Generalised Kundt waves and their physical interpretation

J. B. Griffiths[1,*], P. Docherty[1] and J. Podolský[2†]

[1] Department of Mathematical Sciences, Loughborough University, Loughborough, Leics. LE11 3TU, U.K.

[2] Institute of Theoretical Physics, Charles University in Prague, V Holešovičkách 2, 18000 Prague 8, Czech Republic.

September 26, 2003

Abstract

We present the complete family of space-times with a non-expanding, shear-free, twist-free, geodesic principal null congruence (Kundt waves) that are of algebraic type III and for which the cosmological constant (Λ_c) is non-zero. The possible presence of an aligned pure radiation field is also assumed. These space-times generalise the known vacuum solutions of type N with arbitrary Λ_c and type III with $\Lambda_c = 0$. It is shown that there are two, one and three distinct classes of solutions when Λ_c is respectively zero, positive and negative. The wave surfaces are plane, spherical or hyperboloidal in Minkowski, de Sitter or anti-de Sitter backgrounds respectively, and the structure of the family of wave surfaces in the background space-time is described. The weak singularities which occur in these space-times are interpreted in terms of envelopes of the wave surfaces.

1 Introduction

The families of plane and *pp*-waves are among the best known exact solutions of Einstein's equations. They describe plane-fronted waves with parallel rays which propagate in a Minkowski background. A more general family of plane-fronted waves is also well known [1] and described as Kundt's class. This is characterised by the fact that it possesses a repeated principal null geodesic congruence which is non-expanding and has zero twist and shear. In the original papers of Kundt [2]–[3] it was shown that, for zero cosmological constant but permitting an aligned pure radiation field, these space-times are of algebraic types III, N or O. Together, these are characterised as the only radiation fields which possess plane wave surfaces. The type D space-times which satisfy the above criteria, and some type II space-times, are also known (see [1] and references cited therein).

The introduction of a non-zero cosmological constant (Λ_c) to the above criteria, however, is not so easily achieved as it is for the equivalent (Robinson–Trautman) class of expanding waves. Nevertheless, families of non-expanding type N vacuum space-times with $\Lambda_c \neq 0$ were found by García Díaz and Plebański [4] and analysed further by Ozsváth, Robinson and Rózga [5] and by Bičák and Podolský [6]–[7]. Also, Podolský and Ortaggio [8] have described a large family of explicit Kundt type II and N solutions with $\Lambda_c \neq 0$ and interpreted them as gravitational waves in various type D and O universes. However, no type III solutions with a non-zero cosmological constant have been studied. One purpose of the present paper is to fill this gap. In fact, in

*E–mail: J.B.Griffiths@Lboro.ac.uk

†E–mail: Podolsky@mbox.troja.mff.cuni.cz

doing so, we also find a new family of type N solutions for which the cosmological constant is necessarily non-zero. The main purpose of the paper, however, is to present a comprehensive derivation of all type III solutions within this class with a non-zero cosmological constant and an aligned pure radiation field, and to describe some aspects of their physical interpretation at least in the weak field limit.

2 The line element for Kundt's class

To derive all space-times of this class, we will employ the Newman–Penrose (NP) formalism [9] and the notations associated with it. We start with the assumption that the space-time is of algebraic type III, and choose the tetrad vector ℓ^μ to be everywhere tangent to the repeated principal null congruence. This immediately implies that $\Psi_0 = \Psi_1 = \Psi_2 = 0$. In general, we will also include a cosmological constant using the NP notation $\Lambda = \frac{1}{6}\Lambda_c$ and an aligned pure radiation field such that $R_{\mu\nu} = -2\Phi_{22}\ell_\mu\ell_\nu$.

With the above assumptions, the repeated principal null congruence is necessarily geodesic and shear-free so that $\kappa = \sigma = 0$. To obtain Kundt's class of solutions, we make the additional assumption that the repeated principal null congruence is also expansion-free and twist-free, so that $\rho = 0$. We are free to choose an affine parameter r along the congruence and, since ℓ_μ must be proportional to a gradient, we use the tetrad freedom to set ℓ_μ equal to a gradient. Using also a freedom in another tetrad vector, this produces the further constraints on the spin coefficients $\epsilon = 0$ and $\bar\alpha + \beta = \tau$.

We adopt coordinates which are labelled such that $r = x^1$, $\zeta = \frac{1}{\sqrt{2}}(x^2 + ix^3)$ and $u = x^4$ and such that $\ell_\mu = \delta^4_\mu$ and $\ell^\mu = \delta^\mu_1$. We then use the available coordinate and tetrad freedoms to put $\pi = -\bar\tau$ and $\lambda = 0$ and to simplify the tetrad to the form

$$\begin{aligned}
\ell^\mu &= \delta^\mu_1 & D &= \frac{\partial}{\partial r} \\
m^\mu &= -P\bar W\,\delta^\mu_1 + \tfrac{1}{\sqrt{2}}P\,(\delta^\mu_2 + i\delta^\mu_3) & \delta &= P\left(\frac{\partial}{\partial\zeta} - \bar W\,\frac{\partial}{\partial r}\right) \\
n^\mu &= -H\,\delta^\mu_1 + \delta^\mu_4 & \Delta &= \frac{\partial}{\partial u} - H\,\frac{\partial}{\partial r}
\end{aligned} \tag{1}$$

where P and H are real functions, W is a complex function, and $P_{,r} = 0$. The line element then has the form

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}r + H\mathrm{d}u + W\mathrm{d}\zeta + \bar W\mathrm{d}\bar\zeta) - 2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta. \tag{2}$$

(The details to this point are given in [1].)

We now note that the Ricci identity $D\tau = 0$ implies that $\tau = \tau^\circ$, in which we are using the standard notation that a degree sign indicates any function (except P) which is independent of r. One of the metric equations can then immediately be integrated to give

$$W = \frac{2\bar\tau^\circ}{P}\,r + W^\circ, \tag{3}$$

where W° is arbitrary. Another, together with one of the Ricci identities, implies that

$$H = -(\tau^\circ\bar\tau^\circ + \Lambda)r^2 + 2G^\circ r + H^\circ \tag{4}$$

where G° and H° are real functions. It can also be shown that $\beta = \frac{1}{2}(\tau^\circ - P_{,\bar\zeta})$.

3 Wave surfaces

For any metric in the form (2), the surfaces given by $u =$ const. are spacelike surfaces on which $\mathrm{d}s^2 = -2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta$. The vector fields $P\partial_\zeta$ and $P\partial_{\bar\zeta}$ are tangent to these surfaces which are orthogonal to the repeated principal null direction ℓ^μ. They are therefore regarded as "wave surfaces" associated with waves propagating in the null direction ℓ^μ.

One of the field equations immediately implies that

$$P^2(\log P)_{,\zeta\bar\zeta}=\Lambda, \tag{5}$$

which ensures that the Gaussian curvature of the wave surfaces given by $K=2P^2(\log P)_{,\zeta\bar\zeta}$ is constant with a value that is directly related to the cosmological constant. Also, it can be shown that a general solution of (5) is given by

$$P=(1+\Lambda F\bar F)(F_\zeta\bar F_{\bar\zeta})^{-1/2} \tag{6}$$

where $F=F(\zeta,u)$ is an arbitrary complex function, holomorphic in ζ. It is then possible to use the coordinate freedom $\zeta\to\tilde\zeta(u,\zeta)$ to put $F=\zeta$ so that, in general

$$P=1+\Lambda\zeta\bar\zeta. \tag{7}$$

This form will generally be assumed below.

