Symmetries and geodesics in (anti–)de Sitter spacetimes with nonexpanding impulsive waves

J. Podolský*

Institute of Theoretical Physics, Charles University,
V Holešovičkách 2, 18000 Prague 8, Czech Republic.

and M. Ortaggio†

Dipartimento di Fisica, Università degli Studi di Trento,
and INFN, Gruppo Collegato di Trento, 38050 Povo (Trento), Italy.

August 9, 2001

Abstract

We consider a class of exact solutions which represent nonexpanding impulsive waves in backgrounds with nonzero cosmological constant. Using a convenient 5-dimensional formalism it is shown that these spacetimes admit at least three global Killing vector fields. The same geometrical approach enables us to find all geodesics in a simple explicit form and describe the effect of impulsive waves on test particles. Timelike geodesics in the axially-symmetric Hotta-Tanaka spacetime are studied in detail. It is also demonstrated that for vanishing cosmological constant, the symmetries and geodesics reduce to those for well-known impulsive pp-waves.

PACS: 04.20.Jb; 04.30.Nk
Keywords: Impulsive waves, (anti–)de Sitter space, symmetries, geodesics.

1 Introduction

Impulsive plane waves in Minkowski space have been studied for decades. These can naturally be understood as a limiting case of sandwich pp-waves [1] with wave-profiles approaching the Dirac delta distribution. Furthermore, Aichelburg and Sexl [2] obtained a specific impulsive pp-wave spacetime by boosting a spherically symmetric point source (described by the Schwarzschild metric) to the speed of light. The same approach has subsequently been used by a number of authors who boosted other solutions of the Kerr-Newman or the Weyl families [3]. Penrose [4] has presented yet another geometrical method for the construction of general impulsive pp-waves in a Minkowski background. This is based on cutting the spacetime along a null plane and then re-attaching the two pieces with a suitable warp, prescribed by the Penrose junction conditions. These spacetimes are often presented in terms of a metric which contains the Dirac delta explicitly. However, it is possible to find a coordinate system for impulsive waves, in which the metric is continuous. For the Aichelburg-Sexl solution this has been done by D'Eath [5] (and used for analysis of ultrarelativistic encounters of black holes), a continuous system for general impulsive pp-wave spacetimes has been presented in [6, 7].

*E–mail: `podolsky@mbox.troja.mff.cuni.cz`
†E–mail: `ortaggio@science.unitn.it`

All the above methods can also be used to construct nonexpanding impulsive waves in backgrounds with nonvanishing cosmological constant Λ. Somewhat surprisingly, this has explicitly been done only recently. It has been demonstrated [8] that such spacetimes can be obtained as a distributional limit of sandwich waves of the Kundt class $KN(\Lambda)$ of solutions [9]. In 1993 Hotta and Tanaka, following the Aichelburg and Sexl approach, boosted the Schwarzschild-(anti–)de Sitter spacetime to the speed of light [10]. They obtained a specific solution which represents an impulsive 2-sphere generated by a pair of two null particles at the poles, which propagates through the de Sitter universe. In the anti–de Sitter universe the impulsive wavefront is hyperboloidal and is generated by a single particle moving with the speed of light [11]. A general class of these spacetimes has been presented in [12]. It has also been demonstrated that impulsive pure gravitational waves of this type are generated by null particles with arbitrary multipole structures. Nonexpanding impulsive waves in the (anti–)de Sitter spacetime can alternatively be obtained by the Penrose "cut and paste" method [13], or equivalently using the "shift function" method [14]. In [13] both distributional and continuous metric forms of these solutions have been presented, in which the limit $\Lambda \to 0$ (resulting in impulsive pp-waves) can be explicitly performed.

Various properties of impulsive pp-waves in Minkowski space have been studied. In particular, symmetries (which form a "richer" structure than those for waves with a smooth profile) have been investigated in [15, 6]. Geodesics have been discussed in several works [16]. However, since the corresponding equations of geodesic and geodesic deviation contain highly singular products of distributions, the advanced framework of Colombeau algebras of generalized functions had to be employed to solve these equations in a mathematically rigorous sense [17].

It is the purpose of this paper to present principal results concerning symmetries (section 3) and behaviour of geodesics (sections 4 and 5) in spacetimes which describe nonexpanding impulsive waves in the (anti–)de Sitter background.

2 General forms of the solutions

It has previously been shown [12, 13] that a complete class of nonexpanding impulsive waves in spacetimes with a nonvanishing cosmological constant Λ can conveniently be written using a 5-dimensional formalism as metrics

$$\mathrm{d}s^2 = H(Z_2, Z_3, Z_4)\,\delta(U)\,\mathrm{d}U^2 - 2\,\mathrm{d}U\mathrm{d}V + \mathrm{d}{Z_2}^2 + \mathrm{d}{Z_3}^2 + \epsilon\,\mathrm{d}{Z_4}^2 \ , \tag{1}$$

on the 4-dimensional hyperboloid

$$-2\,UV + {Z_2}^2 + {Z_3}^2 + \epsilon {Z_4}^2 = \epsilon a^2 \ , \qquad a = \sqrt{3/(\epsilon\Lambda)} \ , \tag{2}$$

where $U = \frac{1}{\sqrt{2}}(Z_0 + Z_1)$ and $V = \frac{1}{\sqrt{2}}(Z_0 - Z_1)$. The wave is absent when $H = 0$, in which case (1) with the constraint (2) reduces to the well-known form of the de Sitter space (for $\Lambda > 0$ and $\epsilon = 1$) or the anti–de Sitter space (for $\Lambda < 0$ and $\epsilon = -1$). For a nontrivial H, the solution represents an impulsive wave propagating in the (anti–)de Sitter universe. The impulse is located on the null hypersurface $U = 0$ given by

$${Z_2}^2 + {Z_3}^2 + \epsilon {Z_4}^2 = \epsilon a^2 \ . \tag{3}$$

This is a nonexpanding 2-sphere in the de Sitter universe, or a hyperboloidal 2-surface in the anti–de Sitter universe [11] which can be parametrized as

$$Z_2 = a\sqrt{\epsilon(1-z^2)}\,\cos\phi \ , \quad Z_3 = a\sqrt{\epsilon(1-z^2)}\,\sin\phi \ , \quad Z_4 = a\,z \ . \tag{4}$$

In general, the above metrics describe impulsive gravitational waves and/or impulses of null matter. Pure gravitational waves occur when the *vacuum* field equation

$$(\Delta + \tfrac{2}{3}\Lambda)H = 0 \ , \tag{5}$$

where $\Delta = \frac{1}{3}\Lambda\{\partial_z[(1-z^2)\partial_z] + (1-z^2)^{-1}\partial_\phi\partial_\phi)\}$ is the Laplacian on the impulsive surface, is satisfied [18, 13, 14]. Non-trivial solutions of (5) can be written as

$$H(z,\phi) = \sum_{m=0}^{\infty} b_m H_m(z,\phi) = \sum_{m=0}^{\infty} b_m Q_1^m(z)\cos[m(\phi-\phi_m)] \ , \tag{6}$$

where b_m and ϕ_m are constants representing an arbitrary amplitude and phase of each component, and $Q_1^m(z)$ are associated Legendre functions of the second kind generated by the relation $Q_1^m(z) = (-\epsilon)^m|1-z^2|^{m/2}(\mathrm{d}^m Q_1(z)/\mathrm{d}z^m)$. The first term (for $m=0$) alone

$$Q_1(z) = Q_1^0(z) = \frac{z}{2}\log\left|\frac{1+z}{1-z}\right| - 1 \ , \tag{7}$$

gives the simplest (axially symmetric) solution found by Hotta and Tanaka [10]. The components H_m describe impulsive gravitational waves generated by null point sources with an m-pole structure localized at the singularities $z=\pm1$ on the wave-front, see [12].

Various 4-dimensional parametrizations of the solutions (1), (2) are known. For example,

$$\begin{aligned} &Z_2 = x/\Omega \ , \qquad Z_3 = y/\Omega \ , \\ &Z_4 = a\,(2/\Omega - 1) \ , \\ &U = u/\Omega \ , \qquad V = v/\Omega \ , \end{aligned} \tag{8}$$

with

$$\zeta = \tfrac{1}{\sqrt{2}}(x+\mathrm{i}y) \ , \qquad \Omega = 1 + \tfrac{1}{6}\Lambda(\zeta\bar\zeta - uv) \ , \tag{9}$$

brings the metric to the form

$$\mathrm{d}s^2 = \frac{2\tilde H(\zeta,\bar\zeta)\,\delta(u)\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{[1+\frac{1}{6}\Lambda(\zeta\bar\zeta - uv)]^2} \ , \tag{10}$$

in which $2\tilde H(\zeta,\bar\zeta) = (1+\frac{1}{6}\Lambda\,\zeta\,\bar\zeta)\,H(\zeta,\bar\zeta)$. The solutions (10) are written in unified form both for $\Lambda>0$ and $\Lambda<0$. For $\Lambda=0$ these reduce to the well-known distributional (Brinkmann) form of the *pp*-waves in Minkowski space. The impulsive wave (in any spacetime of constant curvature) is located on the wavefront $u=0$.

