Null limits of the C-metric

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Abstract

The C-metric is usually understood as describing two black holes which accelerate in opposite directions under the action of some conical singularity. Here, we examine all the solutions of this type which represent accelerating sources and investigate the null limit in which the accelerations become unbounded. We show that the resulting space-times represent spherical impulsive gravitational waves generated by snapping or expanding cosmic strings.

1 Introduction

The vacuum C-metric is a well-known solution of Einstein's equations. It is described by the line element

$$\mathrm{d}s^2 = -A^{-2}(x+y)^{-2}\left(F^{-1}\mathrm{d}y^2 + G^{-1}\mathrm{d}x^2 + G\mathrm{d}\phi^2 - F\mathrm{d}t^2\right), \tag{1}$$

where

$$F = -1 + y^2 - 2mAy^3, \qquad G = 1 - x^2 - 2mAx^3, \tag{2}$$

and A, m are arbitrary constants. Kinnersley and Walker [1] (see also [2]) showed that this may represent two black holes, each of mass m, that have uniform acceleration A in opposite directions. The acceleration is caused either by a strut between the black holes or by two semi-infinite strings connecting them to infinity.

The radiative properties of this space-time were investigated by Farhoosh and Zimmerman [3], and its asymptotic properties by Ashtekar and Dray [4]. A transformation of the line element (1) into a form that explicitly exhibits its boost-rotation symmetry [5], and which facilitates its physical interpretation was achieved by Bonnor [2]. To maintain the signature in (1), F and G are required to be positive. Assuming the condition that

$$0 \leq |mA| < \frac{1}{3\sqrt{3}}, \tag{3}$$

the expressions in (2) have three real roots. This gives rise to four possible space-times according to different ranges of the coordinates x and y. This has been discussed by Cornish and Uttley [6]. However, in a recent review of these solutions, Pravda and Pravdová [7] have clarified that only three of these represent space-times with accelerated sources.

The purpose of the present paper is to investigate the null limits of these solutions as $A \to \infty$.

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2 The explicit solutions

Particular solutions for the C-metric depend on the roots of the cubic

$$2A^4z^3 - A^2z^2 + m^2 = 0, \tag{4}$$

which is related to the cubics in (2) by $z = \frac{m}{A}y$ or $z = -\frac{m}{A}x$. The roots of (4) are given by

$$\begin{aligned} z_1 &= \tfrac{1}{6}A^{-2}\left[1 + 2\cos(\varphi + \tfrac{2}{3}\pi)\right], \\ z_2 &= \tfrac{1}{6}A^{-2}\left[1 + 2\cos(\varphi + \tfrac{4}{3}\pi)\right], \\ z_3 &= \tfrac{1}{6}A^{-2}\left[1 + 2\cos\varphi\right], \end{aligned} \tag{5}$$

where $\varphi = \frac{1}{3}\arccos(1 - 54m^2A^2)$. These roots satisfy $z_1 \leq z_2 < z_3$ for all values of mA in the permitted range (3) which corresponds to $\varphi \in [0, \frac{1}{3}\pi)$.

In coordinates adapted to the boost-rotation symmetry [5], the line element (1) takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{\lambda}\mathrm{d}\rho^2 - \rho^2\mathrm{e}^{-\mu}\mathrm{d}\phi^2 + (\zeta^2 - \tau^2)^{-1}\left[\mathrm{e}^{\mu}(\zeta\mathrm{d}\tau - \tau\mathrm{d}\zeta)^2 - \mathrm{e}^{\lambda}(\zeta\mathrm{d}\zeta - \tau\mathrm{d}\tau)^2\right], \tag{6}$$

in which μ and λ are specific functions of ρ^2 and $\zeta^2 - \tau^2$ (see [2]). For convenience of presenting explicit solutions, let us define the following expressions

$$\begin{aligned} & Z_1 = z_1 - z_3 = -\tfrac{1}{\sqrt{3}}A^{-2}\sin(\varphi + \tfrac{1}{3}\pi), \\ & Z_2 = z_2 - z_3 = -\tfrac{1}{\sqrt{3}}A^{-2}\sin(\varphi + \tfrac{2}{3}\pi), \\ & R = \tfrac{1}{2}(\zeta^2 - \tau^2 + \rho^2), \\ & R_1 = \sqrt{(R + Z_1)^2 - 2Z_1\rho^2}, \qquad R_2 = \sqrt{(R + Z_2)^2 - 2Z_2\rho^2}, \\ & S_1 = R\,(R + Z_1 + R_1) - Z_1\rho^2, \qquad S_2 = R\,(R + Z_2 + R_2) - Z_2\rho^2, \\ & S_{12} = (R + Z_1)(R + Z_2) + R_1R_2 - (Z_1 + Z_2)\rho^2. \end{aligned} \tag{7}$$