It may be noted that (5) also admits the particular solution

$$P=\sqrt{-\Lambda}\,(\zeta+\bar\zeta), \tag{8}$$

which is valid only for a negative cosmological constant. However, using tilded coordinates for (8), this is related to the standard form (7) by the transformation

$$\sqrt{-\Lambda}\,\zeta=\frac{1-\tilde\zeta}{1+\tilde\zeta},\qquad\qquad \tilde\zeta=\frac{1-\sqrt{-\Lambda}\,\zeta}{1+\sqrt{-\Lambda}\,\zeta}. \tag{9}$$

The use of the form (8) will be clarified in section 9 below.

4 Coordinate freedoms

There are a number of remaining coordinate freedoms that may still be employed to simplify expressions.

First, there is the ability to alter the origin of the affine parameter on each geodesic

$$r\to\tilde r=r+R(\zeta,\bar\zeta,u), \tag{10}$$

under which

$$\begin{aligned}\tilde W^\circ&=W^\circ-R_{,\zeta}-\frac{2\bar\tau^\circ R}{P},\\ \tilde G^\circ&=G^\circ+(\tau^\circ\bar\tau^\circ+\Lambda)R,\\ \tilde H^\circ&=H^\circ-(\tau^\circ\bar\tau^\circ+\Lambda)R^2-2G^\circ R-R_{,u}.\end{aligned}$$

Secondly, there is a freedom in the choice of null parameter

$$u\to\tilde u=w(u),\qquad r\to\tilde r=\frac{1}{w_{,u}}r \tag{11}$$

under which

$$\tilde W=\frac{1}{w_{,u}}W,\qquad \tilde H=\frac{1}{w_{,u}^2}H+\frac{w_{,uu}}{w_{,u}^3}\,r.$$

Finally, for the choice (7) it may be noted that, when p and q are arbitrary complex functions of u, the transformation

$$\zeta \to \tilde\zeta = \frac{\bar q + p\zeta}{\bar p - \Lambda q\zeta}, \tag{12}$$

leaves the component $2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta$ invariant, but induces changes in H and W. Specifically

$$W^\circ \to \tilde W^\circ = \frac{(\bar p - \Lambda q\zeta)^2}{p\bar p + \Lambda q\bar q}\left[W^\circ + \frac{m + in\bar\zeta + \Lambda\bar m\bar\zeta^2}{P^2}\right], \qquad \tilde\tau^\circ = \left(\frac{p - \Lambda\bar q\bar\zeta}{\bar p - \Lambda q\zeta}\right)\tau^\circ,$$

where

$$m = \frac{(pq_{,u} - qp_{,u})}{p\bar p + \Lambda q\bar q}, \qquad n = i\frac{(\bar p p_{,u} - p\bar p_{,u} + \Lambda\bar q q_{,u} - \Lambda q\bar q_{,u})}{p\bar p + \Lambda q\bar q}.$$

Similarly, for the choice (8), we can use the transformation $\zeta \to \tilde\zeta = p\zeta + iq$, where p and q are then real functions of u.

5 Integration for τ° and its canonical forms

Let us first consider the Ricci identity $\bar\delta\tau = 2\tau(\bar\tau - \bar\beta) + 2\Lambda$, which now becomes

$$\left(\frac{\tau^\circ}{P}\right)_{,\zeta} = \frac{\tau^\circ\bar\tau^\circ + 2\Lambda}{P^2}. \tag{13}$$

It is immediately clear that τ° cannot be zero for a non-zero cosmological constant. The fact that the right hand side is real implies that there exists a real function $\phi(\zeta,\bar\zeta,u)$ such that

$$\tau^\circ = P\,\phi_{,\bar\zeta}. \tag{14}$$

For the general case in which P is given by (7), the Ricci identity $\delta\tau = 2\tau\beta$ can then be integrated to show that

$$\phi = \log P - \log Q, \tag{15}$$

where $Q = a + \bar b\zeta + b\bar\zeta - a\Lambda\zeta\bar\zeta$, and a and b are arbitrary real and complex functions of u respectively. We thus obtain

$$\tau^\circ = \frac{-b + 2a\Lambda\zeta + \bar b\Lambda\zeta^2}{a + \bar b\zeta + b\bar\zeta - a\Lambda\zeta\bar\zeta}. \tag{16}$$

Let us immediately note that, if $b = 0$, we have

$$\textbf{Case 1}: \qquad \tau^\circ = \frac{2\Lambda\zeta}{(1 - \Lambda\zeta\bar\zeta)} \tag{17}$$

Similarly, if $a = 0$, we can always make b real and then

$$\textbf{Case 2}: \qquad \tau^\circ = -\frac{(1 - \Lambda\zeta^2)}{\zeta + \bar\zeta} \tag{18}$$

These clearly reduce to the two standard cases which are well known when $\Lambda = 0$.

It may be seen in general that $\tau^\circ\bar\tau^\circ + \Lambda = (b\bar b + \Lambda a^2)\frac{P^2}{Q^2}$. (This expression occurs in the metric function H.) It is then convenient to separately label the coefficient

$$k = b\bar b + \Lambda a^2. \tag{19}$$

This has been identified by Ozsváth, Robinson and Rózga [5] as a quantity of invariant sign which can be used to assist in the classification of this family of solutions.

The expression (16) can now be simplified by use of the transformation (12). This leaves its general form invariant, but with coefficients transformed as

$$a \to \tilde a = (p\bar p - \Lambda q\bar q)a + \bar p q\,b + p\bar q\,\bar b, \qquad b \to \tilde b = -2\Lambda\bar p\bar q\,a + \bar p^2 b - \Lambda\bar q^2\bar b$$

so that $k \to \tilde k = (p\bar p + \Lambda q\bar q)^2 k$. The character of this transformation, however, depends on the invariant signs of the quantities Λ and k.

When $\Lambda = 0$, $k = b\bar{b}$. In this case, if $b \neq 0$, it is always possible to use the above transformation to put $a = 0$, so that (16) can be transformed to (18) and we arrive at case 2 which, for type N, are the Kundt waves. If $b = 0$, then $k = 0$ and $\tau^\circ = 0$ and, for type N, these are the *pp*-waves. This identifies two canonical types for these solutions:

$$\Lambda = 0 \qquad \begin{cases} k = 0 & \text{generalised } pp\text{-waves} \qquad \text{case 1} \\ k > 0 & \text{generalised Kundt waves} \qquad \text{case 2} \end{cases}$$

These are distinct solutions, which cannot be related to each other.

When $\Lambda > 0$, it is only possible for k to be positive. In this case, it is always possible to choose the function $\frac{p}{q}$ to put either $\tilde{a} = 0$ and $\tilde{b} \neq 0$, or $\tilde{a} \neq 0$ and $\tilde{b} = 0$. A further transformation using the freedom in p, with $q = 0$, can then be used to set either $b = 1$ or $a = 1$ respectively. Thus, τ° can always be transformed into either of the canonical forms (17) or (18). Indeed, it is possible to transform (17) into (18) or vice versa. These two forms, for this case, are completely equivalent. This family of solutions may therefore be considered to be a generalisation of either the Kundt waves or the *pp*-waves in the sense that they reduce to either of these forms for type N solutions in the appropriate limit depending on the particular coordinates adopted.

$$\Lambda > 0 \quad \Rightarrow \quad k > 0 \qquad \text{generalised } pp \text{ and Kundt waves} \qquad \text{cases 1 \& 2}$$

When $\Lambda < 0$: In this case there are three distinct possibilities which are identified by the sign of k.

