Explicit Kundt type II and N solutions as gravitational waves in various type D and O universes

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Abstract

A particular yet large class of non-diverging solutions which admits a cosmological constant, electromagnetic field, pure radiation and/or general non-null matter component is explicitly presented. These spacetimes represent exact gravitational waves of arbitrary profiles which propagate in background universes such as Minkowski, conformally flat (anti-)de Sitter, Edgar–Ludwig, Bertotti–Robinson, and type D (anti-)Nariai or Plebański–Hacyan spaces, and their generalizations. All possibilities are discussed and are interpreted using a unifying simple metric form. Sandwich and impulsive waves propagating in the above background spaces with different geometries and matter content can easily be constructed. New solutions are identified, e.g. type D pure radiation or explicit type II electrovacuum waves in (anti-)Nariai universe. It is also shown that, in general, there are no conformally flat Einstein–Maxwell fields with a non-vanishing cosmological constant.

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1 Introduction

The Kundt class, which is characterized by the geometrical property that it admits a non-expanding and non-twisting null congruence, is a well-known family of solutions to Einstein's equations. Wolfgang Kundt was the first to introduce, emphasize and investigate this large class [1–3] (in the case of vacuum and pure radiation) although some of its important subclasses, in particular the *pp*-waves [4–7] or the Nariai [8] and Bertotti-Robinson [9–11] universe, were discovered and studied previously. Since then a great number of papers have been devoted to the derivation of such spacetimes and an analysis of their properties, see e.g. [12] for the review. More recent articles which are related to the specific topic of this paper are mentioned below in the appropriate context.

Naturally, most works on exact non-diverging spacetimes investigate particular subclasses of the large Kundt family by restricting attention to a specific algebraic Petrov type *and* considering special matter contents (vacuum, cosmological constant, pure radiation, null or non-null electromagnetic fields). In many of these subcases, all solutions of the given type were explicitly obtained. On the other hand, at present it is still impossible to find a general solution which describes different Petrov types and matter fields in an explicit form. Even with an assumption of symmetries such solutions could in most cases only be given implicitly, see e.g. [12, 13].

This contribution focuses on the "gap" which lies between these "extreme" approaches. We present in section 2 a simple explicit metric which contains several arbitrary parameters and functions. For particular choices we recover, in sections 3 and 4 respectively, many Kundt spacetimes with various matter contents which are of the Petrov types O, D, and N, II. Moreover, the latter solutions can be understood as exact gravitational waves of arbitary profile which propagate in the corresponding "background" universes. All possibilities are discussed below in detail and are interpreted by analyzing the geodesic deviation. The paper also contains an appendix in which we prove that there are no conformally flat Einstein–Maxwell fields with $\Lambda \neq 0$.

2 General form of the solutions

Throughout the present paper we consider and discuss the following metric

$$\mathrm{d}s^2 = 2\,\frac{1}{P^2}\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta - 2\,\frac{Q^2}{P^2}\,\mathrm{d}u\,\mathrm{d}v + F\,\mathrm{d}u^2\ , \tag{1}$$

with

$$\begin{aligned} P &= 1+\alpha\,\zeta\bar\zeta\ ,\\ Q &= (1+\beta\,\zeta\bar\zeta)\,\varepsilon + C\,\zeta + \bar C\,\bar\zeta\ ,\\ F &= D\,\frac{Q^2}{P^2}\,v^2 - \frac{(Q^2)_{,u}}{P^2}\,v - \frac{Q}{P}\,H\ , \end{aligned} \tag{2}$$

where α, β, and ε are real constants (without loss of generality we can assume $\varepsilon = 0$ or $\varepsilon = 1$), $C(u)$ and $D(u)$ are arbitrary functions of the null coordinate u (C may be complex), and $H(\zeta, \bar\zeta, u)$ is an arbitrary function of the spatial coordinates ζ, $\bar\zeta$, and of u. In the natural null tetrad

$$\begin{aligned} \mathbf{m} &= P\,\partial_{\bar\zeta}\ ,\\ \bar{\mathbf{m}} &= P\,\partial_{\zeta}\ ,\\ \mathbf{l} &= \frac{P^4}{2Q^4}F\,\partial_v + \frac{P^2}{Q^2}\,\partial_u\ ,\\ \mathbf{k} &= \partial_v\ , \end{aligned} \tag{3}$$

we obtain (using $\varepsilon^2 \equiv \varepsilon$) the following form of the only non-vanishing Weyl and Ricci scalars

$$\begin{aligned}
\Psi_2 &= -\tfrac{1}{6}\left(D+2\varepsilon\beta-2C\bar{C}\right)\frac{P^2}{Q^2}\,, && (4)\\
\Psi_4 &= \tfrac{1}{2}\,(PH)_{,\zeta\zeta}\,\frac{P^4}{Q^3}\,, && (5)\\
R &= 24\alpha-12\varepsilon(\alpha+\beta)\,\frac{P}{Q}+2\left(D+2\varepsilon\beta-2C\bar{C}\right)\frac{P^2}{Q^2}\,, && (6)\\
\Phi_{11} &= \tfrac{1}{2}\varepsilon(\alpha+\beta)\,\frac{P}{Q}-\tfrac{1}{4}\left(D+2\varepsilon\beta-2C\bar{C}\right)\frac{P^2}{Q^2}\,, && (7)\\
\Phi_{22} &= \tfrac{1}{2}\Big\{\Big[\varepsilon(\alpha-\beta)\,(1-\alpha\,\zeta\bar{\zeta})\;+2\alpha\,(C\,\zeta+\bar{C}\,\bar{\zeta})\Big]\,H+P^2Q\,H_{,\zeta\bar{\zeta}}\\
&\qquad -2\varepsilon(\alpha+\beta)\,v\;(\dot{C}\,\zeta+\dot{\bar{C}}\,\bar{\zeta})\Big\}\,\frac{P^3}{Q^4}\,. && (8)
\end{aligned}$$

The spin coefficients ρ, σ and κ vanish identically so that the multiple principal null direction $\mathbf{k}$ is non-expanding, twist-free, shear-free and geodesic. The spacetimes (1), (2) thus belong to the Kundt class and are algebraically special. By a simple transformation $v=(P^2/Q^2)\,w$, these can be put into the Kundt canonical form [12]

$$\mathrm{d}s^2 = 2\,\frac{1}{P^2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}-2\,\mathrm{d}u\,\left(\mathrm{d}w+W\,\mathrm{d}\zeta+\bar{W}\mathrm{d}\bar{\zeta}+\mathcal{H}\,\mathrm{d}u\right)\,, \tag{9}$$

where

$$\begin{aligned}
W &= 2w\,\frac{QP_{,\zeta}-PQ_{,\zeta}}{PQ} \;\equiv\; -2w\,\frac{C-\varepsilon(\alpha-\beta)\bar{\zeta}-\alpha\bar{C}\bar{\zeta}^2}{PQ}\,,\\
\mathcal{H} &= -\tfrac{1}{2}D\,\frac{P^2}{Q^2}\,w^2-\frac{Q_{,u}}{Q}\,w+\frac{Q}{2P}\,H\,. && (10)
\end{aligned}$$

It follows from (4) and (5) that, in general, the above spacetimes are of Petrov type II. In special cases, these degenerate to types D, N, or can be conformally flat. It is convenient to introduce the following two functions,

$$\begin{aligned}
D_0 &= -2\varepsilon\beta+2C\bar{C}\,, && (11)\\
H_0 &= \frac{A_0+A_1\zeta+\bar{A}_1\bar{\zeta}+A_2\zeta\bar{\zeta}}{1+\alpha\,\zeta\bar{\zeta}}\,, && (12)
\end{aligned}$$

where A_0 and A_2 are arbitrary real functions and A_1 is an arbitrary complex function of u. When the functions D and H in (2) take these special forms D_0 and H_0, the scalars Ψ_2 and Ψ_4 vanish, respectively. We may thus easily summarize the possible types in the following table 1.

	$D=D_0$ $(\Psi_2=0)$	$D\neq D_0$ $(\Psi_2\neq 0)$
$H=H_0$ $(\Psi_4=0)$	O	D
$H\neq H_0$ $(\Psi_4\neq 0)$	N	II

Table 1: Possible Petrov types of the spacetimes (1).

