Ultrarelativistic boost of the black ring

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We investigate the ultrarelativistic boost of the five-dimensional Emparan-Reall non-rotating black ring. Following the classical method of Aichelburg and Sexl, we determine the gravitational field generated by a black ring moving "with the speed of light" in an arbitrary direction. In particular, we study in detail two different boosts along axes orthogonal and parallel to the plane of the ring circle, respectively. In both cases, after the limit one obtains a five-dimensional impulsive pp-wave propagating in Minkowski spacetime. The curvature singularity of the original static spacetime becomes a singular source within the wave front, in the shape of a ring or a rod according to the direction of the boost. In the case of an orthogonal boost, the wave front contains also a remnant of the original disk-shaped membrane as a component of the Ricci tensor (which is everywhere else vanishing). We also analyze the asymptotic properties of the boosted black ring at large spatial distances from the singularity, and its behaviour near the sources. In the limit when the singularity shrinks to a point, one recovers the well known five-dimensional analogue of the Aichelburg-Sexl "monopole" solution.

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I. INTRODUCTION

The study of black holes in higher dimensions has been for a long time motivated by unified theories, in particular string theory [1]. In the past few years, extra-dimension models of TeV gravity have raised further interest in view of possible black hole production at colliders [2–5]. According to [6, 7], in semi-classical investigations of such high energy phenomena one can represent the incoming states with black hole metrics boosted "to the speed of light". In the case of the four-dimensional Schwarzschild black hole, the corresponding ultrarelativistic gravitational field is described by the Aichelburg-Sexl impulsive pp-wave [8]. In the spirit of [2–5], however, one clearly needs to consider higher dimensional settings. Indeed, the boosting technique of [8] has been already applied to static (charged) black holes in higher dimensions [9] (and straightforwardly extended to the $D \geq 4$ Schwarzschild black hole in an external magnetic field [10]). Recent analyses of black hole production in high energy collisions [11–13] thus employed the Aichelburg-Sexl solution (or other impulsive waves) in $D \geq 4$ spacetime dimensions (see [14] for a subtler discussion). The very recent work [15] studied the more elaborate ultrarelativistic limit of the Myers-Perry solution [1] (a generalization of the rotating Kerr metric to arbitrary dimensions).

In fact, one of the most remarkable feature of General Relativity in $D > 4$ is the non-uniqueness of the Myers-Perry spherical black holes. In five-dimensional vacuum gravity, there exist also asymptotically flat rotating black rings with an event horizon of topology $S^1 \times S^2$ [16]. It is our purpose to investigate the gravitational field generated by such rings when they move at the speed of light, in the sense of the Aichelburg-Sexl limit. In the present paper we will be focusing on the special subcase of zero angular momentum, i.e. on the static black rings found in [17]. We shall consider spinning rings in a separate subsequent work.

The structure of the paper is as follows. In Sec. II we briefly describe the static black ring of [17], which we intend to Lorentz-boost subsequently. In Sec. III we split the corresponding line element into flat space plus a term that becomes "small" at asymptotic infinity. We also introduce (asymptotically) cartesian coordinates useful for performing the ultrarelativistic boost. Secs. IV–VI contain our main results. In Secs. IV and V we explicitly calculate the metric of the black ring boosted along a direction orthogonal and parallel to the plane of the ring circle, respectively. This leads to two different impulsive pp-waves that naturally "recall" the original curvature and conical singularities of the static ring (in a sense to be made clear later). We analyze several specific properties of such solutions, in particular the Ricci and Weyl tensors, and asymptotic expansions far from and close to the singularities, and near geometrically privileged axes and planes. In Sec. VI we briefly discuss a boost along an arbitrary direction. We again obtain an impulsive pp-wave, whose singular source is described by an ellipse. Our final remarks are presented in Sec. VII. Appendix A summarizes the definitions and properties of the complete elliptic integrals employed in Secs. IV and V, whereas Appendix B provides the explicit tetrad components of the Weyl tensor in the case of the orthogonal boost.

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II. THE STATIC BLACK RING

In this section we briefly summarize the basic properties of the static black ring, referring to [17, 18] for further details. In the coordinates of [18],[1] the line element reads

$$\mathrm{d}s^2 = -\frac{F(y)}{F(x)}\mathrm{d}t^2 + \frac{L^2}{(x-y)^2}F(x)\Big[(y^2-1)\mathrm{d}\psi^2 + \frac{1}{F(y)}\frac{\mathrm{d}y^2}{y^2-1} + \frac{1}{F(x)}\frac{\mathrm{d}x^2}{1-x^2} + (1-x^2)\mathrm{d}\phi^2\Big], \quad (1)$$

where

$$F(\zeta) = \frac{1+\lambda\zeta}{1-\lambda}, \qquad 0 \le \lambda < 1. \quad (2)$$

The parameter λ is dimensionless, and for $\lambda = 0$ [i.e., $F = 1$] the spacetime (1) is flat. The constant $L > 0$ represents a length related to the radius of the "central circle" of the ring. For a physical interpretation of the spacetime (1), we take

$$y \in (-\infty, -1], \qquad x \in [-1, +1], \quad (3)$$

and ψ and ϕ as periodic angular coordinates (see below). Now, y is an "area coordinate" that, loosely speaking, parametrizes "distances" from the ring circle. Surfaces of constant y have topology $S^1 \times S^2$, and area which is monotonically growing with y. The coordinate ψ runs along the S^1 factor, whereas (x, ϕ) parametrize S^2 (see [17, 18] for illustrative pictures). At $y \to -\infty$ the spacetime has a curvature singularity, $y = -1/\lambda$ is a horizon of topology $S^1 \times S^2$, and spatial infinity corresponds to $x, y \to -1$. To avoid conical singularities at the axes $x = -1$ and $y = -1$, the angular coordinates must have the standard periodicity

$$\Delta\phi = 2\pi = \Delta\psi. \quad (4)$$

With this choice, however, there is a conical singularity at $x = +1$. This describes a disk-shaped membrane (with an excess angle) inside the ring which prevents the ring from collapsing under its self-gravity.[2] Nevertheless, the spacetime (1) is asymptotically flat [17], and the black ring has mass

$$M = \frac{3\pi L^2}{4}\frac{\lambda}{1-\lambda}. \quad (5)$$

[1] More precisely, one has to multiply $F(\zeta)$ by $(1-\lambda)$, and to divide ψ and ϕ by $\sqrt{1-\lambda}$ to obtain the corresponding quantities of [18]. The original notation of [17] is recovered with the transformations $y = (y' - \lambda)/(1 - \lambda y')$, $x = (x' - \lambda)/(1 - \lambda x')$, $\psi = \psi'/\sqrt{1+\lambda}$, $\phi = \phi'/\sqrt{1+\lambda}$, $L^2 = (1-\lambda^2)/A^2$.

[2] Alternatively, one can require regularity at $x = +1$ and place the conical singularity at $x = -1$, i.e., outside the ring [17]; we will not consider this case because the singularity would extend to infinity. There is no way to achieve regularity at both $x = -1$ and $x = +1$, unless the ring rotates [16].

Except on the disk membrane at $x = +1$, the metric (1) is a vacuum solution. It clearly admits three commuting orthogonal Killing vector fields $\partial_t, \partial_\psi, \partial_\phi$ and, in fact, it belongs to the generalized Weyl class of [17]. Interestingly, it has been proven recently [19] that vacuum black rings (with or without rotation) differ from the five-dimensional Myers-Perry black holes not only in the horizon topology, but also in the algebraic type of the Weyl tensor: black holes are of type D, whereas black rings are of the more general type I_i (type II on the horizon), according to the higher dimensional classification of [20].

III. SPLITTING OF THE METRIC AND CONVENIENT COORDINATES

For our purposes, it is convenient to decompose the line element (1) as

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \lambda\Delta, \quad (6)$$

in which $\mathrm{d}s_0^2$ is Minkowski spacetime [given by Eq. (1) with $\lambda = 0$, i.e., $F(x) = 1 = F(y)$], and

$$\Delta = \frac{x-y}{1+\lambda x}\mathrm{d}t^2 + \frac{L^2}{(x-y)^2}\left[\frac{x+1}{1-\lambda}(y^2-1)\mathrm{d}\psi^2 + \frac{x-y}{1+\lambda y}\frac{\mathrm{d}y^2}{y^2-1} + \frac{x+1}{1-\lambda}(1-x^2)\mathrm{d}\phi^2\right] \quad (7)$$

measures the deviation from flatness of the full black ring metric (1). Asymptotically $(x, y \to -1)$, Δ becomes "negligible" (in the sense of the Minkowskian metric $\mathrm{d}s_0^2$).

