

# Explicit algebraic classification of Kundt geometries in any dimension

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**Abstract.** We present an algebraic classification, based on the null alignment properties of the Weyl tensor, of the general Kundt class of spacetimes in arbitrary dimension  $D$  for which the non-expanding, non-twisting, shear-free null direction  $\mathbf{k}$  is a (multiple) Weyl aligned null direction (WAND). No field equations are used, so that the results apply not only to Einstein's gravity and its direct extension to higher dimensions, but also to any metric theory of gravity which admits the Kundt spacetimes. By an explicit evaluation of the Weyl tensor in a natural null frame we demonstrate that all Kundt geometries are of type I(b) or more special, and we derive simple necessary and sufficient conditions under which  $\mathbf{k}$  becomes a double, triple or quadruple WAND. All possible algebraically special types, including the refinement to subtypes, are identified, namely II(a), II(b), II(c), II(d), III(a), III(b), N, O, II<sub>*i*</sub>, III<sub>*i*</sub>, D(a), D(b), D(c) and D(d). The corresponding conditions are surprisingly clear and expressed in an invariant geometric form. Some of them are always satisfied in four dimensions. To illustrate our classification scheme, we apply it to the most important subfamilies of the Kundt class, namely the pp-waves, the VSI spacetimes, and generalization of the Bertotti–Robinson, Nariai, and Plebański–Hacyan direct-product spacetimes of any dimension.

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## 1. Introduction

Half a century ago, Wolfgang Kundt [1, 2] introduced and started to study one of the most important classes of exact spacetimes in Einstein's general relativity theory. These spacetimes are defined by a specific geometric property, namely that they admit a null geodesic congruence which is non-expanding, non-twisting, and shear-free, see chapter 31 of [3] or chapter 18 of [4]. Such a wide Kundt class admits various vacuum and pure radiation solutions, possibly with any value of the cosmological constant  $\Lambda$ , electromagnetic field, and other matter fields, including supersymmetry. These spacetimes may be of all algebraic types (namely of the Petrov type N, III, D, II, I, or conformally flat). Interestingly, the Kundt geometries can also be explicitly extended to any number  $D$  of higher dimensions.

Among the famous subclasses of Kundt geometries (in four and higher dimensions) there are pp-waves which admit a covariantly constant null vector field [3–11]. These include relativistic gyratons [12–18] which represent the fields of localised spinning sources that propagate with the speed of light. The Kundt class also contains VSI and CSI spacetimes [5–11, 19–21] for which all polynomial scalar invariants constructed from the Riemann tensor and its derivatives vanish and are constant, respectively.

In four dimensions, conformally flat pure radiation Kundt spacetimes provide an exceptional case for the invariant classification of exact solutions [22–25] and tests of the GHP and GIF formalisms [26–28]. All pure radiation type D solutions are known [29], and all electrovacuum Kundt solutions of type D with an arbitrary cosmological constant  $\Lambda$  were also found and studied [30–37]. These contain a subfamily of direct-product spacetimes, namely the Bertotti–Robinson, (anti-)Nariai, and Plebański–Hacyan spacetimes of type O and D, see chapter 7 of [4]. Together with Minkowski and (anti-)de Sitter spaces they form backgrounds on which non-expanding gravitational waves and gyratons of types N and II propagate [17, 18, 38–47]. It is an interesting open problem to find and analyse possible extensions of all such spacetimes to higher dimensions.

The study of Kundt spacetimes is thus an active research area with a lot of applications, ranging from purely mathematical aspects to investigation of various physical properties and models. In this paper we consider the fully general class of Kundt geometries in an arbitrary dimension  $D \geq 4$ , without *a priori* assuming any field equations. Specifically, we present their explicit and complete classification into the primary algebraic types and corresponding subtypes based on the WAND multiplicity of the optically privileged null vector field  $\mathbf{k}$ . We hope that the analysis may help us to understand the rich and interesting family of Kundt geometries and to elucidate mutual relations between its subclasses. The results apply to *any* dimension, the algebraic classification within standard general relativity is simply obtained by setting  $D = 4$  and applying Einstein's field equations.

