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Chaos in pp-wave spacetimes and its smearing in the impulsive limit

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Abstract. In recent works [1, 2] we demonstrated by a fractal method that geodesic motion in non-homogeneous vacuum pp-wave spacetimes with a constant wave-profile is chaotic. These results were generalized to non-homogeneous sandwich pp-waves [3]. It was shown that chaos smears as the duration of these gravitational waves is reduced: the outcome channels to infinity become fuzzy, and the fractal structure is cut at lower and lower levels. Here we calculate the box-counting dimension of the basin boundaries. In the impulsive limit its value approaches an integer which indicates that the fractal boundaries become regular. Indeed, in the limit of impulsive waves, the motion is fully non-chaotic.

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1 Chaotic motion in pp-waves with a constant profile

The geodesic equations for the widely known class of vacuum pp-waves [4]

$$ds^2 = 2\,d\zeta d\bar\zeta - 2\,du dv - [f(\zeta,u) + \bar f(\bar\zeta,u)]\,du^2\ , \tag{1}$$

reduce to

$$\zeta'' + \tfrac{1}{2}\bar f_{,\bar\zeta} = 0\ , \tag{2}$$

$v(u) = \int [\,\zeta'\bar\zeta' - \frac{1}{2}(f+\bar f) - \frac{1}{2}(\epsilon/U^2)\,]\,du + \tilde V$, $u(\tau) = U\tau + \tilde U$, where prime denotes d/du, τ is a normalized affine parameter, $U, \tilde U, \tilde V$ are constants, and $\epsilon = -1, 0, +1$ for timelike, null or spacelike geodesics, respectively. It is thus crucial to solve (2) for $\zeta(u)$. Introducing real coordinates x and y by $\zeta = x + \mathrm{i}y$, Eq. (2) follows from the Hamiltonian $H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(x,y,u)$, where $V(x,y,u) = \frac{1}{2}\,\mathcal{R}e\,f$. For non-homogeneous pp-waves given by

$$f = \tfrac{2}{n}\,h(u)\,\zeta^n\ , \quad n = 3, 4, \cdots\ , \tag{3}$$

the corresponding potential V is polynomial. Motion in these Hamiltonian systems with $h(u) = C =$ const. is chaotic since the Smale horseshoe map can be embedded along specific homoclinic orbits. In [1, 2] the structure of chaotic motion was investigated by a fractal method [5]. Geodesics starting at $x(0) = \cos\phi_0$, $y(0) = \sin\phi_0$

were evolved. It was observed that they escaped to infinity only along one of the n distinct outcome channels. Moreover, the basin boundaries which separated initial conditions leading to different outcomes were shown to be fractal, exhibiting chaos, i.e. an extremely sensitive dependence on the choice of initial data.

2 Smearing of chaos in the impulsive limit of sandwich pp-waves

In [3] we investigated motion in sandwich waves (3) with a 'square' profile $h(u) = a^{-1}[\Theta(u)-\Theta(u-a)]$, where a is a positive constant. Test particles which are initilly at rest start moving at $u=0$. At $u=a$ the sandwich wave ends and the potential barriers disappear. For $u>a$, the particles move uniformly along straight lines in the flat space and their trajectories are not constrained to the former channels. Thus, the fractal structure of basin boundaries indicating chaos is 'cut' at some level related to a: there is not enough time for particles to bounce arbitrarily many times between the potential walls. As $a \to 0$, the number of bounces tends to zero so that the fractal structure is completely erased. The narrower the sandwich wave, the 'less chaotic' the motion. This effect can be called a 'smearing of chaos'. Its detailed formal description can be found in [3]. In the impulsive limit $a \to 0$ (i.e. for $f = \frac{2}{n}\delta(u)\zeta^n$), the motion is non-chaotic. Indeed, the geodesics in these impulsive pp-wave spacetimes can explicitly be written

$$\begin{aligned} x(u) &= \rho_0 \cos\phi_0 - C\rho_0^{n-1}\cos[(n-1)\phi_0]\,u \ , \\ y(u) &= \rho_0 \sin\phi_0 \ + C\rho_0^{n-1}\sin[(n-1)\phi_0]\,u \ , \end{aligned} \tag{4}$$

where ρ_0, ϕ_0 parametrize the initial position of the particle. At any time $u>0$ the initial circle $\rho=\rho_0$ is deformed into a hypotrochoid, a curve generated by a point attached to a small circle rolling around the inside of a large fixed circle, see [3].

The smearing of chaos can rigorously by characterized by the dependence of the box-counting dimension d of the basin boundaries [5] on a. This can numerically be calculated from the relation $P(\varepsilon) \sim \varepsilon^{1-d}$, where $P(\varepsilon)$ is the fraction of initial conditions which are uncertain as to which escape channel is chosen when there is an initial error ε (i.e. comparing geodesics starting from ϕ_0 and $\phi_0 \pm \varepsilon$). From the dependence of $\ln P(\varepsilon)$ on $\ln\varepsilon$ for $\varepsilon \to 0$, we obtain the following results (for $n=3$): $d=0.2$ for $a=100$, $d=0.3$ for $a=60$, $d=0.6$ for $a=20$, $d=0.9$ for $a=10$. In fact, the box-counting dimension approaches an integer ($d \to 1$ as $a \to 0$) which means that the basin boundaries completely lose their fractal structure in the impulsive limit.

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