Exact non-singular waves in the anti–de Sitter universe

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Abstract

A class of radiative solutions of Einstein's field equations with a negative cosmological constant and a pure radiation is investigated. The space-times, which generalize the Defrise solution, represent exact gravitational waves which interact with null matter and propagate in the anti–de Sitter universe. Interestingly, these solutions have homogeneous and non-singular wave-fronts for all freely moving observers. We also study properties of sandwich and impulsive waves which can be constructed in this class of space-times.

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1 Introduction

There has been a growing interest in radiative space-times which are not asymptotically flat. In the last two decades, new exact solutions of this type, representing gravitational waves in cosmology were found and analyzed, for example in [1]-[18] and elsewhere (review of the works can be found in [4], [19]-[21]). Some of these solutions can be interpreted as spatially inhomogeneous cosmological models in which the homogeneity of the universe is broken due to the presence of gravitational waves.

In this work we concentrate on presenting a physical interpretation of one special class of exact type N solutions with a negative cosmological constant Λ. These are generalizations of the Defrise solution [22] such that the profile of the wave may be arbitrary. Therefore, particular gravitational waves propagating in an everywhere curved anti–de Sitter universe can be constructed.

A unique feature of all such space-times is that (for any bounded wave-profile) they are non-singular, just like the well-known homogeneous *pp*-waves (plane waves) in a flat Minkowski background. This is demonstrated in the next section. In Section 3 we investigate geodesics, and in Section 4 the geodesic deviation in the generalized Defrise solutions. Remarks on the global structure are presented in Section 5. Using these solutions we construct, in Section 6, non-singular sandwich waves in the anti–de Sitter universe and investigate their impulsive limits. In particular, geodesic motion in the Defrise sandwich and impulsive waves is studied in Section 7.

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2 Generalized Defrise space-times

The Defrise solution [22], [23] belongs to the Kundt class of non-twisting, non-expanding and shear-free space-times of type N, in particular, to its interesting subclass which was found by Siklos [9]. For this subclass the quadruple Debever-Penrose null vector field $\mathbf{k}$ is simultaneously the Killing vector field. The metric can be written in the simple form

$$\mathrm{d}s^2 = \frac{\beta^2}{x^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + 2\,\mathrm{d}u\,\mathrm{d}v + H\,\mathrm{d}u^2\right) , \tag{1}$$

where $\beta = \sqrt{-3/\Lambda}$, Λ is a negative cosmological constant, x and y are spatial coordinates, v is an affine parameter along rays generated by $\mathbf{k} = \partial_v$, and u is a retarded time. The function $H(x, y, u)$ for the Defrise solution has the form $H = -x^{-2}$. It admits six Killing vectors.

In the present paper we investigate solutions which generalize the Defrise solution in such a way that the profile of the gravitational wave may be arbitrary: we consider solution which can be written as

$$H = \frac{d(u)}{x^2} , \tag{2}$$

where $d(u)$ is an arbitrary (bounded) function of u. The metric (1), (2) satisfies the Einstein field equations with a negative cosmological constant and a pure radiation field, $T_{\mu\nu} = \Phi k_\mu k_\nu$, where $\Phi = -5\,d(u)/8\pi\beta^4$. This corresponds to null matter propagating along the principal null congruence. For non-constant $d(u)$, the space-time (1), (2) admits three Killing vectors: ∂_v, ∂_y, and $y\partial_v - u\partial_y$.

An interesting property of the solutions given by (2) is that these are *non-singular* in the following sense. By inspecting the components of the curvature tensor with respect to orthonormal frames parallelly propagated along any timelike geodesic (see [24]), it can be observed that all non-vanishing components are proportional to one of the following functions: $\mathcal{A}_+ = -\frac{C^2}{2}\,x^5\,(H_{,x}/x)_{,x}$, $\mathcal{A}_\times = \frac{C^2}{2}\,x^5\,(H_{,x}/x)_{,y}$, or $\mathcal{M} = \frac{C^2}{2}\,x^3(xH_{,yy} - H_{,x})$, where C is a constant (plus, in some cases, also to a constant term $\pm\Lambda/3$). However, for the particular choice (2) of the function H we obtain

$$\mathcal{A}_+ = -4C^2\,d(u) , \qquad \mathcal{M} = C^2\,d(u) , \qquad \mathcal{A}_\times = 0 . \tag{3}$$

Therefore, for an arbitrary bounded profile function $d(u)$, the frame components of the *curvature tensor remains finite*, as seen by *any* timelike observer. In this sense, all the wave-surfaces $u =$ *const* are singularity-free. This is analogous to a similar property for the well-known plane waves in Minkowski space. Thus we may characterize the solution (1), (2) as representing exact waves in the anti–de Sitter universe with non-singular and homogeneous wave-fronts. (In fact, it can easily be shown that the choice (2) is the only non-trivial possibility for which the functions $\mathcal{A}_+$, $\mathcal{M}$ and $\mathcal{A}_\times$ are independent of the spatial coordinates x and y.)

The above solutions reduce to the anti–de Sitter solution in the regions where $d(u) = 0$. Therefore, we can easily construct space-times representing sandwich waves in the anti–de Sitter universe by considering $d(u)$ non-zero on a finite interval of u only, just as for standard sandwich gravitational waves in flat space [25]-[27]. Some properties of these space-times will be discussed later in Sections 6 and 7, together with the possibility of constructing impulsive waves in the anti–de Sitter universe by letting the profiles $d(u)$ approach the Dirac delta function.

3 Geodesics

The geodesic equations for the metric (1), (2) are

$$\begin{aligned}
&\ddot{v} = 2\dot{v}\frac{\dot{x}}{x} + 2C\,d(u)\frac{\dot{x}}{x} - \tfrac{1}{2}C^2 d'(u)x^2\ ,\\
&\frac{\ddot{x}}{x} - \left(\frac{\dot{x}}{x}\right)^2 + B^2x^2 + 2C\dot{v} + 2C^2 d(u) = 0\ ,\\
&\dot{y} = Bx^2\ , \qquad \dot{u} = Cx^2\ ,\\
&\left(\frac{\dot{x}}{x}\right)^2 + B^2x^2 + 2C\dot{v} + C^2 d(u) - \frac{\epsilon}{\beta^2} = 0\ ,
\end{aligned} \tag{4}$$

where the dot denotes the derivative with respect to the affine parameter τ, the prime denotes the derivative with respect to u, and B, C are constants. Here the equations for $\dot{y}$ and $\dot{u}$ have already been integrated (this can easily be achieved due to the existence of corresponding Killing vectors). The last equation in (4) is the normalization condition of the four-velocity, $u_\alpha u^\alpha = \epsilon$, where $\epsilon = -1, 0$, or $+1$, for timelike, null or spacelike geodesics, respectively.