Slightly modifying the notation of [7], the three cases that are of physical interest (denoted by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$) are now given by

$$\mathrm{e}^{\mu}_{\mathcal{A}} = a\,\frac{(R + Z_1 + R_1 - \rho^2)(R + Z_2 + R_2 - \rho^2)}{(\zeta^2 - \tau^2)^2}, \qquad \mathrm{e}^{\lambda}_{\mathcal{A}} = \frac{a}{2}\,\frac{S_1\,S_2}{R_1R_2\,S_{12}}, \tag{8}$$

$$\mathrm{e}^{\mu}_{\mathcal{B}} = a\,\frac{R + Z_1 + R_1 - \rho^2}{R + Z_2 + R_2 - \rho^2}, \qquad \mathrm{e}^{\lambda}_{\mathcal{B}} = \frac{a}{2}\,\frac{S_2\,S_{12}}{R_1R_2\,S_1}, \tag{9}$$

$$\mathrm{e}^{\mu}_{\mathcal{D}} = a\,\frac{R + Z_2 + R_2 - \rho^2}{R + Z_1 + R_1 - \rho^2}, \qquad \mathrm{e}^{\lambda}_{\mathcal{D}} = \frac{a}{2}\,\frac{S_1\,S_{12}}{R_1R_2\,S_2}, \tag{10}$$

where a is a positive constant. The condition that the metric is regular on the "roof" $\zeta^2 - \tau^2 = 0$ has already been inserted. The physical interpretation of these solutions has been given in [7].

The case $\mathcal{A}$ describes two uniformly accelerated black holes with a curvature singularity between them. In general, there is a conical singularity on the axis $\rho = 0$ extending from the black holes to infinity. However, this is absent when $a = 1$.

The physically most interesting case $\mathcal{B}$, which has been widely considered in the literature, describes two uniformly accelerated black holes connected to conical singularities. The axis is regular between the particles when $a = Z_1/Z_2 = 2\left[1 + \sqrt{3}\cot(\varphi + \frac{1}{3}\pi)\right]^{-1}$. In this case, the black holes can be considered to be accelerated by two strings connecting them to infinity. Alternatively, the axis is regular outside the particles if $a = 1$, in which case the black holes are accelerated by a strut between them.

The case $\mathcal{D}$ also has similar conical singularities either between the sources or connecting them to infinity, but the sources are now different types of curvature singularities whose interpretation is unclear. In this case, the axis is regular between the singularities when $a = Z_2/Z_1$, or is regular outside them if $a = 1$.

3 Null limits

The purpose of this paper is to investigate the limits of the above solutions as $A \to \infty$. To maintain the inequality (3), it is necessary to simultaneously scale the parameter m to zero such that mA remains constant. The parameter φ is unchanged by this scaling. However, the parameters Z_1 and Z_2 become zero in these limits, but their ratio remains a finite constant, and all the regularity conditions are thus preserved.

Let us first consider the above null limit for the more familiar case $\mathcal{B}$. In this case, we find (expanding to terms in Z_i^2) that the limits of (9) are

$$e^{\mu}_{\mathcal{B}} \to a, \quad \text{everywhere}; \qquad e^{\lambda}_{\mathcal{B}} \to \begin{cases} a, & \text{outside the null cone.} \\ a\left(\dfrac{Z_2}{Z_1}\right)^2, & \text{inside the null cone.} \end{cases} \tag{11}$$

Thus $e^{\mu}_{\mathcal{B}}$ is constant everywhere, but there is a discontinuity in $e^{\lambda}_{\mathcal{B}}$ on the null cone $\rho^2+\zeta^2-\tau^2=0$. This corresponds to the presence of a spherical impulsive gravitational wave exactly as described in a different context in the previous paper [8]. Also, there is generally a conical singularity on the axis of symmetry $\rho=0$. However, this can be removed either inside or outside the null cone by an appropriate choice of the constant a. For the choice $a=Z_1/Z_2$, the axis is regular inside the null cone, but a conical singularity appears on the axis outside it. This situation can be considered to describe the "snapping" of a cosmic string with deficit angle $(1-\beta)2\pi$, where

$$\beta = Z_2/Z_1 = \tfrac{1}{2}\left[\,1+\sqrt{3}\cot(\varphi+\tfrac{1}{3}\pi)\,\right], \tag{12}$$

so that $\beta \in (0,1]$. Alternatively, for the choice $a=1$, the axis is regular outside the null cone, but there is a conical singularity on the axis inside. This describes a strut, or a conical singularity with excess angle $(1-\beta^{-1})2\pi$, whose length is increasing in both directions at the speed of light.