Another form of the solutions is obtained by performing the transformation

$$\begin{aligned} u &= \mathcal{U} \ , \\ v &= \mathcal{V} + \tilde H\,\Theta(\mathcal{U}) + \tilde H_Z \tilde H_{\bar Z}\,\Theta(\mathcal{U})\,\mathcal{U} \ , \\ \zeta &= Z + \tilde H_{\bar Z}\,\Theta(\mathcal{U})\,\mathcal{U} \ , \end{aligned} \tag{11}$$

where $\Theta(\mathcal{U})$ is the Heaviside step function. This transformation is discontinuous on $u=0$ in exact correspondence with the junction conditions prescribed by Penrose in the "cut and paste" method [4]. The resulting line element,

$$\mathrm{d}s^2 = \frac{-2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2\,|\mathrm{d}Z + \Theta(\mathcal{U})\,\mathcal{U}\,(\tilde H_{Z\bar Z}\,\mathrm{d}Z + \tilde H_{\bar Z\bar Z}\,\mathrm{d}\bar Z)|^2}{[1+\frac{1}{6}\Lambda(Z\bar Z - \mathcal{U}\mathcal{V} + \tilde G\,\Theta(\mathcal{U})\,\mathcal{U})]^2} \ , \tag{12}$$

where $\tilde G = Z\tilde H_Z + \bar Z\tilde H_{\bar Z} - \tilde H$, is continuous across the null hypersurface $\mathcal{U} = 0 = u$. However, discontinuities in derivatives of the metric in general yield components in the curvature and the Weyl tensors proportional to the Dirac δ. Note finally that for $\Lambda=0$ the metric (12) reduces to the Rosen form of impulsive *pp*-waves [7].

3 Symmetries of the impulsive solutions

Symmetries of impulsive *pp*-waves propagating in Minkowski space were investigated by Aichelburg and Balasin [6, 15]. Using the distributional form (given by (10) for $\Lambda = 0$) they demonstrated that spacetimes with impulsive waves admit more symmetries than the same class of waves with a general profile, e.g. sandwich waves. In fact, *all* impulsive *pp*-waves (with *no restriction* on the form of the structural function $\tilde{H}$) admit at least a 3-parameter group of motions generated by the Killing vector fields

$$x\,(\partial/\partial v) + u\,(\partial/\partial x)\ , \quad y\,(\partial/\partial v) + u\,(\partial/\partial y)\ , \quad (\partial/\partial v)\ . \tag{13}$$

Additional symmetries occur for specific forms of the structural function $\tilde{H}$, see [15]. It is the purpose of this part of our contribution to investigate symmetries of exact nonexpanding impulsive waves which propagate in the de Sitter or anti–de Sitter backgrounds.

Spacetimes of constant curvature admit a 10-parameter group of motions. It is convenient to study symmetries of the (anti–)de Sitter spacetime in a 5-dimensional formalism using the coordinates introduced in (1), (2). Obviously, this metric representation (with $H = 0$) is invariant under the $SO(1,4)$ or $SO(2,3)$ group of transformations for $\Lambda > 0$ or $\Lambda < 0$, respectively. In the first case, these are 3 spatial rotations, 1 boost and 6 null rotations, in the second case 1 spatial rotation, 3 boosts and 6 null rotations generated by the following Killing vector fields:

$$Z_2\,(\partial/\partial Z_3) - Z_3\,(\partial/\partial Z_2) \tag{14}$$
$$Z_4\,(\partial/\partial Z_i) - \epsilon Z_i\,(\partial/\partial Z_4) \tag{15}$$
$$U\,(\partial/\partial U) - V\,(\partial/\partial V) \tag{16}$$
$$Z_i\,(\partial/\partial V) + U\,(\partial/\partial Z_i) \tag{17}$$
$$\epsilon Z_4\,(\partial/\partial V) + U\,(\partial/\partial Z_4) \tag{18}$$
$$Z_i\,(\partial/\partial U) + V\,(\partial/\partial Z_i) \tag{19}$$
$$\epsilon Z_4\,(\partial/\partial U) + V\,(\partial/\partial Z_4) \tag{20}$$

in which $i = 2, 3$. Finite transformations corresponding to (14)-(16) have well-known forms. Null rotations generated by (17) are

$$\begin{aligned} &Z_2' = Z_2 + b\,U\ , \\ &Z_3' = Z_3\ , \quad Z_4' = Z_4\ , \quad U' = U\ , \\ &V' = V + b\,Z_2 + \tfrac{1}{2}b^2\,U\ , \end{aligned} \tag{21}$$

for $i = 2$. For $i = 3$, they are obtained by interchanging $Z_2 \leftrightarrow Z_3$. Null rotations corresponding to (18) are

$$\begin{aligned} &Z_4' = Z_4 + b\,U\ , \\ &Z_2' = Z_2\ , \quad Z_3' = Z_3\ , \quad U' = U\ , \\ &V' = V + \epsilon b\,Z_4 + \epsilon\tfrac{1}{2}b^2\,U\ . \end{aligned} \tag{22}$$

The symmetries (19), (20) are obtained from (21), (22) by interchanging $U \leftrightarrow V$.

Now, let us consider impulsive waves in the (anti–)de Sitter universe of the form (1) with an arbitrary H. Since all these solutions reduce to the constant-curvature spacetimes everywhere except on the null hypersurface $U = 0$, it is natural to look for the symmetries of the *complete* solutions (including the impulse localized at $U = 0$) among the group of transformations generated by (14)-(20). It is straightforward to show that there are *at least three Killing vector fields*, even for a *general* H, namely the null rotations (17), (18). Thus the spacetimes representing nonexpanding impulsive waves in the (anti–)de Sitter universe in general admit three symmetries of the form (21), (22).

Using (8) and the inverse relations, $x = \Omega\, Z_2$, $y = \Omega\, Z_3$, $u = \Omega\, U$, $v = \Omega\, V$, in which the conformal factor is $\Omega = 2a/(Z_4 + a)$, we express the three Killing vectors in the coordinates of the metric form (10)

$$\begin{aligned}
&x\,(\partial/\partial v) + u\,(\partial/\partial x)\ ,\\
&y\,(\partial/\partial v) + u\,(\partial/\partial y)\ ,\\
&[1 - \tfrac{1}{12}\Lambda(x^2+y^2)]\,(\partial/\partial v) - \tfrac{1}{6}\Lambda\, u\,[x\,(\partial/\partial x) + y\,(\partial/\partial y) + u\,(\partial/\partial u)]\ .
\end{aligned} \tag{23}$$

Thus, the corresponding transformations which leave the metric (10) unchanged are

$$\begin{aligned}
1.\ \ & x' = x + b_1 u\ , \quad y' = y\ ,\\
& u' = u\ , \quad v' = v + b_1 x + \tfrac{1}{2}b_1^2 u\ .\\
2.\ \ & x' = x\ , \quad y' = y + b_2 u\ ,\\
& u' = u\ , \quad v' = v + b_2 y + \tfrac{1}{2}b_2^2 u\ .\\
3.\ \ & x' = \frac{x}{1+\frac{1}{6}\Lambda b_3\, u}\ , \quad y' = \frac{y}{1+\frac{1}{6}\Lambda b_3\, u}\ , \quad u' = \frac{u}{1+\frac{1}{6}\Lambda b_3\, u}\ ,\\
& v' = \frac{v + \frac{1}{6}\Lambda b_3^2\, u + b_3[1 - \frac{1}{12}\Lambda(x^2+y^2-2uv)]}{1+\frac{1}{6}\Lambda b_3\, u}\ .
\end{aligned} \tag{24}$$

Obviously, the Killing vector fields (23) are exactly the vectors (13) found previously by Aichelburg and Balasin [6, 15] for the case $\Lambda = 0$. In fact, the first two transformations (24) are the same as those of impulsive *pp*-waves, and the third reduces to $v' = v + b_3$ in the Minkowski background.

Other symmetries arise for some specific forms of the structural function H. For example, in the case of the Hotta-Tanaka solution given by (7), H only depends on $Z_4 = \sqrt{a^2 - \epsilon(Z_2^2 + Z_3^2)}$. Thus, there is an additional (fourth) axial symmetry, namely the rotation generated by (14) Again, this is analogous to the axially symmetric Aichelburg-Sexl solution for $\Lambda = 0$.