We may eliminate the terms containing the constant B and the variable $\dot{v}$ by subtracting the last equation in (4) from the second one. It is also convenient to introduce a new variable $\xi = 1/x$, with which we obtain the equation

$$\ddot{\xi} = \left(C^2 d(u) + \frac{\epsilon}{\beta^2}\right)\xi\ . \tag{5}$$

For the particular case when the profile function d is constant, this is a simple equation for ξ. After obtaining $x(\tau) = \xi^{-1}(\tau)$, we can then integrate the equations for the remaining functions $y(\tau)$, $u(\tau)$ and $v(\tau)$, see (7)-(11). However, if $d(u)$ is a non-constant function, then the equation (5) has to be solved simultaneously with the equation

$$\dot{u} = \frac{C}{\xi^2}\ . \tag{6}$$

Let us now present some particular solutions explicitly. The simplest geodesics arise when $\dot{x} = 0$. In this case (5) implies $C^2 d(u) = -\epsilon/\beta^2$. When $C = 0$, we obtain a privileged class of null geodesics $v = A\tau + v_0$, $x = x_0$, $y = y_0$, $u = u_0$, where A, x_0, y_0, u_0, v_0 are constants. These generate the null wave-fronts $u = const$ in any generalized Defrise solution. If C is non-vanishing then $d(u)$ must necessarily be a constant function, $d(u) = D = const$. The corresponding geodesics are

$$v(\tau) = v_0 + A\tau\ , \quad x(\tau) = x_0\ , \quad y(\tau) = y_0 + Bx_0^2\tau\ , \quad u(\tau) = u_0 + Cx_0^2\tau\ , \tag{7}$$

in which the constants have to satisfy the condition $B^2x_0^2 + 2AC = 2\epsilon/\beta^2$. When $D = 0$ these are null geodesics in the anti–de Sitter space-time. For $D > 0$ the geodesics (7) are timelike and for $D < 0$ spacelike. In these last two cases, an additional restriction $C^2 = 1/(\beta^2|D|)$ applies. Note that the metric (1), (2) for $d(u) = D \neq 0$ represents the Defrise solution since an arbitrary non-vanishing constant D can be scaled to ± 1 by $u \to u/\sqrt{|D|}$, $v \to \sqrt{|D|}\,v$.

In the general case $\dot{x} \neq 0$ we have to solve (5) and (6). For the anti–de Sitter solution ($d(u) = D = 0$) and for the Defrise solution ($d(u) = D = const \neq 0$), the system decouples. The equation for ξ can immediately be solved, yielding the following geodesics:

$$\left.\begin{aligned}
v(\tau) = v_0 + \frac{\epsilon\,\tau}{\beta^2 C} + \frac{1 + B^2x_0^2}{2C(\tau-\tau_0)}\ , \qquad x(\tau) = \frac{x_0}{\tau-\tau_0}\ ,\\
y(\tau) = y_0 - \frac{B\,x_0^2}{\tau-\tau_0}\ , \quad u(\tau) = u_0 - \frac{C\,x_0^2}{\tau-\tau_0}\ ,
\end{aligned}\right\} \quad \text{for} \quad C^2 D + \frac{\epsilon}{\beta^2} = 0\ , \tag{8}$$

$$\left.\begin{array}{c} v(\tau) = v_0 - CD\,\tau + \dfrac{Ax_0^2}{a}\tan(a\tau - \tau_0)\ , \qquad x(\tau) = \dfrac{x_0}{\cos(a\tau - \tau_0)}\ , \\ y(\tau) = y_0 + \dfrac{B\,x_0^2}{a}\tan(a\tau - \tau_0)\ , \quad u(\tau) = u_0 + \dfrac{C\,x_0^2}{a}\tan(a\tau - \tau_0)\ , \\ \text{with} \quad (B^2 + 2AC)x_0^2 = C^2 + D\epsilon\beta^{-2}\ , \end{array}\right\} \quad \text{for} \quad C^2D + \frac{\epsilon}{\beta^2} < 0\ , \tag{9}$$

$$\left.\begin{array}{c} v(\tau) = v_0 - CD\,\tau + \dfrac{Ax_0^2}{a}\tanh(a\tau)\ , \qquad x(\tau) = \dfrac{x_0}{\cosh(a\tau)}\ , \\ y(\tau) = y_0 + \dfrac{B\,x_0^2}{a}\tanh(a\tau)\ , \quad u(\tau) = u_0 + \dfrac{C\,x_0^2}{a}\tanh(a\tau)\ , \\ \text{with} \quad (B^2 + 2AC)x_0^2 = C^2 + D\epsilon\beta^{-2}\ , \end{array}\right\} \quad \text{for} \quad C^2D + \frac{\epsilon}{\beta^2} > 0\ , \tag{10}$$

or

$$\left.\begin{array}{l} v(\tau) = v_0 - CD\,\tau - \dfrac{A}{ax_1}[\,x_0 + x_1\tanh(a\tau)\,]^{-1}\ , \\ x(\tau) = [\,x_0\cosh(a\tau) + x_1\sinh(a\tau)\,]^{-1}\ , \\ y(\tau) = y_0 - \dfrac{B}{ax_1}[\,x_0 + x_1\tanh(a\tau)\,]^{-1}\ , \\ u(\tau) = u_0 - \dfrac{C}{ax_1}[\,x_0 + x_1\tanh(a\tau)\,]^{-1}\ , \\ \text{with} \quad x_1 \neq 0\ , \qquad B^2 + 2AC = (x_0^2 - x_1^2)(C^2 + D\epsilon\beta^{-2})\ , \end{array}\right\} \quad \text{for} \quad C^2D + \frac{\epsilon}{\beta^2} > 0\ , \tag{11}$$

where $a = \sqrt{|C^2D + \epsilon\beta^{-2}|}$, and τ_0, x_1 are arbitrary constants.