The null limit for the case $\mathcal{D}$ is, in fact, exactly equivalent to that of the previous case $\mathcal{C}$. In this case, the null limits of (10) are

$$e^{\mu}_{\mathcal{D}} \to a, \quad \text{everywhere}; \qquad e^{\lambda}_{\mathcal{D}} \to \begin{cases} a, & \text{outside the null cone.} \\ a\left(\dfrac{Z_1}{Z_2}\right)^2, & \text{inside the null cone.} \end{cases} \tag{13}$$

Again $e^{\mu}_{\mathcal{D}}$ is constant everywhere, but there is a discontinuity in $e^{\lambda}_{\mathcal{D}}$ on the null cone corresponding to the presence of a spherical impulsive gravitational wave. For the choice $a=Z_2/Z_1$, the axis is regular inside the null cone, but there is a conical singularity on the axis outside corresponding to a snapped strut which has an excess angle $(1-\beta^{-1})2\pi$. Alternatively, for the choice $a=1$, the axis is regular outside the null cone, but there is a conical singularity on the axis inside, corresponding to an expanding cosmic string with deficit angle $(1-\beta)2\pi$.

In the above limit for the remaining case $\mathcal{A}$, the metric functions become

$$e^{\mu}_{\mathcal{A}} \to \begin{cases} a, \\ a\,\dfrac{\rho^4}{(\zeta^2-\tau^2)^2}, \end{cases} \qquad e^{\lambda}_{\mathcal{A}} \to \begin{cases} a, & \text{outside the null cone.} \\ b\,\dfrac{\rho^4(\zeta^2-\tau^2)^2}{(\rho^2+\zeta^2-\tau^2)^8}, & \text{inside the null cone.} \end{cases} \tag{14}$$

Thus, the region outside the null cone reduces to part of Minkowski space, which is regular on the axis if $a=1$ (otherwise a conical singularity remains). However, the equivalent limit for $e^{\lambda}_{\mathcal{A}}$ inside the null cone vanishes. In fact, a curvature singularity appears on the null cone and the space-times in the two regions cannot be connected. In order to obtain a finite limit for $e^{\lambda}_{\mathcal{A}}$ inside the null cone, it is necessary to rescale the parameter a in this component as $a=b\,(4Z_1Z_2)^{-2}=9bA^8(4\cos^2\varphi-1)^{-2}$, where b is held constant. The resulting space-time in this region also contains a curvature singularity on the axis of symmetry. This limit, and indeed the general case $\mathcal{A}$, is not physically significant.

4 Discussion

The null limits of the cases $\mathcal{B}$ and $\mathcal{D}$ described in the previous section are together identical to the equivalent null limit of the Bonnor–Swaminarayan solution which is described in the previous paper [8]. In that paper, we give the transformation of the metric to appropriate continuous forms that are more appropriate for its physical interpretation as a spherical impulsive gravitational wave generated by a snapping or expanding cosmic string (with a deficit or excess angle). Using the transformations (22) and (24) in [8], we obtain the continuous metric in the Gleiser–Pullin form [9]

$$\mathrm{d}s^2 = 4\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} - (\mathcal{V} - P\mathcal{U})^2\,\mathrm{d}\phi^2 - (\mathcal{V} + P\mathcal{U})^2\,\mathrm{d}\psi^2, \tag{15}$$

where, for case $\mathcal{B}$

$$P = \begin{cases} \Theta(\mathcal{U}) + \beta^2\,\Theta(-\mathcal{U}) & \text{for} \quad a = \beta^{-1} \quad : \text{ snapping string} \\ \beta^{-2}\,\Theta(\mathcal{U}) + \Theta(-\mathcal{U}) & \text{for} \quad a = 1 \quad : \text{ expanding strut} \end{cases} \tag{16}$$

The equivalent metric for case $\mathcal{D}$ for a snapping strut or an expanding string is obtained by replacing β by β^{-1} above.

It is also of interest to contrast the solution described here with that of Aichelburg and Sexl [10] in which a single Schwarzschild black hole is boosted to the relativistic limit. In that case, a plane impulsive gravitational wave is generated by a single null particle (the Ricci tensor has a singular point on the wave surface). By contrast in this case, the structure of the two black holes in the C-metric vanishes in the null limit (the Ricci tensor vanishes everywhere on the null cone). However, the strings remain, and the motion of their end points generates impulsive spherical gravitational waves.

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