4 Geodesics

In this section we explicitly derive all geodesics in spacetimes with nonexpanding impulsive waves and nonvanishing cosmological constant. Again, it is useful to employ the 5-dimensional formalism. The spacetimes to be investigated can be understood as 4-dimensional submanifolds $\mathcal{H}$ of the manifold $\mathcal{M}$ which is a 5-dimensional *pp*-wave (1), such that $\mathcal{H}$ is given by the constraint (2). It is well-known (see, e.g. [19]) that a curve with tangent $\mathbf{T}$ in $\mathcal{M}$ lying also on $\mathcal{H}$ is a *geodesic in* $\mathcal{H}$ if and only if its $\mathcal{M}$-acceleration, $\mathbf{A} = \nabla_{\mathbf{T}}\mathbf{T}$, is everywhere normal to $\mathcal{H}$, i.e. its $\mathcal{H}$-acceleration vanishes. If $\mathbf{N}$ is the normal vector to $\mathcal{H}$ satisfying $\mathbf{N}\cdot\mathbf{N} = \epsilon$ (where the dot is the scalar product in $\mathcal{M}$), the above condition $\mathbf{A} \sim \mathbf{N}$ using $\mathbf{N}\cdot\mathbf{T} = 0$ gives

$$\nabla_{\mathbf{T}}\mathbf{T} = -\epsilon\,(\mathbf{T}\cdot\nabla_{\mathbf{T}}\mathbf{N})\,\mathbf{N}\ . \tag{25}$$

In case of (1), (2) the above vectors have the contravariant components

$$\mathbf{T} = (\dot U, \dot V, \dot Z_2, \dot Z_3, \dot Z_4)\ , \quad \mathbf{N} = a^{-1}(U, V, Z_2, Z_3, Z_4)\ . \tag{26}$$

Note that throughout the paper we apply the distributional identity $U\delta(U) = 0$. The nonzero Christoffel symbols for (1) are $\Gamma^V_{UU} = -\frac{1}{2}H\delta'(U)$, $\Gamma^V_{Up} = -\frac{1}{2}H_{,p}\,\delta(U)$, $\Gamma^i_{UU} = -\frac{1}{2}H_{,i}\delta(U)$, $\Gamma^4_{UU} = -\frac{1}{2}\epsilon H_{,4}\delta(U)$, in which we introduced the notation

$$p = 2,3,4 \qquad i = 2,3\ . \tag{27}$$

Thus, $\mathbf{T}\cdot\nabla_{\mathbf{T}}\mathbf{N} = a^{-1}[e+\frac{1}{2}G\delta(U)\,\dot{U}^2]$, where $e \equiv \mathbf{T}\cdot\mathbf{T} = 0, -1, +1$ for null, timelike and spacelike geodesics, respectively, and $G = Z_p\,H_{,p} - H$ (summation convention is used). Consequently, the equations (25) for geodesics in spacetimes with nonexpanding impulsive waves (1), (2) are

$$
\begin{aligned}
\ddot{U} &= -\tfrac{1}{3}\Lambda\, U\, e\ , \\
\ddot{V} - \tfrac{1}{2}H\,\delta'(U)\,\dot{U}^2 - H_{,p}\,\dot{Z}_p\,\delta(U)\,\dot{U} &= -\tfrac{1}{3}\Lambda\, V\left[e + \tfrac{1}{2}G\,\delta(U)\,\dot{U}^2\right]\ , \\
\ddot{Z}_i - \tfrac{1}{2}H_{,i}\,\delta(U)\,\dot{U}^2 &= -\tfrac{1}{3}\Lambda\, Z_i\left[e + \tfrac{1}{2}G\,\delta(U)\,\dot{U}^2\right]\ , \\
\ddot{Z}_4 - \tfrac{1}{2}\epsilon H_{,4}\,\delta(U)\,\dot{U}^2 &= -\tfrac{1}{3}\Lambda\, Z_4\left[e + \tfrac{1}{2}G\,\delta(U)\,\dot{U}^2\right]\ .
\end{aligned}
\tag{28}
$$

It is obvious that for an arbitrary H and any $U \neq 0$ (i.e. everywhere in front of and behind the impulse), these geodesic equations are simple and reduce to those in "pure" (anti–)de Sitter spacetime (for which $H = 0$). In fact, all these equations have the same (decoupled) form, $\ddot{X} = -\epsilon e\, a^{-2}\, X$, where X stands for U, V, Z_i, or Z_4. Thus, for $U \neq 0$ the geodesics are

$$
\begin{aligned}
X &= X^0 + \dot{X}^0\,\tau\ , && \text{when} \quad \epsilon e = 0\ , \\
X &= X^0\cosh(\tau/a) + a\dot{X}^0\sinh(\tau/a)\ , && \text{when} \quad \epsilon e < 0\ , \\
X &= X^0\cos(\tau/a) + a\dot{X}^0\sin(\tau/a)\ , && \text{when} \quad \epsilon e > 0\ .
\end{aligned}
\tag{29}
$$

Recall that the first relation in (29) describes null geodesics, the second represents timelike geodesics in the de Sitter space or spacelike geodesics in the anti–de Sitter space, whereas the third line in (29) corresponds to spacelike/timelike geodesics in the de Sitter/anti–de Sitter space, respectively. In the above equations, τ is an affine parameter, and X^0, $\dot{X}^0$ are constants of integration. These ten constants are constrained by the following three conditions

$$
-2\,\dot{U}^0\dot{V}^0 + (\dot{Z}^0_2)^2 + (\dot{Z}^0_3)^2 + \epsilon(\dot{Z}^0_4)^2 = e\ , \tag{30}
$$
$$
-2\,U^0V^0 + (Z^0_2)^2 + (Z^0_3)^2 + \epsilon(Z^0_4)^2 = \epsilon a^2\ , \tag{31}
$$
$$
-U^0\dot{V}^0 - \dot{U}^0V^0 + Z^0_2\dot{Z}^0_2 + Z^0_3\dot{Z}^0_3 + \epsilon Z^0_4\dot{Z}^0_4 = 0\ . \tag{32}
$$

The first is the normalization of the affine parameter, the equation (31) follows from (2), and (32) from its derivative.

Let us first observe that the equation (28) for U is decoupled and does not involve any distributional term. The solution, which is *everywhere* given by (29), is a *smooth* function. Using a freedom in the choice of the affine parameter $\tau \to \dot{\tau}_0\tau + \tau_0$ for $e = 0$, and $\tau \to \tau + \tau_0$ for $e \neq 0$ ($\dot{\tau}_0$, τ_0 are constants), we can simplify U to the form

$$
U = \tau\ , \qquad U = a\,\dot{U}^0\sinh(\tau/a) \qquad U = a\,\dot{U}^0\sin(\tau/a)\ , \tag{33}
$$

for $\epsilon e = 0$, $\epsilon e < 0$, or $\epsilon e > 0$, respectively. (The only exceptions are trivial null geodesics $U =$const., and some geodesics (29) which do not intersect the impulse on $U = 0$.) The relations (33) allow us to take U as the geodesic parameter, in which case the three different forms of solutions (29) for Z_p and V can be written in a unified form

$$
X = X^0\sqrt{1 - \tfrac{1}{3}\Lambda\, e\,(\dot{U}^0)^{-2}\,U^{2}} + (\dot{X}^0/\dot{U}^0)\,U\ . \tag{34}
$$

Now, it is useful to employ the symmetries found in the previous section. Without loss of generality, we can put a general solution of the geodesic equations *in front* of the impulse to a much simpler form. Considering suitable values of the parameters b of the three finite symmetry transformations (21), (22), we can always achieve that the velocities $\dot{Z}^0_2$, $\dot{Z}^0_3$, $\dot{Z}^0_4$ vanish at $U = 0_-$. Let us note that the corresponding Killing vectors commute and this permits the symmetry transformations to be combined in an arbitrary order. Using the constraints (30)-(32) we can finally fix three of the remaining constants. Since $U^0 = 0$ and $\dot{Z}^0_p = 0$, equation (32) implies $V^0 = 0$. The relation (30) gives $\dot{V}^0 = -\frac{1}{2}(e/\dot{U}^0)$. Thus, without loss of generality, any geodesic (34) in front of the impulse can be written in the form

$$Z_p(U) = Z_p^0 \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2}\ , \qquad V(U) = -\tfrac{1}{2}\, e\, (\dot{U}^0)^{-2}\, U\ , \tag{35}$$

where the constants Z_p^0, as a consequence of (31), are related by

$$\epsilon[(Z_2^0)^2 + (Z_3^0)^2] = a^2 - (Z_4^0)^2\ . \tag{36}$$

We wish to present geodesics in complete spacetimes (1) with the impulsive wave localized on $U = 0$. Geodesics which pass through the wave have the *same* form (34), both in front of the impulse ($U < 0$) and behind it ($U > 0$). However, the constants of integration may have *different values* on both sides. The only remaining problem is to find explicit relations between these constants. This would provide a complete solution of the geodesic equations (28) in the spacetimes studied. In other words, we have to apply appropriate junction conditions on the impulse.