For a non-constant profile $d(u)$ the geodesics can be obtained by solving simultaneously the equations (5) and (6) numerically, and integrating subsequently $v(\tau)$ and $y(\tau)$. However, one important general observation can be made for an arbitrary solution. It is obvious from the equation (6) that for *any* geodesic such that $x(\tau) \to 0$, one obtains $u(\tau) \to u_0$. Therefore, for a solution with an arbitrary wave-profile, it follows that $d(u) \to d(u_0) = const = D$. This means that all geodesics behave asymptotically according to one of the corresponding possibilities described by (8), (10), or (11), as $x(\tau) \to 0$.

4 Geodesic deviation

It has been shown previously in [24] that the equation of geodesic deviation along any timelike geodesic, given by (4), in a suitably chosen orthonormal frame $\{\mathbf{e}_{a'}\}$

$$\begin{aligned} e^{\mu}_{(0)} &= u^{\mu} = \left(\dot{v}, \dot{x}, Bx^2, Cx^2\right)\ , \\ e^{\mu}_{(1')} &= \left(-\frac{1}{\beta C}\frac{\dot{x}}{x}\,, \frac{x}{\beta}, 0, 0\right)\ , \\ e^{\mu}_{(2)} &= \frac{x}{\beta}\left(-\frac{B}{C}\,, 0, 1, 0\right)\ , \\ e^{\mu}_{(3')} &= \left(\dot{v} + \frac{1}{\beta^2 C}\,, \dot{x}, Bx^2, Cx^2\right)\ , \end{aligned} \tag{12}$$

can be written as

$$
\begin{aligned}
\ddot{Z}^{(1')} &= \frac{\Lambda}{3} Z^{(1')} \;-\; \mathcal{A}_+ Z^{(1')} \;, \\
\ddot{Z}^{(2)} &= \frac{\Lambda}{3} Z^{(2)} \;+\; \mathcal{M}\, Z^{(2)} \;, \\
\ddot{Z}^{(3')} &= \frac{\Lambda}{3} Z^{(3')} \;.
\end{aligned}
\tag{13}
$$

The amplitudes $\mathcal{A}_+$ and $\mathcal{M}$ are $\mathcal{A}_+ = -4C^2\, d(u)$ and $\mathcal{M} = C^2\, d(u)$ (see (3)), $Z^{(i)} = e^{(i)}_\mu Z^\mu$ denote frame components of the displacement vector connecting two neighbouring free test particles, and $\ddot{Z}^{(i)} = e^{(i)}_\mu \frac{D^2 Z^\mu}{d\tau^2}$ are their relative accelerations.

Equations (13) suggest the following physical interpretation of the generalized Defrise space-times. In the regions where $d(u) = 0$ the functions $\mathcal{A}_+$ and $\mathcal{M}$ vanish. The solution reduces to the anti–de Sitter space-time in which all test particles move isotropically one with respect to the other, $\ddot{Z}^{(i)} = \frac{\Lambda}{3} Z^{(i)}$. Thus, the terms proportional to Λ in (13) represent the influence of the *anti–de Sitter background*. If the amplitudes $\mathcal{A}_+$ and $\mathcal{M}$ do not vanish (which is for $d(u) \neq 0$), these background motions of particles are influenced also by the *effect of the gravitational wave* combined with that of the *null matter*. Both the gravitational wave and the pure radiation propagate in the spacelike direction of $\mathbf{e}_{(3')}$ and have a *transverse* character since only motions in the perpendicular directions $\mathbf{e}_{(1')}$ and $\mathbf{e}_{(2)}$ are affected.

Note however that the direction of propagation $\mathbf{e}_{(3')}$ in *not* parallelly transported. Instead, it uniformly rotates with angular velocity given by $1/\beta = \sqrt{-\Lambda/3}$ with respect to frames $\{\mathbf{e}_{(a)}\}$ parallelly propagated along any timelike geodesic,

$$
\mathbf{e}_{(1')} = \cos(\tau/\beta)\, \mathbf{e}_{(1)} - \sin(\tau/\beta)\, \mathbf{e}_{(3)} \;, \quad \mathbf{e}_{(3')} = \sin(\tau/\beta)\, \mathbf{e}_{(1)} + \cos(\tau/\beta)\, \mathbf{e}_{(3)} \;.
\tag{14}
$$

This effect has been demonstrated for all solutions of the Siklos class in [24].

It is well-known that the effect of pure vacuum gravitational waves with the '+' polarization mode on relative motions of the test particles can be described by the equations (13) with $\mathcal{M} = \mathcal{A}_+$ (see e.g. [28]). However, in our case $\mathcal{M} \neq \mathcal{A}_+$. Nevertheless, we may interpret the effect by introducing a decomposition $\mathcal{M} = \mathcal{A}_+ + \mathcal{P}$, where $\mathcal{P} = 5C^2 d(u)$. Substituting for $\mathcal{M}$ in (13) we observe that the influence on particles given by the anti–de Sitter background and a 'pure' gravitational wave with the amplitude $\mathcal{A}_+$, superpose with the effect given by the term $\mathcal{P}\, Z^{(2)}$. This is responsible for an additional acceleration in the direction of $\mathbf{e}_{(2)}$ due to the presence of null matter.

