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Geodesic deviation: useful tool for understanding higher dimensional spacetimes

R. Švarc and J. Podolský

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

Abstract. General method which is useful for investigation of geometrical and physical properties of an arbitrarily dimensional spacetime is presented. This is based on the systematic analysis of relative motion of free test particles described by equation of geodesic deviation which is rewritten with respect to a natural orthonormal frame and then decomposed into a canonical set of transverse, longitudinal and Newton-Coulomb-type components, isotropic influence of a cosmological constant, and contributions arising from matter content of the universe. The utility of this method is illustrated on the family of Kundt spacetimes in higher dimensions, in particular pp-waves.

Keywords: Higher dimensional gravity, geodesic deviation, algebraic classification of spacetimes.
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EQUATION OF GEODESIC DEVIATION

Investigation of relative motion of free test particles reveals important informations about geometry of the spacetime. Here, we generalise the classic work [1] to an arbitrary number of dimensions, for more details and references see [2]. Relative motion of closeby free test particles is described by the equation of geodesic deviation

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\,\tau^2} = R^\mu{}_{\alpha\beta\nu}\, u^\alpha u^\beta Z^\nu\,, \tag{1}$$

where $R^\mu{}_{\alpha\beta\nu}$ is the Riemann curvature tensor, u^α are components of the velocity vector of the reference particle, the parameter τ is a proper time of the observer's timelike geodesic, and Z^μ are components of the separation vector which connects the reference particle with another nearby test particle.

To obtain results independent of the choice of the coordinates we introduce orthonormal frame $\{\mathbf{e}_a\}$. The timelike vector is identified with the observer's velocity vector, $\mathbf{e}_{(0)} = \mathbf{u}$, and remaining $\mathbf{e}_{(\mathrm{i})}$ represent $D-1$ perpendicular spacelike vectors, $\mathbf{e}_a \cdot \mathbf{e}_b \equiv g_{\alpha\beta}\, e^\alpha_a e^\beta_b = \eta_{ab}$. We also define a real null frame $\{\mathbf{k}, \mathbf{l}, \mathbf{m}_i\}$ by the relations

$$\mathbf{k} = \tfrac{1}{\sqrt{2}}(\mathbf{u} + \mathbf{e}_{(1)})\,, \quad \mathbf{l} = \tfrac{1}{\sqrt{2}}(\mathbf{u} - \mathbf{e}_{(1)})\,, \quad \mathbf{m}_i = \mathbf{e}_{(i)} \;\text{ for }\; i = 2, \dots, D-1\,, \tag{2}$$

where $\mathbf{k}$ and $\mathbf{l}$ are future oriented null vectors, and $\mathbf{m}_i$ are $D-2$ spatial vectors orthogonal to them, i.e., $\mathbf{k} \cdot \mathbf{l} = -1$, $\mathbf{m}_i \cdot \mathbf{m}_j = \delta_{ij}$, $\mathbf{k} \cdot \mathbf{k} = 0 = \mathbf{l} \cdot \mathbf{l}$ and $\mathbf{k} \cdot \mathbf{m}_i = 0 = \mathbf{l} \cdot \mathbf{m}_i$. Using this definition, for the zeroth frame-component of the equation (1) we immediately obtain $\mathrm{d}^2 Z^{(0)} / \mathrm{d}\,\tau^2 = -R_{\mu\alpha\beta\nu}\, u^\mu u^\alpha u^\beta Z^\nu = 0$. Therefore, we can set $Z^{(0)} = 0$ which means that

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test particles stay in the same spacelike hypersurfaces synchronized by a proper time τ. By projecting the geodesic deviation equation onto spatial frame vectors $\mathbf{e}^{(\mathrm{i})}$ we get