For $U < 0$ the solution is given by (35). For $U > 0$ it must be of a general form (34). By inspecting the character of the distributional terms in the equations (28) it is natural to assume a complete solution in the form

$$\begin{aligned} Z_p(U) &= Z_p^0 \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2}\ +\ A_p\, \Theta(U)\, U\ , \\ V(U) &= -\tfrac{1}{2}\, e\, (\dot{U}^0)^{-2}\, U\ +\ B\, \Theta(U) \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2}\ +\ C\, \Theta(U)\, U\ , \end{aligned} \tag{37}$$

where A_p, B, C are suitable constants to be determined. In the above equation, $Z_p(U)$ is continuous on $U = 0$ but not C^1 in such a way that $\ddot{Z}_p$ contains the Dirac delta. This is consistent with the equations (28) for Z_p. Inserting this ansatz into the geodesic equations (28) and considering the distributional identities $f(U)\,\delta(U) = f(0)\,\delta(U)$, $f(U)\,\delta'(U) = f(0)\,\delta'(U) - f'(0)\,\delta(U)$, one obtains

$$\begin{aligned} A_i &= \tfrac{1}{2}\left[H_{,i}(0) - \tfrac{1}{3}\Lambda\, Z_i^0 G(0)\right]\ , \quad A_4 = \tfrac{1}{2}\left[\epsilon H_{,4}(0) - \tfrac{1}{3}\Lambda\, Z_4^0 G(0)\right]\ , \\ B &= \tfrac{1}{2} H(0)\ , \\ C &= \tfrac{1}{8}\left[H_{,2}^2(0) + H_{,3}^2(0) + \epsilon H_{,4}^2(0) + \tfrac{1}{3}\Lambda\, H^2(0) - \tfrac{1}{3}\Lambda\left(Z_p^0\, H_{,p}(0)\right)^2\right]\ , \end{aligned} \tag{38}$$

where $f(0) \equiv f(Z_p(0)) = f(Z_p^0)$, and the constants Z_p^0 are again constrained by (36). It can be observed that in general there is a discontinuity in V and its derivative on the impulse. In fact, the jump on $U = 0$ is given by $\Delta V = B = \frac{1}{2}H(0)$, which is in full agreement with the Penrose junction conditions in the "cut and paste" method for constructing nonexpanding impulsive gravitational waves [4, 13].

Let us emphasize that the solution (37), (38) describes *any* geodesic in a *privileged* coordinate system such that the transverse velocities $\dot{Z}_p^0$ vanish on $U = 0_-$. This has been achieved by performing null rotations (21), (22) with suitable choice of the parameters $b_p = -\dot{Z}_p^0/\dot{U}^0$. Note that these values do not depend on the positions Z_p^0, so that the coordinates are well-adapted to describe all geodesics with the same velocities. However, for a physical interpretation of the effects of impulsive waves on a *set* of test particles, it is necessary to present a *general* solution which, in a fixed coordinate system, describes all geodesics at once. This general form can easily be obtained from (37) by application of the null rotations with $b_p = \dot{Z}_p^0/\dot{U}^0$, which reintroduce the velocities. Thus, a general solution of geodesic equations can be written as

$$\begin{aligned} Z_p(U) &= Z_p^0 \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2}\ +\ (\dot{Z}_p^0/\dot{U}^0)\, U\ +\ A_p\, \Theta(U)\, U\ , \\ V(U) &= V^0 \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2}\ +\ (\dot{V}^0/\dot{U}^0)\, U\ +\ B\, \Theta(U) \sqrt{1 - \tfrac{1}{3}\Lambda\, e\, (\dot{U}^0)^{-2}\, U^2} \\ &\qquad +\ (\dot{U}^0)^{-1} (\dot{Z}_i^0 A_i + \epsilon \dot{Z}_4^0 A_4)\, \Theta(U)\, U\ +\ C\, \Theta(U)\, U\ , \end{aligned} \tag{39}$$

where the constants are constrained by (30)-(32). Of course, for $\dot{Z}_p^0 = 0$, this reduces to the simpler form (37). It can immediately be seen that the magnitude of the refraction of geodesics in the transverse directions Z_p and the jump in the longitudinal direction V are totally independent of the velocity $\dot{Z}_p^0$. However, the refraction in the longitudinal direction does depend on the velocity.

Let us close this section with few comments on the solution (39). First, for $\Lambda = 0$, the above results also describe all geodesics in impulsive *pp*-wave spacetimes, i.e. in a Minkowski universe with a gravitational and/or null matter impulsive plane wave-front. Setting $\epsilon = 0$ in (1), dropping the constraint (2), and considering $H = H(Z_2, Z_3)$, we obtain the standard metric for *pp*-waves. In such a case the general solution (39), (38) takes the explicit form of geodesics in impulsive *pp*-waves, as discussed previously in a number of works, see e.g. [16], [17]. In fact it can be observed that, for $\Lambda = 0$, the geodesic equations (28) exactly reduce to the system which has recently been *rigorously* solved by Kunzinger and Steinbauer [17] using Colombeau algebras within the context of which products of distributions can be handled.

Note that geodesics in spacetimes representing (anti–)de Sitter universe with nonexpanding impulsive waves have already been briefly discussed in [14] using coordinates introduced by Dray and 't Hooft [16]. However, this approach leads not only to δ, but also δ^2 terms in the coefficients of the geodesic equations. Similar problems occur with other coordinate systems, e.g. (10).

The main advantage of the geometrical approach presented above is that this leads to a simpler system (28) which includes only "weakly" singular terms, in particular there is no square of δ. In addition, the decoupled equation for $U(\tau)$ can explicitly and globally be solved. It is a smooth function of the affine parameter τ, which permits the solutions of the remaining equations for V and Z_p to be expressed in terms of U (analogously to *pp*-waves).

It can be observed that with the form (39) of the solution, the system (28) contains products of Θ and δ distributions only in the equation for V. Thus, continuous (but not C^1) functions Z_p are consistent solutions. On the other hand, the solution (39) for V has to be considered to be only heuristic. However, similar problems appearing in the *pp*-wave case have recently been rigorously resolved in [17]. Moreover, we can alternatively replace the equation for V in (28) by the constant-norm condition, $2\,\dot{U}\,\dot{V} = \dot{Z}_2^2 + \dot{Z}_3^2 + \epsilon\dot{Z}_4^2 + H\,\delta(U)\,\dot{U}^2 - e$. In this case neither $\Theta\,\delta$ nor δ' terms appear. This approach leads to the *same* form of the solution (37).

5 Discussion of the geodesics

It has been demonstrated above that geodesics in the spacetimes studied are influenced by the impulsive wave in such a way that $Z_p(U)$ are everywhere continuous functions, but there is a jump $\Delta V = \frac{1}{\sqrt{2}}(\Delta Z_0 - \Delta Z_1) = B$ on $U = \frac{1}{\sqrt{2}}(Z_0 + Z_1) = 0$. Thus, for generic geodesics there is a discontinuity on the impulse, both in time Z_0 and longitudinal spatial coordinate Z_1, such that $\Delta Z_0 = -\Delta Z_1 = \frac{\sqrt{2}}{4}H(0)$.

In general, there are changes in all velocity components when free test particles pass through the impulsive wave, given by $\Delta\dot{Z}_p = A_p\,\dot{U}^0$ and $\Delta\dot{V} = A_i\dot{Z}_i^0 + \epsilon A_4\dot{Z}_4^0 + C\,\dot{U}^0$ (i.e. $\Delta\dot{Z}_0 = -\Delta\dot{Z}_1 = \frac{1}{\sqrt{2}}\Delta\dot{V}$). This effect leads to a refraction of trajectories. Introducing angles α_p and β_p by $\cot\alpha_p \equiv (\mathrm{d}Z_p/\mathrm{d}U)(U = 0_-) = \dot{Z}_p^0/\dot{U}_0$ and $-\cot\beta_p \equiv (\mathrm{d}Z_p/\mathrm{d}U)(U = 0_+) = \dot{Z}_p^0/\dot{U}_0 + A_p$, we obtain the relation $\cot\alpha_p + \cot\beta_p = -A_p$, which is a generalization of the "refraction formula" for deflection of geodesics, previously introduced for impulsive *pp*-waves [16, 21].

The behaviour of geodesics can be visualized in suitable sections of the 5-dimensional space. There are two natural and, in a sense, complementary figures. The first shows the transverse directions (Z_2, Z_3, Z_4) in which trajectories are refracted only. Another section visualizes geodesics in the (U, V)-space corresponding to the time coordinate Z_0 and the longitudinal spatial direction Z_1. As discussed above, in such a diagram the geodesics suffer a refraction and also a jump. Let us now investigate the behaviour of specific timelike and null geodesics in some detail.

5.1 Timelike geodesics

In this part of our contribution we investigate the effect of nonexpanding impulsive waves on privileged families of timelike observers in the (anti–)de Sitter spacetimes.