As in the case of vacuum Siklos space-times, we can rewrite the equation of geodesic deviation in a form that is suitable for integration (note that $\ddot{Z}^{(i)}$ does not represent the total time derivative of $Z^{(i)}(\tau)$ for $i = 1', 3'$ since $\mathbf{e}_{(1')}, \mathbf{e}_{(3')}$ are not parallelly transported). Using the relations (23) given in [24], the system (13) can be written as

$$
\begin{aligned}
\frac{d^2 Z^{(1')}}{d\tau^2} + 4\left[\frac{1}{\beta^2} - C^2\, d(u(\tau))\right] Z^{(1')} &= -\frac{2}{\beta} C_1 \;, \\
\frac{d^2 Z^{(2)}}{d\tau^2} \;+ \left[\frac{1}{\beta^2} - C^2\, d(u(\tau))\right] Z^{(2)} &= 0 \;, \\
Z^{(3')} &= \int \left(\frac{2}{\beta} Z^{(1')} + C_1\right) d\tau \;,
\end{aligned}
\tag{15}
$$

where C_1 is a constant. These decoupled equations can be integrated provided the geodesic function $u(\tau)$ is known. However, there exists a solution, along *any* geodesic in *any* generalized Defrise solution, given by $Z^{(1')} = 0 = Z^{(2)}$, $Z^{(3')} = Z_0 = const$, i.e. using (14),

$$Z^{(1)} = Z_0 \sin(\tau/\beta), \quad Z^{(2)} = 0, \quad Z^{(3)} = Z_0 \cos(\tau/\beta) \ . \tag{16}$$

The particles may corotate uniformly in *circles* with constant angular velocity $\sqrt{-\Lambda/3}$.

Note also that the equations (15) are *independent* of $x(\tau)$, $y(\tau)$ and $v(\tau)$ which again demonstrates the homogeneity of the wave-fronts $u = const$. Thus all timelike observers on a given u will view the same relative motions of the surrounding test particles.

Let us finally present the complete solution of (15) for the case when $d(u) = D = const$:

$$\left.\begin{aligned} Z^{(1')}(\tau) &= -\frac{C_1}{\beta}\,\tau^2 + C_2\,\tau + C_3 \ , \\ Z^{(2)}(\tau) &= C_4\,\tau + C_5 \ , \\ Z^{(3')}(\tau) &= -\frac{2C_1}{3\beta^2}\,\tau^3 + \frac{C_2}{\beta}\,\tau^2 + \left(\frac{2C_3}{\beta} + C_1\right)\,\tau + C_6 \ , \end{aligned}\right\} \quad \text{for} \quad C^2 D - \frac{1}{\beta^2} = 0 \ , \tag{17}$$

$$\left.\begin{aligned} Z^{(1')}(\tau) &= -\frac{C_1}{2a^2\beta} + C_2 \cos(2a\tau) + C_3 \sin(2a\tau) \ , \\ Z^{(2)}(\tau) &= C_4 \cos(a\tau) + C_5 \sin(a\tau) \ , \\ Z^{(3')}(\tau) &= -\frac{C_1 C^2 D}{a^2}\,\tau + \frac{C_2}{a\beta} \sin(2a\tau) - \frac{C_3}{a\beta} \cos(2a\tau) + C_6 \ , \end{aligned}\right\} \quad \text{for} \quad C^2 D - \frac{1}{\beta^2} < 0 \ , \tag{18}$$

$$\left.\begin{aligned} Z^{(1')}(\tau) &= \frac{C_1}{2a^2\beta} + C_2 \cosh(2a\tau) + C_3 \sinh(2a\tau) \ , \\ Z^{(2)}(\tau) &= C_4 \cosh(a\tau) + C_5 \sinh(a\tau) \ , \\ Z^{(3')}(\tau) &= \frac{C_1 C^2 D}{a^2}\,\tau + \frac{C_2}{a\beta} \sinh(2a\tau) + \frac{C_3}{a\beta} \cosh(2a\tau) + C_6 \ , \end{aligned}\right\} \quad \text{for} \quad C^2 D - \frac{1}{\beta^2} > 0 \ , \tag{19}$$

where C_i are constants and a is again given by $a = \sqrt{|C^2 D - \beta^{-2}|}$. These relations describe all possible relative motions of nearby particles in the anti–de Sitter and the Defrise space-times.

In particular, for the anti–de Sitter universe, $D = 0$, so that only the motions given by (18) are allowed. Using the relation (14), these can be written in a parallelly propagated frame as $Z^{(i)}(\tau) = A_i \cos(\tau/\beta + \delta_i)$, where A_i and δ_i, $i = 1, 2, 3$, are constants.

On the other hand, we may consider the limit $\Lambda \to 0$, i.e. $1/\beta \to 0$, in which case the rotation of the frame (14) vanishes, $Z^{(1')} \to Z^{(1)}$, $Z^{(3')} \to Z^{(3)}$. Assuming $D = -\omega^2$, where ω is some positive constant, we get $a \to |C|\omega$, and equations (18) become

$$\begin{aligned} Z^{(1)} &\approx A_1 \cos(\, 2|C|\omega\,\tau + \delta_1 \,) \ , \\ Z^{(2)} &= A_2 \cos(\, |C|\omega\,\tau + \delta_2 \,) \ , \\ Z^{(3)} &\approx A_3\,\tau + \delta_3 \ . \end{aligned} \tag{20}$$

The particles move freely — as in Minkowski space — along the direction $\mathbf{e}_{(3)}$ which is the direction of propagation of the waves. In the transverse plane the relative motions of nearby test particles follow the famous closed Lissajous figures.

5 On the global structure

The metric (1), (2) indicates that the space-times are regular everywhere except possibly at $x = 0$ and/or $x = \infty$. We shall investigate these regions in detail. Let us perform the transformation

$$
\begin{aligned}
\eta &= -\beta\cos(T/\beta)/\mathcal{D}\ , \quad & x &= \beta\cos\chi/\mathcal{D}\ , \\
y &= \beta\sin\chi\cos\vartheta/\mathcal{D}\ , \quad & z &= \beta\sin\chi\sin\vartheta\cos\varphi/\mathcal{D}\ ,
\end{aligned}
\tag{21}
$$

where $\eta = (u-v)/\sqrt{2}$, $z = (u+v)/\sqrt{2}$, and $\mathcal{D} = \sin(T/\beta) + \sin\chi\sin\vartheta\sin\varphi$. This puts the metric of generalized Defrise space-times into the form

$$
\begin{aligned}
\mathrm{d}s^2 \;=\;& \frac{\beta^2}{\cos^2\chi}\left\{-\frac{\mathrm{d}T^2}{\beta^2} + \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\right\} \\
+\;& \frac{d(u(T,\chi,\vartheta,\varphi))}{2\cos^4\chi}\Big\{[1-\cos(T/\beta+\varphi)\sin\chi\sin\vartheta]\frac{\mathrm{d}T}{\beta} + \sin(T/\beta+\varphi)\cos\chi\sin\vartheta\,\mathrm{d}\chi \\
& + \sin(T/\beta+\varphi)\sin\chi\cos\vartheta\,\mathrm{d}\vartheta + \sin\chi\sin\vartheta[\cos(T/\beta+\varphi) - \sin\chi\sin\vartheta]\,\mathrm{d}\varphi\Big\}^2\ ,
\end{aligned}
\tag{22}
$$

where the argument of the profile function d is

$$
u(T,\chi,\vartheta,\varphi) = (\beta/\sqrt{2})\frac{-\cos(T/\beta) + \sin\chi\sin\vartheta\cos\varphi}{\sin(T/\beta) + \sin\chi\sin\vartheta\sin\varphi}\ . \tag{23}
$$