1. If $k > 0$, then $b\bar{b} > -\Lambda a^2$, and it is always possible to use the above transformation to put $a = 0$ and hence to obtain generalised Kundt waves (case 2).
2. Similarly, if $k < 0$, $-\Lambda a^2 > b\bar{b}$ and it is always possible to use the transformation to put $b = 0$ to obtain generalised *pp*-waves (case 1).
3. However, another distinct case arises here when $k = 0$. This occurs, when $b = \sqrt{-\Lambda}e^{i\theta}a$ for any arbitrary function $\theta(u)$. In this case, the above transformation reduces to the following form:

$$\tilde{a} = (p + \sqrt{-\Lambda}e^{i\theta}q)(\bar{p} + \sqrt{-\Lambda}e^{-i\theta}\bar{q})\, a \qquad\qquad \tilde{b} = \sqrt{-\Lambda}e^{i\theta}(\bar{p} + \sqrt{-\Lambda}e^{-i\theta}\bar{q})^2 a$$

so that it is obviously not possible to reduce either a or b separately to zero. This particular case corresponds to that in which Q factorises, and hence τ° is given by

$$\tau^\circ = -\sqrt{-\Lambda}e^{i\theta}\left(\frac{1 + \sqrt{-\Lambda}e^{-i\theta}\zeta}{1 + \sqrt{-\Lambda}e^{i\theta}\bar{\zeta}}\right).$$

However, it is clearly possible to remove the phase $e^{i\theta}$, so that this additional case can be expressed in the canonical form

$$\textbf{Case 3}_{(\Lambda<0)}: \qquad \tau^\circ = -\sqrt{-\Lambda}\left(\frac{1 + \sqrt{-\Lambda}\,\zeta}{1 + \sqrt{-\Lambda}\,\bar{\zeta}}\right). \tag{20}$$

These solutions generalise those of Siklos [10] using a different coordinate system.
Thus, in this case, there exist three distinct canonical types:

$$\Lambda < 0 \qquad \begin{cases} k < 0 & \text{generalised } pp\text{-waves} \qquad \text{case 1} \\ k = 0 & \text{generalised Siklos waves} \qquad \text{case 3} \\ k > 0 & \text{generalised Kundt waves} \qquad \text{case 2} \end{cases}$$

For an analysis of local exact solutions, it is sufficient to consider separately the three particular cases listed above: i.e. P is given by (7) and τ by (17), (18) or (20). However, since $a(u)$ and $b(u)$ in (16) are arbitrary functions, it is possible also to construct composite space-times in which these functions are non-zero for different ranges of u. In the remainder of this paper, however, equations are expressed in forms which apply to any of the particular cases above. It is only assumed that τ° is independent of u.

6 Integration for G° and W°

The metric equations imply that γ and μ have the forms $\gamma = -(\tau^\circ\bar\tau^\circ + \Lambda)r + G^\circ + \frac{1}{2}im^\circ$ and $\mu = im^\circ$ where

$$m^\circ = \tfrac{1}{2}iP^2[W^\circ{}_{,\bar\zeta} - \bar W^\circ{}_{,\zeta}] + iP(\tau^\circ W^\circ - \bar\tau^\circ \bar W^\circ). \tag{21}$$

The r component of the Ricci identity $\delta\gamma - \Delta\beta = \mu\tau$ for this case vanishes identically, and the remaining part becomes

$$\left(G^\circ + \tfrac{1}{2}im^\circ\right)_{,\bar\zeta} = -(\tau^\circ\bar\tau^\circ + \Lambda)\bar W^\circ + \frac{im^\circ\tau^\circ}{P}.$$

Using the Ricci identity $(P\tau^\circ)_{,\bar\zeta} = \tau^{\circ 2}$, this equation can be rewritten in the form

$$\left[G^\circ - \frac{P^2}{4}\left[W^\circ{}_{,\bar\zeta} - \bar W^\circ{}_{,\zeta}\right] + \frac{P}{2}(\tau^\circ W^\circ + \bar\tau^\circ \bar W^\circ)\right]_{,\bar\zeta} = \frac{\tau^\circ P}{2}\left[W^\circ{}_{,\bar\zeta} + \bar W^\circ{}_{,\zeta}\right] - \Lambda \bar W^\circ. \tag{22}$$

At this point, we can use the coordinate freedom (10) to set

$$W^\circ{}_{,\bar\zeta} + \bar W^\circ{}_{,\zeta} = 0. \tag{23}$$

It can be seen that any further freedom in R must be constrained by the condition

$$P^2 R_{,\zeta\bar\zeta} + P(\tau^\circ R_{,\zeta} + \bar\tau^\circ R_{,\bar\zeta}) + 2(\tau^\circ\bar\tau^\circ + 2\Lambda)R = 0. \tag{24}$$

The condition (23) implies that there exists a real function $X(\zeta,\bar\zeta,u)$ such that

$$W^\circ = -i\,X_{,\zeta}.$$

With this, the identity (22) can be integrated to give

$$G^\circ + \tfrac{1}{2}P(\tau^\circ W^\circ + \bar\tau^\circ\bar W^\circ) + \tfrac{1}{2}i\left[P^2 X_{,\zeta\bar\zeta} + 2\Lambda X\right] = -2\Lambda f(\zeta,u), \tag{25}$$

where $f(\zeta,u)$ is an arbitrary function, and the factor -2Λ has been added for convenience. (It may be noted that any function f which is independent of ζ may be removed by an appropriate transformation.) The real and imaginary parts of (25) are respectively

$$G^\circ = -\frac{P}{2}(\tau^\circ W^\circ + \bar\tau^\circ \bar W^\circ) - \Lambda(f + \bar f) \tag{26}$$

and

$$P^2 X_{,\zeta\bar\zeta} + 2\Lambda X = 2i\Lambda(f - \bar f),$$

but the remaining freedom (10) satisfying (24) can be used (provided $\tau^\circ \neq 0$) to set $X_{,\zeta\bar\zeta} = 0$. Thus, provided $\Lambda \neq 0$, we can set $X = i(f - \bar f)$ and

$$W^\circ = f_{,\zeta}, \tag{27}$$

which is independent of $\bar\zeta$. In fact, except for the case 1 solutions with $\Lambda = 0$, these solutions are described by the arbitrary holomorphic function $W^\circ(\zeta,u)$ (or $f(\zeta,u)$), together with the arbitrary real function $H^\circ(\zeta,\bar\zeta,u)$.

7 The curvature tensor components

With the above conditions, we now have $\mu = im^\circ = -P(\tau^\circ W^\circ - \bar\tau^\circ \bar W^\circ)$. The only spin coefficient which has not yet been specified is ν, which can be expressed as $\nu = n^\circ r + \nu^\circ$, where

$$n^\circ = -\tau^\circ(P^2W^\circ)_{,\zeta} - 2\Lambda PW^\circ + P\bar\tau^\circ(\tau^\circ W^\circ - \bar\tau^\circ\bar W^\circ)$$

and

$$\nu^\circ = PH^\circ_{,\zeta} + 2\bar\tau^\circ H^\circ - PW^\circ_{,u} + P^2W^\circ(\tau^\circ W^\circ + \bar\tau^\circ\bar W^\circ) + 2\Lambda PW^\circ(f+\bar f).$$