In section 2, we start with fully general Kundt metric, introduce a suitable null frame, and project the Weyl tensor onto it. The corresponding Weyl scalars of all boost weights are employed for the algebraic classification. In subsequent sections 3–7 we derive the necessary and sufficient conditions for the specific algebraic types II, III, N, O, D, including their subtypes. These are summarized in section 8. The subclasses of pp-waves, VSI spacetimes, and generalized direct-product spacetimes are discussed in the final sections 9–11, respectively. Explicit coordinate components of the Riemann, Ricci, and Weyl tensors for the generic Kundt geometry are given in Appendix A, and for the algebraically special Kundt spacetimes in Appendix B.

## 2. The Weyl tensor of a general Kundt geometry

In an arbitrary dimension  $D$ , the Kundt class is defined by admitting a non-expanding, twist-free, and shear-free null congruence. Such a geometric definition can be naturally expressed in terms of the optical scalars  $\Theta$  (expansion),  $A^2$  (twist), and  $\sigma^2$  (shear) [5, 6, 48, 49] which, for affinely parameterized geodesic null congruence generated by a null vector field  $\mathbf{k}$ , are

$$\Theta = \frac{1}{D-2} k^a{}_{;a}, \quad A^2 = -k_{[a;b]} k^{a;b}, \quad \sigma^2 = k_{(a;b)} k^{a;b} - \frac{1}{D-2} (k^a{}_{;a})^2. \quad (1)$$

For the Kundt family  $\Theta = 0$ ,  $A = 0$ , and  $\sigma = 0$ , in which case there exist suitable coordinates such that any Kundt spacetime can be written as [1–4, 50–52]

$$ds^2 = g_{pq}(u, x) dx^p dx^q + 2 g_{up}(r, u, x) du dx^p - 2 du dr + g_{uu}(r, u, x) du^2. \quad (2)$$

The coordinate  $r$  is the affine parameter along the ‘‘optically privileged’’ null congruence ( $\mathbf{k} = \partial_r$ ),  $u = \text{const.}$  label null (wave)surfaces, and  $x \equiv (x^2, x^3, \dots, x^{D-1})$  are  $(D-2)$  spatial coordinates in the transverse Riemannian space. Notice that the corresponding spatial part  $g_{pq}$  of the metric must be independent of  $r$ , all other metric components  $g_{up}$  and  $g_{uu}$  can, in principle, be functions of all the coordinates  $(r, u, x)$ .

For such most general Kundt line element (2) the Christoffel symbols and the coordinate components of the Riemann, Ricci, and Weyl curvature tensors are presented in Appendix A.

As in [5, 6, 9], algebraic classification of spacetimes here refers to determining specific properties of the Weyl tensor components (of different boost weights) with respect to a null frame  $\{\mathbf{k}, \mathbf{l}, \mathbf{m}_i\}$  whose vectors satisfy the normalization conditions  $\mathbf{k} \cdot \mathbf{l} = -1$ ,  $\mathbf{m}_i \cdot \mathbf{m}_j = \delta_{ij}$ , and  $\mathbf{k} \cdot \mathbf{k} = 0 = \mathbf{l} \cdot \mathbf{l}$ ,  $\mathbf{m}_i \cdot \mathbf{k} = 0 = \mathbf{m}_i \cdot \mathbf{l}$ . In this work we will denote these Weyl tensor components as

$$\begin{aligned} \Psi_{0ij} &= C_{abcd} k^a m_i^b k^c m_j^d, & \Psi_{1T^i} &= C_{abcd} k^a l^b k^c m_i^d, & \Psi_{1ijk} &= C_{abcd} k^a m_i^b m_j^c m_k^d, \\ \Psi_{2S} &= C_{abcd} k^a l^b l^c k^d, & \Psi_{2^{ijk}l} &= C_{abcd} m_i^a m_j^b m_k^c m_l^d, \\ \Psi_{2T^{ij}} &= C_{abcd} k^a m_i^b l^c m_j^d, & \Psi_{2^{ij}} &= C_{abcd} k^a l^b m_i^c m_j^d, \\ \Psi_{3T^i} &= C_{abcd} l^a k^b l^c m_i^d, & \Psi_{3^{ijk}} &= C_{abcd} l^a m_i^b m_j^c m_k^d, \\ \Psi_{4^{ij}} &= C_{abcd} l^a m_i^b l^c m_j^d, \end{aligned} \quad (3)$$