For $d \equiv 0$, this is the well-known form of the anti–de Sitter solution in global coordinates (cf. §5.2 in [29] where $\cosh r = 1/\cos\chi$) which is used in the literature to construct the Penrose diagram. Choosing the conformal factor $\Omega = \beta^{-1}\cos\chi$, the boundary $\Omega = 0$ of the anti–de Sitter manifold (corresponding to $\chi = \pi/2$) represents null and spacelike infinity which can be thought of as a timelike surface with topology $R \times S^2$.

The metric form (22) demonstrates explicitly that, for bounded profiles $d(u)$, the space-times are regular everywhere, except at $\chi = \pi/2$. Therefore, $x = \infty$ (which corresponds to $\mathcal{D} = 0$, $\chi \neq \pi/2$) is only a *coordinate* singularity. In fact, by inspecting the particular geodesics (8), (9), (11) it can be seen that $x = \infty$ is reached at *finite* values of the affine parameters. This indicates that $x = \infty$ is not a boundary of the manifold, which can thus be extended beyond $\mathcal{D} = 0$. This continuation is achieved by putting the solution into the form (22) and considering the full range of the coordinates, $T \in (-\infty, +\infty)$, $\chi \in [0, \pi/2)$, $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. (Let us also remark that even with the help of the coordinate $\xi = 1/x$, the geodesics (8)-(11) can analytically be extended through $x = \infty$ which corresponds to $\xi = 0$.)

We now investigate the singularity at $x = 0$. This is mapped to $\chi = \pi/2$, i.e. it is located at the "anti–de Sitter-like" infinity given by the boundary $\Omega = 0$. We have already emphasized (see end of Section 3) that *all* geodesics approaching $x = 0$ behave asymptotically according to (8), (10), or (11). Therefore, an *infinite* value of the affine parameter τ is required to reach $x = 0$. This supports our observation that this singularity is located at the very boundary of the manifold. Moreover, all components of the curvature tensor in the orthonormal frame parallelly propagated along timelike geodesics are given by (3), and obviously *remain finite* even as $x \to 0$. This indicates that the singularity at $x = 0$ is quasiregular (according to the classification scheme introduced in [30]), i.e. it has a "topological" rather than a "curvature" character.

Finally, let us transform the generalized Defrise solutions (1), (2) using

$$2x = \frac{\pm 1}{\cosh\theta + \sinh\theta\cos\phi}\ , \qquad\qquad 2y = \frac{\sinh\theta\sin\phi}{\cosh\theta + \sinh\theta\cos\phi}\ , \tag{24}$$

to obtain

$$\begin{aligned} \mathrm{d}s^2 = {} & \beta^2\,(\mathrm{d}\theta^2 + \sinh\theta\,\mathrm{d}\phi^2) + 8\beta^2\,(\cosh\theta + \sinh\theta\cos\phi)^2\,\mathrm{d}u\,\mathrm{d}v \\ & +16\beta^2\,(\cosh\theta + \sinh\theta\cos\phi)^4\ d(u)\,\mathrm{d}u^2\ , \end{aligned} \tag{25}$$

where $\theta \in [0,\infty)$, $\phi \in [0, 2\pi)$, $u, v \in (-\infty, +\infty)$. The singularity at $x = 0$ is now given by $\theta = \infty$. The form (25) of the solutions exhibits explicitly the geometry of the wave-surfaces $u = const$: these are two-dimensional *hyperboloidal surfaces* of constant negative curvature $-\beta$.

6 Sandwich and impulsive waves in the anti–de Sitter universe

Using the above results, we may now consider the construction of sandwich (gravitational plus null matter) waves in the anti–de Sitter space. Obviously, these are described by the metric (1) , (2), or equivalently by (25), if the wave-profile function $d(u)$ is non-vanishing on a *finite* interval, say $u \in [u_1, u_2]$, only. In such a case, the sandwich wave has a finite duration and extends between two hyperboloidal surfaces u_1 and u_2 representing the front and the end of the wave. In front of the propagating sandwich wave of type N , for $u < u_1$, and also behind it, for $u > u_2$, there are two anti–de Sitter regions which are conformally flat and maximally symmetric. The situation is analogous to the well-known case in Minkowski universe in which, however, the plane waves propagate through the flat space [25]-[27].

To obtain a better understanding of the geometry of these sandwich waves let us recall that the anti–de Sitter universe can be seen as a four-dimensional hyperboloid $-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 - Z_4^2 = -\beta^2$, embedded in a five-dimensional flat space-time $\mathrm{d}s_0^2 = -\mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 - \mathrm{d}Z_4^2$, with two time coordinates Z_0 and Z_4 (see e.g. [29]). This is shown in Fig. 1. The most natural global parametrization is

$$\begin{aligned} Z_0 &= -\beta\ \cos(T/\beta)/\cos\chi\ , & Z_4 &= \beta\ \sin(T/\beta)/\cos\chi\ , \\ Z_1 &= \beta\ \tan\chi\sin\vartheta\cos\varphi\ , & Z_2 &= \beta\ \tan\chi\cos\vartheta, \\ Z_3 &= \beta\ \tan\chi\sin\vartheta\sin\varphi\ , \end{aligned} \tag{26}$$

which gives the coordinate system (22) for $d = 0$. The beginning and the end of the sandwich wave, given by $u = u_1$ and $u = u_2$, can now be visualized on the above hyperboloid. (Note that when d is small, the waves can be considered to represent a perturbation of the anti–de Sitter hyperboloid so that also the "inner" wave-surfaces $u = const \in (u_1, u_2)$ can be treated similarly.) Using (23) and (26) we obtain