The remaining Ricci identities give expressions for the non-zero components of the curvature tensor. After substituting the above relations, we obtain

$$\Psi_3 = -\tau^\circ(P^2W^\circ)_{,\zeta} - 2\Lambda PW^\circ \tag{28}$$

$$\begin{aligned}\Psi_4 = \Big[&- P\tau^\circ(P^2W^\circ)_{,\zeta\zeta} + 2(\tau^\circ\bar\tau^\circ - \tau^\circ P_{,\zeta} - 2\Lambda)(P^2W^\circ)_{,\zeta} + 6\Lambda P\bar\tau^\circ W^\circ\Big] r \\ &+P^2H^\circ_{,\zeta\zeta} + 2P(\bar\tau^\circ + P_{,\zeta})H^\circ_{,\zeta} + 2\bar\tau^{\circ 2}H^\circ - (P^2W^\circ)_{,u\zeta} \\ &+[3P\tau^\circ W^\circ + P\bar\tau^\circ\bar W^\circ + 2\Lambda(f+\bar f)](P^2W^\circ)_{,\zeta} \\ &+2P^2\bar\tau^{\circ 2}W^\circ\bar W^\circ + 6\Lambda P^2W^{\circ 2}\end{aligned} \tag{29}$$

$$\begin{aligned}\Phi_{22} = P^2H^\circ_{,\zeta\bar\zeta} &+ P\tau^\circ H^\circ_{,\zeta} + P\bar\tau^\circ H^\circ_{,\bar\zeta} + 2(\tau^\circ\bar\tau^\circ + 2\Lambda)H^\circ \\ &+P\tau^\circ\bar W^\circ(P^2W^\circ)_{,\zeta} + P\bar\tau^\circ W^\circ(P^2\bar W^\circ)_{,\bar\zeta} + 2P^2(\tau^\circ\bar\tau^\circ + 3\Lambda)W^\circ\bar W^\circ\end{aligned} \tag{30}$$

In these expressions, P is given by (7), and τ° can take any of its canonical forms (17), (18) or (20). They contain an arbitrary holomorphic function $W^\circ(\zeta,u)$ (or $f(\zeta,u)$), and an arbitrary real function $H^\circ(\zeta,\bar\zeta,u)$. For case 2, these reduce to the known expressions for the Kundt waves in the limit as $\Lambda\to 0$.

For the case 1 solutions, the limit as $\Lambda\to 0$ of the above expressions only leads to the type N *pp*-waves. Since $\tau^\circ = 0$ in this limit, it is not possible as above to set W° independent of $\bar\zeta$. In this case, however, the transformation (10) can alternatively be used to set $W^\circ = W^\circ(\bar\zeta,u)$ and the curvature tensor components with $\Lambda = 0$ and $\tau^\circ = 0$ can be expressed as

$$\begin{aligned}\Psi_3 &= \tfrac{1}{2}\bar W^\circ{}_{,\zeta\zeta} \\ \Psi_4 &= \tfrac{1}{2}\bar W^\circ{}_{,\zeta\zeta\zeta}\, r + H^\circ{}_{,\zeta\zeta} - W^\circ\bar W^\circ{}_{,\zeta\zeta} \\ \Phi_{22} &= H^\circ{}_{,\zeta\bar\zeta} - \tfrac{1}{2}(W^\circ W^\circ{}_{,\bar\zeta\bar\zeta} + \bar W^\circ\bar W^\circ{}_{,\zeta\zeta}) - \tfrac{1}{2}(W^\circ{}_{,u\bar\zeta} + \bar W^\circ{}_{,u\zeta}) - \tfrac{1}{2}\left(W^\circ{}_{,\bar\zeta}{}^2 + \bar W^\circ{}_{,\zeta}{}^2\right)\end{aligned} \tag{31}$$

8 The type N reductions with $\Lambda \neq 0$

It may first be noticed that the expression (28) for Ψ_3 can be written as

$$\Psi_3 = -\frac{P^2}{Q}(Q\tau^\circ W^\circ)_{,\zeta}. \tag{32}$$

This clearly vanishes (and the solution reduces to type N) when $Q\tau^\circ W^\circ = c\Lambda$, where c is an arbitrary complex function of u, so that

$$W^\circ = \frac{c\Lambda}{-b + 2a\Lambda\zeta + \bar b\Lambda\zeta^2}. \tag{33}$$

The multiple Λ has been included since this term can obviously be transformed away when $\Lambda = 0$. Further, when c is real and $k\neq 0$, the transformation (10) can be used to set both W° and G° to zero simultaneously. Thus, for case 1 and case 2 solutions, c may be purely imaginary. However, c may be generally complex for the case 3 solutions which occur when $\Lambda_c < 0$.

For this type N limit, it can be shown that the r component of Ψ_4 vanishes, as is required by the Bianchi identities, and general expressions for the non-zero components of the curvature tensor are

$$\begin{aligned}\Psi_4 = P^2H^\circ_{,\zeta\zeta} + 2P(\bar\tau^\circ + P_{,\zeta})H^\circ_{,\zeta} + 2\bar\tau^{\circ 2}H^\circ - (P^2W^\circ)_{,u\zeta} \\ +2P^2\frac{\bar\tau^\circ}{\tau^\circ}(\tau^\circ\bar\tau^\circ - \Lambda)W^\circ\bar W^\circ + 2\Lambda(f+\bar f)(P^2W^\circ)_{,\zeta}\end{aligned} \tag{34}$$

$$\Phi_{22} = P^2H^\circ_{,\zeta\bar\zeta} + P\tau^\circ H^\circ_{,\zeta} + P\bar\tau^\circ H^\circ_{,\bar\zeta} + 2(\tau^\circ\bar\tau^\circ + 2\Lambda)H^\circ + \frac{2kP^4}{Q^2}W^\circ\bar W^\circ \tag{35}$$

with W° given by (33). When $W^\circ = 0$ (i.e. when $c = 0$) and $\Lambda \neq 0$, these are the solutions that are described in [5] and [6]. However, when $c\Lambda \neq 0$, this identifies a new family of type N solutions that has not been identified in previous literature.

9 Alternative form for P when $\Lambda < 0$

Let us now consider the special case in which $\Lambda < 0$ and P is given by (8), i.e. $P = \sqrt{-\Lambda}\,(\zeta+\bar\zeta)$. In this case, the integration for τ° can be performed in a similar way to that above, and we obtain

$$\tau^\circ = \frac{\sqrt{-\Lambda}(\tilde a + 2i\tilde b\zeta - \tilde c\zeta^2)}{\tilde a + i\tilde b\zeta - i\tilde b\bar\zeta + \tilde c\zeta\bar\zeta}, \tag{36}$$

where $\tilde a$, $\tilde b$ and $\tilde c$ are arbitrary real functions of u. We can then use the coordinate freedom $\zeta \to \tilde\zeta = p\zeta + iq$, where p and q are real functions of u to simplify this expression. Specifically, if $\tilde c \neq 0$, we can set

$$\textbf{Case 4}: \qquad \tau^\circ = \frac{\sqrt{-\Lambda}(\epsilon - \zeta^2)}{\epsilon + \zeta\bar\zeta}, \tag{37}$$

where $\epsilon = 1, 0, -1$. Alternatively, if $\tilde c = 0$ and $\tilde b \neq 0$, we can set

$$\textbf{Case 5}: \qquad \tau^\circ = \frac{2\sqrt{-\Lambda}\,\zeta}{\zeta - \bar\zeta}. \tag{38}$$

Finally, if $\tilde c$ and $\tilde b$ both vanish, we obtain

$$\textbf{Case 6}: \qquad \tau^\circ = \sqrt{-\Lambda}. \tag{39}$$

These forms, however, can be related directly to those above using the transformation (9). This yields the expression (16) for τ° where

$$a = \tilde a + \tilde c, \qquad b = \sqrt{-\Lambda}(\tilde a - \tilde c + 2i\tilde b).$$

For **case 4**$_{(\Lambda<0)}$, we have $\tilde a = \epsilon$, $\tilde b = 0$, $\tilde c = 1$.
It is therefore evident that $a = \epsilon + 1$ and $b = \sqrt{-\Lambda}(\epsilon - 1)$. Thus

- the case when $\epsilon = 1$ belongs to **case 1** with $\Lambda < 0$.
- the case when $\epsilon = 0$ corresponds to **case 3**$_{(\Lambda<0)}$.
- the case when $\epsilon = -1$ belongs to **case 2** with $\Lambda < 0$.

For **case 5**$_{(\Lambda<0)}$, we have $\tilde a = 0$, $\tilde b = 1$, $\tilde c = 0$, so that $a = 0$ and $b = 2i\sqrt{-\Lambda}$.