where the indices  $i, j, k, l = 2, \dots, D-1$  label the spatial Cartesian vectors  $\mathbf{m}_2, \mathbf{m}_3, \dots, \mathbf{m}_{D-1}$ . The scalars (3), listed by their boost weight, directly generalize the standard Newman–Penrose complex scalars  $\Psi_A$  known from the  $D = 4$  case [53, 54]. All such scalars respect the standard symmetries of the Weyl tensor, for example

$$\Psi_{0ij} = \Psi_{0(ij)}, \quad \Psi_{0^k} = 0, \quad \Psi_{4^{ij}} = \Psi_{4(ij)}, \quad \Psi_{4^k} = 0. \quad (4)$$

There are relations between the scalars in the left and right columns of (3), namely

$$\begin{aligned} \Psi_{1T^i} &= \Psi_{1^k k i}, \\ \Psi_{2S} &= \Psi_{2T^k k}, \quad \Psi_{2T^{[ij]}} = \frac{1}{2} \Psi_{2^{ij}}, \quad \Psi_{2T^{(ij)}} = \frac{1}{2} \Psi_{2^{ikj} k}, \\ \Psi_{3T^i} &= \Psi_{3^k k i}. \end{aligned} \quad (5)$$

We should also emphasize that our notation, which in any dimension uses the symbols  $\Psi_{A\dots}$ , is equivalent to the notations employed elsewhere, namely in [5, 9], [55, 56], and [6, 57]. The identifications for the corresponding components are summarized

	ref. [5, 9]	ref. [55, 56]	ref. [6, 57]
$\Psi_{0^{ij}}$	$C_{0i0j}$		$\Omega_{ij}$
$\Psi_{1T^j}$	$-C_{010j}$		$-\Psi_j$
$\Psi_{1^{ijk}}$	$C_{0ijk}$		$\Psi_{ijk}$
$\Psi_{2S}$	$-C_{0101}$	$-\Phi$	$-\Phi$
$\Psi_{2T^{ij}}$	$-C_{0i1j}$	$-\Phi_{ij}$	$-\Phi_{ij}$
$\Psi_{2T^{(ij)}}$			$-\Phi_{ij}^S$
$\Psi_{2T^{[ij]}}$			$-\Phi_{ij}^A$
$\Psi_{2^{ij}}$	$-C_{01ij}$		$-2\Phi_{ij}^A$
$\Psi_{2^{ijkl}}$	$C_{ijkl}$		$\Phi_{ijkl}$
$\Psi_{3T^j}$	$C_{101j}$	$\Psi_j$	$\Psi'_j$
$\Psi_{3^{ijk}}$	$-C_{1ijk}$		$-\Psi'_{ijk}$
$\Psi_{4^{ij}}$	$C_{1i1j}$	$2\Psi_{ij}$	$\Omega'_{ij}$

**Table 1.** Different equivalent notations used in the literature for the Weyl scalars, in particular the GHP formalism [6, 57].

in table 1. The different signs arise from the opposite orientation of the null vector  $\mathbf{l} \rightarrow -\mathbf{l}$  resulting in a different normalization condition  $\mathbf{k} \cdot \mathbf{l} = +1$  (see [53]).

To evaluate all the scalars (3) for the general Kundt spacetime (2), we project the coordinate components of the Weyl tensor (A.30) onto the natural null frame ‡

$$\begin{aligned}
 \mathbf{k} &= \partial_r, \\
 \mathbf{l} &= \frac{1}{2}g_{uu}\partial_r + \partial_u, \\
 \mathbf{m}_i &= m_i^p(g_{up}\partial_r + \partial_p),
 \end{aligned} \tag{6}$$