$$Z_0 + Z_1 = (\sqrt{2}/\beta)\,u\,(Z_3 + Z_4)\ . \tag{27}$$

Each wave-front is thus located on the two-dimensional intersection of the hyperboloid with the null hyperplane (27) for a fixed u. In particular, the wave-surface $u = 0$ corresponds to $Z_0 + Z_1 = 0$, which is a two-dimensional hyperboloidal surface $Z_4^2 - Z_2^2 - Z_3^2 = \beta^2$. The wave-fronts $u = \pm\infty$ are given by $Z_3+Z_4=0$, corresponding to $Z_0^2-Z_1^2-Z_2^2=\beta^2$. The intersections for general u given by (27) are more difficult to visualize. In Fig. 1 we draw them after supressing two space coordinates, $Z_2=0=Z_3$, and we also assume that $u_1<0$ and $u_2>0$. Nevertheless, this has a disadvantage that the null character of these intersections is not seen explicitly, except for the wave-surface $u=0$. However, the picture still gives a useful insight into the geometry of the sandwich waves in the anti–de Sitter universe. We can also easily observe that the complete picture should contain *two* sandwich waves, first at $Z_4>0$ and another one at $Z_4<0$. Any observer moving around the anti–de Sitter hyperboloid in closed timelike loops would first observe a sandwich wave propagating in one direction, then the second propagating in the opposite direction, then again the first one, and so on in an endless cycle. (Alternatively, this can be considered to represent only one wave which "bounces" back and forth from one side of the universe to the other.) These sandwich waves are also shown in the conformal diagram in Fig. 2.

The five-dimensional formalism also enables us to construct the Defrise-type *impulsive* waves in the anti–de Sitter universe. By combining the transformation (21) with the parametrization (26), we may write the generalized Defrise solutions as $\mathrm{d}s^2=\mathrm{d}s_0^2+(\beta^2/x^4)\,d(u)\,\mathrm{d}u^2$, where $\mathrm{d}s_0^2$ is the metric on the anti–de Sitter hyperboloid. Let us now consider a sequence of wave-profiles $d(u)$ approaching the Dirac delta-distribution $\delta(u)$ localized on the null hypersurface $u=(\beta/\sqrt{2})(Z_0+Z_1)/(Z_3+Z_4)=0$. Straightforward calculation gives the distributional limit

$$ds^2=-\mathrm{d}Z_0^2+\mathrm{d}Z_1^2+\mathrm{d}Z_2^2+\mathrm{d}Z_3^2-\mathrm{d}Z_4^2\ +\ H\,\delta(Z_0+Z_1)\,(\mathrm{d}Z_0+\mathrm{d}Z_1)^2\ , \qquad \text{where}\qquad \sqrt{2}\beta^5\,H=(Z_3+Z_4)^3\ , \tag{28}$$

which describes the metric of the implusive gravitational wave plus the null-matter wave. This is the particular solution which belongs to a general class on non-expanding impulsive waves in the anti–de Sitter universe, presented in [31], [32]. The geometry of the impulsive surfaces $Z_0+Z_1=0$ has been discussed in detail in [33].

7 Geodesics in sandwich and impulsive Defrise waves

Finally, we present the simplest example of these sandwich waves given by profile functions of the form

$$d(u)\ =\ D\,[\,\Theta(u)\ -\ \Theta(\,u-u_2\,)\,]\ , \tag{29}$$

where u_2 is a positive constant and Θ is the Heaviside step function. Since $d(u)=0$ for $u<0$ and for $u>u_2$, whereas $d(u)=D$ for $u\in[0,u_2]$, the solutions given by (29) represent the Defrise sandwich waves with *constant* amplitudes propagating in the anti–de Sitter universe.

Using the explicit forms of the geodesics (7)-(11) it is possible to find motion in these space-times. We concentrate here on a privileged class of timelike geodesics which in the (complete) anti–de Sitter spacetime are given by

$$Z_0\ =\ -\beta\,\cos(\tau/\beta)\ ,\quad Z_1\ =\ \beta\,\sin(\tau/\beta)\,\sinh\psi\ ,\quad Z_4\ =\ \beta\,\sin(\tau/\beta)\,\cosh\psi\ , \qquad Z_2\ =\ 0\ =\ Z_3\ , \tag{30}$$

where τ is the proper time and $\psi\in(-\infty,+\infty)$ is an arbitrary constant which parametrizes the specific geodesic from the above family. Observers following these geodesics move around the hyperboloid in closed timelike loops given by the intersections of the hyperboloid with the planes $Z_1 = \tanh\psi\, Z_4$, as indicated in Fig. 1. At $\tau = 0$ all observers are located at one point, $Z_0 = -\beta$, $Z_1 = 0 = Z_4$, and start moving with different velocities. At $\tau = (\pi/2)\,\beta$ they reach their maximum distance Z_1 from the "space origin" $Z_1 = 0$. Subsequently, they converge back and all meet again simultaneously at the point $Z_0 = \beta$, $Z_1 = 0 = Z_4$ at $\tau = \pi\,\beta$. Then continue on the other side of the anti–de Sitter hyperboloid ($Z_4 < 0$) symmetrically, and return back to the starting point. The existence of these specific geodesics is caused by the presence of a negative cosmological constant which has the effect of a universal attractive force.

Our objective here is to investigate how the "focusing" effect described above is changed when the observers pass through a sandwich wave of the Defrise type. We assume the wave-profile $d(u)$ has the form (29) so that there are three regions: I. The anti–de Sitter region $u < 0$ in front of the wave, II. The Defrise wave for $0 < u < u_2$, and III. Another anti–de Sitter region $u > u_2$ behind the wave (see Fig. 3).