- This case belongs to **case 2** with $\Lambda < 0$, but in a form in which ζ is replaced by $i\zeta$.

For **case 6**$_{(\Lambda<0)}$, we have $\tilde b = 0$, and $\tilde c = 0$, so that $a = \tilde a$ and $b = \sqrt{-\Lambda}\,\tilde a$.

- This case corresponds to **case 3**$_{(\Lambda<0)}$.

It may be noted that the apparent difference between case 4 with $\epsilon = 0$ and case 6 corresponds only to the phase of ζ which may always be reduced to zero. Similarly, a transformation exists which relates case 4 with $\epsilon = -1$ and case 5. (The equivalence of cases $4_{(\epsilon=-1)}$ and 5 is similar to the equivalence of cases 1 and 2 when $\Lambda > 0$.) Thus it is sufficient to consider just three distinct canonical types. These can be taken as the three types of case 4 with different values of ϵ or, preferably, case 4 with $\epsilon = 1$, case 5 and case 6. These canonical types were identified by Siklos [10], although he only investigated case 6.

10 Interpretation of the solutions

The solutions described above are radiative space-times in which the rays are non-expanding (as well has having zero shear and twist). Moreover, the wave surfaces have constant curvature proportional to the cosmological constant. However, the fact that τ is non-zero indicates that subsequent wave surfaces are locally rotated relative to each other. These general properties will be illustrated below.

Apart from the quantities Λ, P and τ°, the above family of solutions is essentially represented by an arbitrary holomorphic function $W^\circ(\zeta,u)$ (or $f(\zeta,u)$), and an arbitrary real function $H^\circ(\zeta,\bar\zeta,u)$. Our general method of approach will be to initially consider the weak field limit in which H° and W° (and f) are taken to be zero. This identifies the background in which weak fields of this type propagate. In this limit, the space-time is conformally flat and is Minkowski, de Sitter or anti-de Sitter according as the cosmological constant is respectively zero, positive or negative.

The background metric is given by (2) with

$$H = -(\tau^\circ\bar\tau^\circ + \Lambda)r^2, \qquad W = \frac{2\bar\tau^\circ}{P}\,r \tag{40}$$

for different values of Λ, and hence differing expressions for P and τ°. In these background space-times, it is possible to explicitly investigate the geometry of the waves surfaces and the way in which they foliate the space-time. This will also enable us to investigate the location and character of the singularities of the solutions in this weak field background.

This approach is also appropriate in the study of exact sandwich waves as it explicitly determines the geometries of the shock fronts and backs of the waves.

11 Waves in a Minkowski background

We first consider the case in which $\Lambda = 0$. Here, space-times of case 1, for which $\tau = 0$, are the *pp*-waves which have plane wave surfaces and parallel rays. They include plane waves. These solutions are well known and need not be further discussed here.

We therefore only need consider case 2, for which $\tau^\circ = -(\zeta+\bar\zeta)^{-1}$. Putting $\zeta = \frac{1}{\sqrt{2}}(x+iy)$, the line element (2) in the conformally flat limit (40) with $\Lambda = 0$ takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\left(\mathrm{d}r - \frac{r^2}{2x^2}\,\mathrm{d}u - \frac{2\,r}{x}\,\mathrm{d}x\right) - \mathrm{d}x^2 - \mathrm{d}y^2, \tag{41}$$

which is related to the cartesian form of Minkowski space

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2$$

by the transformation

$$\begin{aligned}
T &= \frac{1}{2x}\left(r + u^2r + 2ux^2\right) & \qquad u &= \frac{X \mp \sqrt{X^2+Z^2-T^2}}{T+Z} \\
Z &= \frac{1}{2x}\left(r - u^2r - 2ux^2\right) & r &= \pm(T+Z)\sqrt{X^2+Z^2-T^2} \\
X &= x + \frac{u\,r}{x} & x &= \pm\sqrt{X^2+Z^2-T^2} \\
Y &= y & y &= Y
\end{aligned} \tag{42}$$

It can be seen from (42) that the wave surfaces $u = u_0 = \text{const.}$ in this background space-time are given by

$$(1+u_0^2)T - 2u_0 X - (1-u_0^2)Z = 0,$$

which describes a family of null hyperplanes whose orientation varies for different values of u_0. Putting $u_0 = \tan(\alpha/2)$, where α is a constant on any wave surface, these can be written in the form

$$\sin\alpha\, X + \cos\alpha\, Z = T. \tag{43}$$

This clearly demonstrates that, at any time, successive wave surfaces $u = u_0$ are rotated about the Y axis as u_0 increases from $-\infty$ to $+\infty$ (or as α goes from $-\pi$ to $+\pi$). The rotation of these planes for different values of u_0 is consistent with the non-zero value of τ for these metrics.

For all this family of solutions, it may be noticed that the curvature components Ψ_3, Ψ_4 and Φ_{22} are unbounded when τ° is unbounded. This occurs when $\zeta + \bar\zeta = 0$ (or $x = 0$ in the above notation). In the background space-time, this singularity occurs on the hypersurface

$$X^2 + Z^2 = T^2, \tag{44}$$

which is an expanding cylinder centred on the Y-axis and whose radius expands with the speed of light. It may be noticed that each wave surface touches this cylinder on the line $X = \sin\alpha\, T$, $Z = \cos\alpha\, T$. Thus, the singularity on the expanding cylinder (44) can be interpreted as the caustic formed from the envelope of the family of wave surfaces.

It may also be observed that two plane wave surfaces (43) pass through each point outside the cylinder (44). However, this repetition must be excluded according to the assumption that the tangent vector ℓ^μ is unique. The complete family of wave surfaces should therefore be taken as the family of half-planes for which $x \geq 0$. These are illustrated in figure 1. The geometry of these wave surfaces in the background space-time is exactly the same as that of the pure radiation conformally flat case described in [11].

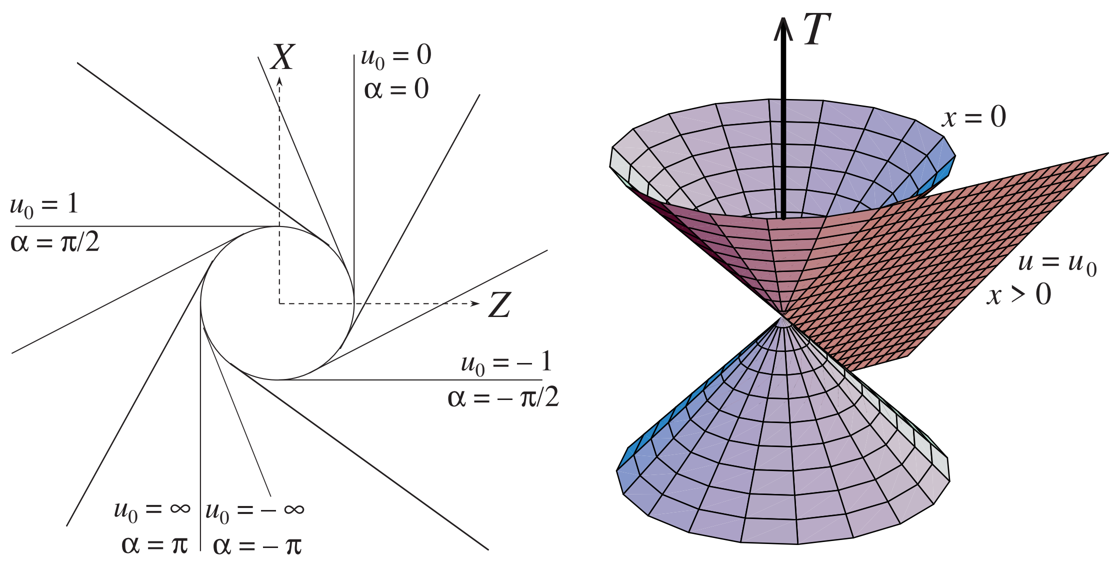

Figure 1: The geometry of the wave surfaces in the background Minkowski space-time.
(Left) For constant T and Y, the wave surfaces $u = u_0 = \text{const.}$ can be represented as a family of half-lines at a perpendicular distance T from an origin as indicated. The envelope of the lines forms a circle corresponding to the coordinate singularity at $x = 0$. As T increases, the circle expands and each wave surface propagates perpendicular to the tangent to that surface.
(Right) The half-plane wave surfaces are tangent to a 2-dimensional null cone on which $x = 0$. The picture illustrates the singularity and a single wave surface in a section with constant Y.