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})}\, Z^{(\mathrm{j})}\,, \tag{3}$$

where $\ddot{Z}^{(\mathrm{i})} = e^{(\mathrm{i})}_{\mu}\, \frac{\mathrm{D}^2 Z^{\mu}}{\mathrm{d}\,\tau^2}$ and $R_{(\mathrm{i})(0)(0)(\mathrm{j})} \equiv R_{\mu\alpha\beta\nu}\, e^{\mu}_{(\mathrm{i})} u^{\alpha} u^{\beta} e^{\nu}_{(\mathrm{j})}$, where $\mathrm{i},\mathrm{j} = 1,2,\ldots,\mathrm{D}-1$. Next, it is natural to decompose the curvature tensor into the traceless Weyl part C_{abcd} and specific combinations of the Ricci tensor R_{ab} and Ricci scalar R. Using also the Einstein equations we obtain

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = \frac{2\Lambda\,\delta_{\mathrm{ij}}}{(D-1)(D-2)} + C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \frac{8\pi}{D-2}\left[T_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}}\left(T_{(0)(0)} + \frac{2\,T}{D-1}\right)\right]. \tag{4}$$

The components of the Weyl tensor in null frame $\{\mathbf{k}, \mathbf{l}, \mathbf{m}_i\}$ are fully determined by the following scalars (grouped by their boost weight),

$$\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}\, k^a m_i^b k^c m_j^d\,, && \\
\Psi_{1^{ijk}} &= C_{abcd}\, k^a m_i^b m_j^c m_k^d\,, &\qquad \Psi_{1T^i} &= C_{abcd}\, k^a l^b k^c m_i^d\,,\\
\Psi_{2^{ijkl}} &= C_{abcd}\, m_i^a m_j^b m_k^c m_l^d\,, &\qquad \Psi_{2S} &= C_{abcd}\, k^a l^b l^c k^d\,,\\
\Psi_{2^{ij}} &= C_{abcd}\, k^a l^b m_i^c m_j^d\,, &\qquad \Psi_{2T^{ij}} &= C_{abcd}\, k^a m_i^b l^c m_j^d\,,\\
\Psi_{3^{ijk}} &= C_{abcd}\, l^a m_i^b m_j^c m_k^d\,, &\qquad \Psi_{3T^i} &= C_{abcd}\, l^a k^b l^c m_i^d\,,\\
\Psi_{4^{ij}} &= C_{abcd}\, l^a m_i^b l^c m_j^d\,, &&
\end{aligned} \tag{5}$$

where $i,j,k,l = 2,\ldots,D-1$. The scalars in the left column are independent, up to the obvious constraints, while those in the right column can be expressed as their contractions. All other frame components can be obtained using the symmetries of the Weyl tensor. Our notation which uses $\Psi_{A^{\cdots}}$ in any dimension is simply related to the notations employed, e.g., in the works [3, 4], [5, 6] or [7], as summarized in Table 1. Using the relations (2) and (5), a straightforward calculation leads to

$$\begin{aligned}
C_{(1)(0)(0)(j)} &= \tfrac{1}{\sqrt{2}}\left(\Psi_{1T^j} - \Psi_{3T^j}\right), &\qquad C_{(i)(0)(0)(1)} &= \tfrac{1}{\sqrt{2}}\left(\Psi_{1T^i} - \Psi_{3T^i}\right),\\
C_{(1)(0)(0)(1)} &= \Psi_{2S}\,, &\qquad C_{(i)(0)(0)(j)} &= -\tfrac{1}{2}\left(\Psi_{0^{ij}} + \Psi_{4^{ij}}\right) - \Psi_{2T^{(ij)}}\,.
\end{aligned} \tag{6}$$

Invariant general form of the equation of geodesic deviation (3) can thus be rewritten as