We start in the region I. with the privileged geodesics (30) which can be written in the coordinates of the anti–de Sitter metric (1) using the corresponding parametrization

$$\eta = \frac{\beta Z_0}{Z_3 + Z_4}\ , \quad x = \frac{\beta^2}{Z_3 + Z_4}\ , \quad y = \frac{\beta Z_2}{Z_3 + Z_4}\ , \quad z = \frac{\beta Z_1}{Z_3 + Z_4}\ , \tag{31}$$

(which follows from (21) and (26)) as

$$\begin{aligned}
x(\tau) &= \frac{\beta}{\cosh\psi\,\sin(\tau/\beta)}\ ,\\
y(\tau) &= 0\ ,\\
u(\tau) &= \frac{\beta}{\sqrt{2}}\tanh\psi\ -\ \frac{\beta}{\sqrt{2}}\,\frac{\cot(\tau/\beta)}{\cosh\psi}\ ,\\
v(\tau) &= \frac{\beta}{\sqrt{2}}\tanh\psi\ +\ \frac{\beta}{\sqrt{2}}\,\frac{\cot(\tau/\beta)}{\cosh\psi}\ .
\end{aligned} \tag{32}$$

These can easily be identified in the general class of timelike geodesics (9). The geodesics (32) start at $\tau = 0$ on the hypersurface $u = -\infty$ and continue through the anti–de Sitter region $u < 0$ until they reach the front $u = 0$ of the sandwich wave, as indicated in Fig. 3. Different observers with their specific values of the parameter ψ reach the wave in different times τ_f which are given by

$$\cot(\,\tau_f/\beta\,) = \sinh\psi\ . \tag{33}$$

This implies that observers with higher values of ψ encounter the wave sooner, so that the wave propagates from right to left (from positive to negative values of Z_1).

Now, we wish to extend the geodesics (32) into the sandwich-wave region II. We assume that the geodesic functions $x(\tau)$, $y(\tau)$, $u(\tau)$ and $v(\tau)$ are *continuous* across $u = 0$, i.e. at $\tau = \tau_f$. For simplicity we consider here only geodesics for which $y(\tau) \equiv 0$ at any τ. In addition, we require that $\dot{x}(\tau)$ is also a continuous function of the proper time. Note that $\dot{u}$ is continuous as a consequence of the relation $\dot{u} = Cx^2$, provided the constant C has the same value in all the three regions. However, we *cannot* require $\dot{v}$ to be continuous. In fact, by inspecting the geodesic equations (4), in particular the last equation representing the normalization condition, it is obvious that such an additional assumption would be inconsistent with (29). Instead, we have to prescribe a *discontinuity* in $\dot{v}$ at $u = 0$ and $u = u_2$ given by

$$\begin{aligned} \dot{v}(u \to 0_+) &= \dot{v}(u \to 0_-) \, - \tfrac{1}{2} C D \ , \qquad (34) \\ \dot{v}(u \to u_{2+}) &= \dot{v}(u \to u_{2-}) + \tfrac{1}{2} C D \ . \qquad (35) \end{aligned}$$

It is now straightforward to find among (9) the explicit forms of the geodesics in the Defrise wave-zone region II. given by $0 \le u \le u_2$. These are

$$\begin{aligned} x(\tau) &= \sqrt{\frac{1-(D/2\beta^2)\cosh^2\psi}{1-(D/2\beta^2)}}\,\frac{\beta}{\cosh\psi\,\cos(a\tau-\tau_0)} \ , \\ u(\tau) &= \frac{\beta\tanh\psi}{\sqrt{2}[1-(D/2\beta^2)]}\left[1+\frac{\sqrt{1-(D/2\beta^2)\cosh^2\psi}}{\sinh\psi}\,\tan(a\tau-\tau_0)\right] \ , \qquad (36) \\ v(\tau) &= \frac{D}{\sqrt{2}\beta^2}\cosh\psi\,(\tau_f-\tau) \ + \ \frac{\beta}{\sqrt{2}}\tanh\psi\,\left[1-\frac{\sqrt{1-(D/2\beta^2)\cosh^2\psi}}{\sinh\psi}\,\tan(a\tau-\tau_0)\right] \ , \end{aligned}$$

in which the constant τ_0 is given by

$$\tan(\,a\tau_f-\tau_0\,) = -\frac{\sinh\psi}{\sqrt{1-(D/2\beta^2)\cosh^2\psi}} \ , \qquad (37)$$

and $a = \sqrt{1-(D/2\beta^2)\cosh^2\psi}\,/\,\beta$. The observers with specific ψ move along the geodesics (36) until they reach the end of the Defrise sandwich wave, $u = u_2$, at their proper times τ_e given by

$$\tan(\,a\tau_e-\tau_0\,) = \frac{(\sqrt{2}/\beta)[1-(D/2\beta^2)]\,u_2\cosh\psi-\sinh\psi}{\sqrt{1-(D/2\beta^2)\cosh^2\psi}} \ . \qquad (38)$$

Using (35) we may similarly extend the geodesic across $u = u_2$ into the anti–de Sitter region III. behind the wave:

$$\begin{aligned} x(\tau) &= \frac{\beta\sqrt{K}}{\cosh\psi\,\cos(\tau/\beta-\tau_0^*)} \ , \\ u(\tau) &= \Big(1-K[1-(D/2\beta^2)]\Big)\,u_2+\frac{\beta K}{\sqrt{2}}\tanh\psi\,\Big[1+\frac{\tan(\tau/\beta-\tau_0^*)}{\sinh\psi}\Big] \ , \qquad (39) \\ v(\tau) &= \frac{D}{\sqrt{2}\beta^2}\cosh\psi\,(\tau_f-\tau_e) \ + \ \frac{\beta}{\sqrt{2}}\tanh\psi\,\left[1-\frac{\tan(\tau/\beta-\tau_0^*)}{\sinh\psi}\right] \ , \end{aligned}$$

in which the constant τ_0^* is given by

$$\tan(\,\tau_e/\beta-\tau_0^*\,) = (\sqrt{2}/\beta)[1-(D/2\beta^2)]\,u_2\cosh\psi-\sinh\psi \ , \qquad (40)$$

and

$$K = \frac{1+(2/\beta^2)[1-(D/2\beta^2)]\,u_2^2-(2\sqrt{2}/\beta)\,u_2\tanh\psi}{1+(2/\beta^2)[1-(D/2\beta^2)]^2\,u_2^2-(2\sqrt{2}/\beta)[1-(D/2\beta^2)]\,u_2\tanh\psi} \ . \qquad (41)$$

It is obvious that the geodesics (36) and (39) reduce to (32) when $D = 0$ (implying $a = 1/\beta$, $K = 1$, and $\tau_0 = \pi/2 = \tau_0^*$), and also for $u_2 \to 0$ with finite D (in which case $K \to 1$, $\tau_e \to \tau_f$, $\tau_0^* \to \pi/2$, and the Defrise wave-region II. disappears).