It may be observed that no wave surfaces pass through points that are inside the expanding cylinder (44). The coordinates used in (2) do not cover this part of the space-time.

Any observer outside the cylindrical envelope could detect a gravitational wave whose direction of propagation will rotate until it is reached by the singularity itself.

For this family of solutions, the scalar polynomial invariants all vanish. The singularity which occurs when $\zeta+\bar\zeta=0$ is therefore not a polynomial curvature singularity. Nevertheless, some curvature tensor components diverge and an observer approaching this singularity would experience unbounded tidal forces.

Finally, it should be pointed out that the complete solution is time-symmetric, so that the envelope of wave surfaces is a cylinder whose radius decreases to zero at the speed of light and then increases.

12 The de Sitter and anti-de Sitter backgrounds for case 2 solutions

Let us next consider the case 2 solutions in which $\Lambda\neq 0$ and τ° has the canonical form (18). In this case, the line element (2) in the conformally flat limit (40) takes the form

$$\mathrm{d}s^2=2\mathrm{d}u\left[\mathrm{d}r-\frac{(1+\Lambda\zeta\bar\zeta)^2r^2}{(\zeta+\bar\zeta)^2}\mathrm{d}u-\frac{2(1-\Lambda\bar\zeta^2)r}{(1+\Lambda\zeta\bar\zeta)(\zeta+\bar\zeta)}\mathrm{d}\zeta-\frac{2(1-\Lambda\zeta^2)r}{(1+\Lambda\zeta\bar\zeta)(\zeta+\bar\zeta)}\mathrm{d}\bar\zeta\right]-\frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{(1+\Lambda\zeta\bar\zeta)^2}.$$

It is then convenient to put $r=(Q^2/P^2)v$ (with $Q=\zeta+\bar\zeta$) and $\zeta=\frac{1}{\sqrt2}(x+iy)$, so that the background line element (which is de Sitter or anti-de Sitter space) can be expressed as

$$\mathrm{d}s^2=\frac{4x^2}{P^2}\,(\mathrm{d}u\,\mathrm{d}v-v^2\,\mathrm{d}u^2)-\frac{1}{P^2}\,(\mathrm{d}x^2+\mathrm{d}y^2). \tag{45}$$

Now, it is well known that the (anti-)de Sitter space can be represented as a four-dimensional hyperboloid

$$Z_0{}^2-Z_1{}^2-Z_2{}^2-Z_3{}^2-\varepsilon Z_4{}^2=-\varepsilon a^2, \tag{46}$$

embedded in a five-dimensional Minkowski space

$$\mathrm{d}s^2=\mathrm{d}Z_0{}^2-\mathrm{d}Z_1{}^2-\mathrm{d}Z_2{}^2-\mathrm{d}Z_3{}^2-\varepsilon\mathrm{d}Z_4{}^2, \tag{47}$$

where $a^2=1/(2|\Lambda|)$, $\varepsilon=1$ for a de Sitter background ($\Lambda>0$), and $\varepsilon=-1$ for an anti-de Sitter background ($\Lambda<0$). The two forms of the metric (45) and (47) are related by the transformation

$$\begin{aligned}
Z_0&=\frac{x}{\sqrt2\,P}(v+2u+2u^2v)\\
Z_1&=\frac{x}{\sqrt2\,P}(v-2u-2u^2v)\\
Z_2&=\frac{x}{P}(1+2uv)\\
Z_3&=\frac{y}{P}\\
Z_4&=a\frac{(2-P)}{P}
\end{aligned}
\quad\Leftrightarrow\quad
\begin{aligned}
u&=\pm\frac{1}{\sqrt2}\,\frac{Z_2-\sqrt{-Z_0{}^2+Z_1{}^2+Z_2{}^2}}{Z_0+Z_1}\\
v&=\pm\frac{1}{\sqrt2}\,\frac{Z_0+Z_1}{\sqrt{-Z_0{}^2+Z_1{}^2+Z_2{}^2}}\\
x&=\pm\frac{2a\sqrt{\varepsilon a^2-Z_3{}^2-\varepsilon Z_4{}^2}}{a+Z_4}\\
y&=\frac{2a\,Z_3}{a+Z_4}
\end{aligned} \tag{48}$$

where $P=1+\frac{1}{4a^2}(x^2+y^2)$.

It can immediately be seen that the singularity which occurs when $\zeta+\bar\zeta=0$ (or $x=0$) is located on

$$Z_3{}^2+\varepsilon Z_4{}^2=\varepsilon a^2\qquad\text{and}\qquad Z_1{}^2+Z_2{}^2=Z_0{}^2.$$

For the de Sitter background (for which $\varepsilon = 1$), this is an expanding torus. The sections in the Z_1, Z_2 plane are circles which are expanding at the speed of light with the time coordinate Z_0, and sections in the Z_3, Z_4 plane are circles of constant circumference $2\pi a$ that may be considered to be the circumference of the universe. (This is exactly as expected since the expanding cylinder described above in Minkowski space must become an expanding torus in the closed de Sitter space.)

For the anti-de Sitter background ($\varepsilon = -1$), however, the singularity is located on an expanding hyperboloid in which the sections in the Z_1, Z_2 plane are circles which are expanding at the speed of light with the time coordinate Z_0, and sections in the Z_3, Z_4 plane are hyperbolae. (Again, this is the negative curvature equivalent of the expanding cylinder.)

For both cases, it can also be seen that the wave surfaces $u = u_0$ are given in this five-dimensional representation by the intersection of the hyperboloid (46) with the plane

$$(1 + 2u_0^2)Z_0 - (1 - 2u_0^2)Z_1 \mp 2\sqrt{2}\, u_0\, Z_2 = 0.$$

Putting $\sin\alpha = \pm\frac{2\sqrt{2}\,u_0}{1+2u_0^2}$, $\cos\alpha = \frac{1-2u_0^2}{1+2u_0^2}$, this becomes

$$\cos\alpha\, Z_1 + \sin\alpha\, Z_2 = Z_0.$$

These are a family of planes which rotate relative to each other in the Z_1, Z_2 plane. They cut the four-dimensional hyperboloid at

$$(\sin\alpha\, Z_1 - \cos\alpha\, Z_2)^2 + {Z_3}^2 + \varepsilon {Z_4}^2 = \varepsilon a^2.$$

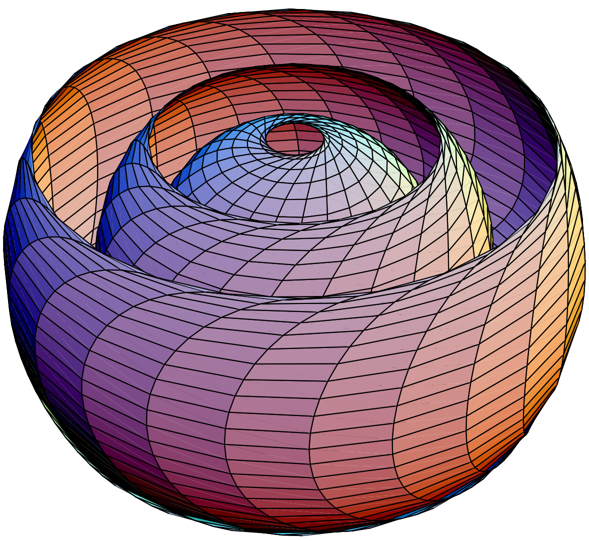

Figure 2: Portions of the de Sitter universe covered by the coordinates of (45) for three values of the time coordinate Z_0. Using Z_1, Z_2, Z_3 coordinates (Z_3 being the symmetry axis) with $Z_4 = 0$, the expanding 3-sphere of the universe reduces to an expanding 2-sphere. The wave surfaces $u = u_0$ are represented here by semicircles of constant radius a which are all tangent to the two expanding circles at $Z_3 = \pm a$ which are particular sections of the expanding torus corresponding to the singularity $x = 0$. It can be seen that the wave surfaces cover a decreasing portion of each successive 2-sphere.