A boost is now naturally defined with respect to the flat background $\mathrm{d}s_0^2$ (as well as with respect to asymptotic infinity), namely by its isometries. We wish to visualize this in standard cartesian coordinates. In order to introduce them, it is first convenient to replace the "C-metric" coordinates (y, x) with (ξ, η) via the substitution[3]

$$y = -\frac{\xi^2+\eta^2+L^2}{\sqrt{(\xi^2+\eta^2-L^2)^2+4L^2\eta^2}}, \qquad x = -\frac{\xi^2+\eta^2-L^2}{\sqrt{(\xi^2+\eta^2-L^2)^2+4L^2\eta^2}}. \quad (8)$$

The flat term $\mathrm{d}s_0^2$ in Eq. (6) then takes the form of Minkowski space in double cylindrical coordinates

$$\mathrm{d}s_0^2 = -\mathrm{d}t^2 + \mathrm{d}\eta^2 + \eta^2\mathrm{d}\phi^2 + \mathrm{d}\xi^2 + \xi^2\mathrm{d}\psi^2, \quad (9)$$

and the additional quantity Δ reads

$$\Delta = \frac{2L^2}{\Sigma(1+\lambda x)}\mathrm{d}t^2 + 2L^2\frac{\left[(\xi^2-\eta^2-L^2)\mathrm{d}\xi + 2\eta\xi\mathrm{d}\eta\right]^2}{\Sigma^3(1+\lambda y)} + \frac{\Sigma-\eta^2-\xi^2+L^2}{\Sigma(1-\lambda)}\left(\xi^2\mathrm{d}\psi^2 + \eta^2\mathrm{d}\phi^2\right), \quad (10)$$

[3] We have simply inverted the relations $\xi = L\sqrt{y^2-1}/(x-y)$ and $\eta = L\sqrt{1-x^2}/(x-y)$ of [17].

where we have denoted

$$\Sigma = \sqrt{(\xi^2+\eta^2-L^2)^2+4L^2\eta^2}. \qquad (11)$$

In Eq. (10) we have kept an explicit simple dependence on the old coordinates (y,x) for brevity and for later convenience [but one can readily substitute Eq. (8) into Eq. (10) if necessary].

Cartesian coordinates are finally given by

$$\begin{aligned} x_1 &= \eta\cos\phi, \qquad & y_1 &= \xi\cos\psi, \\ x_2 &= \eta\sin\phi, \qquad & y_2 &= \xi\sin\psi, \end{aligned} \qquad (12)$$

so that $\eta = \sqrt{x_1^2+x_2^2}$, $\xi = \sqrt{y_1^2+y_2^2}$, and the background is $\mathrm{d}s_0^2 = -\mathrm{d}t^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}y_1^2+\mathrm{d}y_2^2$.

In principle, one could now study a boost along a general direction. Since the original spacetime (1) is symmetric under (separate) rotations in the (x_1,x_2) and (y_1,y_2) planes, such a direction can be specified by a single parameter α, namely introducing the rotated axes

$$z_1 = x_1\cos\alpha + y_1\sin\alpha, \quad z_2 = -x_1\sin\alpha + y_1\cos\alpha. \quad (13)$$

Defining suitable double null coordinates (u',v') by

$$t = \frac{-u'+v'}{\sqrt{2}}, \qquad z_1 = \frac{u'+v'}{\sqrt{2}}, \qquad (14)$$

a Lorentz boost along z_1 takes the simple form

$$u' = \epsilon^{-1}u, \qquad v' = \epsilon v. \qquad (15)$$

The parameter $\epsilon > 0$ is related to the standard Lorentz factor via $\gamma = (\epsilon+\epsilon^{-1})/2$.

In the following, we will study in detail two different boosts of the black ring along the privileged axes x_1 ($\alpha = 0$) and y_1 ($\alpha = \pi/2$), which are respectively "orthogonal" and "parallel" to the ring. But we will also discuss a boost in a general direction. In particular, we will consider "ultrarelativistic" boosts to the speed of light, i.e. the transformation (15) in the limit $\epsilon \to 0$. Along with that, we will perform the standard mass rescaling [8]

$$M = \gamma^{-1}p = 2\epsilon(1+\epsilon^2)^{-1}p, \qquad (16)$$

which keeps the total energy finite ($p > 0$ is a constant). From Eq. (5), in term of the dimensionless parameter λ the rescaling (16) becomes

$$\lambda = \lambda_\epsilon = \epsilon\frac{8p}{3\pi L^2+\epsilon(8p+3\pi L^2\epsilon)}, \qquad (17)$$

so that when $\epsilon \to 0$ then $\lambda_\epsilon \approx \epsilon\frac{8p}{3\pi L^2} \to 0$.

IV. ORTHOGONAL BOOST: $\alpha = 0$

A. Evaluation of the impulsive limit of the metric

For $\alpha = 0$ in Eq. (13), Eq. (14) reduces to

$$t = \frac{-u'+v'}{\sqrt{2}}, \qquad x_1 = \frac{u'+v'}{\sqrt{2}}, \qquad (18)$$

so that the transformation (15) describes a Lorentz boost along the x_1 axis, which lies in the 2-plane spanned by (η,ϕ) [cf. Eq. (12)]. The latter is orthogonal to the 2-plane (ξ,ψ), which contains the ring circle. We wish now to evaluate how the black ring metric (1) [that is, Eq. (6) with Eqs. (9) and (10)] transforms under the boost (15) with $\alpha = 0$. Since the coordinates ξ and ψ remain unchanged in this case, it suffices to substitute only the first column of Eq. (12) into Eqs. (9) and (10). Then, we put Eq. (18) into the thus obtained expressions for $\mathrm{d}s_0^2$ and for Δ (we omit the intermediate expressions, which are cumbersome and not of particular significance). Finally, we perform the boost (15). This leaves $\mathrm{d}s_0^2$ invariant ($2\mathrm{d}u'\mathrm{d}v' = 2\mathrm{d}u\mathrm{d}v$), i.e.

$$\mathrm{d}s_0^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x_2^2 + \mathrm{d}\xi^2 + \xi^2\mathrm{d}\psi^2, \qquad (19)$$

and makes Δ dependent parametrically on ϵ. Using the shortcut

$$z_\epsilon = \frac{1}{\sqrt{2}}(\epsilon^{-1}u+\epsilon v), \qquad (20)$$

one obtains[4]

$$\begin{aligned}\Delta_\epsilon = {}& \frac{L^2(\epsilon^{-1}\mathrm{d}u-\epsilon\mathrm{d}v)^2}{\Sigma_\epsilon(1+\lambda_\epsilon x)} \\ &+ \frac{2L^2}{\Sigma_\epsilon^3(1+\lambda_\epsilon y)}\Big(\left[\xi^2-z_\epsilon^2-x_2^2-L^2\right]\mathrm{d}\xi \\ &\qquad + 2\xi\left[\tfrac{1}{\sqrt{2}}z_\epsilon(\epsilon^{-1}\mathrm{d}u+\epsilon\mathrm{d}v)+x_2\mathrm{d}x_2\right]\Big)^2 \\ &+ \frac{\Sigma_\epsilon - z_\epsilon^2 - x_2^2 - \xi^2 + L^2}{\Sigma_\epsilon(1-\lambda_\epsilon)}\Bigg(\xi^2\mathrm{d}\psi^2 \\ &\qquad + \frac{\frac{1}{2}\left[\sqrt{2}z_\epsilon\mathrm{d}x_2 - x_2(\epsilon^{-1}\mathrm{d}u+\epsilon\mathrm{d}v)\right]^2}{z_\epsilon^2+x_2^2}\Bigg). \end{aligned} \qquad (21)$$

Here, the quantity Σ_ϵ comes from the expression (11) using the above described coordinate transformations and the boost (15), and it can be rewritten as

$$\Sigma_\epsilon = \sqrt{\left[z_\epsilon^2+x_2^2+(\xi+L)^2\right]\left[z_\epsilon^2+x_2^2+(\xi-L)^2\right]}. \quad (22)$$