We start in front of the impulse ($U<0$) with natural comoving geodesic observers connected to the standard global parametrization of the *de Sitter universe*,

$$\begin{aligned}
Z_0 &= a\,\sinh(t/a)\ ,\\
Z_1 &= a\,\cosh(t/a)\cos\chi_0\ ,\\
Z_2 &= a\,\cosh(t/a)\sin\chi_0\sin\vartheta_0\cos\varphi_0\ ,\\
Z_3 &= a\,\cosh(t/a)\sin\chi_0\sin\vartheta_0\sin\varphi_0\ ,\\
Z_4 &= a\,\cosh(t/a)\sin\chi_0\cos\vartheta_0\ ,
\end{aligned}\tag{40}$$

where t is the global "cosmic time" and $\chi_0,\vartheta_0,\varphi_0$ are arbitrary constant parameters on the 3-sphere S^3, which at any slice $t=$const. has radius $a\cosh(t/a)$. Considering

$$t=\tau+t_0\ ,\quad \text{where}\quad \sinh(t_0/a)=-\cot\chi_0\ ,\tag{41}$$

in the relations (40), we obtain an explicit form of the privileged family of timelike observers. For all of them $U=a\dot U^0\sinh(\tau/a)$ with $\dot U^0=\frac{1}{\sqrt2}\sin\chi_0$, so that the normalization (33) is valid. This means that *every* observer reaches the impulse $U=0$ at $\tau=0$, where τ is the *individual proper time*. In view of (41) we conclude that different observers cross the impulse at different values of the cosmic time $t=t_0$, which depends on χ_0. Note that $t_0=0$ for observers in the equatorial plane $\chi_0=\frac{\pi}{2}$, whereas $t_0=-\infty$ and $t_0=+\infty$ for observers localized at the north ($\chi_0=0$) and the south pole ($\chi_0=\pi$) of S^3. Now, it follows immediately that the relations (40) are of the form (34), in which the constants are

$$\begin{aligned}
&Z_2^0=a\,\sin\vartheta_0\cos\varphi_0\ ,\quad Z_3^0=a\,\sin\vartheta_0\sin\varphi_0\ ,\quad Z_4^0=a\,\cos\vartheta_0\ ,\\
&\dot Z_p^0=-(\cos\chi_0/a)\,Z_p^0\ ,\quad V^0=-\sqrt2 a\,\cot\chi_0\ ,\quad \dot V^0=\frac{1+\cos^2\chi_0}{\sqrt2\,\sin\chi_0}\ .
\end{aligned}\tag{42}$$

Consequently, the complete behavior of this family of timelike geodesics for all U is described by the explicit solution (39) with the above values of the constant parameters. This can be used for a physical interpretation of the influence of the impulse on motion of these observers. In particular, for the *Hotta-Tanaka solution* (7), $H=b_0Q_1(z)$, we obtain using (42)

$$\begin{aligned}
&A_2=-\frac{b_0}{2a}\frac{\cos\varphi_0}{\sin\vartheta_0}\ ,\quad A_3=-\frac{b_0}{2a}\frac{\sin\varphi_0}{\sin\vartheta_0}\ ,\quad A_4=\frac{b_0}{2a}\log\left(\cot\frac{\vartheta_0}{2}\right)\ ,\\
&B=\frac{b_0}{2}\left[\cos\vartheta_0\log\left(\cot\frac{\vartheta_0}{2}\right)-1\right]\ ,\quad \dot Z_i^0A_i+\epsilon\dot Z_4^0A_4=-(\cos\chi_0/a)\,B\ ,\\
&C=\frac{b_0^2}{8a^2}\left[\log^2\left(\cot\frac{\vartheta_0}{2}\right)+\frac{1}{\sin^2\vartheta_0}\right]\ .
\end{aligned}\tag{43}$$

Notice that the coefficients A_p,B,C are independent of χ_0, and diverge for $\vartheta_0=0$, $\vartheta_0=\pi$ where the two singular null particles generating the impulsive wave are localized. The explicit geodesics are given by (39), (42), (43). These demonstrate the axial symmetry of the spacetime in the transverse directions, $Z_2=Z\,\cos\varphi_0$, $Z_3=Z\,\sin\varphi_0$, where

$$Z(U)=a\,\sin\vartheta_0\left[\sqrt{1+\frac{2\,U^2}{a^2\sin^2\chi_0}}\;-\;\frac{\sqrt2}{a}\cot\chi_0\,U\right]\;-\;\frac{b_0}{2a\sin\vartheta_0}\,\Theta(U)\,U\ .\tag{44}$$

The character of geodesics in the "equatorial" plane $\chi_0=\frac{\pi}{2}$ is visualized in Fig. 1. Obviously, there is a focussing effect. In general, a ring of test particles having fixed values of χ_0,ϑ_0 with

$\varphi_0 \in [0, 2\pi)$, is *focused* ($Z_2 = 0 = Z_3$) at the value of U_f for which $Z(U_f) = 0$, i.e.

$$U_f = a \,/ \sqrt{\frac{b_0^2}{4a^2 \sin^4 \vartheta_0} + \frac{\sqrt{2}\, b_0 \cot \chi_0}{a \sin^2 \vartheta_0} - 2}\ . \tag{45}$$

The argument of the square root is quadratic in b_0. Thus, for each nonsingular χ_0, ϑ_0, there are two values of b_0 such that $U_f = \infty$. For $|b_0|$ larger than the corresponding roots, the value of U_f is finite. For smaller values, there is a defocussing effect.

Motion in the direction Z_4 in de Sitter universe with the Hotta-Tanaka impulse is given by

$$Z_4(U) = a \cos \vartheta_0 \left[\sqrt{1 + \frac{2\,U^2}{a^2 \sin^2 \chi_0}} \;-\; \frac{\sqrt{2}}{a} \cot \chi_0\, U \right] \;+\; \frac{b_0}{2a} \log \left(\cot \frac{\vartheta_0}{2} \right) \Theta(U)\, U\ . \tag{46}$$

It can be observed that all test particles with $\vartheta_0 = \frac{\pi}{2}$ stay in the physically privileged plane $Z_4 = 0$ ("perpendicular" to the sources), not only in front on the impulse, but also behind it. In general, in front of the impulse ($U < 0$) the test particles (40) for fixed χ_0 form a two-sphere $Z_2^2 + Z_3^2 + Z_4^2 = R^2$ with radius $R = a \left[\sqrt{1 + (2/a^2) \sin^{-2} \chi_0\, U^2} - (\sqrt{2}/a) \cot \chi_0\, U \right] = a \cosh(t/a) \sin \chi_0$. As the de Sitter universe contracts and then re-expands, the sphere of co-moving particles undergoes the same behaviour. At $U = 0$ the particles are hit by the impulse and the sphere starts to deform. The deformation due to the Hotta-Tanaka impulse (in the "equatorial" plane $\chi_0 = \frac{\pi}{2}$) is shown in Fig. 2. The focussing of particles on $Z = 0$, which occurs at U_f given by (45), is related to the formation of specific caustic shapes.

The geodesic motion in the time direction Z_0 and longitudinal spatial direction Z_1 is given by $V(U)$. For the Hotta-Tanaka spacetime this is demonstrated in Fig. 3. Indeed, in the above natural coordinates, the geodesics are refracted and also broken by the impulse.

Similarly, it is possible to study geodesics in the de Sitter universe with an arbitrary non-expanding impulsive wave. The impulse is purely gravitational when the vacuum field equation (5) is satisfied. Note that this field equation and the corresponding Weyl tensor can also be calculated using the 5-dimensional geometrical approach, as shown in the appendix. In particular, it can be observed from (A-3), (A-5) that there exist special conformally flat solutions with pure radiation given by $H = H_0 =$const. A simple explicit form of the timelike geodesics (40) influenced by such an impulse is

$$Z_2 = Z \sin \vartheta_0 \cos \varphi_0\ , \quad Z_3 = Z \sin \vartheta_0 \sin \varphi_0\ , \quad Z_4 = Z \cos \vartheta_0, \tag{47}$$

where

$$Z(U) = a \sqrt{1 + \frac{2\,U^2}{a^2 \sin^2 \chi_0}} \;-\; \sqrt{2} \cot \chi_0\, U \;+\; \frac{H_0}{2a} \Theta(U)\, U\ . \tag{48}$$

It is obvious that there is *perfect focussing* of all the test particles with arbitrary ϑ_0 and φ_0 (which initially spanned a two-sphere) when $Z(U_f) = 0$. This occurs at $U = U_f = 2a^2/\sqrt{H_0^2 - 4\sqrt{2}a\, H_0 \cot \chi_0 - 8a^2}$. Thus, all observers (40) with the same value of χ_0 are hit by the above impulse at the same time t_0 given by (41), and then exactly focus in a *single event* $Z_p = 0$, $V = -a^2/(2U_f)$. The same effect has already been described by Ferrari, Pendenza and Veneziano [16] for null geodesics influenced by impulsive pp-waves with constant energy density of null matter on the wave-front.