where the coefficients  $m_i^p$  satisfy  $g_{pq}m_i^p m_j^q = \delta_{ij}$  to fulfil the normalization conditions  $\mathbf{m}_i \cdot \mathbf{m}_j = \delta_{ij}$  and  $\mathbf{k} \cdot \mathbf{l} = -1$ . The vector  $\mathbf{k}$  is oriented along the optically privileged null congruence  $\partial_r$  defining the Kundt family. Direct calculation yields

$$\Psi_{0^{ij}} = 0, \tag{7}$$

$$\Psi_{1T^j} = m_j^p C_{rp ru}, \tag{8}$$

$$\Psi_{2S} = -C_{ruru}, \tag{9}$$

$$\Psi_{2T^{ij}} = m_i^p m_j^q (-C_{rp ru} g_{uq} + C_{rp uq}), \tag{10}$$

$$\Psi_{3T^j} = m_j^p \left(-\frac{1}{2}C_{rp ru} g_{uu} + C_{ruru} g_{up} - C_{ruup}\right), \tag{11}$$

$$\begin{aligned}
 \Psi_{4^{ij}} &= m_i^p m_j^q \left[ C_{ruru} g_{up} g_{uq} - \frac{1}{2}g_{uu}(C_{rp ru} g_{uq} + C_{rq ru} g_{up}) \right. \\
 &\quad \left. + \frac{1}{2}g_{uu}(C_{rp uq} + C_{rq up}) - (C_{ruup} g_{uq} + C_{ruuq} g_{up}) + C_{up uq} \right],
 \end{aligned} \tag{12}$$

$$\Psi_{1^{ijk}} = m_i^p m_j^m m_k^q C_{rp mq}, \tag{13}$$

$$\Psi_{2^{ij}} = m_i^p m_j^q (C_{rq ru} g_{up} - C_{rp ru} g_{uq} + C_{rupq}), \tag{14}$$

$$\Psi_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q (C_{rp nq} g_{um} - C_{rm nq} g_{up} + C_{rq mp} g_{un} - C_{rn mp} g_{uq} + C_{mp nq}), \tag{15}$$

$$\begin{aligned}
 \Psi_{3^{ijk}} &= m_i^p m_j^m m_k^q \left( \frac{1}{2}C_{rp mq} g_{uu} + C_{rm ru} g_{up} g_{uq} - C_{rq ru} g_{up} g_{um} \right. \\
 &\quad \left. + C_{rq up} g_{um} - C_{rm up} g_{uq} - C_{rumq} g_{up} + C_{up mq} \right).
 \end{aligned} \tag{16}$$

‡ In this paper,  $i, j, k, l$  are frame labels, whereas the indices  $p, q, m, n$  denote the spatial coordinate components. For example,  $m_i^p$  stands for the  $p^{\text{th}}$  spatial coordinate component of the vector  $\mathbf{m}_i$ .

To determine the specific algebraic type of any of the Kundt spacetimes we have to investigate the values and mutual relations of these scalars.

It is well known that for the general Kundt geometry (2) the privileged null vector  $\mathbf{k} = \partial_r$  is a WAND, so that the spacetime is of *algebraic type* I or more special [6, 51, 58]. Indeed, we immediately see from (7) that  $\Psi_{0ij} = 0$ , i.e., the Weyl tensor components of the highest boost weight +2 are completely missing.

Using the frame (6) the boost weight +1 scalars  $\Psi_{1Tj}$  and  $\Psi_{1ijk}$  for all frame labels  $i, j, k = 2, \dots, D-1$  are, in view of (8), (13) and (A.30), (A.14), (A.24), explicitly

$$\Psi_{1Tj} = -\frac{1}{2} \frac{D-3}{D-2} m_j^p g_{up,rr}, \quad (17)$$

$$\Psi_{1ijk} = \frac{1}{D-3} (\delta_{ij} \Psi_{1Tk} - \delta_{ik} \Psi_{1Tj}). \quad (18)$$

Clearly,  $\Psi_{1Tk} = \Psi_{1i^i k}$  which confirms (5). Moreover, we observe that there are no Kundt spacetimes of genuine subtype I(a), see the definition in section 2.3 of [6]. We thus proved that *all Kundt geometries are actually of algebraic type* I(b) $\equiv$ I, or more special.