In general, the above spacetimes do not have a constant Ricci curvature R. They also contain non-uniform pure radiation described by Φ_{22} plus a non-null matter component Φ_{11} (which may be associated to a fluid with anisotropic pressure, see [14], with the 4-velocity of the fluid $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{l})$, and $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{l})$ being the vector responsible for the anisotropy), cf. (6)–(8). In particular cases, a constant scalar R may correspond to the cosmological constant Λ, and the matter components Φ_{11}, Φ_{22} could represent an electromagnetic field. In the latter case, the Maxwell equations (see, e.g. [12]) for the metric (1), (2) reduce to

$$D\Phi_1 = 0\ , \qquad D\Phi_2 - \bar\delta\Phi_1 = 2\pi\,\Phi_1\ , \qquad \delta\Phi_1 = 2\tau\,\Phi_1\ , \qquad \delta\Phi_2 - \Delta\Phi_1 = (\tau-2\beta)\,\Phi_2\ , \quad (13)$$

since $\Phi_0 = 0$ because $\Phi_{ij} \equiv \Phi_i\bar\Phi_j = 0$ for $i=0,\ j=0,1,2$. The non-vanishing spin coefficients we need here are given by $\tau = -\bar\pi = Q\,(P/Q)_{,\bar\zeta}$ and $\tau - 2\beta = \left(Q^2/P^2\right)\left(P^3/Q^2\right)_{,\bar\zeta}$. Moreover, $\Phi_{12} = 0$ so that either Φ_1 or Φ_2 must vanish. There are thus *only two* decoupled possibilities, and the equations (13) can explicitly be integrated as

$$Case\ 1: \qquad \Phi_1(\zeta,\bar\zeta) = E_1\frac{P^2}{Q^2}\ , \qquad \Phi_2 = 0\ , \qquad (14)$$

$$Case\ 2: \qquad \Phi_1 = 0\ , \qquad \Phi_2(\zeta,\bar\zeta,u) = E_2\frac{P^3}{Q^2}\ , \qquad (15)$$

where $E_1(u)$ and $E_2(u,\zeta)$ are complex functions. However, for the purely electromagnetic case there are additional constraints since we have to satisfy the Einstein–Maxwell system. This requires that the Ricci scalar R given by (6) is equal to 4Λ, and the expressions for Φ_{11} and Φ_{22} given by (7), (8) have to satisfy

$$\Phi_{11} = |E_1|^2\frac{P^4}{Q^4}\ , \qquad \Phi_{22} = |E_2|^2\frac{P^6}{Q^4}\ , \qquad (16)$$

respectively. In fact, it can be seen that these restrictions rule out the presence of a purely electromagnetic field in general spacetimes of the form (1), (2). Only for some special choices of α, β, ε, C, D and H are the Einstein–Maxwell field equations satisfied.

In the next sections we shall describe all the particular possibilities in detail. We start with various backgrounds which may be either conformally flat or of the type D. Then we shall introduce gravitational waves into all these background spacetimes by considering the corresponding type N or II solutions.

3 The possible backgrounds

3.1 Conformally flat spacetimes

The above solutions (1), (2) are conformally flat if and only if $D = D_0$ and $H = H_0$, in which case $\Psi_2 = 0 = \Psi_4$. For this choice, the expressions (6)–(8) reduce to

$$R = 24\alpha - 12\varepsilon(\alpha+\beta)\,\frac{P}{Q}\ , \qquad (17)$$

$$\Phi_{11} = \tfrac{1}{2}\varepsilon(\alpha+\beta)\,\frac{P}{Q} \;\equiv\; \alpha - \tfrac{1}{24}R\ , \qquad (18)$$

$$\Phi_{22} = \tfrac{1}{2}\Big\{(\alpha A_0 + A_2)\,Q - \varepsilon(\alpha+\beta)(A_0 + A_1\zeta + \bar A_1\bar\zeta + A_2\zeta\bar\zeta) - 2\varepsilon(\alpha+\beta)\,v\,\big(\dot C\,\zeta + \dot{\bar C}\,\bar\zeta\big)\Big\}\,\frac{P^3}{Q^4}\ . \qquad (19)$$

In order to obtain only a *pure radiation* field, one has to set $\beta = -\alpha$ so that $R = 24\alpha$, $\alpha = \frac{1}{6}\Lambda$, and

$$\Phi_{22} = \tfrac{1}{2}(\alpha A_0 + A_2)\,\frac{P^3}{Q^3}\ . \qquad (20)$$

Consequently, we obtain explicit solutions in the form (1) in which

$$
\begin{aligned}
P &= 1+\tfrac{1}{6}\Lambda\,\zeta\bar\zeta\ , \qquad Q = (1-\tfrac{1}{6}\Lambda\,\zeta\bar\zeta)\,\varepsilon + C\,\zeta + \bar C\,\bar\zeta\ , \\
F &= \left(\tfrac{1}{3}\Lambda\varepsilon + 2C\bar C\right)\frac{Q^2}{P^2}\,v^2 - \frac{(Q^2)_{,u}}{P^2}\,v - \frac{Q}{P^2}\left(A_0 + A_1\zeta + \bar A_1\bar\zeta + A_2\zeta\bar\zeta\right)\ ,
\end{aligned}
\tag{21}
$$

where A_i and C are arbitrary functions of u. (Note that Φ_{11} in the expression (18) vanishes also for $\varepsilon = 0$, which however gives just a subcase of (21).) This complete family of pure radiation, conformally flat spacetimes with Λ was mentioned already in [15]. As discussed in [15, 16] there exist several invariant subclasses of this, namely (conformally flat) *pp* and Kundt waves with $\Lambda = 0$, and their generalizations to non-vanishing Λ. For *pp*-wave spacetimes ($\Lambda = 0$, $\varepsilon = 1$, $C = 0$ so that $P = 1 = Q$) or the solution of the Siklos type [17, 18] ($\Lambda < 0$, $\varepsilon = 1$, $C = \sqrt{-\frac{1}{6}\Lambda}$), one obtains $D_0 = 0$ which implies that ∂_v is a Killing vector. In particular, this includes the only conformally flat pure radiation solution of the Einstein–Maxwell equations with $\Phi_2(u) = \sqrt{\frac{1}{2}A_2}\,e^{i\varphi(u)}$ in (15), namely the plane waves introduced by Baldwin, Jeffery [5] and Brdička [6] (see the appendix). Another special subclass of (2) arises for $\varepsilon = 0$ and $C = 1$, in which case $Q = \zeta + \bar\zeta$. In the canonical coordinates (9) this corresponds to the metric coefficients (10)

$$
\begin{aligned}
W &= -\frac{2w\left(1-\frac{1}{6}\Lambda\,\bar\zeta^2\right)}{(\zeta+\bar\zeta)\left(1+\frac{1}{6}\Lambda\,\zeta\bar\zeta\right)}\ , \\
\mathcal{H} &= \tfrac{1}{2}\left(A_0 + A_1\zeta + \bar A_1\bar\zeta + A_2\zeta\bar\zeta\right)\frac{\zeta+\bar\zeta}{\left(1+\frac{1}{6}\Lambda\,\zeta\bar\zeta\right)^2} - \left(\frac{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}{\zeta+\bar\zeta}\right)^2 w^2\ .
\end{aligned}
\tag{22}
$$

For vanishing Λ, this exactly reduces to the interesting class of conformally flat pure radiation metrics found by Edgar and Ludwig [19,20] and discussed e.g. in [21–23].

The complementary situation in which there is no pure radiation ($\Phi_{22} = 0$) but *only a non-vanishing* Φ_{11} *component* of the matter field, requires $\alpha + \beta \neq 0$, $\varepsilon = 1$, $C \equiv \gamma =$const., $D = D_0 = -2\beta + 2\gamma\bar\gamma$, $A_2 = \beta A_0$ and $A_1 = \gamma A_0$, so that $H = A_0 Q/P$. However, this function $H(u)$ can always be removed by the coordinate transformation $v \to Bv + \dot B/D$, $u \to \int B^{-1}du$, for a suitable choice of $B(u)$. Without loss of generality we may thus write the solutions of this subclass as the metric (1) in which

$$
P = 1+\alpha\,\zeta\bar\zeta\ , \qquad Q = 1+\beta\,\zeta\bar\zeta+\gamma\,\zeta+\bar\gamma\,\bar\zeta\ , \qquad F = (2\gamma\bar\gamma - 2\beta)\,\frac{Q^2}{P^2}\,v^2\ . \tag{23}
$$

By performing a simple transformation $v = r/[\,1+(\beta-\gamma\bar\gamma)ur\,]$, we can put this into the form

$$
\mathrm{d}s^2 = \frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{P^2} - \frac{Q^2}{P^2}\,\frac{2\,\mathrm{d}u\,\mathrm{d}r}{[\,1+(\beta-\gamma\bar\gamma)\,ur\,]^2}\ , \tag{24}
$$

in which P and Q are given by (23). (There is still a coordinate freedom given by $\zeta \to (\bar b + a\zeta)/(\bar a - \alpha b\zeta)$, a and b being constants, and rescaling of u and/or r, which can be used to modify β and γ.) The matter content in these spacetimes is described by $\Phi_{11} = \frac{1}{2}(\alpha+\beta)P/Q$ which, in general, is a function of the spatial coordinate ζ. However, it immediately follows from (18) that the (invariant) *additional condition* $R =$const. necessarily implies $R = 0$, $\alpha = \beta$, $\gamma = 0$, $P = Q$. Thus, for a constant R we obtain the Bertotti–Robinson universe [9–12]

$$
\mathrm{d}s^2 = \frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{\left(1+\alpha\,\zeta\bar\zeta\right)^2} - 2\,\mathrm{d}u\,\mathrm{d}v - 2\alpha\,v^2\,\mathrm{d}u^2\ . \tag{25}
$$

This homogeneous space is the unique conformally flat solution of the Einstein–Maxwell equations with a (uniform) non-null electromagnetic field $\Phi_1 = \sqrt{\alpha}\,e^{i\varphi}$ in (14) where $\alpha > 0$ (see the appendix).