For the de Sitter background ($\varepsilon = 1$), these intersections representing the wave surfaces are a family of spheres with constant area $4\pi a^2$. Moreover, the plane cuts are clearly tangent to the expanding torus, so that the singularity can again be interpreted as a caustic formed from the envelope of wave surfaces. Also, since two spheres pass through each point within the region covered by these coordinates, it is appropriate to restrict the family of wave surfaces to the hemispheres on which $x \geq 0$ whose boundary is located on the expanding torus. (The situation is obviously the positive curvature equivalent of the half-plane wave surfaces for the Kundt waves in a Minkowski background.) This is illustrated in figure 2. (Notice that the view along the axis corresponds to the first picture in figure 1.)

For the anti-de Sitter background ($\varepsilon = -1$), the wave surfaces are hyperboloidal. They are tangent to the singularity which is an expanding hyperboloid that can again be interpreted as an envelope of wave surfaces. Also, since it is only possible for one wave surface to pass through any point, it is appropriate to take the wave surfaces as the family of semi-infinite hyperboloids on which $x \geq 0$ (obviously generalising the half-planes of the Minkowski background).

For this case ($\Lambda < 0$), an additional apparent singularity occurs when $P = 0$, i.e. when $\zeta\bar{\zeta} = 1/(-\Lambda)$. It can easily be seen that this simply corresponds to anti-de Sitter infinity.

Let us finally observe that, just as for the case 2 waves in the Minkowski background, the above wave surfaces in de Sitter and anti-de Sitter backgrounds are outside the expanding torus or hyperboloid respectively. They therefore foliate a decreasing portion of the complete background for increasing time coordinate Z_0.

13 The de Sitter and anti-de Sitter backgrounds for case 1 solutions

Let us now consider the case 1 solutions in which $\Lambda \neq 0$ and τ° has the canonical form (17). In this case, the line element (2) in the conformally flat limit (40) takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}r - \frac{(1+\Lambda\zeta\bar{\zeta})^2}{(1-\Lambda\zeta\bar{\zeta})^2}\Lambda r^2\mathrm{d}u + \frac{4\Lambda\, r\,(\bar{\zeta}\mathrm{d}\zeta + \zeta\mathrm{d}\bar{\zeta})}{(1-\Lambda\zeta\bar{\zeta})(1+\Lambda\zeta\bar{\zeta})}\right] - \frac{2\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}}{(1+\Lambda\zeta\bar{\zeta})^2}.$$

Putting $r = (Q^2/P^2)v$ (with $Q = 1 - \Lambda\zeta\bar{\zeta}$), this can be written as

$$\mathrm{d}s^2 = 2\frac{Q^2}{P^2}\,(\mathrm{d}u\,\mathrm{d}v - \Lambda\, v^2\,\mathrm{d}u^2) - \frac{2}{P^2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}.$$

Clearly, this must be (anti-)de Sitter space which can be represented as the four-dimensional hyperboloid (46) embedded in the five-dimensional Minkowski space (47). In this case, the appropriate parametrization is given by

$$\begin{aligned} Z_0 &= \frac{Q}{\sqrt{2}\,P}(u+v+\Lambda u^2 v) \\ Z_1 &= \frac{Q}{\sqrt{2}\,P}(u-v+\Lambda u^2 v) \\ Z_2 + iZ_3 &= \sqrt{2}\frac{\zeta}{P} \\ Z_4 &= a\frac{Q}{P}(1+2\Lambda uv) \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} u &= \sqrt{2}\,a\frac{\varepsilon(Z_4 - \sqrt{-\varepsilon {Z_0}^2 + \varepsilon {Z_1}^2 + {Z_4}^2})}{Z_0 - Z_1} \\ v &= \frac{a}{\sqrt{2}}\frac{Z_0 - Z_1}{\sqrt{-\varepsilon {Z_0}^2 + \varepsilon {Z_1}^2 + {Z_4}^2}} \\ \zeta &= \frac{\sqrt{2}\,a(Z_2 + iZ_3)}{a + \sqrt{a^2 - \varepsilon {Z_2}^2 - \varepsilon {Z_3}^2}} \end{aligned} \qquad (49)$$

For the case in which $\Lambda > 0$ ($\varepsilon = 1$), it can be seen that the wave surfaces, which are given by $u = u_0$, are identical to those for case 2 with $\Lambda > 0$, but with the roles of Z_2 and Z_4 interchanged. (For this de Sitter background, Z_2 and Z_4 are both spacelike coordinates and their interchange is trivial.) This result is consistent with the fact that these two cases are equivalent for a positive cosmological constant. The wave surfaces are therefore a family of hemispheres of common constant area $4\pi a^2$ which are tangent to the expanding torus

$${Z_2}^2 + {Z_3}^2 = a^2 \qquad \text{and} \qquad {Z_1}^2 + {Z_4}^2 = {Z_0}^2,$$

which here corresponds to the coordinate singularity $1 - \Lambda\zeta\bar{\zeta} = 0$.

The case 1 space-times, however, are generalisations of the *pp*-waves in the sense that they reduce to the *pp*-waves as $\Lambda \to 0$. In this case, for $\Lambda > 0$ the background universe is closed, and the analogue of the plane wave surfaces which occur in a Minkowski background are spherical wave surfaces of equal constant area $4\pi a^2$. These spherical surfaces must clearly intersect each other. So it is not surprising that it is appropriate to restrict them to hemispheres, although it is perhaps unexpected that this leads to the same foliation of the space-time as that for case 2.

Now consider the case for which $\Lambda < 0$. In this case, the wave surfaces $u = u_0$ are now given in this five-dimensional representation by the intersection of the hyperboloid (46) with $\varepsilon = -1$ with the hyperplane

$$(1 - \tfrac{u_0^2}{2a^2})Z_0 + (1 + \tfrac{u_0^2}{2a^2})Z_1 - \tfrac{\sqrt{2}\,u_0}{a}\, Z_4 = 0.$$

Putting

$$\sin\alpha = \frac{-\frac{\sqrt{2}\,u_0}{a}}{1 + \frac{u_0^2}{2a^2}}, \qquad \cos\alpha = \frac{1 - \frac{u_0^2}{2a^2}}{1 + \frac{u_0^2}{2a^2}},$$

this becomes

$$Z_1 + \cos\alpha\, Z_0 + \sin\alpha\, Z_4 = 0.$$

This corresponds to a family of planes which are rotated relative to each other in the section of the timelike coordinates (Z_0, Z_4). These planes intersect the four-hyperboloid in a way which parametrizes the complete space-time.