### 3. Algebraically special (type II) Kundt spacetimes

Kundt spacetimes are *algebraically special*, that is of type II (or more special), if the corresponding Weyl scalars  $\Psi_{A\dots}$  of the boost weight +1 (or lower) also vanish.

In this paper we restrict ourselves to the geometrically privileged large subclass§ of Kundt spacetimes which are of type II or more special *with respect to the WAND*  $\mathbf{k}$ .

From (17), (18) it follows that the simple condition  $\Psi_{1Tj} = 0$ , that is explicitly  $m_j^p g_{up,rr} = 0$ , is the necessary and sufficient condition for a Kundt spacetime of any dimension to be of type II $\equiv$ I(a), i.e., algebraically special with double WAND  $\mathbf{k} = \partial_r$ . Since the  $(D-2)$  spatial vectors  $\mathbf{m}_j$  of (6) forming the local Cartesian frame in the transverse Riemannian space are linearly independent,  $g_{up}$  must be of the form

$$g_{up} = e_p(u, x) + f_p(u, x) r, \quad (19)$$

where for any  $p = 2, \dots, D-1$  the functions  $e_p, f_p$  are independent of the coordinate  $r$ , confirming the results of [6, 52]. *The most general Kundt spacetime of algebraic type II with respect to double WAND*  $\mathbf{k}$  in any dimension  $D$  can be thus written as

$$ds^2 = g_{pq} dx^p dx^q + 2(e_p + f_p r) du dx^p - 2 du dr + g_{uu}(r, u, x) du^2, \quad (20)$$

where  $g_{pq}, e_p, f_p$  are functions of  $u$  and  $x$  only. Any Kundt spacetime of Weyl type II with respect to  $\mathbf{k}$  *necessarily* implies  $g_{up,rr} = 0$  and thus  $R_{rp} = 0$ , see (A.24), which is equivalent to  $R_{abk}^b \propto k_a$ . Consequently, the corresponding Ricci tensor must be of type II, or more special, without any additional geometrical or physical condition (see Proposition 7.1 of [6]).

For the algebraically special Kundt geometries with multiple WAND  $\mathbf{k}$ , we can evaluate all the remaining Weyl scalars of boost weights 0, -1, -2. First, after substituting (19) into (A.13)–(A.30), a straightforward but very lengthy calculation yields the explicit forms of the Riemann, Ricci, and Weyl tensors for an *arbitrary such algebraically special Kundt spacetime* of dimension  $D \geq 4$ . The complete results are presented in Appendix B. Next, using the expressions (9)–(16) and the Weyl tensor

§ There may exist “non-aligned” Kundt geometries for which the WAND  $\mathbf{k}$  (oriented along the geodesic, non-expanding, non-twisting, and shear-free congruence) is *not* multiple while they admit *another* non-Kundt WAND that *is* multiple, cf. [52].

coordinate components (B.21)–(B.30), after surprising and non-trivial cancelation of various terms we obtain the following explicit expressions:

$$\Psi_{2S} = \frac{D-3}{D-1} \left[ \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p + \frac{1}{D-2} \left( \frac{{}^S R}{D-3} + f \right) \right], \quad (21)$$

$$\begin{aligned} \Psi_{2T^{ij}} &= \frac{g_{pq} m_i^p m_j^q}{(D-1)(D-2)} \left[ \frac{1}{2} (D-3) g_{uu,rr} - \frac{1}{4} (D-3) f^m f_m - {}^S R - \frac{1}{2} (D-5) f \right] \\ &\quad + \frac{1}{D-2} m_i^p m_j^q \left[ {}^S R_{pq} + \frac{1}{2} (D-4) f_{pq} + \frac{1}{2} (D-2) F_{pq} \right], \end{aligned} \quad (22)$$