Of course, conformally flat spacetimes of the above form exist which combine Φ_{11} and Φ_{22}. For example, one can introduce a pure radiation in the Bertotti–Robinson universe by adding the term $-H_0\,\mathrm{d}u^2$ to the metric (25).

Finally, the conformally flat spaces with *no matter* (Einstein spaces) are given by (21) with the additional constraint $A_2 = -\frac{1}{6}\Lambda A_0$, see (20). In such a case, the function H_0 takes the form $H_0 = [(1-\frac{1}{6}\Lambda\,\zeta\bar\zeta)A_0 + A_1\zeta + \bar A_1\bar\zeta]/P$, which can be removed completely by a suitable coordinate transformation [16]. These constant curvature vacuum solutions with $H=0$ can then be put into the standard form of Minkowski, de Sitter or anti-de Sitter spacetimes [24]. For example, assuming $\varepsilon=1$, $C=0$, we obtain

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{\left(1+\frac{1}{6}\Lambda\,\zeta\bar\zeta\right)^2} - 2\left(\frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\right)^2 \mathrm{d}u\,\mathrm{d}v + \tfrac{1}{3}\Lambda\left(\frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\right)^2 v^2\,\mathrm{d}u^2\,. \tag{26}$$

For the $\Lambda=0$ case, this is obviously a flat space. For $\Lambda\neq0$, the parametrization of the (anti-) de Sitter universe represented geometrically as a hyperboloid $-Z_0^2+Z_1^2+Z_2^2+Z_3^2+\epsilon Z_4^2 = 3/\Lambda$ ($\epsilon\equiv\Lambda/|\Lambda|$) imbedded in a flat five-dimensional space $\mathrm{d}s^2 = -\mathrm{d}Z_0^2+\mathrm{d}Z_1^2+\mathrm{d}Z_2^2+\mathrm{d}Z_3^2+\epsilon\mathrm{d}Z_4^2$ is

$$\begin{aligned}
Z_0 &= \frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\left[\tfrac{1}{2}\left(\tfrac{1}{3}\Lambda\right)^2 u^2 v + \tfrac{1}{3}\Lambda\,u + \tfrac{1}{2}\left(\tfrac{1}{3}\Lambda\right)^{-1} v\right]\,,\\
Z_1 &= \frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\left[\tfrac{1}{2}\left(\tfrac{1}{3}\Lambda\right)^2 u^2 v + \tfrac{1}{3}\Lambda\,u - \tfrac{1}{2}\left(\tfrac{1}{3}\Lambda\right)^{-1} v\right]\,,\\
Z_2 &= \tfrac{1}{\sqrt2}\frac{\zeta+\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\,,\qquad Z_3 = \tfrac{\mathrm{i}}{\sqrt2}\frac{\zeta-\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\,,\\
Z_4 &= \left(\tfrac{1}{3}|\Lambda|\right)^{-1/2}\frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\left(1+\tfrac{1}{3}\Lambda\,uv\right)\,.
\end{aligned} \tag{27}$$

Analogous parameterizations can be found for other (canonical) choices of ε and C.

3.2 Type D spacetimes

In this case, the vanishing of Ψ_4 requires $H=H_0$, and the curvature scalars (4)–(8) reduce to

$$\Psi_2 = -\tfrac{1}{6}\left(D-D_0\right)\frac{P^2}{Q^2}\,, \tag{28}$$

$$R = 24\alpha - 12\varepsilon(\alpha+\beta)\frac{P}{Q} + 2\left(D-D_0\right)\frac{P^2}{Q^2}\,, \tag{29}$$

$$\Phi_{11} = \tfrac{1}{2}\varepsilon(\alpha+\beta)\,\frac{P}{Q} - \tfrac{1}{4}\left(D-D_0\right)\,\frac{P^2}{Q^2}\,, \tag{30}$$

$$\begin{aligned}
\Phi_{22} = \tfrac{1}{2}\Big\{&(\alpha A_0 + A_2)\,Q - \varepsilon(\alpha+\beta)(A_0 + A_1\zeta + \bar A_1\bar\zeta + A_2\zeta\bar\zeta)\\
&- 2\varepsilon(\alpha+\beta)\,v\,\left(\dot C\,\zeta + \dot{\bar C}\,\bar\zeta\right)\Big\}\frac{P^3}{Q^4}\,.
\end{aligned} \tag{31}$$

Notice that the Φ_{22} component (31) has the same form as in the conformally flat case (19) but the other scalars differ since $D\neq D_0$.

A *pure radiation* field requires $\Phi_{11}=0$, which implies that $\alpha=\beta=\frac{1}{2}\Lambda$, $\varepsilon=1$, $C=0$, $D=\Lambda$, and the metric takes the form

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{\left(1+\frac{1}{2}\Lambda\zeta\bar\zeta\right)^2} - 2\,\mathrm{d}u\,\mathrm{d}v + \left(\Lambda v^2 - H_0\right)\,\mathrm{d}u^2\,, \tag{32}$$

the scalars being $\Psi_2 = -\frac{1}{3}\Lambda$, $R = 4\Lambda$, and

$$\Phi_{22} = \tfrac{1}{2}\left(A_2 - \tfrac{1}{2}\Lambda\,A_0\right)\left(1-\tfrac{1}{2}\Lambda\,\zeta\bar\zeta\right) - \tfrac{1}{2}\Lambda\left(A_1\zeta + \bar A_1\bar\zeta\right)\,. \tag{33}$$

When $\Lambda>0$ ($\Lambda<0$) this solution represents the (anti-)Nariai universe [8] with pure radiation which can not be an electromagnetic field (see the general theorem in [25]). However, for $\Lambda=0$ this becomes the conformally flat electromagnetic plane wave mentioned in the previous section. Note also that the metric (32) is identical to the Kundt canonical form (9) since $P=Q$ in the case of pure radiation solutions, and consequently $v=w$, $W=0$, $\mathcal{H}=-\frac{1}{2}\Lambda w^2+\frac{1}{2}H_0$. In particular, it belongs to the invariant subclass defined by $W_{,w}=0$, and thus seems to be a counterexample to a conjecture of [26] that the solutions exhibited there (for which $W_{,w}\neq 0$) are the only type D pure radiation metrics of the Kundt class.

The situation in which there is no pure radiation ($\Phi_{22}=0$), but *only the* Φ_{11} *component* of the matter field, admits two possibilities. If $\varepsilon(\alpha+\beta)\neq 0$ then $\varepsilon=1$, $C=\gamma$, $A_2=\beta A_0$, and $A_1=\gamma A_0$ so that $H=A_0Q/P$. As in the conformally flat case, this function H can always be removed by a coordinate transformation. Thus, all the solutions of this subclass can be written as the metric (1) in which

$$P=1+\alpha\,\zeta\bar\zeta\,,\qquad Q=1+\beta\,\zeta\bar\zeta+\gamma\,\zeta+\bar\gamma\,\bar\zeta\,,\qquad F=D(u)\,\frac{Q^2}{P^2}\,v^2\ ,\tag{34}$$

With $v=r/[\,1+U(u)\,r\,]$, where $U(u)=-\frac{1}{2}\int D(u)\,du$, we put the solution into the form

$$\mathrm{d}s^2=\frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{P^2}-\frac{Q^2}{P^2}\,\frac{2\,\mathrm{d}u\,\mathrm{d}r}{[\,1+U(u)\,r\,]^2}\ ,\tag{35}$$