We are now interested in taking the ultrarelativistic limit $\epsilon \to 0$, i.e. in finding the resulting metric

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \lim_{\epsilon\to 0}\lambda_\epsilon\Delta_\epsilon. \qquad (23)$$

Recalling Eq. (17), one easily sees that $\lim_{\epsilon\to 0}(\lambda_\epsilon\Delta_\epsilon) = 0$ at any given spacetime point with $u \neq 0$ (and away from

[4] Again, for convenience in Eq. (21) we have left some expressions containing the old coordinates y and x [cf. Eq. (8)], which now depend on ϵ. However, these terms will not contribute to the final result in the limit $\epsilon \to 0$.

the ring singularity $y=-\infty$). At $u=0$ this limit diverges, but in fact it represents a sound distribution supported on $u=0$. By inspecting the various quantities in Eq. (21), it suffices to retain only the terms proportional to $\mathrm{d}u^2$, as the remaining ones become negligible for $\epsilon\to 0$. Similarly, we drop the factors $1+\lambda_\epsilon x$, $1+\lambda_\epsilon y$ and $1-\lambda_\epsilon$, since $\lambda_\epsilon\to 0$ for $\epsilon\to 0$. Using Eqs. (17), (21) and (20), for $\epsilon\sim 0$ we can thus write

$$\lambda_\epsilon\Delta_\epsilon \approx \frac{8p}{3\pi L^2}\frac{1}{\epsilon}h_\perp(z_\epsilon)\mathrm{d}u^2, \tag{24}$$

where

$$h_\perp(z_\epsilon) = \frac{2L^2-x_2^2}{2\Sigma_\epsilon}+\frac{4L^2\xi^2z_\epsilon^2}{\Sigma_\epsilon^3}+\frac{x_2^2(L^2-\xi^2)}{2(z_\epsilon^2+x_2^2)\Sigma_\epsilon} + \frac{x_2^2}{2(z_\epsilon^2+x_2^2)}. \tag{25}$$

We have emphasized the dependence of $h_\perp$ on z_ϵ (which gives the only dependence on ϵ) because this is essential in our limit (of course, $h_\perp$ depends on the coordinates x_2 and ξ as well). In taking the limit $\epsilon\to 0$ of Eq. (24), we can now apply the distributional identity [recall Eq. (20)]

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}f(z_\epsilon)=\sqrt{2}\,\delta(u)\int_{-\infty}^{+\infty}f(z)\mathrm{d}z. \tag{26}$$

With this, the final metric is [cf. Eqs. (23) and (19)]

$$\mathrm{d}s^2=2\mathrm{d}u\mathrm{d}v+\mathrm{d}x_2^2+\mathrm{d}\xi^2+\xi^2\mathrm{d}\psi^2+H_\perp(x_2,\xi)\delta(u)\mathrm{d}u^2, \tag{27}$$

with a profile function given by

$$H_\perp(x_2,\xi)=\frac{8\sqrt{2}p}{3\pi L^2}\left(\int_{-\infty}^{+\infty}h_\perp(z)\mathrm{d}z\right). \tag{28}$$

It only remains to explicitly perform the integration in Eq. (28), with $h_\perp$ given by Eq. (25) with Eq. (22). The last term in Eq. (25) gives rise to the simple integral $\int_{-\infty}^{+\infty}(z^2+x_2^2)^{-1}\mathrm{d}z=\pi|x_2|^{-1}$. The first three terms lead to the elliptic integrals (A13), (A14) and (A15) of Appendix A. Combining the various quantities, we finally obtain

$$\begin{aligned}H_\perp(x_2,\xi) &= \frac{8\sqrt{2}p}{3\pi L^2}\\ &\times\Bigg[\left(3L^2+\xi^2+x_2^2\frac{\xi+L}{\xi-L}\right)\frac{K(k)}{\sqrt{(\xi+L)^2+x_2^2}}\\ &-\sqrt{(\xi+L)^2+x_2^2}\,E(k)\\ &-\frac{\xi+L}{\xi-L}\frac{(\xi-L)^2+x_2^2}{\sqrt{(\xi+L)^2+x_2^2}}\Pi(\rho_0,k)+\frac{\pi}{2}|x_2|\Bigg],\end{aligned} \tag{29}$$

with

$$k=\sqrt{\frac{4\xi L}{(\xi+L)^2+x_2^2}}, \qquad \rho_0=-\frac{(\xi-L)^2}{x_2^2}. \tag{30}$$

One can reexpress the elliptic integral $\Pi(\rho_0,k)$ using identities (A4) and (A5), and obtain an alternative form of $H_\perp$, which will be useful for subsequent discussions,

$$\begin{aligned}H_\perp(x_2,\xi) &= \frac{8\sqrt{2}p}{3\pi L^2}\\ &\times\Bigg[\frac{3L^2+\xi^2}{\sqrt{(\xi+L)^2+x_2^2}}K(k)-\sqrt{(\xi+L)^2+x_2^2}\,E(k)\\ &+\frac{\xi-L}{\xi+L}\frac{x_2^2}{\sqrt{(\xi+L)^2+x_2^2}}\Pi(\rho,k)+\pi|x_2|\Theta(L-\xi)\Bigg],\end{aligned} \tag{31}$$

where

$$\rho=\frac{4\xi L}{(\xi+L)^2}, \tag{32}$$

and $\Theta(L-\xi)$ denotes the step function.

Let us observe that no singular coordinate transformation of the type of [8] had to be performed in the calculation above, since all the required integrals are convergent.

B. Properties of the solution

A static black ring boosted to the speed of light in a direction orthogonal to the ring circle is thus described by the metric (27), with $H_\perp$ given explicitly by Eq. (31). This is evidently a five-dimensional impulsive *pp*-wave propagating along the x_1 direction [see Eq. (18)]. Such a spacetime is flat everywhere except on the null hyperplane $u=0$, which represents the impulsive wave front. In particular, the line element (27) is singular at the points satisfying $u=0=x_2$ and $\xi=L$ [$k=1$ in Eq. (30)], i.e. on a *circle of radius* L contained within the wave front. This is a remnant of the curvature singularity ($y=-\infty$) of the original static black ring (1). Since the boost performed above was orthogonal to the ring circle, the latter has not Lorentz-contracted. We have plotted the profile function $H_\perp$ in Fig. 1.

1. Killing vectors

The *pp*-wave line element (27), (31) is obviously invariant under the transformations generated by the vector fields ∂_v and ∂_ψ. It has been demonstrated in four spacetime dimensions [21] that impulsive *pp*-waves admit more isometries than the same class of waves with a general profile. Similarly, it is easy to see that, thanks to the presence of $\delta(u)$, the line element (27) admits also the three commuting Killing vectors

$$u\partial_{x_2}-x_2\partial_v, \quad u\partial_{y_1}-y_1\partial_v, \quad u\partial_{y_2}-y_2\partial_v. \tag{33}$$

[Recall the simple relation (12) between (y_1,y_2) and (ξ,ψ).] These are generators of null rotations. Incidentally, we observe that impulsive waves in the four-dimensional (anti-)de Sitter universe can be described