$$\Psi_{2^{ij}} = m_i^p m_j^q F_{pq}, \quad (23)$$

$$\begin{aligned} \Psi_{2^{ijkl}} &= m_i^m m_j^p m_k^n m_l^q \left[ {}^S R_{mpnq} - \frac{1}{D-2} \left( g_{mn} ({}^S R_{pq} - f_{pq}) - g_{mq} ({}^S R_{pn} - f_{pn}) \right. \right. \\ &\quad \left. \left. + g_{pq} ({}^S R_{mn} - f_{mn}) - g_{pn} ({}^S R_{mq} - f_{mq}) \right) \right. \\ &\quad \left. + \frac{1}{(D-1)(D-2)} \left( g_{uu,rr} + {}^S R - 2f - \frac{1}{2} f^s f_s \right) (g_{mn} g_{pq} - g_{mq} g_{pn}) \right], \end{aligned} \quad (24)$$

$$\begin{aligned} \Psi_{3T^j} &= -m_j^p \frac{D-3}{D-2} \left[ \frac{1}{2} \left( r f_p g_{uu,rr} + g_{uu,rp} - f_{p,u} \right) + e_p \left( \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^q f_q \right) \right. \\ &\quad \left. + \frac{1}{4} f^q e_q f_p - \frac{1}{2} f^q E_{qp} - \frac{1}{D-3} X_p - r \left( \frac{1}{2} f^q F_{qp} + \frac{1}{D-3} Y_p \right) \right], \end{aligned} \quad (25)$$

$$\Psi_{3^{ijk}} = \frac{1}{D-3} (\delta_{ij} \Psi_{3T^k} - \delta_{ik} \Psi_{3T^j}) + \tilde{\Psi}_{3^{ijk}}, \quad (26)$$

$$\begin{aligned} \tilde{\Psi}_{3^{ijk}} &= m_i^p m_j^m m_k^q \left[ \left( X_{pmq} - \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m) \right) \right. \\ &\quad \left. + r \left( Y_{pmq} - \frac{1}{D-3} (g_{pm} Y_q - g_{pq} Y_m) \right) \right], \end{aligned} \quad (27)$$

$$\Psi_{4^{ij}} = m_i^p m_j^q \left[ W_{pq} - \frac{1}{D-2} g_{pq} W \right], \quad (28)$$

where  $g_{pq} m_i^p m_j^q = \delta_{ij}$  and

$$\begin{aligned} X_{pmq} &\equiv e_{[q|m]||p} + F_{qm} e_p + \frac{1}{2} (F_{pm} e_q - F_{pq} e_m) \\ &\quad + \frac{1}{2} (e_{pm} f_q - e_{pq} f_m) - \frac{1}{2} (f_{pm} e_q - f_{pq} e_m) + g_{p[m,u||q]}, \end{aligned} \quad (29)$$

$$Y_{pmq} \equiv f_{[q|m]||p} + F_{qm} f_p + \frac{1}{2} (F_{pm} f_q - F_{pq} f_m), \quad (30)$$

$$\begin{aligned} W_{pq} &\equiv -\frac{1}{2} (g_{uu})_{||p||q} + \frac{1}{2} g_{uu} f_{(p||q)} + \frac{1}{4} (g_{uu,p} f_q + g_{uu,q} f_p) \\ &\quad - \frac{1}{2} g_{uu,r} (r f_{(p||q)} + e_{pq}) - \frac{1}{2} g_{uu,rr} (r^2 f_p f_q + r (f_p e_q + f_q e_p) + e_p e_q) \\ &\quad + \frac{1}{2} [(f_{p,u} - g_{uu,rp}) (r f_q + e_q) + (f_{q,u} - g_{uu,rq}) (r f_p + e_p)] \\ &\quad + r^2 g^{mn} F_{mp} F_{nq} + r [f_{(p,u||q)} + g^{mn} (E_{mp} F_{nq} + E_{mq} F_{np}) \\ &\quad + \frac{1}{2} [f^m (F_{mp} e_q + F_{mq} e_p) - e^m (F_{mp} f_q + F_{mq} f_p)]] \\ &\quad + (e_{(p,u||q)} - \frac{1}{2} g_{pq,uu}) + g^{mn} E_{mp} E_{nq} \\ &\quad + \frac{1}{2} [f^m (E_{mp} e_q + E_{mq} e_p) - e^m (E_{mp} f_q + E_{mq} f_p)] \\ &\quad + \frac{1}{4} (e^m e_m f_p f_q + f^m f_m e_p e_q) - \frac{1}{4} f^m e_m (f_p e_q + f_q e_p). \end{aligned} \quad (31)$$

