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Exact solutions for impulsive gravitational waves

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Abstract. All impulsive waves (with no tails) in vacuum backgrounds of constant curvature (i.e. Minkowski, de Sitter and anti-de Sitter) are classified according to whether they are expanding or non-expanding. It is demonstrated that the expanding waves may, in some sense, be regarded as impulsive limits of Robinson–Trautman type N solutions.

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1 Introduction

Many exact solutions of Einstein's equations are known which describe pure impulsive gravitational waves in Minkowski and (anti-) de Sitter backgrounds. These belong to two distinct families in which the waves are either expanding or non-expanding. In each case, solutions can be constructed by three distinct methods. The non-expanding case is briefly reviewed. Using an analogous approach, it is argued that the expanding cases may be regarded as impulsive limits of Robinson–Trautman type N solutions.

2 Non-expanding impulsive waves with $\Lambda = 0$

We first use Penrose' [1] geometrical method for constructing impulsive gravitational waves in which space-time is cut along a null hypersurface and the two pieces re-attached with a suitable warp. Consider Minkowski space in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2\mathrm{d}\zeta\mathrm{d}\bar\zeta, \tag{1}$$

and perform the coordinate transformation $u = U$, $v = V - H + UH_Z H_{\bar Z}$, $\zeta = Z - UH_{\bar Z}$, where $H = H(Z,\bar Z)$ is an arbitrary real function, to obtain

$$\mathrm{d}s^2 = 2\mathrm{d}U\,\mathrm{d}V - 2\left|\mathrm{d}Z - U(H_{Z\bar Z}\mathrm{d}Z + H_{\bar Z\bar Z}\mathrm{d}\bar Z)\right|^2. \tag{2}$$

Following Penrose's "cut and paste" method, we now take (1) with $u = U$, $v = V$ and $\zeta = Z$ for $U < 0$ and combine this with (2) for $U > 0$. The resulting line element

$$\mathrm{d}s^2 = 2\mathrm{d}U\,\mathrm{d}V - 2\left|\mathrm{d}Z - U\Theta(U)(H_{Z\bar Z}\mathrm{d}Z + H_{\bar Z\bar Z}\mathrm{d}\bar Z)\right|^2, \tag{3}$$

where $\Theta(u)$ is the Heaviside step function, is continuous across the null hypersurface $U=0$. However, the discontinuity in the derivatives of the metric produces impulsive components in the curvature tensor.

Let us now consider the combination of the transformations above as applied to the two regions of space-time. This can be expressed in the explicit form

$$u=U, \qquad v=V-H\,\Theta(U)+U\,\Theta(U)\,H_Z H_{\bar Z}, \qquad \zeta=Z-U\,\Theta(U)\,H_{\bar Z}. \tag{4}$$

This is obviously discontinuous at $u=0$ in such a way that $(u=0,v,\zeta,\bar\zeta)_{M^-} = (u=0,v+H(\zeta,\bar\zeta),\zeta,\bar\zeta)_{M^+}$. This is the Penrose junction condition for re-attaching the two halves of the space-time $M^-(u<0)$ and $M^+(u>0)$ with a "warp".

Significantly, the transformation (4), relates the continuous form of the impulsive wave metric (3), not to the initial metric (1), but to the following metric which also includes an impulsive component explicitly located on the wavefront $u=0$

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2\mathrm{d}\zeta\,\mathrm{d}\bar\zeta + 2H(\zeta,\bar\zeta)\,\delta(u)\,\mathrm{d}u^2. \tag{5}$$

It may be observed that this general solution may also be obtained as the impulsive limit of the well known class of *pp*-waves expressed in standard Brinkmann form. In fact, the solutions (5) also arise [2] as impulsive limits of the more general class of Kundt waves. Using a natural tetrad, the nonzero components of the curvature tensor are given by $\Psi_4 = H_{\zeta\zeta}\delta(u)$ and $\Phi_{22} = H_{\zeta\bar\zeta}\delta(u)$.

The problems associated with the discontinuity in the transformation (4) have been investigated by Kunzinger and Steinbauer [3]. By considering the impulsive solution as limits of sandwich waves in the two coordinate systems they have given a precise meaning to the "physical equivalence" of the two metrics (3) and (5). They have also shown that the transformation can be defined rigorously within Colombeau's theory of generalized functions.

Particular cases of this family of solutions can also be obtained by boosting some well known space-times to the ultrarelativistic limit. Interesting examples have been obtained by boosting the Schwarzschild space-time [4], the Kerr solution [5, 6, 7], the Kerr–Newman family [8], and the Weyl solutions [9]. In addition, since the main field equation is linear [10], separate solutions can always be superposed, either on separate wave surfaces $u=$ const. thus forming a sequence of plane-fronted impulsive waves, or by superposing different sources on the same wave surface $u=0$.

3 Non-expanding impulsive waves with $\Lambda \neq 0$

Non-expanding impulsive waves in a de Sitter (or anti-de Sitter) background can be obtained in exactly analogous ways to those described above. They can be derived using Penrose's "cut and paste" method [11], or as the impulsive limit of Kundt waves with non-zero cosmological constant [2]. In addition, they can be derived by boosting sources [12, 13] by elements of the (anti-) de Sitter group for solutions with $\Lambda \neq 0$.

However, the global properties of the wavefront is different in these cases [13, 14]. When $\Lambda > 0$, the impulsive wave is located on the spherical horizon of the de Sitter universe. In an anti-de Sitter background, the space-time contains closed timelike lines and the wavefront is hyperboloidal and propagates from one side of the universe to the other and is then reflected back in an endless cycle.