$$\begin{aligned}
\ddot{Z}^{(1)} &= \tfrac{2\Lambda}{(D-1)(D-2)}\, Z^{(1)} + \Psi_{2S}\, Z^{(1)} + \tfrac{1}{\sqrt{2}}\left(\Psi_{1T^j} - \Psi_{3T^j}\right) Z^{(j)}\\
&\quad + \tfrac{8\pi}{D-2}\left[T_{(1)(1)}\, Z^{(1)} + T_{(1)(j)}\, Z^{(j)} - \left(T_{(0)(0)} + \tfrac{2}{D-1}\, T\right) Z^{(1)}\right],\\
\ddot{Z}^{(i)} &= \tfrac{2\Lambda}{(D-1)(D-2)}\, Z^{(i)} - \Psi_{2T^{(ij)}}\, Z^{(j)} + \tfrac{1}{\sqrt{2}}\left(\Psi_{1T^i} - \Psi_{3T^i}\right) Z^{(1)} - \tfrac{1}{2}\left(\Psi_{0^{ij}} + \Psi_{4^{ij}}\right) Z^{(j)}\\
&\quad + \tfrac{8\pi}{D-2}\left[T_{(i)(1)}\, Z^{(1)} + T_{(i)(j)}\, Z^{(j)} - \left(T_{(0)(0)} + \tfrac{2}{D-1}\, T\right) Z^{(i)}\right].
\end{aligned} \tag{7}$$

In the *vacuum case*, i.e. $T_{ab} = 0$, the effect of the gravitational field on particles consists of cosmological constant Λ and the Weyl scalars $\Psi_{0^{ij}}$, Ψ_{1T^i}, Ψ_{2S}, $\Psi_{2T^{ij}}$, Ψ_{3T^i} and $\Psi_{4^{ij}}$.

TABLE 1. Equivalent notations used for the Weyl scalars.

	Ψ_{2S}	$\Psi_{2T^{ij}}$	Ψ_{1T^j}	Ψ_{3T^j}	$\Psi_{0^{ij}}$	$\Psi_{4^{ij}}$
refs. [3, 4]	$-C_{0101}$	$-C_{0i1j}$	$-C_{010j}$	C_{101j}	C_{0i0j}	C_{1i1j}
refs. [5, 6]	$-\Phi$	$-\Phi_{ij}$		Ψ_j		$2\Psi_{ij}$
ref. [7]	$-\Phi$	$-\Phi_{ij}$	$-\Psi_j$	Ψ'_j	Ω_{ij}	Ω'_{ij}

OUTLINE OF THE SPECIFIC EFFECTS

The presence of the *cosmological constant* Λ is encoded in the term

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \frac{2\Lambda}{(D-1)(D-2)} \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}. \tag{8}$$

These isotropic relative motions of test particles are characteristic for spacetimes of constant curvature, namely Minkowski space, de Sitter space and anti-de Sitter space.

The terms Ψ_{2S} and $\Psi_{2T^{(ij)}}$ represent *Newton–Coulomb components* of a gravitational field and the motion of test particles is given by

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \begin{pmatrix} \Psi_{2S} & 0 \\ 0 & -\Psi_{2T^{(ij)}} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \tag{9}$$

where $\Psi_{2S} = \Psi_{2T^k{}^k}$. These terms are typically present in type D spacetimes.

Ψ_{3T^i} and Ψ_{1T^i} represent the *longitudinal components* of a gravitational field with respect to directions $+\mathbf{e}_{(1)}$ and $-\mathbf{e}_{(1)}$, respectively. Such terms cause deformations

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \Psi_{\mathrm{A}T^j} \\ \Psi_{\mathrm{A}T^i} & 0 \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \tag{10}$$

where $\Psi_{\mathrm{A}T^i}$ represents Ψ_{3T^i} or $-\Psi_{1T^i}$ which are equivalent under $\mathbf{k} \leftrightarrow \mathbf{l}$. These $D-2$ scalars combine motion in the privileged spatial direction $\mathbf{e}_{(1)}$ with motion in the transverse directions $\mathbf{e}_{(i)}$. Longitudinal effects given by Ψ_{3T^i} occur in spacetimes of type III.