The corresponding contractions  $X_q \equiv g^{pm} X_{pmq}$ ,  $Y_q \equiv g^{pm} Y_{pmq}$ ,  $W \equiv g^{pq} W_{pq}$  are

$$X_q = g^{pm} e_{[q||p]||m} - \frac{3}{2} F_{pq} e^p + \frac{1}{2} (f_{pq} e^p - e_{pq} f^p) + \frac{1}{2} (g^{pm} e_{pm} f_q - f e_q) + g^{pm} g_{p[m,u]||q}, \quad (32)$$

$$Y_q = g^{pm} f_{[q||p]||m} - \frac{3}{2} F_{pq} f^p, \quad (33)$$

$$\begin{aligned} W = & -\frac{1}{2} \Delta g_{uu} + \frac{1}{2} g_{uu} f^p{}_{||p} + \frac{1}{2} g_{uu,p} f^p \\ & - \frac{1}{2} g_{uu,r} g^{pq} (r f_{(p||q)} + e_{pq}) - \frac{1}{2} g_{uu,rr} (r^2 f^p f_p + 2 r f^p e_p + e^p e_p) \\ & + (f_{p,u} - g_{uu,rp}) (r f^p + e^p) + r^2 g^{pq} g^{mn} F_{pm} F_{qn} \\ & + r [g^{pq} f_{(p,u||q)} + 2 g^{pq} g^{mn} E_{pm} F_{qn} + 2 f^p e^q F_{pq}] \\ & + g^{pq} (e_{(p,u||q)} - \frac{1}{2} g_{pq,uu}) + g^{pq} g^{mn} E_{pm} E_{qn} + 2 f^p e^q e_{[p||q]} \\ & + \frac{1}{2} e^p e_p f^q f_q - \frac{1}{2} f^p e_p f^q e_q. \end{aligned} \quad (34)$$

We have introduced the convenient geometric quantities

$$f^p \equiv g^{pq} f_q, \quad (35)$$

$$f_{p||q} \equiv f_{p,q} - \Gamma_{pq}^m f_m, \quad (36)$$

$$f^p{}_{||p} \equiv g^{pq} f_{p||q}, \quad (37)$$

$$f_{pq} \equiv f_{(p||q)} + \frac{1}{2} f_p f_q, \quad (38)$$

$$f \equiv g^{pq} f_{pq} = f^p{}_{||p} + \frac{1}{2} f^p f_p, \quad (39)$$

$$F_{pq} \equiv f_{[p||q]} = f_{[p,q]}, \quad (40)$$

$$f_{[m||q]||p} \equiv f_{[m,q],p} - \Gamma_{pm}^n f_{[n,q]} - \Gamma_{pq}^n f_{[m,n]}, \quad (41)$$

$$f_{p,u||q} \equiv (f_{p,u})_{||q} = f_{p,uq} - f_{n,u} \Gamma_{pq}^n, \quad (42)$$

$$f_{(p,u||q)} \equiv f_{(p,q),u} - f_{n,u} \Gamma_{pq}^n, \quad (43)$$

$$e^p \equiv g^{pq} e_q, \quad (44)$$

$$e_{p||q} \equiv e_{p,q} - \Gamma_{pq}^m e_m, \quad (45)$$

$$e_{pq} \equiv e_{(p||q)} - \frac{1}{2} g_{pq,u}, \quad (46)$$

$$E_{pq} \equiv e_{[p||q]} + \frac{1}{2} g_{pq,u}, \quad (47)$$

$$e_{[m||q]||p} \equiv e_{[m,q],p} - \Gamma_{pm}^n e_{[n,q]} - \Gamma_{pq}^n e_{[m,n]}, \quad (48)$$

$$e_{p,u||q} \equiv (e_{p,u})_{||q} = e_{p,uq} - e_{n,u} \Gamma_{pq}^n, \quad (49)$$

$$e_{(p,u||q)} \equiv e_{(p,q),u} - e_{n,u} \Gamma_{pq}^n, \quad (50)$$

and

$$g_{p[m,u]||q} \equiv \frac{1}{2} [(g_{pm,u})_{||q} - (g_{pq,u})_{||m}] \quad (51)$$