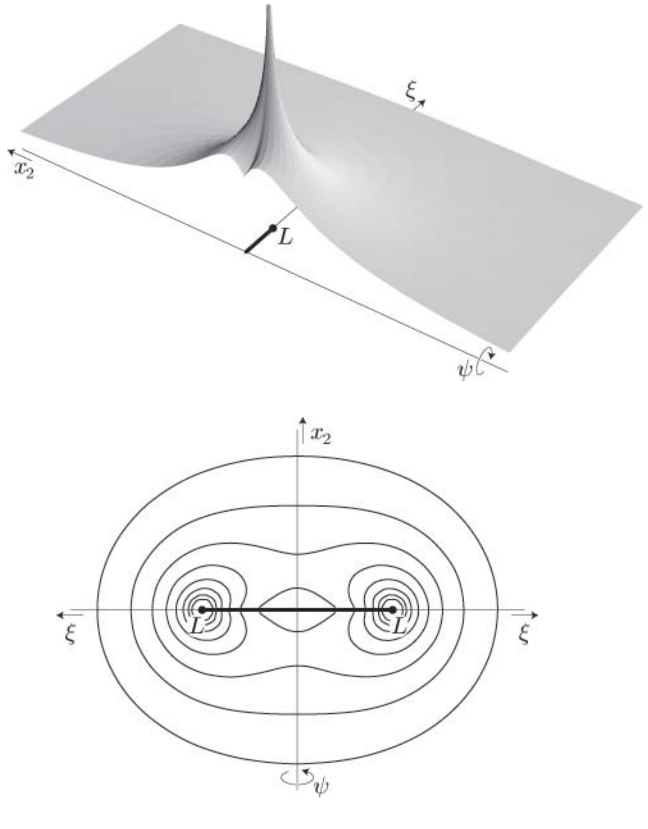

FIG. 1: The profile function $H_\perp(x_2,\xi)$ given by Eq. (31). The upper picture represents the values taken by $H_\perp$ over the plane (x_2,ξ), whereas the lower one displays curves along which $H_\perp$ is constant. This is the case of a static black ring boosted to the speed of light in a direction orthogonal to the plane (ξ,ψ), which contains the ring circle. The coordinates (x_2,ξ,ψ) span spatial sections of the impulsive wave front $u=0$, cf. Eq. (27). Here the coordinate ψ is suppressed, since it just describes the orbits of a Killing vector field. The profile function $H_\perp$ diverges (only) at the *ring* singularity $x_2=0$, $\xi=L$, as indicated by the thick point(s) in the pictures. In addition, there is a disk membrane within the ring, i.e. at $x_2=0$, $\xi<L$ [cf. Eq. (34)], which manifests itself as a jump in $\partial H_\perp/\partial x_2$. This is drawn above as a thick line.

as a submanifold of five-dimensional impulsive *pp*-waves, and they admit symmetries very similar to the above [22].

2. The Ricci tensor

The static ring (1) is a vacuum spacetime ($R_{\mu\nu}=0$) everywhere except on the disk membrane $x=+1$ (and of course on the ring singularity $y=-\infty$). Therefore, one would expect also the ultrarelativistic boosted ring to be a vacuum solution except at a possible remnant (after the boost) of the original disk membrane. To check the results, we have verified that the Ricci tensor associated to the spacetime (27), (31) is indeed zero everywhere but at $u=0=x_2$, $\xi<L$, i.e. inside a two-dimensional disk lying on the wave front. Namely, using Eqs. (A6) one finds

$$R_{uu}=-\frac{1}{2}\boldsymbol{\Delta}H_\perp\,\delta(u)=-\frac{8\sqrt{2}p}{3L^2}\,\Theta(L-\xi)\delta(x_2)\delta(u) \quad (34)$$

[the symbol $\boldsymbol{\Delta}$ denotes the Laplace operator over the transverse flat space (x_2,ξ,ψ)]. This nonvanishing component arises only due to the last term in Eq. (31), a typical term associated to boosted conical singularities [23]. On the disk rim $u=0=x_2$, $\xi=L$ the metric (27), (31) is singular, and its exact structure may be not reflected correctly by Eq. (34).

3. The Weyl tensor

For any five-dimensional *pp*-wave written in the form $\mathrm{d}s^2=2\mathrm{d}u\mathrm{d}v+\mathrm{d}x_2^2+\mathrm{d}\xi^2+\xi^2\mathrm{d}\psi^2+H_\perp\mathrm{d}u^2$, in the null/orthonormal frame

$$\begin{aligned}&\boldsymbol{k}=\mathrm{d}u, \qquad \boldsymbol{l}=-\mathrm{d}v-\frac{1}{2}H_\perp\mathrm{d}u,\\ &\boldsymbol{m}_{(1)}=\mathrm{d}\xi, \quad \boldsymbol{m}_{(2)}=\xi\mathrm{d}\psi, \quad \boldsymbol{m}_{(3)}=\mathrm{d}x_2, \end{aligned} \quad (35)$$

the Weyl tensor is

$$\begin{aligned}\boldsymbol{C}=\Psi_{ij}[&(\boldsymbol{k}\wedge\boldsymbol{m}_{(i)})\otimes(\boldsymbol{k}\wedge\boldsymbol{m}_{(j)})\\ &+(\boldsymbol{k}\wedge\boldsymbol{m}_{(j)})\otimes(\boldsymbol{k}\wedge\boldsymbol{m}_{(i)})],\end{aligned} \quad (36)$$

where summation over $i,j=1,2,3$ is understood. This is the canonical form of type N spacetimes [20, 24], and $\boldsymbol{k}$ is the unique principal null direction. The symbols [24]

$$\Psi_{ij}=\frac{1}{2}C_{\mu\nu\rho\sigma}l^\mu m^\nu_{(i)}l^\rho m^\sigma_{(j)}, \qquad i,j=1,2,3 \quad (37)$$

define a 3×3 symmetric traceless matrix that expresses the independent frame components of the Weyl tensor, which are in general five in $D=5$.[5] In particular, this demonstrates that in the ultrarelativistic boost studied above the original type I_i [19] of the static ring (1) has degenerated to the type N on the wave front of our specific *pp*-wave (27), (31). Moreover, for such a solution the symmetry under ∂_ψ implies $\Psi_{12}=0=\Psi_{23}$. One is thus left with

$$\begin{aligned}\Psi_{11}&=-\frac{1}{2}\left(\frac{1}{2}\frac{\partial^2H_\perp}{\partial\xi^2}\delta(u)+\frac{1}{3}R_{uu}\right),\\ \Psi_{13}&=-\frac{1}{4}\frac{\partial^2H_\perp}{\partial\xi\partial x_2}\delta(u),\\ \Psi_{22}&=-\frac{1}{2}\left(\frac{1}{2\xi}\frac{\partial H_\perp}{\partial\xi}\delta(u)+\frac{1}{3}R_{uu}\right),\\ \Psi_{33}&=-\frac{1}{2}\left(\frac{1}{2}\frac{\partial^2H_\perp}{\partial x_2^2}\delta(u)+\frac{1}{3}R_{uu}\right).\end{aligned} \quad (38)$$

[5] The quantities Ψ_{ij} can be understood as a generalization of the complex scalar Ψ_4, which fully characterizes type N spacetimes in the well known $D=4$ theory.

The above components of the Weyl tensor confirm the presence of an impulsive gravitational wave at $u = 0$. For $H_\perp$ given by Eq. (31), the explicit form of the scalars Ψ_{ij} is presented in Appendix B. There one can observe that the elliptic integral $\Pi(\rho, k)$ disappears from such expressions.

4. *Asymptotic behaviour*

The spacetime (27) is flat everywhere except on the wave front $u = 0$. If we restrict within the latter, it is interesting to analyze how the gravitational field generated by the boosted black ring behaves *at a large spatial distance from the centre of the ring singularity* (given by $\xi = 0 = x_2$). Spatial sections of the wave front are three-dimensional spaces, in which we can introduce standard spherical coordinates (r, θ, ψ) by

$$x_2 = r\cos\theta, \qquad \xi = r\sin\theta. \tag{39}$$

Since r is a radial coordinate from the centre of the ring ($r^2 = x_2^2 + \xi^2$) and L is the radius of the ring, we consider an expansion for small values of the dimensionless parameter L/r. This means considering Eq. (31) for k and ρ approaching zero. Using Eqs. (A7), (A8) and (A9), we obtain

$$H_\perp = \frac{1}{\sqrt{2}}\frac{8p}{3L}\left[3\frac{L}{r} - \frac{7}{8}(3\cos^2\theta - 1)\frac{L^3}{r^3} + O\left(\frac{L^5}{r^5}\right)\right]. \tag{40}$$

We recognize the standard multipole terms [indeed, for $\xi > L$, $H_\perp$ is a solution of a three-dimensional Laplace equation, cf. Eq. (34)]. Notice that the dipole term is missing, due to the geometry of the source. In the limit when the ring shrinks to a point, i.e. $L \to 0$, the expansion reduces just to the monopole term,

$$H_\perp^0 = \lim_{L\to 0} H_\perp = \frac{1}{\sqrt{2}}\frac{8p}{r}. \tag{41}$$

This exhibits the "Newtonian" $1/r$ fall-off in three-dimensional space, with a "mass" proportional to p. The metric (27) with a profile function given by $H_\perp^0$ coincides with the five-dimensional analogue of the Aichelburg-Sexl solution, obtained by boosting the Schwarzschild line element to the speed of light [9] (cf. also, e.g., [10, 11, 13]).