An analogous investigation can be performed for geodesic observers in the *anti–de Sitter* universe. The privileged family of timelike geodesics is given by

$$\begin{aligned}
Z_0 &= -a \cos(t/a)\ , \\
Z_1 &= a \sin(t/a)\ \sinh \psi_0 \cos \vartheta_0\ , \\
Z_2 &= a \sin(t/a)\ \sinh \psi_0 \sin \vartheta_0 \cos \varphi_0\ , \\
Z_3 &= a \sin(t/a)\ \sinh \psi_0 \sin \vartheta_0 \sin \varphi_0\ , \\
Z_4 &= a \sin(t/a)\ \cosh \psi_0\ ,
\end{aligned} \tag{49}$$

where $\psi_0, \vartheta_0, \varphi_0$ are arbitrary constant parameters which specify particular geodesics. These observers move around the anti–de Sitter hyperboloid in closed timelike loops along the intersections with the planes $Z_1 \sim Z_p$. Again, for

$$t = \tau + t_0 \ , \quad \text{where} \quad \cot(t_0/a) = \sinh\psi_0 \cos\vartheta_0 \ , \tag{50}$$

the relations (49) represents a family of timelike observers with $U = a\,\dot{U}^0 \sin(\tau/a)$ and $\dot{U}^0 = [\sqrt{2}\sin(t_0/a)]^{-1}$. Again, for all observers, $U = 0$ at $\tau = 0$. The relations (49) corresponds to (34) when we set the constants

$$\begin{aligned}
Z_2^0 &= a\ \sin(t_0/a)\sinh\psi_0 \sin\vartheta_0 \cos\varphi_0 \ ,\\
Z_3^0 &= a\ \sin(t_0/a)\sinh\psi_0 \sin\vartheta_0 \sin\varphi_0 \ ,\\
Z_4^0 &= a\ \sin(t_0/a)\cosh\psi_0 \ ,\\
\dot{Z}_p^0 &= a^{-1}\sinh\psi_0 \cos\vartheta_0\, Z_p^0 \ ,\\
V^0 &= -(a/\dot{U}^0)\ \sinh\psi_0 \cos\vartheta_0 \ ,\\
\dot{V}^0 &= (2\,\dot{U}^0)^{-1}(1 - \sinh^2\psi_0 \cos^2\vartheta_0) \ .
\end{aligned} \tag{51}$$

The general solution (39) with the above choice of parameters describes the complete behavior of this family of timelike geodesics in the anti–de Sitter universe with a nonexpanding wave.

In particular, geodesics in the "equatorial plane" $\vartheta_0 = \frac{\pi}{2}$ are given by

$$\begin{aligned}
&Z_2^0 = a\ \sinh\psi_0 \cos\varphi_0 \ , \quad Z_3^0 = a\ \sinh\psi_0 \sin\varphi_0 \ , \quad Z_4^0 = a\ \cosh\psi_0 \ ,\\
&\dot{Z}_p^0 = 0 \ , \quad V^0 = 0 \ , \quad \dot{U}^0 = \dot{V}^0 = \tfrac{1}{\sqrt{2}} \ ,
\end{aligned} \tag{52}$$

which for the *Hotta-Tanaka* solution $H = b_0 Q_1(z)$ in the anti–de Sitter universe yields

$$\begin{aligned}
A_2 &= -\frac{b_0}{2a}\frac{\cos\varphi_0}{\sinh\psi_0} \ , \quad A_3 = -\frac{b_0}{2a}\frac{\sin\varphi_0}{\sinh\psi_0} \ , \quad A_4 = -\frac{b_0}{2a}\log\left(\coth\frac{\psi_0}{2}\right) \ ,\\
B &= \frac{b_0}{2}\left[\cosh\psi_0 \log\left(\coth\frac{\psi_0}{2}\right) - 1\right] \ ,\\
C &= -\frac{b_0^2}{8a^2}\left[\log^2\left(\coth\frac{\psi_0}{2}\right) - \frac{1}{\sinh^2\psi_0}\right] \ .
\end{aligned} \tag{53}$$

Discussion of the behaviour of these geodesics and corresponding figures would be analogous to those for the Hotta-Tanaka solution with $\Lambda > 0$. However, there are some important differences. For example, the impulsive wave-surface is not spherical but hyperboloidal. Also, there are not two singular point sources on the impulse but only one localized at $\psi_0 = 0$.

Another interesting example of nonexpanding impulsive waves in the anti–de Sitter universe is a nonsingular *Defrise-type impulse* given by the metric (1) with $H = ca^{-5}(Z_3 + Z_4)^3$, c is a constant. With the parametrization

$$u = \frac{a\,U}{Z_3 + Z_4} \ , \quad v = \frac{a\,V}{Z_3 + Z_4} \ , \quad x = \frac{a^2}{Z_3 + Z_4} \ , \quad y = \frac{aZ_2}{Z_3 + Z_4} \ , \tag{54}$$

this takes the 4-dimensional form

$$\mathrm{d}s^2 = \frac{a^2}{x^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}v + c\frac{\delta(u)}{x^2}\,\mathrm{d}u^2\right) \ . \tag{55}$$

This solution represents a gravitational plus null-matter impulse. Specific particular geodesics (39), (51) in this spacetime for $\vartheta_0 = 0$ are

$$\begin{aligned}
&Z_i^0 = 0 \ , \quad Z_4^0 = a \ , \quad \dot{Z}_i^0 = 0 \ , \quad \dot{Z}_4^0 = -\sinh\psi_0 \ ,\\
&V^0 = -\sqrt{2}a\ \tanh\psi_0 \ , \quad \dot{U}^0 = \tfrac{1}{\sqrt{2}}\cosh\psi_0 \ , \quad \dot{V}^0 = \tfrac{1}{\sqrt{2}}(1 - \sinh^2\psi_0)/\cosh\psi_0 \ ,
\end{aligned} \tag{56}$$

for which

$$A_2 = 0 \ , \ A_3 = \tfrac{3}{2}\, ca^{-3} \ , \ A_4 = -\tfrac{1}{2}\, ca^{-3} \ , \ B = \tfrac{1}{2}\, ca^{-2} \ , \ C = c^2 a^{-6} \ . \tag{57}$$

These geodesics are continuous in the transverse directions x, y such that $x = a$, $y = 0$ on the impulse $u = 0$. There is a discontinuity in the longitudinal direction $\Delta v = c/(2a^2)$. This agrees with the results presented in [22], in which various properties of the Defrise sandwich and impulsive wave spacetimes have been analyzed.

5.2 Null geodesics

For an arbitrary null geodesic, the explicit solution (39) simplifies to

$$\begin{aligned} Z_p(U) &= Z_p^0 + \dot{Z}_p^0\, U + A_p\, \Theta(U)\, U \ , \\ V(U) &= V^0 \ + \ \dot{V}^0\, U + B\, \Theta(U) + (\dot{Z}_i^0\, A_i + \epsilon \dot{Z}_4^0\, A_4)\, \Theta(U)\, U + C\, \Theta(U)\, U \ . \end{aligned} \tag{58}$$

in which U is an affine parameter, see (33). In the above coordinates, the trajectories are simple straight lines, both in front of and behind the impulse. As in the timelike case, at $U = 0$ these null geodesics are refracted and broken (in the longitudinal direction). One could easily study the (possible) focussing of these null geodesics. It is obvious from (58) that each null particle crosses $Z_p = 0$ at $U_{f\,p} = -Z_p^0/(\dot{Z}_p^0 + A_p)$. This, of course, depends on the initial position Z_p^0 of a particle, its velocity parameter $\dot{Z}_p^0$, and the specific solution given by H. Thus, the focussing is in general astigmatic. However (as already described above for timelike geodesics), for $H =$ const. and particular families of observers, the focussing is exact. This effect was previously observed for special impulsive *pp*-waves which are conformal to the above solution. This is not surprising since the form of all *null* geodesics in conformally related spacetimes is the *same* (up to reparametrization). Thus, the geodesics (58) in the spacetime (10) with $\Lambda \neq 0$ have the same form as in the spacetime (10) with $\Lambda = 0$, i.e. in impulsive *pp*-waves.

