_{ij})^{TF} = W_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl}W_{kl} = W_{ij} - \frac{1}{3}\bar{\gamma}_{ij}\bar{\gamma}^{kl}W_{kl}$. (Note that for linear perturbation $\bar{\gamma}_{ij} = \delta_{ij} + \bar{h}_{ij}$ we get $(\bar{\gamma}_{ij})^{\mathrm{TF}} = \bar{h}_{ij}$.) The BSSN approach uses the partition $R_{ij} = \bar{R}_{ij} + R^{\phi}_{ij}$ where

$$\bar{R}_{ij} \equiv -\frac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}\left(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj}\right) \tag{10}$$

and

$$R^{\phi}_{ij} \equiv -2\bar{D}_i\bar{D}_j\phi - 2\bar{\gamma}_{ij}\bar{D}^2\phi + 4\bar{D}_i\phi\bar{D}_j\phi - 4\bar{\gamma}_{ij}\left(\bar{D}\phi\right)^2\,, \tag{11}$$

where $\bar{D}$ is the covariant derivative associated with $\bar{\gamma}_{ij}$ (this detail is not important for linearisation around a flat background metric).

Problem 5b

Show that all linear perturbation of this system with vanishing matter sources and $\beta^i \equiv 0$ evolve as waves with speeds $\pm 1, \pm\sqrt{f}$ or stay constant. (You may start with the code
`https://utf.mff.cuni.cz/~ledvinka/PrNumrel/2020/ADM-BM-v2.nb`).

Note: The constraints $\mathrm{tr}\bar{A}_{ij} = 0$ and $\det\bar{\gamma}_{ij} = 1$ are conserved by the evolution equations and so you can take all components of $\bar{A}_{ij}$ and $\bar{\gamma}_{ij}$ independent.

The constraint equations of this system read

$$\mathcal{H} \equiv \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho = 0$$

and

$$\mathcal{M}^i \equiv \bar{D}_j(e^{6\phi}\tilde{A}^{ij}) - \frac{2}{3}e^{6\phi}\bar{D}^i K - 8\pi e^{10\phi}S^i = 0\,. \tag{12}$$

Problem 5c

How do perturbations of all variables corresponding to a harmonic gravitational wave (with polarization and direction you choose) look like? Check that they satisfy constraints.

Example: For electromagnetic waves you can say, that such wave with amplitude a propagating in direction $\vec{e}_x$ may have nonzero components of $B_z = E_y = ae^{i\omega(x-t)}$ (but E_x must be zero so that the constraint $\mathrm{div}\vec{E} = 0$ holds).