In order to gain further physical insight, one can similarly consider other expansions near "special places". For example, *near the axis* $\xi = 0$ we obtain

$$\begin{aligned} H_\perp = {} & \frac{8\sqrt{2}p}{3L^2}\left[\frac{L^2 - x_2^2}{\sqrt{L^2 + x_2^2}} + |x_2| \right. \\ & \left. + \frac{3L^2}{4}\frac{L^2 - x_2^2}{\sqrt{(L^2 + x_2^2)^5}}\,\xi^2 + O\left(\xi^4\right)\right]. \end{aligned} \tag{42}$$

Near the plane of the ring $x_2 = 0$,

$$\begin{aligned} H_\perp = {} & \frac{1}{\sqrt{2}}\frac{8p}{3\pi L^2}\left[2\frac{3L^2 + \xi^2}{L + \xi}K(\tilde{k}) - 2(L + \xi)E(\tilde{k}) \right. \\ & + 2\pi|x_2|\Theta(L - \xi) - \left(\frac{1}{L + \xi}K(\tilde{k}) \right. \\ & \left.\left. + \frac{5L^2 - \xi^2}{(L - \xi)^2(L + \xi)}E(\tilde{k})\right)x_2^2 + O\left(x_2^4\right)\right], \end{aligned} \tag{43}$$

where $\tilde{k} = k(x_2 = 0) = \sqrt{4L\xi}/(L + \xi)$.

If we introduce suitable coordinates "centred on the ring"

$$x_2 = \tilde{r}\sin\tilde{\theta}, \qquad \xi = L + \tilde{r}\cos\tilde{\theta}, \tag{44}$$

using Eqs. (A10), (A11) and (A12), the expansion [of Eq. (29)] *near the singular ring* $\tilde{r} = 0$ is

$$\begin{aligned} H_\perp = {} & \frac{1}{\sqrt{2}}\frac{8p}{3\pi L}\left\{-4\left(1 + \log\frac{\tilde{r}}{8L}\right) + 2\tilde{\theta}\sin\tilde{\theta}\frac{\tilde{r}}{L} \right. \\ & \left. - \frac{1}{4}\left[1 + \cos 2\tilde{\theta}\left(1 + 3\log\frac{\tilde{r}}{8L}\right)\right]\frac{\tilde{r}^2}{L^2} + O\left(\frac{\tilde{r}^3}{L^3}\right)\right\}. \end{aligned} \tag{45}$$

The ring-shaped singularity is explicitly visible in the first logarithmic term.

V. PARALLEL BOOST: $\alpha = \pi/2$

A. Evaluation of the impulsive limit of the metric

For $\alpha = \pi/2$ in Eq. (13), Eq. (14) becomes

$$t = \frac{-u' + v'}{\sqrt{2}}, \qquad y_1 = \frac{u' + v'}{\sqrt{2}}, \tag{46}$$

so that the transformation (15) describes a boost in the y_1 direction, i.e. in the 2-plane (ξ, ψ) [cf. Eq. (12)] containing the ring circle. As for the previous orthogonal boost, we need to calculate how the black ring metric (1) transforms under the boost (15), and then take the limit $\epsilon \to 0$. In the present case ($\alpha = \pi/2$) the coordinates η and ϕ remain unchanged, hence we substitute the second column of Eq. (12) into Eqs. (9) and (10). Apart from this, we follow the same steps as in Sec. IV, and the derivation here will be therefore shortened in its straightforward parts. The flat, boost-invariant part of the decomposition (6) can now be written as

$$\mathrm{d}s_0^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}y_2^2 + \mathrm{d}\eta^2 + \eta^2\mathrm{d}\phi^2. \tag{47}$$

For the additional term $\lambda\Delta$, as $\epsilon \sim 0$ we get an expression analogous to Eq. (24), but with $h_\perp(z_\epsilon)$ replaced by

$$\begin{aligned} h_{||}(z_\epsilon) = {} & \frac{4L^2 - y_2^2}{2\Sigma_\epsilon} - \frac{4L^2\eta^2 z_\epsilon^2}{\Sigma_\epsilon^3} - \frac{y_2^2(L^2 + \eta^2)}{2(z_\epsilon^2 + y_2^2)\Sigma_\epsilon} \\ & + \frac{y_2^2}{2(z_\epsilon^2 + y_2^2)}, \end{aligned} \tag{48}$$

and

$$\Sigma_\epsilon = \sqrt{z_\epsilon^4 + 2(y_2^2 + \eta^2 - L^2)z_\epsilon^2 + a^4}, \tag{49}$$

with

$$a = \left[(\eta^2 + y_2^2 - L^2)^2 + 4\eta^2 L^2\right]^{1/4}. \tag{50}$$

Again employing identity (26), the final boosted metric is now

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}y_2^2 + \mathrm{d}\eta^2 + \eta^2\mathrm{d}\phi^2 + H_{||}(y_2,\eta)\delta(u)\mathrm{d}u^2, \tag{51}$$

with a profile function

$$H_{||}(y_2,\eta) = \frac{8\sqrt{2}p}{3\pi L^2}\left(\int_{-\infty}^{+\infty} h_{||}(z)\mathrm{d}z\right). \tag{52}$$

We employ the elliptic integrals (A17), (A18) and (A19) of Appendix A in order to perform the integration in Eq. (52) [with $h_{||}$ given by Eqs. (48) and (49)]. Combining all the terms, and using identity (A5) to reexpress the elliptic integral Π in a more convenient form, we obtain

$$H_{||}(y_2,\eta) = \frac{8\sqrt{2}p}{3\pi L^2}\left[\frac{5L^2+\eta^2+2a^2}{2a}K(k) - 2aE(k) \right.$$
$$\left. + \frac{\eta^2+L^2}{2a}\frac{a^2+y_2^2}{a^2-y_2^2}\Pi(\rho,k) + \pi|y_2|\Theta\left(y_2^2-a^2\right)\right], \tag{53}$$

where

$$k = \frac{\left(a^2-\eta^2-y_2^2+L^2\right)^{1/2}}{\sqrt{2}a},$$
$$\rho = -2y_2^2\frac{a^2-\eta^2-y_2^2+L^2}{\left(a^2-y_2^2\right)^2}, \tag{54}$$

and a as in Eq. (50).

B. Properties of the solution

A static black ring boosted to the speed of light in a direction contained in the plane of the ring circle is thus represented by the metric (51), (53). As in the case of the orthogonal boost of Sec. IV, this is a five-dimensional impulsive pp-wave. It propagates along the y_1 direction, and it is singular at the points satisfying $u = 0 = \eta$ and $|y_2| \le L$ [$k = 1$ in Eq. (54)], i.e. on a *rod of length* $2L$ contained within the wave front. This is a remnant of the curvature singularity of the original static black ring (1), which has Lorentz-contracted because of the ultrarelativistic boost in the plane of the ring. On the contrary, notice that the apparent divergence of $H_{||}$ at $y_2^2 = a^2$ is only a fictitious effect: the singular behaviour of the coefficient of Π in Eq. (53) is compensated if one takes into account the form of ρ [Eq. (54)] and the step function in the last term. The profile function $H_{||}$ is plotted in Fig. 2.