4 Expanding impulsive waves with $\Lambda = 0$

We now consider the expanding impulsive waves that can be obtained using Penrose's method of cutting Minkowski space-time on a null cone and re-attaching with "warp". This is exactly analogous to the case for non-expanding impulsive waves. However, there is one important difference. Although these solutions may also contain sources on the wave-front, additional sources necessarily appear in this case due to the global structure of the space-times. For the non-expanding case, the Minkowski space on either side of the impulse is the entire space-time cut along a null hypersurface. For spherical waves, on the other hand, points on one side of the null cone do not match to the entire null cone on the other side. This incompleteness can be interpreted in terms of deficit angles, cosmic strings, or other forms of topological structure in the Minkowski space on at least one side of the wave surface.

Applying the transformation $u \to u + v\xi\bar{\xi}$, $\zeta = v\xi$, the line element (1) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2v^2\mathrm{d}\xi\,\mathrm{d}\bar{\xi}, \tag{6}$$

in which the hypersurfaces $u =$ const. are a family of null cones. To this we apply the additional transformation $u = U|h'|/\mathcal{A}$, $v = V\mathcal{A}/|h'|$, $\xi = h(Z) + Uh'\bar{\beta}/2V\mathcal{A}$, where $h(Z)$ is an arbitrary holomorphic function, $h' = h_{,Z}$, $\beta = h''/h'$ and $\mathcal{A} = 1 - U\beta\bar{\beta}/4V$. This takes (6) to the form

$$\mathrm{d}s^2 = 2\mathrm{d}U\,\mathrm{d}V - 2\left|V\mathrm{d}Z - U\bar{H}\,\mathrm{d}\bar{Z}\right|^2, \tag{7}$$

where $H = \frac{1}{2}\beta' - \frac{1}{4}\beta^2 = \frac{1}{2}(h'''/h') - \frac{3}{4}(h''/h')^2$. Using the Penrose method, we take (6) for $u < 0$ and join this to (7) for $U > 0$ to produce a spherical impulsive wave on the null cone $u = 0 = U$ given by the continuous metric ([15]–[17])

$$\mathrm{d}s^2 = 2\mathrm{d}U\,\mathrm{d}V - 2\left|V\mathrm{d}Z - U\Theta(U)\bar{H}\,\mathrm{d}\bar{Z}\right|^2. \tag{8}$$

As in the nonexpanding case, it is possible to combine the two transformations either side of the wavefront (for details see [18]). Putting $F(\xi) = h(Z) - Z$ at $U = 0$, this relates the continuous form of the line element (8), not to (6), but to the form

$$\begin{aligned}\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2v^2\mathrm{d}\xi\,\mathrm{d}\bar{\xi} + 2v^2\bar{F}\delta(u)\mathrm{d}\xi\,\mathrm{d}u + 2v^2F\delta(u)\mathrm{d}\bar{\xi}\,\mathrm{d}u \\ - \left[2v^2F\bar{F}\delta^2(u) - v(F_\xi + \bar{F}_{\bar{\xi}})\delta(u)\right]\,\mathrm{d}u^2,\end{aligned} \tag{9}$$

which includes an impulsive component explicitly located on the spherical wavefront $u = 0$. This can immediately be seen to be identical to the impulsive limit of the simplest type N Robinson–Trautman solution $RTN(\Lambda = 0, \epsilon = 0)$ which, expressed using García–Plebański coordinates, has the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - 2v^2\mathrm{d}\xi\,\mathrm{d}\bar{\xi} + 2v^2\bar{f}\mathrm{d}\xi\,\mathrm{d}u + 2v^2f\mathrm{d}\bar{\xi}\,\mathrm{d}u - \left[2v^2f\bar{f} - v(f_\xi + \bar{f}_{\bar{\xi}})\right]\mathrm{d}u^2, \tag{10}$$

where, $f = f(\xi, u)$ is an arbitrary complex function holomorphic in ξ, and the impulsive limit is obtained by putting $f(\xi, u) = F(\xi)\delta(u)$.

It must be observed that this impulsive limit includes the square of the Dirac δ-function, and so this form should not be used in practice. Nevertheless, it explicitly demonstrates that the spherical impulsive waves obtained using the "cut and paste" method are equivalent to the impulsive limit of Robinson–Trautman type N solutions. Further analysis of the distributional features of this solution is obviously required.

It may also be observed that Bičák and Schmidt [19] obtained a particular case of this class of solutions as a limiting case of a solution with boost-rotation symmetry. In this limit, two null particles recede from a common point generating an impulsive spherical gravitational wave and there are either cosmic strings attaching each particle to infinity, or there is an expanding cosmic string along the axis of symmetry separating the two particles. This solution was given independently in [20] and discussed in [21].

5 Expanding impulsive waves with $\Lambda \neq 0$

As in the previous sections, it can be shown that there are again three possible approaches to the construction of expanding impulsive waves in de Sitter and anti-de Sitter backgrounds. (1) General solutions can be obtained using Penrose's "cut and paste" method with a suitable "warp", cutting (anti-) de Sitter space-time on a null cone. This was performed in [22]. (2) We can also take the impulsive limit of the type N Robinson–Trautman solutions with cosmological constant [18]. (3) Particular cases may be obtained as limits of space-times with boost-rotation symmetry and non-zero cosmological constant. However, these have not yet been considered.

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