(For $D=D_0$ the metric (34) reduces to the conformally flat spacetime (23), i.e. (35) gives (24).) Looking for the electrovacuum solutions we require $R=4\Lambda$ which implies $\alpha=\beta$, $\gamma=0$, $D=2(\Lambda-\alpha)$, and the scalars are $\Psi_2=-\frac{1}{3}\Lambda$, $\Phi_{11}=\alpha-\frac{1}{2}\Lambda$. In such a case, the metric (1) simplifies considerably to

$$\mathrm{d}s^2=\frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{(1+\alpha\,\zeta\bar\zeta)^2}-2\,\mathrm{d}u\,\mathrm{d}v+2(\Lambda-\alpha)\,v^2\,\mathrm{d}u^2\ .\tag{36}$$

These are well-known electrovacuum spacetimes with the geometry of a direct product of two constant curvature 2-spaces [10], see also [12, 27]. This is obvious from the form (35) since $P=Q$, $U(u)=-\frac{1}{2}Du$. Of course, for $\Lambda=0$ we recover the conformally flat solution (25). The second possibility for the case $\Phi_{22}=0$ is $\varepsilon(\alpha+\beta)=0$ which implies $A_2=-\alpha A_0$, i.e. $H_0=[(1-\alpha\,\zeta\bar\zeta)A_0+A_1\zeta+\bar A_1\bar\zeta]/P$. The corresponding scalars are $R=24\alpha+2(D-D_0)P^2/Q^2$, where $D_0=2\varepsilon\alpha+2C\bar C$, $\Psi_2=2\alpha-\frac{1}{12}R$, $\Phi_{11}=3\alpha-\frac{1}{8}R$. For electrovacuum solutions, $R=4\Lambda$ so that $\alpha=0=\beta$, $\varepsilon=1$, $C=0$, $D=2\Lambda<0$. We thus uniquely obtain the exceptional Plebański–Hacyan universe [28]

$$\mathrm{d}s^2=2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta-2\,\mathrm{d}u\,\mathrm{d}v+\left(2\Lambda\,v^2-A_1\zeta-\bar A_1\bar\zeta\right)\,\mathrm{d}u^2\ ,\tag{37}$$

for which $\Phi_1=\sqrt{-\frac{1}{2}\Lambda}\,e^{i\varphi}$ (the function A_0 can be removed by a coordinate transformation [28]). When $A_1=0$, this reduces to the form (36) of the direct product spacetimes.

Again, type D spacetimes which combine Φ_{11} and Φ_{22} can be considered. For example, there exist solutions which represent pure radiation in the electrovacuum universes (36) and (37), see e.g. [29].

Finally, the type D Einstein spaces with *no matter* are given by (32)-(33) with the constraints $A_2=\frac{1}{2}\Lambda A_0$, $A_1=0$. In such a case, the function $H_0=A_0$ can again be removed by a suitable coordinate transformation. Thus, without loss of generality one obtains

$$\mathrm{d}s^2=\frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{\left(1+\frac{1}{2}\Lambda\zeta\bar\zeta\right)^2}-2\,\mathrm{d}u\,\mathrm{d}v+\Lambda v^2\,\mathrm{d}u^2\ ,\tag{38}$$

which is the (anti-)Nariai universe [8], discussed recently in [30]. Obviously, this is also the direct product spacetime (36) for $\alpha=\frac{1}{2}\Lambda$. There is no type D vacuum solution of the form (1) for $\Lambda=0$ since (38) reduces to Minkowski space in such a case.

The main results presented above are summarized in the table 2.

	type O	type D
no matter $\Phi_{11}=0, \Phi_{22}=0$	Minkowski ($\Lambda=0$) de Sitter ($\Lambda>0$) anti-de Sitter ($\Lambda<0$)	$\longleftarrow$ none for $\Lambda=0$ Nariai ($\Lambda>0$) anti-Nariai ($\Lambda<0$)
pure radiation $\Phi_{11}=0, \Phi_{22}\neq 0$	Baldwin–Jeffery–Brdička ($\Lambda=0$) Edgar–Ludwig ($\Lambda=0$) Ozsváth–Robinson–Rózga ($\Lambda\neq 0$)	$\longleftarrow$ none for $\Lambda=0$ (32) for $\Lambda\neq 0$
no pure radiation $\Phi_{11}\neq 0, \Phi_{22}=0$	$(23), (24)$ Bertotti–Robinson ($\Lambda=0$)	$(34), (35)$ direct product spacetimes Plebański–Hacyan ($\Lambda<0$)
general $\Phi_{11}\neq 0, \Phi_{22}\neq 0$	(1) with $H=H_0$, $D=D_0$	(1) with $H=H_0$, $D\neq D_0$

Table 2: Summary of the background spacetimes and the most important subcases.

4 Exact gravitational waves of arbitrary profile on the above backgrounds

By considering an arbitrary function $H(\zeta,\bar\zeta,u)$ in the metric (1), (2), different from H_0 as introduced in (12), the scalar Ψ_4 representing gravitational radiation becomes non-vanishing, see (5). As the coordinate u plays the role of the retarded time, *gravitational waves of arbitrary profiles* $H^w\equiv H-H_0$ can thus be introduced into the above spacetimes. When the backgrounds are taken to be conformally flat, exact radiative spacetimes of Petrov type N are obtained. For the backgrounds of type D one gets explicit gravitational waves of type II, see table 1. It is obvious from (1) that the wave-fronts $u=\text{const.}$ are *non-expanding* 2-spaces with a *constant curvature* equal to 2α.

It can be observed from (4)–(8) that the function H does not appear in the scalars Ψ_2, R, and Φ_{11}. In fact, introducing the additional term $-(Q/P)\,H^w\,\mathrm{d}u^2$ in the metric corresponds to a generalised Kerr–Schild transformation (see e.g. [31, 32]) of the background geometries[1]. Consequently, the radiative spacetimes (1) of type N and II have *exactly the same values* of the Weyl component Ψ_2, the Ricci curvature R, and the non-null matter component Φ_{11} as the corresponding backgrounds described in the previous section (see table 2). The only difference, apart from the introduction of the Ψ_4 component, may be in the pure radiation field. Indeed, the component Φ_{22}, which is *linear* in H, can now be understood as a superposition of the background term Φ^b_{22} given by (19), identical to (31), with the term Φ^w_{22} related to the presence of gravitational radiation,

$$\Phi^w_{22}=\tfrac{1}{2}\Big\{\Big[\varepsilon(\alpha-\beta)\,(1-\alpha\,\zeta\bar\zeta)\;+2\alpha\,(C\,\zeta+\bar C\,\bar\zeta)\Big]\,H^w+P^2Q\,H^w_{,\zeta\bar\zeta}\Big\}\,\frac{P^3}{Q^4}\;. \tag{39}$$

In general, a gravitational wave ($\Psi_4\neq 0$) is thus accompanied by the (additional) pure radiation field Φ^w_{22}.

However, in special situations when $\Phi^w_{22}=0$, the gravitational wave is *not* related to the above pure radiation. For example when $P=Q=1+\alpha\,\zeta\bar\zeta$, i.e. for spacetimes (36) which are direct product of two constant curvature 2-spaces (Minkowski, Bertotti–Robinson, (anti-)Nariai, Plebański–Hacyan spaces), the relation $\Phi^w_{22}=0$ is satisfied if and only if $H^w=f(\zeta,u)+\bar f(\bar\zeta,u)$.

[1]The authors are grateful to the referee for this observation.

This is just the Einstein equation for purely gravitational waves propagating in Einstein spaces, in which case $\Phi^b_{22}=0=\Phi_{11}$.

Let us now discuss in some detail the particular subclasses which include gravitational waves.

4.1 Type N spacetimes

A complete class of non-expanding type N *vacuum* solutions with (possibly) non-vanishing cosmological constant was found by Ozsváth, Robinson and Rózga [15], and later studied also from a physical point of view [33]. These Einstein spaces represent exact pure gravitational waves which propagate in constant curvature backgrounds, i.e. in Minkowski, de Sitter or anti-de Sitter universe (for $\Lambda=0$, $\Lambda>0$ or $\Lambda<0$, respectively). For example, considering the background $\mathrm{d}s^2_b$ given by (26), one obtains radiative solutions of the form

$$\mathrm{d}s^2=\mathrm{d}s^2_b-\left(\frac{1-\frac{1}{6}\Lambda\,\zeta\bar\zeta}{1+\frac{1}{6}\Lambda\,\zeta\bar\zeta}\right)H^w\,\mathrm{d}u^2\ . \tag{40}$$

For vanishing cosmological constant we obtain exactly the well-known class of *pp*-waves [12]. Alternatively, with other forms of constant curvature backgrounds corresponding to different canonical choices of the parameters ε and C other classes are obtained, namely the specific Kundt spacetimes for $\Lambda=0$ [1–3] or the Siklos family [17] for $\Lambda<0$, see [15, 16]. When $\Phi^w_{22}=0$ these are pure gravitational waves, otherwise they are accompanied by a pure radiation component.