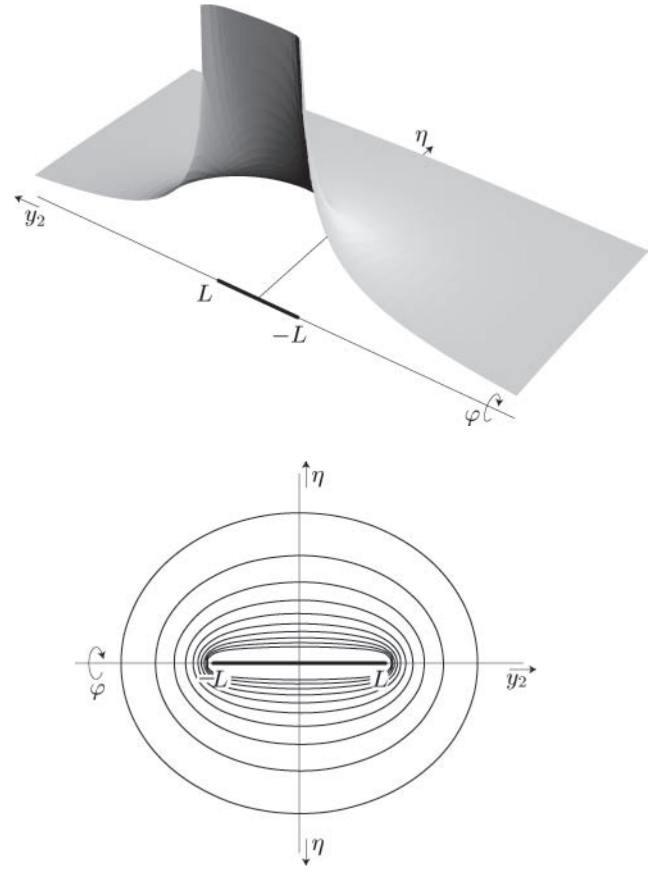

FIG. 2: Plot of the profile function $H_{||}(y_2,\eta)$ given by Eq. (53). This is the case of a static black ring boosted to the speed of light along a direction contained within the plane of the ring circle, i.e. (ξ,ψ). The coordinates (y_2,η,ϕ) span spatial sections of the impulsive wave front $u = 0$, cf. Eq. (51). The Killing coordinate ϕ is suppressed in the figures. The profile function $H_{||}$ diverges (only) at the *rod* singularity $\eta = 0$, $|y_2| \le L$, as indicated by the thick line in the pictures.

The discussion of further properties of the solution (51), (53) is now shortened, since it follows the similar one in Sec. IV. There exist isometries generated by the Killing vector fields ∂_v, ∂_ϕ, $u\partial_{y_2} - y_2\partial_v$, $u\partial_{x_1} - x_1\partial_v$ and $u\partial_{x_2} - x_2\partial_v$ [cf. Eq. (33)].

During the parallel boost, also the original disk membrane has Lorentz-contracted, and it is now located on the singular region $u = 0 = \eta$, $|y_2| \le L$. We will not discuss the behaviour of the solution there. Except on this singular rod, the Ricci tensor associated to the spacetime (51), (53) is vanishing, as we verified using identities (A6).

Similarly as in Sec. IV, one can cast the Weyl tensor in the type N canonical form using the frame (35) with the replacements $\xi \to \eta$, $\psi \to \phi$, $x_2 \to y_2$ and, of course, $H_\perp \to H_{||}$. Analogously, one obtains the corresponding Weyl components from Eq. (38). In this case, we omit the explicit form of the scalars Ψ_{ij}, which is rather complicated and does not provide any immediate physical insight. We just notice that, again, the elliptic integral Π disappears.

Following the corresponding analysis of Sec. IV, we can analyze how the gravitational field generated by the boosted black ring (51) behaves *at a large spatial distance from the centre of the rod singularity* (given by $\eta = 0 = y_2$). With spherical coordinates defined on the wave front by

$$y_2 = r\cos\theta, \qquad \eta = r\sin\theta, \tag{55}$$

we now obtain

$$H_{_{||}} = \frac{1}{\sqrt{2}}\frac{8p}{3L}\left[3\frac{L}{r} + \frac{5}{8}(3\cos^2\theta - 1)\frac{L^3}{r^3} + O\left(\frac{L^5}{r^5}\right)\right]. \tag{56}$$

The monopole term coincides with that obtained in the case of $H_{_\perp}$, cf. Eqs. (40) and (41), and for $L \to 0$ (i.e., when the rod shrinks to a point) it gives rise to the five-dimensional Aichelburg-Sexl solution. Again, there is no dipole, but the quadrupole term is different from that of Eq. (40).

In addition, we can consider an expansion of $H_{_{||}}$ *near the axis* $\eta = 0$. The rod singularity lies exactly at $\eta = 0$, for $|y_2| \le L$. Therefore, we have to study the two cases $|y_2| > L$ and $|y_2| < L$ separately. For $|y_2| > L$, one has

$$\begin{aligned} H_{_{||}} = \; & \frac{8\sqrt{2}p}{3L^2}\left[\frac{L^2}{\sqrt{y_2^2 - L^2}} + |y_2| - \sqrt{y_2^2 - L^2}\right. \\ & \left. - \frac{3}{4}\frac{L^2 y_2^2}{\sqrt{(y_2^2 - L^2)^5}}\,\eta^2 + O\left(\eta^4\right)\right]. \end{aligned} \tag{57}$$

The case $|y_2| < L$ is more delicate and one has to employ expansions (A10), (A11) and (A12). At the end,

$$\begin{aligned} H_{_{||}} = \; & \frac{8\sqrt{2}p}{3\pi L^2}\left[\frac{y_2^2 - 2L^2}{\sqrt{L^2 - y_2^2}}\log\frac{L^2\eta^2}{16(L^2 - y_2^2)^2} - 2\sqrt{L^2 - y_2^2}\right. \\ & \left. + |y_2|\mathrm{arccot}\frac{L^2 - 2y_2^2}{2|y_2|\sqrt{L^2 - y_2^2}} + O\left(\eta^2\right)\right], \end{aligned} \tag{58}$$

where “arccot” takes values in $[0, \pi]$. The first term carries the singular behaviour at the rod $\eta = 0$.

VI. GENERAL BOOST: AN ARBITRARY α

We finally consider the boost in a general direction z_1, which is characterized by the angular parameter α, see Eqs. (13)–(15). We employ the method of the previous sections, and after straightforward calculations we again obtain an impulsive *pp*-wave

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x_2^2 + \mathrm{d}y_2^2 + \mathrm{d}z_2^2 + H(x_2, y_2, z_2)\delta(u)\mathrm{d}u^2. \tag{59}$$

Now the profile function

$$H(x_2, y_2, z_2) = \frac{8\sqrt{2}p}{3\pi L^2}\left(\int_{-\infty}^{+\infty} h(z)\mathrm{d}z\right), \tag{60}$$

is an integral of the function

$$\begin{aligned} h(z) = \; & \frac{L^2}{\Sigma} + \frac{L^2}{\Sigma^3}\left[(\xi^2 - \eta^2 - L^2)\frac{y_1}{\xi}\sin\alpha + 2\xi x_1\cos\alpha\right]^2 \\ & + \frac{1}{2}\left(1 - \frac{\xi^2 + \eta^2 - L^2}{\Sigma}\right)\left(\frac{y_2^2}{\xi^2}\sin^2\alpha + \frac{x_2^2}{\eta^2}\cos\alpha\right). \end{aligned} \tag{61}$$

Here the dependence on z is contained in

$$\begin{aligned} y_1 &= z\sin\alpha + z_2\cos\alpha, & \xi^2 &= y_1^2 + y_2^2, \\ x_1 &= z\cos\alpha - z_2\sin\alpha, & \eta^2 &= x_1^2 + x_2^2, \end{aligned} \tag{62}$$

and in Σ, given by Eq. (11). In order to perform the above integration, it is convenient to factorize Σ as

$$\Sigma = \sqrt{[(z - r_1)^2 + s_1^2]\,[(z + r_1)^2 + s_2^2]}, \tag{63}$$

where the parameters s_1 and s_2 are defined by

$$s_1^2 = r_1^2 + A - \frac{2B}{r_1}, \qquad s_2^2 = r_1^2 + A + \frac{2B}{r_1}, \tag{64}$$

and r_1 by the equation (of third order in r_1^2)

$$r_1^6 + Ar_1^4 + \tfrac{1}{4}(A^2 - C)r_1^2 - B^2 = 0, \tag{65}$$

with

$$\begin{aligned} A &= x_2^2 + y_2^2 + z_2^2 - L^2 + 2L^2\cos^2\alpha, \\ B &= L^2 z_2\sin\alpha\cos\alpha, \\ C &= (x_2^2 + y_2^2 + z_2^2 - L^2)^2 + 4L^2(x_2^2 + z_2^2\sin^2\alpha). \end{aligned} \tag{66}$$

Using Cardano’s formula we may write the root r_1^2 as

$$r_1^2 = -\frac{A}{3} + \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}, \tag{67}$$

where

$$\begin{aligned} Q &= \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \\ p &= -\tfrac{1}{3}A^2 + \tfrac{1}{4}(A^2 - C), \\ q &= \tfrac{2}{27}A^3 - \tfrac{1}{12}A(A^2 - C) - B^2. \end{aligned} \tag{68}$$