Instead of presenting a specific analysis of behaviour of null geodesics in particular impulsive spacetimes (such as the Hotta-Tanaka solution (7), or the Defrise solution (55)), we finally present an alternative and equivalent derivation of the solution (58). In fact, this explicit form of null geodesics can also be found using the 4-dimensional coordinates for the (anti–)de Sitter spacetime with impulsive waves. It can easily be shown that there exist privileged null geodesics given by $\mathcal{V} = 0$ with constant values $Z = Z^0$ of the complex transverse spatial coordinate of the *continuous* form of the metric (12). The remaining geodesic equation, which reduces to $\ddot{\mathcal{U}} + \Gamma^{\mathcal{U}}_{\mathcal{U}\mathcal{U}}\dot{\mathcal{U}}^2 = 0$, for the initial conditions $\mathcal{U}(0) = 0$, $\dot{\mathcal{U}}(0) = 1$ yields the solution

$$\mathcal{U}(\sigma) = \frac{(1 + \frac{1}{6}\Lambda Z^0 \bar{Z}^0)\, \sigma}{1 + \frac{1}{6}\Lambda[Z^0\bar{Z}^0 - \tilde{G}_0 \Theta(\sigma)\, \sigma]} \ , \tag{59}$$

where σ is an affine parameter, and $\tilde{G}_0 = \tilde{G}(Z_0, \bar{Z}_0)$. Consequently, the conformal factor introduced in (9) evaluated along the above null geodesics, $\Omega = 1 + \frac{1}{6}\Lambda[Z^0\bar{Z}^0 + \tilde{G}_0\Theta(\mathcal{U})\, \mathcal{U}\,] > 0$, takes the form

$$\Omega = \frac{(1 + \frac{1}{6}\Lambda Z^0\bar{Z}^0)^2}{1 + \frac{1}{6}\Lambda[Z^0\bar{Z}^0 - \tilde{G}_0\Theta(\sigma)\, \sigma]} = (1 + \tfrac{1}{6}\Lambda Z^0\bar{Z}^0)\frac{\mathcal{U}}{\sigma} \ . \tag{60}$$

Using the relations (8) and (11) we immediately obtain

$$\sigma = (1 + \tfrac{1}{6}\Lambda Z^0 \bar{Z}^0)\, U \ , \qquad \frac{1}{\Omega} = \frac{1 - \frac{1}{6}\Lambda\tilde{G}_0\Theta(U)\, U}{1 + \frac{1}{6}\Lambda Z^0\bar{Z}^0} \ , \tag{61}$$

and

$$\begin{aligned} Z_p(U) &= Z_p^0 \ + \ A_p\, \Theta(U)\, U \ , \\ V(U) &= B\, \Theta(U) \ + \ C\, \Theta(U)\, U \ . \end{aligned} \tag{62}$$

The constants Z_p^0 are

$$Z_2^0 + \mathrm{i}Z_3^0 = \frac{\sqrt{2}\, Z^0}{1+\frac{1}{6}\Lambda Z^0 \bar{Z}^0}\ , \qquad Z_4^0 = a\frac{1-\frac{1}{6}\Lambda Z^0\bar{Z}^0}{1+\frac{1}{6}\Lambda Z^0\bar{Z}^0}\ , \tag{63}$$

and the parameters A_p, B, C are given by (38). The explicit solution (62) is identical to the solution obtained previously using the 5-dimensional formalism. Note again that a general form of null geodesics (58) is obtained form (62) by performing null rotations (21), (22) with a suitable choice of the parameters b_p.

6 Concluding remarks

Symmetries and geodesics in spacetimes which represent nonexpanding impulsive waves in backgrounds with nonzero cosmological constant were investigated. One of the main results is that the coordinates (10), (12), or the Dray and 't Hooft metric form [14], are not useful for finding explicit geodesics since the corresponding equations are complicated and contain highly singular terms such as δ^2. Instead, a geometrical approach based on embedding of the 5-dimensional form of solutions (1) on the (anti–)de Sitter hyperboloid (2) leads to a more convenient system (28). This is similar to geodesic equations for impulsive pp-waves [17] and permits all geodesics to be found in a simple explicit form (39) for arbitrary value of the cosmological constant Λ.

Discussion of these solutions, describing the effect of impulsive waves on timelike and null test particles, was then presented. The coefficients (38) were calculated for privileged families of observers in de Sitter and anti-de Sitter universe, in particular for the axially symmetric Hotta-Tanaka impulse, conformally flat pure radiation impulse, and the Defrise-type impulse. These are responsible both for a refraction of trajectories and an additional jump in the longitudinal direction. The focussing effect of geodesics was also described.

Acknowledgments

The authors wish to thank Roland Steinbauer and Jerry Griffiths for their useful comments and suggestions. The work was supported by the grant GACR-202/99/0261 of the Czech Republic, and the grant GAUK 141/2000 of the Charles University. The stay of M.O. at the Institute of Theoretical Physics in Prague in 1999 was also enabled by a scholarship of the Czech Ministry of Education, and in 2000 by Istituto Nazionale di Fisica Nucleare.

Appendix: Field equation and the Weyl tensor

The 5-dimensional formalism and embedding enabled us to derive the geodesic equations in a particularly convenient form. Here we demonstrate that the same approach can also be used to derive the curvature, Ricci and Weyl tensors. We start with the well-known Gauss equation (see, e.g. [20])

$$R_{\mu\nu\rho\sigma} = g_\mu{}^\alpha g_\nu{}^\beta g_\rho{}^\gamma g_\sigma{}^\delta\, \mathcal{R}_{\alpha\beta\gamma\delta} + \epsilon\,(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho})\ , \tag{A-1}$$

in which $\mathcal{R}_{\alpha\beta\gamma\delta}$ is the curvature tensor of the 5-dimensional metric $\gamma_{\mu\nu}$ on $\mathcal{M}$ given by (1), the induced metric on $\mathcal{H}$ defined by (2), with the normal vector $\mathbf{N} = N^\mu\partial_\mu$, is $g_{\mu\nu} = \gamma_{\mu\nu} - \epsilon N_\mu N_\nu$, and $K_{\mu\nu} \equiv g_\mu{}^\kappa \nabla_\kappa N_\nu = a^{-1}[g_{\mu\nu} + \frac{1}{2}\,\delta_\mu^U\,\delta_\nu^U\, G\,\delta(U)]$ is the extrinsic curvature of $\mathcal{H}$ in $\mathcal{M}$. The equation (A-1) allows us to express the 4-Riemann and 4-Ricci tensors of the metric $g_{\mu\nu}$. The only nontrivial components of the curvature tensor are $\mathcal{R}_{UpUq} = -\frac{1}{2}H_{,pq}\,\delta(U)$, so that

$$R_{UrUs} = \tfrac{1}{3}\Lambda(g_{UU}g_{rs} - g_{Us}g_{rU}) + \tfrac{1}{2}\left[\tfrac{1}{3}\Lambda\, g_{rs}\, G - g_r{}^p g_s{}^q H_{,pq}\right]\delta(U)\ , \tag{A-2}$$

where $p, q, r, s = 2, 3, 4$, other components have the standard constant-curvature form, $R_{\mu\nu\rho\sigma} = \frac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. It is obvious that, for $U \neq 0$, this is the Riemann tensor of a constant curvature spacetime, whereas in the limit $\Lambda \to 0$ we recover its form for impulsive pp-waves. Since $g^{UU}\delta(U) = 0$, it turns out that $R = 4\Lambda$ and it is sufficient to consider only the traceless part of the Ricci tensor, $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}R\, g_{\mu\nu}$. Then, the Einstein equations read $S_{\mu\nu} = 8\pi\, T_{\mu\nu}$. Straightforward calculation shows that the only nonvanishing component of $S_{\mu\nu}$ is

$$S_{UU} = -\tfrac{1}{2}\left[H_{,22} + H_{,33} + \epsilon H_{,44} - \tfrac{1}{3}\Lambda\, Z_p\, Z_q\, H_{,pq} - \tfrac{2}{3}\Lambda\, G\right]\delta(U)\ . \tag{A-3}$$

It is obvious that the spacetime is everywhere a vacuum solution, except possibly on the impulse localized at $U = 0$. In general, the impulse consists of a gravitational-wave and null matter components. Notice that S_{UU} is linear in H so that the superposition of arbitrary sources is also allowed. Purely gravitational impulsive waves arise when

$$H_{,22} + H_{,33} + \epsilon H_{,44} - \tfrac{1}{3}\Lambda\, Z_p Z_q\, H_{,pq} + \tfrac{2}{3}\Lambda\,(H - Z_p\, H_{,p}) = 0\ . \tag{A-4}$$

Considering the parametrization (4) of the two-dimensional impulsive wave-surface (3), the equation (A-4) gives exactly (5), which is the form already derived using different approaches in [18, 13, 14]. In the limit $\Lambda = 0$, $\epsilon = 0$ we recover the well-known vacuum field equation for pp-waves.

By means of (A-2), (A-3) and the standard decomposition of the Riemann tensor, we can calculate the Weyl tensor, which represents a contribution of pure gravitational waves to curvature. The only nontrivial nonvanishing components are

$$C_{UrUs} = \left[-\tfrac{1}{2}g_r{}^p g_s{}^q H_{,pq} + \tfrac{1}{4}g_{rs}\left(H_{,22} + H_{,33} + \epsilon H_{,44} - \tfrac{1}{3}\Lambda\, Z_p\, Z_q\, H_{,pq}\right)\right]\delta(U)\ . \tag{A-5}$$

These are linear and homogeneous in the second derivatives of H. It follows that H which is either constant or linear in the coordinates Z_p corresponds to a *globally conformally flat* spacetime. For H =const. one gets $T^{VV} = -\frac{1}{24\pi}\Lambda\, H\,\delta(U)$, the energy-momentum tensor representing a homogeneous distribution of null matter forming the impulse. For H linear in Z_p, the spacetime is *everywhere* a conformally flat vacuum solution, which is the (anti–)de Sitter spacetime, i.e. a trivial constant curvature universe with no impulse.