Another possibility is to consider the conformally flat *pure radiation* ($\Phi^b_{22}\neq0$, $\Phi_{11}=0$) backgrounds $\mathrm{d}s^2_b$ of the form (21) and introduce H^w. Again the *pp*-waves and gravitational waves in the Edgar–Ludwig type backgrounds (22) with any Λ are thus obtained. In fact, these radiative spacetimes are contained in the Ozsváth–Robinson–Rózga family [15].

On the other hand, considering the conformally flat backgrounds (24) *without pure radiation* ($\Phi^b_{22}=0$, $\Phi_{11}\neq0$) gravitational waves (plus possibly pure radiation if $\Phi^w_{22}\neq0$) are generated. In particular, one obtains the type N solutions representing gravitational radiation in the Bertotti–Robinson electrovacuum universe

$$\mathrm{d}s^2=\mathrm{d}s^2_b-H^w\,\mathrm{d}u^2\ . \tag{41}$$

where $\mathrm{d}s^2_b$ is given by (25).

Of course, gravitational waves in more general conformally flat backgrounds (17)–(19) with $\Phi_{11}\neq0$, $\Phi^b_{22}\neq0$ are also possible.

4.2 Type II spacetimes

All type II solutions of the form (1), (2) which represent exact gravitational waves propagating in backgrounds with the matter field component Φ_{11} *vanishing* can be written in the form (41) where $\mathrm{d}s^2_b$ is now the metric (32). These describe gravitational radiation in the (anti-)Nariai (type D) universe filled with a pure radiation field $\Phi_{22}=\Phi^b_{22}+\Phi^w_{22}$, where Φ^b_{22} is given by (33) and Φ^w_{22} by (39), which now reduces to a simple expression $\Phi^w_{22}=\frac{1}{2}(1+\frac{1}{2}\Lambda\zeta\bar\zeta)^3H^w_{\zeta\bar\zeta}$.

As an important special subcase we easily obtain *pure gravitational waves* of the above form by considering, not only $\Phi_{11}=0$, but also $\Phi_{22}=0$, i.e. Einstein spaces of the Petrov type II without matter. Obviously, these solutions can explicitly be written as

$$\mathrm{d}s^2=\mathrm{d}s^2_b-\left[\,f(\zeta,u)+\bar f(\bar\zeta,u)\,\right]\,\mathrm{d}u^2\ , \tag{42}$$

where $\mathrm{d}s^2_b$ is the metric (38) of the (anti-)Nariai vacuum universe with $\Lambda>0$ ($\Lambda<0$), and $f(\zeta,u)$ is an arbitrary function (holomorphic in ζ) which characterizes the profile of the gravitational wave. Note that the solution (42) of the Einstein vacuum equations with cosmological constant is included in the class of solutions that was investigated from a different point of view by Lewandowski [34].

The complementary situation in which $\Phi^b_{22} = 0$, but $\Phi_{11} \neq 0$, corresponds to introducing type II waves into the background (35). In particular, considering only the *non-null electrovacuum* background universes we obtain gravitational waves in the spacetimes $\mathrm{d}s^2_b$ which are a direct product of two constant curvature 2-spaces (36). The second possibility for the electrovacuum case $\Phi^b_{22} = 0$ is the exceptional Plebański–Hacyan background universe (37). In both these cases the radiative metric has the form (42) with the corresponding form of $\mathrm{d}s^2_b$.

Again, these gravitational waves are in general accompanied by a pure radiation contribution Φ^w_{22}. As particular cases of the type II spacetimes *without pure radiation*, $\Phi^b_{22} = 0 = \Phi^w_{22}$, we obtain the special electrovacuum solutions with gravitational waves that were found by García and Alvarez [35]. For $\alpha = \beta = C = 0$, $\varepsilon = 1$, $D = 2\Lambda$, this is their special II-$E_{(+)}$ solution with $\Lambda < 0$ which can be written in the form (42) where $\mathrm{d}s^2_b$ is the Plebański–Hacyan metric (36). For $\alpha = \beta = \Lambda$, $C = 0$, $\varepsilon = 1$, $D = 0$ we obtain the (non-twisting subclass of) II-$E_{(-)}$ solution with $\Lambda > 0$ in the form (42) where $\mathrm{d}s^2_b$ is the other direct product Plebański–Hacyan metric of the form (36). In both cases, $\Psi_2 = -\frac{1}{3}\Lambda$, $R = 4\Lambda$, $\Phi_{11} = \frac{1}{2}|\Lambda|$. Note that all the above mentioned metrics representing pure gravitational waves in electromagnetic universes are in fact specific subcases of a solution presented by Khlebnikov [36].

Finally, we note that gravitational waves (plus possibly pure radiation Φ^w_{22}) in general type D backgrounds (28)–(31) for which both $\Phi_{11} \neq 0$ and $\Phi^b_{22} \neq 0$ are easy to construct.

4.3 Electrovacuum solutions

It may be useful to summarize all the possible cases in which the solution of the form (1), (2) represents a spacetime containing an *electromagnetic field but no other matter fields.* In section 3 we concentrated on the background spaces of type O and D. We now investigate the corresponding situations when gravitational waves are present, i.e. the spacetimes of type N or II, respectively.

We have demonstrated (see also the appendix) that the only *conformally flat* spacetimes (including Λ) which satisfy the Einstein–Maxwell equations are some special *pp*-waves with a null Maxwell field, and the Bertotti–Robinson universe with a non-null Maxwell field. For *type N pp*-waves, corresponding to $P = 1 = Q$, $F = -H$ in (1), it is well known [12] that there is a general combination of electromagnetic and gravitational waves when $H = 2\,|\int E_2\,\mathrm{d}\zeta|^2 + f(u,\zeta) + \bar{f}(u,\bar{\zeta})$, where $E_2(u,\zeta)$ represents the null Maxwell field (15). On the other hand, one has type N gravitational waves propagating in the Bertotti–Robinson electrovacuum universe, with $P = 1 + \alpha\zeta\bar{\zeta} = Q$, $F = -2\alpha v^2 - H$ in (1), for $H = f(u,\zeta) + \bar{f}(u,\bar{\zeta})$, cf. [29, 30, 36].

For *type D* solutions we may have only a non-null Maxwell field, represented by spacetimes which are the direct product of two constant curvature 2-spaces (36) plus the exceptional Plebański–Hacyan universe (37). Pure *type II* gravitational waves in such type D electrovacuum spacetimes arise for $H = f(u,\zeta) + \bar{f}(u,\bar{\zeta})$, see [29, 30, 35, 36].

Finally, we consider *vacuum backgrounds* and possible *electromagnetic waves propagating on these.* Conformally flat vacuum spacetimes are just spaces of constant curvature. Electromagnetic waves in the Minkowski spacetime are simply the *pp*-waves discussed above. Electromagnetic waves in the (anti-)de Sitter space were discussed in [15], and are necessarily accompanied by gravitational waves (see [15] for explicit formulae). The only remaining possibility is thus given by waves in vacuum backgrounds of type D, i.e. in the (anti-)Nariai universe (38). Interestingly, also in this case the Einstein–Maxwell equation (16), (39) for the profile function H can be integrated explicitly. Exactly as in the case of *pp*-waves, this admits the general solution $H = 2\,|\int E_2\,\mathrm{d}\zeta|^2 + f(u,\zeta) + \bar{f}(u,\bar{\zeta})$. However, these electromagnetic waves are now necessarily accompanied by a gravitational wave component (see section 3 and [25]). Such an explicit solution for *electromagnetic (plus gravitational) waves in the (anti-)Nariai universe* seems to have remained unnoticed in the literature so far (whereas pure gravitational waves were already considered [30, 36]).