Notice that for the particular case of the orthogonal boost ($\alpha = 0$) we obtain $r_1 = 0$, $s_1^2 = x_2^2 + (\xi + L)^2$, $s_2^2 = x_2^2 + (\xi - L)^2$, which coincides with Eq. (22), while for the parallel boost ($\alpha = \frac{\pi}{2}$) one has $2r_1^2 = a^2 - y_2^2 - \eta^2 + L^2$ and $2s_1^2 = 2s_2^2 = a^2 + y_2^2 + \eta^2 - L^2$, which is equivalent to Eq. (49) [a is defined in Eq. (50)]. For any α, the integral (60) could now be expressed using elliptic integrals, in principle (because Σ is a square root of a

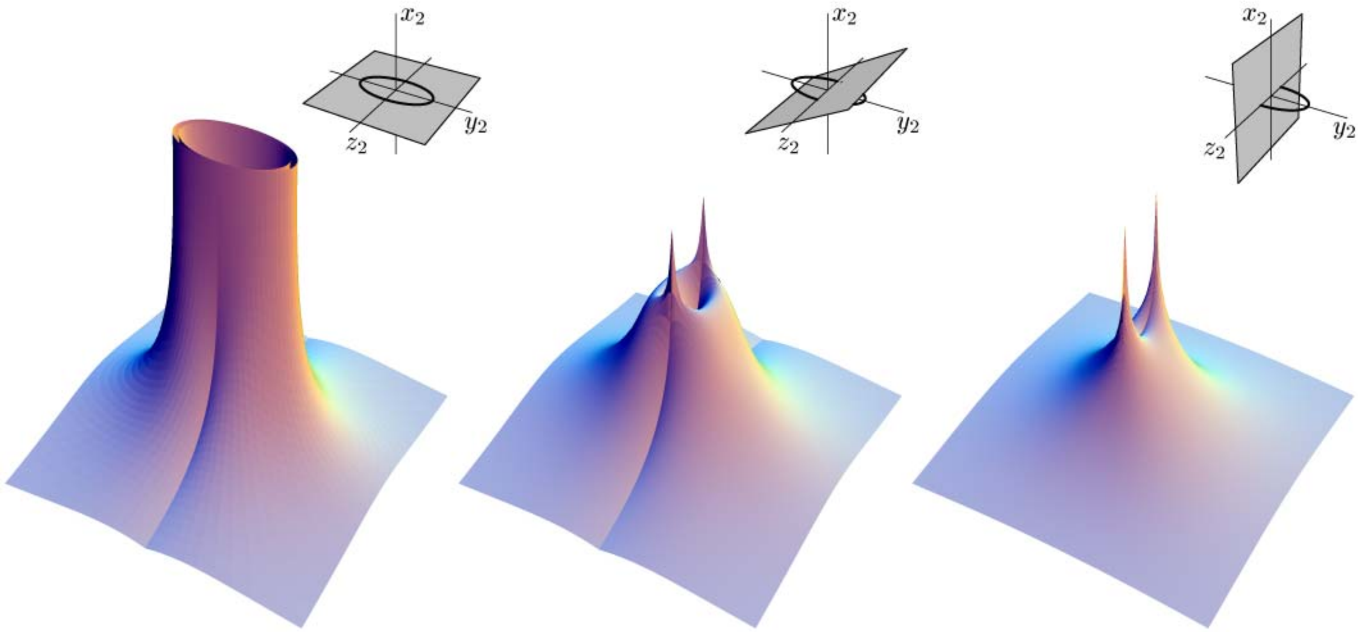

FIG. 3: Plot of the profile function $H(x_2, y_2, z_2)$ given by Eqs. (60) and (61). The integration in Eq. (60) has been performed numerically. The coordinates (x_2, y_2, z_2) span spatial sections of the impulsive wave front $u = 0$, cf. Eq. (59). Since this plot corresponds to a boost in a general direction z_1, the function H is not axially symmetry. We have thus depicted representative plots of the values taken by H over different sections of the three space (x_2, y_2, z_2). In the left figure, in particular, it is evident the ellipse-shaped singularity, cf. Eq. (72).

fourth order polynomial in z) [25, 26]. For example, the simplest first term in Eq. (61) leads to [26]

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\Sigma(z)} = \frac{2}{\sqrt{s_1 s_2 k_1}}\, K(k), \tag{69}$$

where

$$k^2 = \frac{k_1^2 - 1}{k_1^2}, \quad k_1 = \sqrt{D^2 - 1} + D, \quad D = \frac{4r_1^2 + s_1^2 + s_2^2}{2 s_1 s_2}. \tag{70}$$

We can investigate the location of the singularity of the expression (69). This occurs when $k = 1$, i.e. for $s_1 s_2 = 0$. From (64) one gets $s_1^2 s_2^2 = -3r_1^4 - 2Ar_1^2 + C$, so that the singularity is at $r_1^2 = -\frac{1}{3}(A + \sqrt{A^2 + 3C})$. This exactly corresponds to the explicit expression (67) for $Q = 0$, i.e. $27q^2 = -4p^3$. Using the relations (68) and (66), it is straightforward (but somewhat lengthy) to demonstrate that this polynomial condition is satisfied for

$$x_2 = 0, \qquad z_2^2 = (L^2 - y_2^2)\cos^2\alpha. \tag{71}$$

This singular behaviour of the term (69) suggests that there is a singular source located on the wave front ($u = 0$) of the metric (59), precisely in the plane $x_2 = 0$ on the ellipse

$$\left(\frac{y_2}{L}\right)^2 + \left(\frac{z_2}{L\cos\alpha}\right)^2 = 1. \tag{72}$$

Of course the above argument is not conclusive. Rigorously, we should integrate also all the other terms in Eq. (61). This could in principle be done, but it would lead to an involved expression without much practical use. We rather prefer to integrate numerically the full function (61), and display the thus obtained profile H in Fig. 3, which indeed confirms the presence of a singular ellipse within the wave front. This also corresponds to intuitive expectations, since the original static circular source has been boosted in a general direction. [Moreover, it agrees with the following argument: the source of the black ring (1) was located at $y = -\infty$, i.e. $\eta = 0$ and $\xi = L$. In view of Eq. (62), these conditions become Eq. (71), which is unchanged under the boost (15).]

VII. CONCLUSIONS

We have derived the gravitational field generated by a five-dimensional static black ring moving "with the speed of light". More precisely, we have calculated how the Emparan-Reall line element transforms under appropriate boosts, and studied the ultrarelativistic limit when the boost velocity approaches the speed of light. In particular, we have studied in detail two complementary boosts along privileged directions, namely those orthogonal and parallel to the plane containing the ring circle. The resulting line elements represent impulsive pp-waves. These are exact vacuum solutions everywhere except at singular points that are a remnant of the original curvature singularity of the static black ring. In addition, in the case of the orthogonal boost, there is a disk-shaped membrane inside the ring directly inherited from the conical singularity of the static Emparan-Reall spacetime. [Notice that the profile functions obtained via the boosting procedure ultimately provide solutions to equivalent problems of three-dimensional electrostatics (or Newtonian gravity) with a disk or a nonuniform rod source.] Further analysis of the solutions has been supplemented via graphical plots and via suitable expansions of the metric functions. We may also observe here that, if necessary, one could introduce a coordinate system in which the metric coefficients take a continuous form, using the general transformation presented in [6].

It is also worth remarking that, in contradistinction to the well known situation in four dimensions [8], we did not need to perform any infinite subtractions during our calculations. This is essentially due to the faster fall-off of the gravitational potential of a "monopole" in $D>4$, which ensures that all the required integrals are finite. The same simplification occurred in previous investigations of ultrarelativistic boosts in higher dimensions [9, 10, 15], as well as in the case of the boost of particles with multipole moments in $D=4$ (Weyl solutions) [27].

We have concentrated on a static ring containing a disk membrane at $x=+1$, for which there is no conical singularity at infinity. A generalization to the case of a ring with a deficit membrane at $x=-1$ (which extends to infinity) would be straightforward. It would be more interesting to extend our results to the case of rotating black rings. Such work is currently in progress.

Acknowledgments

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APPENDIX A: ELLIPTIC INTEGRALS

In this Appendix we summarize the definitions and the properties of the complete elliptic integrals employed in the main text, following references [25, 26].