The above geodesics (32) in the anti–de Sitter universe reconverge to the "space" origin $Z_1 = 0$ all at the same time $\tau = \pi\beta$. Indeed, using (31) we obtain $Z_1 = \beta\,z/x = (\beta/\sqrt{2})\,(u+v)/x$, but from (32) it follows that $1/x$ is proportional to $\sin(\tau/\beta)$ which vanishes *independently* of ψ. However, if the motion of the observers is influenced by the sandwich wave, it follows from (39) that $Z_1 \sim 1/x \sim \cos(\tau/\beta - \tau_0^*)$. Thus, the observers return back to $Z_1 = 0$ at times $\tau = (\tau_0^* + \pi/2)\beta$. However, these are now generally different and individual since the parameter τ_0^*, given by (40), (38), (37) and (33), is a complicated function of D, u_2 and, in particular, of ψ.

Let us finally consider the geodesic motion in the Defrise-type *impulsive* wave (28) in the anti–de Sitter universe. To this end we assume a sequence of sandwich gravitational plus null matter waves given by (29), in which the parameters D and u_2 satisfy the normalization condition $D\,u_2 = -1$ (we require $D < 0$, but this is the physically interesting case for which the amplitude $\Phi = -5\,D/8\pi\beta^4$ in the pure radiation energy-momentum tensor $T_{\mu\nu} = \Phi k_\mu k_\nu$ is positive). The geodesics in the impulsive Defrise wave can now be obtained from geodesics in the corresponding sandwich waves (32)-(41) by assuming the limit $u_2 \to 0$, i.e. $D \to -\infty$, for which the sequence of $d(u)$ approaches the Dirac distribution $\delta(u)$. Straightforward calculations using (37) and (38) yield

$$\tau_e - \tau_f = \frac{1}{a}\arctan\left(\frac{(\sqrt{2}/\beta)\,u_2\sqrt{1-(D/2\beta^2)\cosh^2\psi}}{\cosh\psi - (\sqrt{2}/\beta)\,u_2\sinh\psi}\right) \sim \frac{\sqrt{2}\,u_2}{\cosh\psi} \to 0\ , \tag{42}$$

so that $\tau_e \to \tau_f$. This is expected since the sandwich wave region II. vanishes in the limit $u_2 \to 0$. The geodesics in the anti–de Sitter region III. behind the impulse ($u > 0$ in the limit) are given by (39). In particular, we can evaluate the functions at the time τ_e at which the specific observers stop interacting with the impulsive wave localized at $u = 0$,

$$x(\tau_e) = \beta\ , \qquad y(\tau_e) = 0\ , \qquad u(\tau_e) = 0\ , \qquad v(\tau_e) = \sqrt{2}\beta\tanh\psi\ +\ \frac{1}{2\beta^2}\ . \tag{43}$$

This can now be compared with the corresponding values obtained from the geodesics (32) in the region I. in front of the wave ($u < 0$) at $\tau = \tau_f$,

$$x(\tau_f) = \beta\ , \qquad y(\tau_f) = 0\ , \qquad u(\tau_f) = 0\ , \qquad v(\tau_f) = \sqrt{2}\beta\tanh\psi\ . \tag{44}$$

Since $\tau_f = \tau_e$, the relations (43) and (44) give the *junction conditions* for geodesics crossing the impulsive wave. It is obvious that the space coordinates x and y are continuous across the impulsive hyperboloidal surface $u = 0$, whereas the parameter v (along the null rays generated by the Debever-Penrose vector field $\mathbf{k} = \partial_v$) suffers a discontinuity $\Delta v = v(\tau_e) - v(\tau_f) = 1/(2\beta^2)$. This behaviour is in full agreement with a general junction condition for the construction of non-expanding impulsive waves in Minkowski, de Sitter and anti–de Sitter space-times by the 'cut and paste' method [34], [35].

8 Conclusions

We have investigated a class of exact solutions which describe gravitational and null-matter waves propagating in the anti–de Sitter universe. By analyzing geodesic and geodesic deviation, we were able to give a physical interpretation of these space-times.

We have also demonstrated that these space-times appear non-singular for all geodesic observers. This is a unique feature for cosmological waves. Therefore, the solutions may be considered as an interesting analogue of the well-known plane gravitational waves in flat Minkowski universe which exhibit the same property.

Moreover, arbitrary profiles of these waves in the anti–de Sitter universe can be prescribed so that sandwich gravitational plus null-matter waves can easily be obtained. We have investigated some of their properties including the geometry of the wave-surfaces and geodesic motion. This enabled us to construct explicitly impulsive waves of the Defrise type. The junction conditions across the impulsive hyperboloidal null surface, which we have derived from the geodesics, are consistent with those discussed in the literature previously.

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Figure Caption

Figure 1. The anti–de Sitter universe, represented as a four-dimensional hyperboloid in a five-dimensional flat space-time with two time coordinates Z_0 and Z_4, is globaly parametrized by the coordinates T, χ, ϑ, φ. Sandwich waves which propagate in the anti–de Sitter universe are bounded by two-dimensional hyperboloidal null surfaces $u = u_1$ and $u = u_2$. Privileged timelike geodesics $\psi = const$ in the background are also indicated.

Figure 2. The conformal diagram of the anti–de Sitter universe, with the global coordinate chart T, χ, in which $\chi = \pi/2$ represents null and spacelike infinity. The wave-surfaces $u = const$ of sandwich waves are indicated. Any timelike observer $\psi = const$ encounters first the sandwich wave propagating to the left and then the wave propagating in the opposite direction.

Figure 3. Part of the conformal diagram representing a sandwich wave localized at $u \in [0, u_2]$. Privileged geodesics $\psi = const$ start at $\tau = 0$, $u = -\infty$ at one point in the anti–de Sitter region I., given by $u < 0$. At τ_f these enter the Defrise wave-region II., and at τ_e emerge into the anti–de Sitter region III. behind the wave, $u > u_2$.