4.4 Effects on test particles

In order to analyze the effects of the gravitational and matter fields of the above solutions, it is natural to investigate the specific influence of various components of these fields on the relative motion of free test particles. Such a local characterization of spacetimes, based on the equation of geodesic deviation, was described in the pioneering works by Pirani [37, 38] and Szekeres [39, 40]. Following the notation introduced in [33] we set up an orthonormal frame $\{\mathbf{e}_{(a)}\}$, $\mathbf{e}_{(a)}\cdot\mathbf{e}_{(b)} = \eta_{(a)(b)} \equiv \mathrm{diag}(-1,1,1,1)$, such that the timelike vector $\mathbf{e}_{(0)}$ coincides at a given event with the four-velocity $\mathbf{u}$ of a geodesic test observer. By projecting the equation of geodesic deviation onto this frame we obtain

$$\ddot{Z}^{(i)} = -R^{(i)}_{\ \ (0)(j)(0)}\, Z^{(j)} \ , \tag{43}$$

where $i,j=1,2,3$. The frame components of the displacement vector $Z^{(j)} \equiv e^{(j)}_{\mu} Z^{\mu}$ determine directly the distance between close test particles, and $\ddot{Z}^{(i)} \equiv e^{(i)}_{\mu}(D^2Z^{\mu}/d\tau^2)$ are their physical relative accelerations. From the standard decomposition of the curvature tensor (see, e.g., Eqs. (3.44)–(3.47) in [12]), we immediately obtain $R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} + \frac{1}{2}(\,\delta_{ij}S_{(0)(0)} - S_{(i)(j)}\,) + \frac{1}{12}R\,\delta_{ij}$, where $C_{(i)(0)(j)(0)}$ are components of the Weyl tensor and R and $S_{(a)(b)}$ respectively denote the trace and the traceless part of the Ricci tensor. These frame components can conveniently be expressed using the corresponding null complex tetrad

$$\mathbf{m} = \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(1)} + \mathrm{i}\,\mathbf{e}_{(2)}\right) \ , \qquad \mathbf{l} = \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} - \mathbf{e}_{(3)}\right) \ , \quad \mathbf{k} = \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} + \mathbf{e}_{(3)}\right) \ . \tag{44}$$

In the natural tetrad (3), the only non-vanishing scalars are those given by (4)–(8) so that

$$\begin{aligned}
&C_{(1)(0)(1)(0)} = \tfrac{1}{2}\,\mathcal{R}e\,\Psi_4 - \Psi_2 \ , \quad C_{(2)(0)(2)(0)} = -\tfrac{1}{2}\,\mathcal{R}e\,\Psi_4 - \Psi_2 \ ,\\
&C_{(1)(0)(2)(0)} = -\tfrac{1}{2}\,\mathcal{I}m\,\Psi_4 \ , \quad C_{(3)(0)(3)(0)} = 2\,\Psi_2 \ ,\\
&S_{(0)(0)} = \Phi_{22} + 2\,\Phi_{11} \ , \quad S_{(1)(1)} = 2\,\Phi_{11} \ ,\\
&S_{(3)(3)} = \Phi_{22} - 2\,\Phi_{11} \ , \quad S_{(2)(2)} = 2\,\Phi_{11} \ .
\end{aligned} \tag{45}$$

Therefore, the equation of geodesic deviation (43) takes the form

$$\begin{aligned}
\ddot{Z}^{(1)} &= \tfrac{1}{12}R\,Z^{(1)} - \tfrac{1}{2}\Phi_{22}\,Z^{(1)} + \mathcal{C}Z^{(1)} - \mathcal{A}_{+}\,Z^{(1)} + \mathcal{A}_{\times}\,Z^{(2)} \ ,\\
\ddot{Z}^{(2)} &= \tfrac{1}{12}R\,Z^{(2)} - \tfrac{1}{2}\Phi_{22}\,Z^{(2)} + \mathcal{C}Z^{(2)} + \mathcal{A}_{+}\,Z^{(2)} + \mathcal{A}_{\times}\,Z^{(1)} \ ,\\
\ddot{Z}^{(3)} &= \tfrac{1}{12}R\,Z^{(3)} - 2\Phi_{11}\,Z^{(3)} - 2\mathcal{C}Z^{(3)} \ ,
\end{aligned} \tag{46}$$

where

$$\mathcal{C} = \Psi_2 \ , \qquad \mathcal{A}_{+} = \tfrac{1}{2}\,\mathcal{R}e\,\Psi_4 \ , \qquad \mathcal{A}_{\times} = \tfrac{1}{2}\,\mathcal{I}m\,\Psi_4 \ . \tag{47}$$

Equations (46) are well suited for physical interpretation. Clearly, the relative motions of nearby test particles depend on:

1. the Ricci scalar R, which is responsible for overall background *isotropic* motions (in the case of pure radiation or vacuum spacetimes this is a constant factor proportional to the cosmological constant, $\frac{1}{12}R = \frac{1}{3}\Lambda$) ;

2. the terms describing the matter-content which consist of two different components: the *pure radiation* field Φ_{22} affects the motion only in the *transverse* plane spanned by the vectors $\mathbf{e}_{(1)}, \mathbf{e}_{(2)}$, whereas the *non-null* field influences only the *longitudinal* direction $\mathbf{e}_{(3)}$;

3. the terms which depend on the local free gravitational field. There is a *Coulomb* component with the amplitude $\mathcal{C}$ for type D and II spacetimes, and transverse components with amplitudes given by $\mathcal{A}_{+}$ and $\mathcal{A}_{\times}$ representing the effect of *gravitational waves* on the particles in type N and II spacetimes.

It can be observed that gravitational radiation is present if and only if $H \neq H_0$. The corresponding effects are typical for both linearized and exact gravitational waves (cf. e.g. [33,39,41]). There are two polarization modes "+" and "×" of a transverse gravitational wave with amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$. A ring of test particles is deformed into an ellipse in the plane perpendicular to the direction of propagation, the axes of different polarizations being shifted one with respect to the other by $\frac{\pi}{4}$. The structure of the equations (46) thus supports our interpretation of the solutions (1) as exact gravitational waves which propagate in various background universes. Notice that the coefficients R, Φ_{11} and Ψ_2 remain the same as for the corresponding background, whereas the amplitudes Φ_{22} and Ψ_4 which represent radiation depend on the specific form of the structural function H.

The above analysis only applies to the reference frame (44) related to the natural null tetrad (3). However, this can easily be generalized to a case in which the orthonormal interpretation frame is adapted to an *arbitrary observer* passing along a timelike geodesic through the given event. Performing the Lorentz transformation

$$\begin{aligned}
\mathbf{k}' &= A^{-1}\,\mathbf{k}\ ,\\
\mathbf{l}' &= A\,\mathbf{l} + B\,\bar{\mathbf{m}} + \bar{B}\,\mathbf{m} + B\bar{B}A^{-1}\,\mathbf{k}\ ,\\
\mathbf{m}' &= \mathbf{m} + BA^{-1}\,\mathbf{k}\ ,
\end{aligned} \tag{48}$$

where $A = \sqrt{2}\,(Q^2/P^2)\,\dot{u}\,,\ B = \sqrt{2}\,(1/P)\,\dot{\zeta}$, we obtain from (3) the null tetrad

$$\begin{aligned}
\mathbf{m}' &= \frac{P\,\dot{\zeta}}{Q^2\,\dot{u}}\,\partial_v + P\,\partial_{\bar{\zeta}}\ , \qquad \bar{\mathbf{m}}' = \frac{P\,\dot{\bar{\zeta}}}{Q^2\,\dot{u}}\,\partial_v + P\,\partial_\zeta\ ,\\
\mathbf{l}' &= \Big(\sqrt{2}\,\dot{v} - \frac{P^2}{\sqrt{2}\,Q^2\,\dot{u}}\Big)\,\partial_v + \sqrt{2}\,\dot{\zeta}\,\partial_\zeta + \sqrt{2}\,\dot{\bar{\zeta}}\,\partial_{\bar{\zeta}} + \sqrt{2}\,\dot{u}\,\partial_u\ ,\\
\mathbf{k}' &= \frac{P^2}{\sqrt{2}\,Q^2\,\dot{u}}\,\partial_v\ .
\end{aligned} \tag{49}$$

Obviously, $\mathbf{e}'_{(0)} \equiv \frac{1}{\sqrt{2}}\,(\mathbf{l}' + \mathbf{k}') = \dot{v}\,\partial_v + \dot{\zeta}\,\partial_\zeta + \dot{\bar{\zeta}}\,\partial_{\bar{\zeta}} + \dot{u}\,\partial_u$ represents the local value of the four-velocity of a general geodesic observer (with the dot indicating a differentiation with respect to its proper time) normalized as

$$\mathbf{e}'_{(0)} \cdot \mathbf{e}'_{(0)} = 2\,\frac{1}{P^2}\,\dot{\zeta}\,\dot{\bar{\zeta}} - 2\,\frac{Q^2}{P^2}\,\dot{u}\,\dot{v} + F\,\dot{u}^2 = -1\ . \tag{50}$$

After the Lorentz transformation (48), the corresponding (non-vanishing) coefficients take the form

$$\begin{aligned}
&\Psi'_2 = \Psi_2\ , \qquad \Psi'_3 = 3\bar{B}\,\Psi_2\ , \qquad \Psi'_4 = A^2\Psi_4 + 6\bar{B}^2\,\Psi_2\ ,\\
&\Phi'_{11} = \Phi_{11}\ , \qquad \Phi'_{12} = 2B\,\Phi_{11}\ , \qquad \Phi'_{22} = A^2\,\Phi_{22} + 4B\bar{B}\,\Phi_{11}\ .
\end{aligned} \tag{51}$$