In this case, there is no singularity as the term $a^2 + {Z_2}^2 + {Z_3}^2$ cannot be zero. The apparent coordinate singularity that occurs in the metric when $P = 1 + \Lambda\zeta\bar\zeta = 0$ again corresponds to anti-de Sitter infinity. These space-times are generalisations of the *pp*-waves and, for $\Lambda < 0$, the background universe is open and the analogue of the plane wave surfaces are hyperboloids which foliate the entire universe.

14 The anti-de Sitter background for case 3 solutions

Finally, let us consider the case 3 solutions in which Λ is necessarily negative, $k = 0$ and τ° has the canonical form (20). In this case, the line element (2) in the conformally flat limit (40) is

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}r + W\mathrm{d}\zeta + \bar W\mathrm{d}\bar\zeta) - 2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar\zeta,$$

where $W = 2(\bar\tau^\circ/P)\,r$. Putting $r = (Q^2/P^2)v$ where $Q = (1 + \sqrt{-\Lambda}\,\zeta)(1 + \sqrt{-\Lambda}\,\bar\zeta)$, this becomes

$$\mathrm{d}s^2 = 2\frac{Q^2}{P^2}\,\mathrm{d}u\,\mathrm{d}v - \frac{2\mathrm{d}\zeta\,\mathrm{d}\bar\zeta}{P^2}.$$

This must be the anti-de Sitter space which can be represented as a four-dimensional hyperboloid (46) with $\varepsilon = -1$ embedded in a five-dimensional Minkowski space-time (47). In this case, the parametrization can be expressed in the form

$$\begin{aligned} Z_0 &= \frac{(u+v)}{\sqrt2}\frac{Q}{P} \\ Z_1 &= \frac{(u-v)}{\sqrt2}\frac{Q}{P} \\ Z_2 &= \frac{(\zeta+\bar\zeta)}{\sqrt2\,P} + \frac{uv}{a}\frac{Q}{P} \\ Z_3 &= -\frac{i(\zeta-\bar\zeta)}{\sqrt2\,P} \\ Z_4 &= \left(a - \frac{uv}{a}\right)\frac{Q}{P} - \frac{(\zeta+\bar\zeta)}{\sqrt2\,P} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} u &= \frac{a}{\sqrt2}\frac{(Z_0+Z_1)}{(Z_2+Z_4)} \\ v &= \frac{a}{\sqrt2}\frac{(Z_0-Z_1)}{(Z_2+Z_4)} \\ \zeta &= \sqrt2\,a\left[\frac{(Z_2+iZ_3+Z_4)^2 - a^2}{(a+Z_2+Z_4)^2 + {Z_3}^2}\right] \end{aligned} \qquad (50)$$

The wave surfaces $u =$ const. are located on the intersection of the hyperboloid with the null hyperplane

$$Z_0 + Z_1 = \frac{\sqrt{2}u_0}{a}(Z_2 + Z_4).$$

The complete family of these hyperboloidal wave surfaces foliate the entire background space-time and, again, the apparent singularity at $P = 0$ simply corresponds to anti-de Sitter infinity.

The type N cases of these solutions have been described in detail by Siklos [10] and by Podolský [12], [13] in different coordinate systems.

15 Conclusions

We have presented the complete family of type III solutions of Kundt's class with a non-zero cosmological constant. These are new and contain all the previously known special cases. We have also found a more general type N solution that had previously been overlooked. A classification of all these space-times has been presented in terms of the signs of the cosmological constant and the parameter k. These give rise to different canonical forms for the spin-coefficient τ. We have described the physical interpretation of these solutions in terms of the global geometry of the family of wave surfaces, at least in the weak field limit, and we have argued that the weak singularities which arise in the space-times can be interpreted in terms of the caustics formed as the envelopes of wave surfaces.

Acknowledgements

We are grateful to Pavel Krtouš of Charles University in Prague for helpful discussions on these topics. JBG is grateful for hospitality at Charles University. This and the work of JP was partly supported by the grants GAČR-202/02/0735 and GAUK 166/2003.

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Note: It is not intended that the following appendix be included in any published version of this paper. It is included here for the benefit of the referees.

Appendix

With the assumptions introduced in section 2, namely $\kappa=\sigma=\rho=\epsilon=\lambda=0$, $\alpha=\bar\tau-\bar\beta$, $\pi=-\bar\tau$ and the only nonzero components of the curvature tensor being Ψ_3, Ψ_4, Φ_{22} and Λ, the Ricci identities become

$$
\begin{aligned}
D\tau &= 0\\
D\beta &= 0\\
D\gamma &= -\tau\bar\tau-\Lambda\\
\delta\tau &= 2\tau\beta\\
D\mu &= 0\\
D\nu+\Delta\bar\tau &= -\bar\tau(\gamma-\bar\gamma)+\Psi_3\\
\bar\delta\nu &= -\nu(\bar\tau-2\bar\beta)+\Psi_4\\
\bar\delta\beta+\delta\bar\beta &= \tau\bar\tau+\tau\bar\beta+\bar\tau\beta-4\beta\bar\beta+\Lambda\\
\bar\delta\mu &= -\bar\tau\bar\mu+\Psi_3\\
\delta\nu-\Delta\mu &= \mu^2+\mu(\gamma+\bar\gamma)+\bar\nu\bar\tau-2\beta\nu+\Phi_{22}\\
\delta\gamma-\Delta\beta &= \mu\tau-\beta(\gamma-\bar\gamma-\mu)\\
\bar\delta\tau &= 2\tau(\bar\tau-\bar\beta)+2\Lambda\\
\Delta\bar\tau-\Delta\bar\beta-\bar\delta\gamma &= (\bar\tau-\bar\beta)(\bar\gamma-\bar\mu)+\gamma(\bar\beta-\bar\tau)-\Psi_3
\end{aligned}
$$

For the tetrad (1) and the line element (2), the metric equations are

$$
\begin{aligned}
DH &= \gamma+\bar\gamma\\
D(P\bar W) &= 2\tau\\
\delta H-\Delta(P\bar W) &= \bar\nu+(\mu-\gamma+\bar\gamma)P\bar W\\
\delta(PW)-\bar\delta(P\bar W) &= \bar\mu-\mu+(\tau-2\beta)PW-(\bar\tau-2\bar\beta)P\bar W\\
DP &= 0\\
\Delta P &= (\mu-\gamma+\bar\gamma)P\\
\delta P &= (\tau-2\beta)P
\end{aligned}
$$

These, with the radial equations above, imply that the P is independent of r and the other metric functions have the structures

$$
\begin{aligned}
H &= -(\tau^\circ\bar\tau^\circ+\Lambda)r^2+2G^\circ r+H^\circ\\
W &= \frac{2\bar\tau^\circ}{P}r+W^\circ
\end{aligned}
$$

With $\alpha=\bar\tau-\bar\beta$ and $\pi=-\bar\tau$, the other non-zero spin coefficients are thus given by

$$
\begin{aligned}
\tau &= \tau^\circ\\
\beta &= \tfrac12(\tau^\circ-P_{,\bar\zeta})\\
\gamma &= -(\tau^\circ\bar\tau^\circ+\Lambda)r+G^\circ+\tfrac12 im^\circ\\
\mu &= im^\circ-(\log P)_{,u}\\
\nu &= 2\Big[PG^\circ_{,\zeta}+(\tau^\circ\bar\tau^\circ+\Lambda)PW^\circ-\bar\tau^\circ_{,u}+\bar\tau^\circ(\log P)_{,u}\Big]r\\
&\quad +PH^\circ_{,\zeta}+2\bar\tau^\circ H^\circ-2PG^\circ W^\circ-PW^\circ_{,u}
\end{aligned}
$$

where

$$im^\circ=-\tfrac12 P[W^\circ_{,\bar\zeta}-\bar W^\circ_{,\zeta}]-P(\tau^\circ W^\circ-\bar\tau^\circ\bar W^\circ).$$

The remaining Ricci identities still need to be satisfied.