The components $\Psi_{4^{ij}}$ and $\Psi_{0^{ij}}$ can be interpreted as a *transverse gravitational waves* propagating in the direction $+\mathbf{e}_{(1)}$ and $-\mathbf{e}_{(1)}$. These parts of a gravitational field are fully equivalent under $\mathbf{k} \leftrightarrow \mathbf{l}$ and influence the test particles as

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \Psi_{\mathrm{A}^{ij}} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \tag{11}$$

where $\Psi_{\mathrm{A}^{ij}}$ represents $\Psi_{4^{ij}}$ or $\Psi_{0^{ij}}$, and causes a purely transverse effect because there is no acceleration in the privileged spatial direction $\mathbf{e}_{(1)}$. The set of scalars $\Psi_{\mathrm{A}^{ij}}$ forms a symmetric and traceless matrix of dimension $(D-2)\times(D-2)$. Spacetimes of algebraic type N can be thus interpreted as exact gravitational waves. The amplitude matrix $\Psi_{\mathrm{A}^{ij}}$ describing gravitational waves has $N \equiv \frac{1}{2}D(D-3)$ independent components corresponding to polarization modes. Freedom in a choice of the frame is given by spatial rotations $\tilde{\mathbf{m}}_i = \Phi_i{}^j\,\mathbf{m}_j$ with $\Phi_i{}^j\,\Phi_k{}^l\,\delta_{jl} = \delta_{ik}$ ($\tilde{\mathbf{k}} = \mathbf{k}$, $\tilde{\mathbf{l}} = \mathbf{l}$), which has generally $N_{rot} \equiv \frac{1}{2}(D-2)(D-3)$ independent parameters representing the generators of

$SO(D-2)$. The number of physical degrees of freedom is thus $N - N_{rot} = D-3$ which corresponds to number of independent eigenvalues of $\Psi_{A^{ij}}$. With respect to the signum of the eigenvalues there can be $\binom{D-2}{2} = \frac{1}{2}(D-2)(D-3)$ physically different cases.

EXAMPLE: HIGHER DIMENSIONAL VSI PP-WAVES

We assume a vacuum spacetime admitting a covariantly constant null vector field **k** with all scalar invariants vanishing. It belongs to pp-waves subclass of the Kundt family, see [8]. In natural coordinates the metric takes the form

$$\mathrm{d}s^2 = \delta_{ij}(x^k,u)\,\mathrm{d}x^i\mathrm{d}x^j + 2\,e_i(x^k,u)\,\mathrm{d}x^i\mathrm{d}u - 2\,\mathrm{d}u\,\mathrm{d}r + c(x^k,u)\,\mathrm{d}u^2\,. \tag{12}$$

The interpretation frame adapted to an observer with the general velocity **u** is

$$\begin{aligned} \mathbf{k} &= \tfrac{1}{\sqrt{2}\dot u}\,\partial_r\,,\\ \mathbf{l} &= \left(\sqrt{2}\,\dot r - \tfrac{1}{\sqrt{2}\dot u}\right)\partial_r + \sqrt{2}\,\dot u\,\partial_u + \sqrt{2}\,\dot x^2\,\partial_{x^2} + \ldots + \sqrt{2}\,\dot x^{D-1}\partial_{x^{D-1}}\,,\\ \mathbf{m}_i &= \tfrac{1}{\dot u}(e_k\dot u + g_{jk}\,\dot x^j)\,m_i^k\,\partial_r + m_i^2\,\partial_{x^2} + \ldots + m_i^{D-1}\partial_{x^{D-1}}\,, \end{aligned} \tag{13}$$

and the only nonvanishing Weyl component is $\Psi_{4^{ij}}$,

$$\Psi_{4^{ij}} = \left[4e_{[k,m],l}\,\dot x^m\dot u + \left(2e_{(k,l),u} - c_{,kl} + 2\delta^{pq}e_{[p,k]}\,e_{[q,l]}\right)\dot u^2\right] m^k_{(i}m^l_{j)}\,. \tag{14}$$

If the functions e_i can be globally removed, the frame (13) is paralelly transported and (14) becomes simply $\Psi_{4^{ij}} = -\dot u^2 c_{,ij}$. Einstein's equation, $\delta^{ij}c_{,ij} = 0$, guarantees its tracelessness. Using $m_i^j = \delta_i^j$, the equations of geodesic deviation (7) thus reduces to

$$\ddot Z^{(1)} = 0\,, \qquad \ddot Z^{(i)} = \tfrac{1}{2}\,\dot u^2\,c_{,ij}\,Z^{(j)}\,. \tag{15}$$

Detailed discussion and explicit solutions can be found in [2].

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