Therefore, the effect on the amplitudes introduced in (46), (47) is such that the Coulomb components $\mathcal{C}$ and Φ_{11} of the fields remain the same, whereas the amplitudes $\mathcal{A}$ and Φ_{22} representing gravitational and pure radiation measured by the corresponding observer in the frame $\{\mathbf{e}'_{(a)}\}$ are rescaled by the factor A^2. Moreover, for geodesics with $\dot{\zeta} \neq 0$ there are "kinematic" contributions to $\mathcal{A}$ and Φ_{22} which involve B. Additional terms Ψ'_3 and Φ'_{12} proportional to B also enter the equation of geodesic deviation (43) in such a case through the contributions

$$\begin{aligned}
C'_{(1)(0)(3)(0)} &= \mathcal{R}e\,\Psi'_3\ , \quad C'_{(2)(0)(3)(0)} = -\mathcal{I}m\,\Psi'_3\ ,\\
S'_{(1)(3)} &= -2\,\mathcal{R}e\,\Phi'_{12}\ , \quad S'_{(2)(3)} = -2\,\mathcal{I}m\,\Phi'_{12}\ .
\end{aligned} \tag{52}$$

These are responsible for effects which relate the displacement in the longitudinal direction $\mathbf{e}'_{(3)}$ to acceleration in the transverse directions $\mathbf{e}'_{(1)}$, $\mathbf{e}'_{(2)}$, and vice versa.

Note finally (cf. the general analysis in [33]) that the interpretation frame (49) is *parallelly transported* along the geodesic if and only if $(Q/P)_{\bar\zeta} = 0 = (Q/P)_{\zeta}$. This condition is equivalent to vanishing of the spin coefficients $\tau = 0 = \pi$, i.e. to the choice $P = Q = 1 + \alpha\,\zeta\bar\zeta$. Therefore, the tetrad is parallelly transported along all timelike geodesics in the spacetimes of the form (36) which are direct product of two constant curvature 2-spaces, in particular in Minkowski, Bertotti–Robinson, (anti-)Nariai, Plebański–Hacyan spaces, and in all the corresponding spacetimes discussed in this section, representing radiation on these backgrounds.

5 Concluding remarks

An explicit family (1), (2) of the Kundt non-diverging class has been presented which depends on three constant parameters α, β, ε and three structural functions $C(u)$, $D(u)$ and $H(\zeta,\bar\zeta,u)$. This contains many particular solutions of Petrov types O, D, N and II which, for specific choices of the constants and functions, admit a cosmological constant, electromagnetic field, pure radiation and/or general non-null matter component. Among these are, for example, the conformally flat Minkowski and (anti-)de Sitter Einstein spaces, the Edgar–Ludwig pure radiation solutions or the Bertotti–Robinson electrovacuum universe. The type D solutions contain the (anti-)Nariai, Plebański–Hacyan or other direct product electrovacuum spacetimes, and their generalizations (see section 3 and table 2). Moreover, all these O and D solutions can naturally be understood as "backgrounds" for the corresponding type N and II radiative spacetimes, respectively. As described in section 4, this is achieved simply by considering an additional term $-(Q/P)\,H^w\,\mathrm{d}u^2$ in the metric, which is proportional to the function $H^w(\zeta,\bar\zeta,u)$. Since the coordinate u is the retarded time, exact gravitational waves of *arbitrary profiles* which propagate in the corresponding background universes can thus easily be introduced.

In particular, it is straightforward to construct *sandwich waves* by considering the function of the form $H^w = h(\zeta,\bar\zeta)\,d(u)$, where the profile $d(u)$ is non-vanishing only on some finite interval representing the wave-zone between two non-radiative background parts in front and behind the wave. Of course, the background need not be flat as in the case of well-known sandwich pp-waves [7], but can be of *any* kind summarized in table 2 — the universe in which the sandwich wave propagates can be curved and filled with various matter contents, such as the electromagnetic field and/or pure radiation.

Also, it is possible to construct *impulsive limits* of such sandwich waves by considering the profile $d(u)$ to approach the Dirac delta distribution $\delta(u)$. When the background is Minkowski or (anti-)de Sitter universe, non-expanding impulsive gravitational waves in spacetimes of constant curvature are recovered, see e.g. [42] for a review. Impulsive waves of this type in the Nariai universe have recently been found and studied in [30], and in all direct product spacetimes (including anti-Nariai and Bertotti–Robinson [30]) in [29].

The family of Kundt's solutions presented in this paper thus represents an interesting generalization of some previously known but "separated" results. Its main advantage is that various particular background and radiative spacetimes are described using a unified, simple and explicit metric form. Of course, most of the special subclasses are well-known. However, we have discussed here all the possible Petrov types and matter contents systematically, identifying some spacetimes which have been overlooked in the literature so far. For example, the type D solution (32) representing pure radiation in (anti-)Nariai universe, which seems to be a counterexample to the conjecture proposed in [26], has been found. In subsection 4.3 we have also presented a new explicit type II electrovacuum solution which describes electromagnetic and gravitational waves propagating in the (anti-)Nariai background.

Finally, it is worth mentioning that some of the solutions presented above belong to the recently obtained class of spacetimes for which all of the scalar curvature invariants vanish [43]. They may play an important role since these are exact solutions in string theory to all perturbative orders [44]. In fact, *all* type N and type O spacetimes with vanishing invariants with $\Phi_{12} = 0$ are contained in our family of solutions (1), (2).

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Appendix. Conformally flat Einstein–Maxwell fields with Λ

All conformally flat electrovac metrics with a vanishing cosmological constant Λ have been known for a long time [45–48]. Demanding a spacetime to have a vanishing Weyl tensor and to satisfy the Einstein–Maxwell equations is so a severe restriction that one is left with only two possibilities, according to the algebraic type of the Maxwell field. For a *non-null* Maxwell field one obtains the Bertotti–Robinson [9–11] universe. This is given by the metric (25) or, alternatively, in a different coordinate system such that $\zeta = \sqrt{2}\, a\, e^{i\phi}\tan(\theta/2)$, $a^2 = 1/2\alpha$,

$$\begin{aligned} t &= \frac{a(-vu^2+2ua^2+2va^2)}{-vu^2+2ua^2-2va^2-2\sqrt{2}\,avu+2\sqrt{2}\,a^3}\ , \\ r &= \frac{2\sqrt{2}\,a^4}{-vu^2+2ua^2-2va^2-2\sqrt{2}avu+2\sqrt{2}\,a^3}\ , \end{aligned}$$

it takes form

$$\mathrm{d}s^2 = \frac{a^2}{r^2}\left[\,-\mathrm{d}t^2+\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,\right]\ , \tag{A1}$$

which is explicitly conformally flat. Alternatively, for a *null* field the solution is represented by the special pp-wave [5,6] discussed in section 3.1

$$\mathrm{d}s^2 = 2\,\mathrm{d}\zeta\mathrm{d}\bar\zeta - 2\,\mathrm{d}u\mathrm{d}v - A_2(u)\zeta\bar\zeta\,\mathrm{d}u^2\ , \tag{A2}$$

where $A_2(u)$ is an arbitrary (positive) function.

Here we consider a more general problem by admitting $\Lambda \neq 0$, i.e. to find all possible conformally flat solutions of the Einstein–Maxwell system

$$\begin{aligned} &R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab} = \tfrac{1}{2}F_a{}^c\bar F_{bc}\ , \\ &F^{ab}{}_{;b} = 0\ , \end{aligned} \tag{A3}$$

where F_{ab} denotes the complex self-dual Maxwell tensor. We prove, in fact, that any such solution necessarily requires $\Lambda = 0$, and thus it reduces to either (A1) or (A2). Note that this was already asserted by Khlebnikov and Shelkovenko [49] but their result seems to have been somewhat overlooked. Moreover, they did not present an explicit proof. We employ the Newman–Penrose (NP) formalism, and refer to [12] for the complete set of general equations. Of course, conformal flatness is equivalent to setting all components of the Weyl tensor to zero, $\Psi_i = 0$ for $i = 0,1,2,3,4$, whereas the tracelessness of T_{ab} requires $R = 4\Lambda =$ constant. We now separately treat the two possible cases of non-null and null Maxwell fields.