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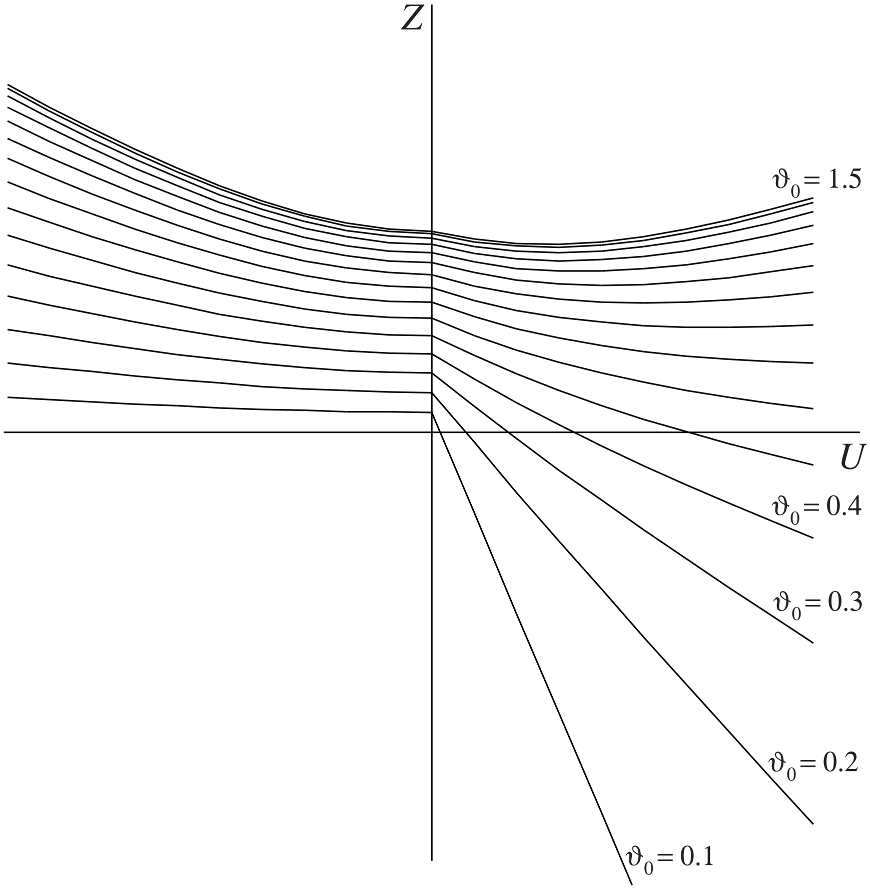

Figure 1: Refraction in the transverse directions $Z_2 = Z\cos\varphi_0$, $Z_3 = Z\sin\varphi_0$ of trajectories (44) of timelike geodesics by the axially symmetric Hotta-Tanaka impulse propagating in the de Sitter universe. The privileged observers are comoving with fixed values of $\chi_0 = \frac{\pi}{2}$, ϑ_0 and φ_0 in front of the spherical impulse.

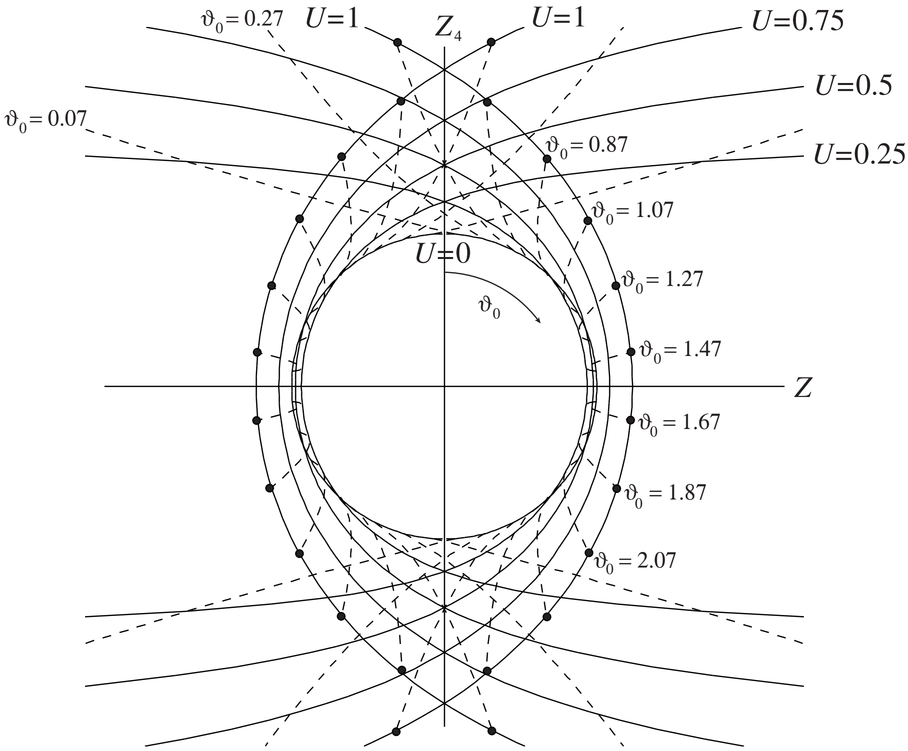

Figure 2: Refraction of timelike geodesics in the transverse directions Z_4 and Z by the Hotta-Tanaka impulse in the de Sitter universe (dashed lines) as given by (46) and (44) for $\chi_0 = \frac{\pi}{2}$. Also, the deformation of a sphere of free test particles is indicated for some values of U. The sphere, which is initially contracting with the contracting de Sitter universe, starts to deform at $U = 0$ by the impulse into shapes with caustic points on the axis of symmetry Z_4.

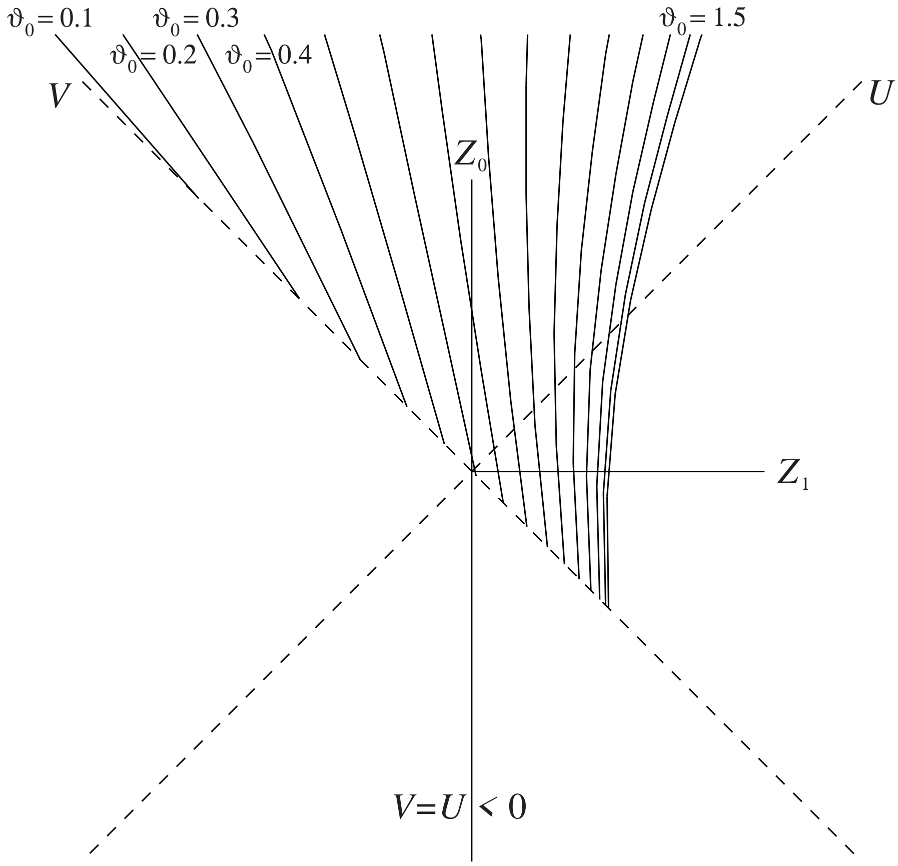

Figure 3: Behaviour of timelike geodesics ($\chi_0 = \frac{\pi}{2}$) in the longitudinal direction in the Hotta-Tanaka impulsive spacetime as given by $V(U)$ in (39) with (43). In addition to refraction of trajectories, there is also a discontinuity in the natural coordinates.