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Construction of Auxiliary Hilbert Space for Refined Algebraic Quantization of Spherically Symmetric Gravitational Collapse

Let us consider spacetime with topology $\mathbb{R} \times \Sigma \times S^2$. Matter fields are ignored, but will be considered later, for example scalar field or null-dust. Now, we will consider only gravitational part of the problem. After ADM foliation of spacetime we work only with "spatial part of the metric".

Space metric

$g = \lambda^2 \mathrm{dr}^2 + \mathrm{R}^2 \mathrm{d}\Omega^2$

observables

$\ell(V) = \int\limits_V \lambda \mathrm{dr}$

$\omega(\mathrm{V}) = \int\limits_{\mathrm{V}} \lambda \mathrm{R}^2 \mathrm{dr}$

$\mathrm{p}(\mathrm{V}) = \int\limits_{\mathrm{V}} (\lambda \Pi_\lambda - \frac{1}{2} \mathrm{R} \Pi_{\mathrm{R}}) \mathrm{dr}$

$\pi(\mathrm{V}) = \int\limits_{\mathrm{V}} \frac{1}{2} \mathrm{R} \Pi_{\mathrm{R}} \mathrm{dr}$

where Π's are canonically conjugated momenta
Canonical commutation algebra

$\{\ell(V), p(U)\} = \ell(V \cap U)$

$\{\omega(V), \pi(U)\} = \omega(V \cap U)$

U, V are compact sets on Σ

Basic quantum algebra

$[\mathfrak{L}(V), \mathfrak{p}(U)] = i\, \mathfrak{L}(V \cap U)$

where $\mathfrak{L}$ is positive measure on Σ which is equivalent with Lebesgue measure[a]. In order to find a representation of such algebra one can construct measure ν on $\mathcal{M}^L(\Sigma)$(set of all measures equivalent with Lebesgue measure L on Σ), then one can construct $L^2(\mathcal{M}^L(\Sigma), \nu)$.

The measure ν could be constructed by using projective techniques [1],[2].

[a] This means that null sets are the same in both cases.

Projective techniques
$[a] = \{b \subset \mathcal{M}^L(\Sigma); a \text{ is compact set} : L((a \setminus b) \cup (b \setminus a)) = 0\}$

$$A \in \mathfrak{B}(\Sigma) \quad \Leftrightarrow \quad \begin{array}{rl} i) & \text{number of elements of } A \in \mathbb{N} \\ ii) & \forall [a], [b] \in A : [a \cap b] = [\emptyset] \\ iii) & \forall [a] \in A : [a] \neq [\emptyset] \end{array}$$

Let us define $\forall A, B \in \mathfrak{B}(\Sigma) : A \prec B \Leftrightarrow \forall [a] \in A \exists [b_i] \in B : [a] = [\cup_i b_i]$,

$$\mathfrak{X}_A := \{\mathfrak{L}_A : A \to (0, \infty)\}$$

and projections $p_{AB} : \mathfrak{X}_B \to \mathfrak{X}_A, A \prec B$

$$p_{AB}(\mathfrak{L}_B)([a]) := \sum_i \mathfrak{L}_B([b_i]).$$

The projective limit of these projective spaces is $\mathcal{M}^L(\Sigma)$. Let $(\nu_A)_{A \in \mathfrak{B}(\Sigma)}$ be the family of regular Borel probability measures on $\mathfrak{X}_A$. If $p_{AB*}\nu_B = \nu_A$ then $(\nu_A)_{A \in \mathfrak{B}(\Sigma)}$ introduce regular Borel probability measure ν on $\mathcal{M}^L(\Sigma)$.

Let $f_A \in C_0(\mathfrak{X}_A)$ then f_A can be viewed as function on $\mathcal{M}^L(\Sigma)$. It can be proven that such type functions are dense in $C_0(\mathcal{M}^L(\Sigma))$. This type of functions is usually called "cylindrical". We can define on cylindrical functions action of operators of quantum algebra:

$$\begin{aligned} \mathfrak{L}(\hat{V}) f_A(\mathfrak{L}) &:= \sum_i \mathfrak{L}(V_i) f_A(\mathfrak{L}) \\ \mathfrak{p}(\hat{V}) f_A(\mathfrak{L}) &:= -i \sum_i \mathfrak{L}(V_i) \frac{\partial f_A}{\partial \mathfrak{L}(V_i)}(\mathfrak{L}) + \Psi(\mathfrak{L}) f_A(\mathfrak{L}) \end{aligned}$$

whenever exist such $V_i \in A$ that $[V] = [\cup_i V_i]$. The concrete type of Ψ depends on the measure ν, but is not determined uniquely .

Open questions
The concrete type of measures ν_A is unfortunately currently unknown to the author. So, there is question whether such measures exist. It's easy to see, that e.g. $\nu(U) = 1$ whenever $L \in U$ or $\nu(U) = 0$, is correct measure, but unfortunately in this case there doesn't exist representation of our algebra on $L^2(\mathcal{M}^L(\Sigma), \nu)$. It may turn out that such measures do not exist.

Literature
[1] A.Ashtekar, J.Lewandowski: Projective techniques and functional integration for gauge theories; J.Math.Phys. **36** (5), May 1995
[2] T.Thiemann: Introduction to Modern Canonical Quantum General Relativity; gr-qc/0110034