Non-null fields

Given an arbitrary non-null electromagnetic field, a null tetrad $(\mathbf{m},\bar{\mathbf{m}},\mathbf{l},\mathbf{k})$ can always be chosen in which the Maxwell tensor takes the simple form $F_{ab} = 4\Phi_1(m_{[a}\bar m_{b]} - k_{[a}l_{b]})$. Then, the only non-trivial Ricci scalar is $\Phi_{11} \equiv \Phi_1\bar\Phi_1 \neq 0$. From the Bianchi identities one obtains that the following spin coefficients vanish,

$$\kappa = \sigma = \lambda = \nu = \rho = \mu = \pi = \tau = 0\ . \tag{A4}$$

Substituting these into the NP equations, it immediately follows $R = 0$. Thus, there are no such solutions with $\Lambda \neq 0$. Notice also that the Maxwell equations have not been actually used, but only the specific algebraic structure of the energy-momentum tensor.

Null fields

In this case it is possible to generalize the approach used in [48]. Without loss of generality, we introduce a convenient null tetrad in which an arbitrary null Maxwell field takes the form $F_{ab} = 4\Phi_2 k_{[a}m_{b]}$, so that $\Phi_{22} \equiv \Phi_2\bar{\Phi}_2 \neq 0$ is the only scalar. The Bianchi identities now imply

$$\kappa = \sigma = \rho = 0 \ , \tag{A5}$$
$$D\Phi_{22} = -2(\varepsilon + \bar{\varepsilon})\Phi_{22} \ , \tag{A6}$$
$$\bar{\delta}\Phi_{22} = (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22} \ . \tag{A7}$$

For our purposes, only three of the NP equations turn out to be necessary. Taking (A5) into account, these read

$$\delta\alpha - \bar{\delta}\beta = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \varepsilon(\mu - \bar{\mu}) + R/24 \ , \tag{A8}$$
$$\delta\tau = (\tau + \beta - \bar{\alpha})\tau \ , \tag{A9}$$
$$\bar{\delta}\tau = (\bar{\tau} - \bar{\beta} + \alpha)\tau + R/12 \ . \tag{A10}$$

The Maxwell equations simplify to

$$D\Phi_2 = -2\varepsilon\Phi_2 \ , \tag{A11}$$
$$\delta\Phi_2 = (\tau - 2\beta)\Phi_2 \ . \tag{A12}$$

From (A7) and (A12), it follows

$$\bar{\delta}\Phi_2 = -2\alpha\Phi_2 \ . \tag{A13}$$

With (A10)–(A13) and (A8), the commutator $(\bar{\delta}\delta - \delta\bar{\delta}) = (\bar{\mu} - \mu)D - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta$ applied on Φ_2 gives

$$\tau\bar{\tau} = -R/6 \ . \tag{A14}$$

Since $\tau\bar{\tau}$ must thus be a constant, it follows $\delta(\tau\bar{\tau}) = 0$. Recalling (A9) and (A10), this immediately leads to $0 = (2\tau\bar{\tau} + R/12)\tau = -R\,\tau/4$ and therefore to $R = 0$, which completes the proof for null electromagnetic fields.

The above results, which generalize the Theorems 32.16 and 32.17 of [12][2] to include a cosmological constant Λ, can be summarized in the following

Theorem. *There are no conformally flat solutions of the Einstein–Maxwell equations (A3) for $\Lambda \neq 0$. When $\Lambda = 0$, the only solutions are given by (A1) and (A2).*

Let us emphasize that the Maxwell equations (contrary to the case of non-null field) do play a key role in preventing the existence of conformally flat solutions with a *null* electromagnetic field for $\Lambda \neq 0$. Indeed, there exist conformally flat metrics with an arbitrary Λ and generic (i.e., non-electromagnetic) pure radiation, as demonstrated in section 3.1. A complete class of these spacetimes was presented in [15] (see also [51] for the case $\Lambda > 0$). For $\Lambda = 0$, such a general family splits into two invariant sub-classes, namely the Edgar–Ludwig metric [19] (still not representing electromagnetic fields) and the electromagnetic plane waves (A2).

References

[1] Kundt W 1961 *Z. Phys.* **163** 77

[2] Kundt W 1962 *Proc. R. Soc.* A **270** 328

[2]Notice that Theorem 32.17 in [12] by mistake includes not only null electromagnetic fields but also generic pure radiation. This was pointed out in [50], whereas the complete class of conformally flat pure radiation metrics with $\Lambda = 0$ was given in [19].

[3] Ehlers J and Kundt W 1962 *Gravitation: An Introduction to Current Research*, ed L Witten (New York: Wiley) pp 49–101

[4] Brinkmann H W 1925 *Math. Ann.* **94** 119

[5] Baldwin O R and Jeffery G B 1926 *Proc. R. Soc.* A **111** 95

[6] Brdička M 1951 *Proc. Roy. Irish Acad.* **A54** 137

[7] Bondi H, Pirani F A E and Robinson I 1959 *Proc. R. Soc.* A **251** 519

[8] Nariai H 1951 *Sci. Rep. Tôhoku Univ.* **35** 62

[9] Levi-Civita T 1917 *Rend. Acc. Lincei* **26** 519

[10] Bertotti B 1959 *Phys. Rev.* **116** 1331

[11] Robinson I 1959 *Bull. Acad. Polon.* **7** 351

[12] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

[13] MacCallum M A H and Siklos S T C 1992 *J. Geom. Phys.* **8** 221

[14] Carminati J and Zakhary E 1999 *Class. Quantum Grav.* **16** 3221

[15] Ozsváth I, Robinson I and Rózga K 1985 *J. Math. Phys.* **26** 1755

[16] Bičák J and Podolský J 1999 *J. Math. Phys.* **40** 4495

[17] Siklos S 1985 *Galaxies, Axisymmetric Systems and Relativity*, ed M A H MacCallum (Cambridge: Cambridge University Press) pp 247–274

[18] Podolský J 1998 *Class. Quantum Grav.* **15** 719

[19] Edgar S B and Ludwig G 1997 *Class. Quantum Grav.* **14** L65

[20] Edgar S B and Ludwig G 1997 *Class. Quantum Grav.* **14** 3453

[21] Skea J E F 1997 *Class. Quantum Grav.* **14** 2393

[22] Griffiths J B and Podolský J 1998 *Class. Quantum Grav.* **15** 3863

[23] Barnes A 2001 *Class. Quantum Grav.* **18** 5287

[24] Podolský J 1993 *On exact radiative space-times with cosmological constant* PhD dissertation Charles University in Prague

[25] Van den Bergh N 1989 *Class. Quantum Grav.* **6** 1871

[26] Wils P and Van den Bergh N 1990 *Class. Quantum Grav.* **7** 577

[27] Cahen M and Defrise L 1968 *Commun. Math. Phys.* **11** 56

[28] Plebański J F and Hacyan S 1979 *J. Math. Phys.* **20** 1004

[29] Ortaggio M and Podolský J 2002 *Class. Quantum Grav.* **19** 5221

[30] Ortaggio M 2002 *Phys. Rev.* D **65** 084046

[31] Taub A H 1981 *Ann. Phys.* **134** 326

[32] Balasin H 2000 *Class. Quantum Grav.* **17** 1913

[33] Bičák J and Podolský J 1999 *J. Math. Phys.* **40** 4506

[34] Lewandowski J 1992 *Class. Quantum Grav.* **9** L147

[35] García D A and Alvarez C M 1984 *Nuovo Cimento* B **79** 266

[36] Khlebnikov V I 1986 *Class. Quantum Grav.* **3** 169

[37] Pirani F A E 1956 *Acta Phys. Polon.* **15** 389

[38] Pirani F A E 1957 *Phys. Rev.* **105** 1089

[39] Szekeres P 1965 *J. Math. Phys.* **6** 1387

[40] Szekeres P 1966 *J. Math. Phys.* **7** 751

[41] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)

[42] Podolský J 2002 *Gravitation: Following the Prague Inspiration*, eds O Semerák, J Podolský and M Žofka (Singapore: World Scientific) pp 205–246 [gr-qc/0201029]

[43] Pravda V, Pravdová A, Coley A and Milson R 2002 *Class. Quantum Grav.* **19** 6213

[44] Coley A A 2002 *Phys. Rev. Lett.* **89** 281601

[45] Cahen M and Leroy J 1966 *J. Math. Mech.* **16** 501

[46] Stephani H 1967 *Commun. Math. Phys.* **5** 337

[47] Tariq N and Tupper B O J 1974 *J. Math. Phys.* **15** 2232

[48] McLenaghan R G, Tariq N and Tupper B O J 1975 *J. Math. Phys.* **16** 829

[49] Khlebnikov V I and Shelkovenko A É 1976 *Sov. Phys. J.* **19** 960

[50] Wils P 1989 *Class. Quantum Grav.* **6** 1243

[51] Van den Bergh N, Gunzig E and Nardone P 1990 *Class. Quantum Grav.* **7** L175