1. Definitions

The complete elliptic integrals in trigonometric form are defined by [25]

$$K(k)=\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}}, \tag{A1}$$

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\,\mathrm{d}\alpha, \tag{A2}$$

$$\Pi(\rho,k)=\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{(1-\rho\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}}. \tag{A3}$$

2. Useful identities

They satisfy the identities [26]

$$(k^2-\rho)\Pi(\rho,k)=k^2K(k)-\frac{\rho(1-k^2)}{1-\rho}\Pi\left(\frac{k^2-\rho}{1-\rho},k\right), \tag{A4}$$

$$\Pi(\rho,k)=K(k)-\Pi\left(\frac{k^2}{\rho},k\right)+\frac{\pi}{2}\sqrt{\frac{-\rho}{(1-\rho)(k^2-\rho)}},\quad \text{with}\ \ \rho(1-\rho)^{-1}(k^2-\rho)^{-1}<0. \tag{A5}$$

3. Differential relations

Derivatives of elliptic integrals lead to combinations of the same integrals:

$$\begin{aligned}
\frac{\mathrm{d}K(k)}{\mathrm{d}k}&=\frac{E(k)}{k(1-k^2)}-\frac{K(k)}{k},\\
\frac{\mathrm{d}E(k)}{\mathrm{d}k}&=\frac{E(k)-K(k)}{k},\\
\frac{\partial\Pi(\rho,k)}{\partial k}&=\frac{k}{k^2-\rho}\left[-\Pi(\rho,k)+\frac{E(k)}{1-k^2}\right],\\
\frac{\partial\Pi(\rho,k)}{\partial\rho}&=\frac{1}{2\rho(1-\rho)}\\
&\quad\times\left[\frac{k^2-\rho^2}{k^2-\rho}\Pi(\rho,k)-K(k)-\frac{\rho}{k^2-\rho}E(k)\right].
\end{aligned} \tag{A6}$$

4. Series representations

The behaviour near $k=0$ is given by

$$K(k)=\frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+O(k^6)\right), \tag{A7}$$

$$E(k)=\frac{\pi}{2}\left(1-\frac{1}{4}k^2-\frac{3}{64}k^4+O(k^6)\right), \tag{A8}$$

$$\Pi(\rho,k)=\frac{\pi}{2}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\mu}\frac{(2\mu-1)!!(2\nu-1)!!}{(2\mu)!!(2\nu)!!}k^{2\nu}\rho^{\mu-\nu},\quad \text{with}\ \ |\rho|<1. \tag{A9}$$

Near the singular point $k=1$ one has

$$K(k) = -\frac{1}{2}\log\frac{1-k^2}{16} - \frac{1}{8}\left(2+\log\frac{1-k^2}{16}\right)(1-k^2) - \frac{9}{128}\left(\frac{7}{3}+\log\frac{1-k^2}{16}\right)(1-k^2)^2 + O\left((1-k^2)^3\right), \quad \text{(A10)}$$

$$E(k) = 1 - \frac{1}{4}\left(1+\log\frac{1-k^2}{16}\right)(1-k^2) - \frac{3}{32}\left(\frac{13}{6}+\log\frac{1-k^2}{16}\right)(1-k^2)^2 + O\left((1-k^2)^3\right), \quad \text{(A11)}$$

$$\Pi(\rho,k) = \frac{1}{1-\rho}\log\frac{4}{\sqrt{1-k^2}} + \frac{\sqrt{-\rho}}{1-\rho}\arctan\sqrt{-\rho} + O(1-k^2), \qquad \text{with } \rho<0. \quad \text{(A12)}$$

5. Useful integrals: orthogonal boost

In Sec. IV we employed the following integrals:

$$\int_0^\infty \frac{\mathrm{d}z}{\sqrt{(z^2+a^2)(z^2+b^2)}} = \frac{1}{a}K(k), \quad \text{(A13)}$$

$$\int_0^\infty \frac{z^2\mathrm{d}z}{\sqrt{(z^2+a^2)^3(z^2+b^2)^3}} = \frac{a^2+b^2}{a(a^2-b^2)^2}K(k) - \frac{2a}{(a^2-b^2)^2}E(k), \quad \text{(A14)}$$

$$\int_0^\infty \frac{\mathrm{d}z}{(z^2+c^2)\sqrt{(z^2+a^2)(z^2+b^2)}} = \frac{1}{a(b^2-c^2)}\left[\frac{b^2}{c^2}\Pi\left(-\frac{b^2-c^2}{c^2},k\right) - K(k)\right], \quad \text{(A15)}$$

where

$$k = \frac{\sqrt{a^2-b^2}}{a}, \quad a>b>0, \quad c\neq 0. \quad \text{(A16)}$$

6. Useful integrals: parallel boost

The integrals used in Sec. V are:

$$\int_0^\infty \frac{\mathrm{d}z}{\sqrt{z^4+2b^2z^2+a^4}} = \frac{1}{a}K(k), \quad \text{(A17)}$$

$$\int_0^\infty \frac{z^2\mathrm{d}z}{\sqrt{(z^4+2b^2z^2+a^4)^3}} = \frac{a}{a^4-b^4}E(k) - \frac{1}{2a(a^2-b^2)}K(k), \quad \text{(A18)}$$

$$\int_0^\infty \frac{\mathrm{d}z}{(z^2+c^2)\sqrt{z^4+2b^2z^2+a^4}} = \quad \text{(A19)}$$
$$\frac{1}{a(a^2-c^2)}\left[\frac{a^2+c^2}{2c^2}\Pi\left(-\frac{(a^2-c^2)^2}{4c^2a^2},k\right) - K(k)\right],$$

with

$$k = \frac{\sqrt{a^2-b^2}}{\sqrt{2}a}, \quad a^2>b^2>-\infty, \quad a^2>0, \quad c\neq 0. \quad \text{(A20)}$$

APPENDIX B: THE WEYL TENSOR FOR $H_\perp$

Here we present explicitly the frame components of the Weyl tensor in the case of the metric (27), (31) describing a black ring boosted in an orthogonal direction. Using Eq. (A6), from Eq. (38) with Eq. (31) we obtain

$$\Psi_{11} = \frac{2\sqrt{2}p}{3\pi L^2}\frac{1}{\xi^2\sqrt{[(\xi+L)^2+x_2^2]^3}} \times\Big\{-\left[L^6+(x_2^2-2\xi^2)L^4+(\xi^4-x_2^4+8\xi^2x_2^2)L^2 - x_2^2(x_2^2+\xi^2)^2\right]\frac{K(k)}{(\xi-L)^2+x_2^2} + \left[L^8+(2x_2^2-7\xi^2)L^6+\xi^2(11\xi^2+7x_2^2)L^4 - (x_2^2+\xi^2)(2x_2^4-13\xi^2x_2^2+5\xi^4)L^2 - x_2^2(x_2^2+\xi^2)^3\right] \times\frac{E(k)}{[(\xi-L)^2+x_2^2]^2}\Big\}\delta(u) - \frac{1}{6}R_{uu}, \quad \text{(B1)}$$

$$\Psi_{13} = \frac{2\sqrt{2}p}{3\pi L^2}\frac{x_2}{\xi\sqrt{[(\xi+L)^2+x_2^2]^3}} \times\Big\{-\frac{3L^4+4(x_2^2-\xi^2)L^2+(\xi^2+x_2^2)^2}{(\xi-L)^2+x_2^2}K(k) + \left[3L^6+(11\xi^2+7x_2^2)L^4+5(x_2^2-3\xi^2)(x_2^2+\xi^2)L^2 + (x_2^2+\xi^2)^3\right]\frac{E(k)}{[(\xi-L)^2+x_2^2]^2}\Big\}\delta(u), \quad \text{(B2)}$$

$$\Psi_{22} = \frac{4\sqrt{2}p}{3\pi L^2}\frac{1}{\xi\sqrt{(\xi+L)^2+x_2^2}} \times\left\{(L^2-\xi^2-x_2^2)K(k) - \frac{L^4-(\xi^2+x_2^2)^2}{(\xi-L)^2+x_2^2}E(k)\right\}\delta(u) - \frac{1}{6}R_{uu}, \quad \text{(B3)}$$

$$\Psi_{33} = -(\Psi_{11}+\Psi_{22}). \quad \text{(B4)}$$

The last equation follows from the tracelessness of the Weyl tensor.

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