$$= g_{p[m,q],u} + \frac{1}{2} (\Gamma_{pm}^n g_{nq,u} - \Gamma_{pq}^n g_{nm,u}), \quad (52)$$

$$\Delta g_{uu} \equiv g^{pq} (g_{uu})_{||p||q}, \quad (53)$$

in which the symbol  $||$  indicates the covariant derivative with respect to the spatial metric  $g_{pq}$  in the transverse  $(D-2)$ -dimensional Riemannian space, whose metric is  $g_{pq}$  and the corresponding Riemann and Ricci curvature tensors are  ${}^S R_{mpnq}$  and  ${}^S R_{pq}$ , respectively.

We should emphasize that all the quantities (35)–(51) are *independent* of the coordinate  $r$ . It is also important to observe that  $W_{pq} = W_{qp}$ , while  $X_{pmq} = -X_{pqm}$ ,

$Y_{pmq} = -Y_{pqm}$ . Consequently,  $X_q \equiv g^{pm}X_{pmq}$  and  $Y_q \equiv g^{pm}Y_{pmq}$ , given in (32), (33), are the only non-trivial contractions of  $X_{pmq}$  and  $Y_{pmq}$ , respectively.

By taking the symmetric and antisymmetric parts of  $\Psi_{2T^{ij}}$  given by expression (22) we immediately obtain

$$\begin{aligned} \Psi_{2T^{(ij)}} &= \frac{g_{pq} m_i^p m_j^q}{(D-1)(D-2)} \left[ \frac{1}{2}(D-3) g_{uu,rr} - \frac{1}{4}(D-3) f^m f_m - {}^S R - \frac{1}{2}(D-5)f \right] \\ &\quad + \frac{1}{D-2} m_i^p m_j^q \left[ {}^S R_{pq} + \frac{1}{2}(D-4) f_{pq} \right], \end{aligned} \quad (54)$$

$$\Psi_{2T^{[ij]}} = \frac{1}{2} m_i^p m_j^q F_{pq}, \quad (55)$$

which explicitly confirm the relations  $\Psi_{2S} = \Psi_{2T^{kk}} \equiv \Psi_{2T^{(ij)}} \delta^{ij}$  and  $\Psi_{2T^{[ij]}} = \frac{1}{2} \Psi_{2^{ij}}$ , see (5). For the corresponding irreducible components defined as

$$\tilde{\Psi}_{2T^{(ij)}} \equiv \Psi_{2T^{(ij)}} - \frac{1}{D-2} \delta_{ij} \Psi_{2S}, \quad (56)$$

$$\begin{aligned} \tilde{\Psi}_{2^{ijkl}} &\equiv \Psi_{2^{ijkl}} - \frac{2}{D-4} (\delta_{ik} \Psi_{2T^{(lj)}} - \delta_{il} \Psi_{2T^{(kj)}} - \delta_{jk} \Psi_{2T^{(li)}} + \delta_{jl} \Psi_{2T^{(ki)}}) \\ &\quad + \frac{2}{(D-3)(D-4)} \Psi_{2S} (\delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj}), \end{aligned} \quad (57)$$

we then get using (21) and (24) that

$$\tilde{\Psi}_{2T^{(ij)}} = \frac{m_i^p m_j^q}{D-2} \left[ \left( {}^S R_{pq} - \frac{g_{pq}}{D-2} {}^S R \right) + \frac{1}{2}(D-4) \left( f_{pq} - \frac{g_{pq}}{D-2} f \right) \right], \quad (58)$$

$$\tilde{\Psi}_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q {}^S C_{mpnq}. \quad (59)$$

In general, all these Weyl scalars of boost weight 0 are non-trivial. However, if some of them vanish, we obtain an *explicit refinement* of the algebraic classification of type II Kundt spacetimes with double WAND  $\mathbf{k} = \partial_r$  into the corresponding *subtypes* defined in [6] (see also [9, 59, 60]). To be specific:

- The Kundt spacetime (20) is of *algebraic type II(a)*  $\Leftrightarrow \Psi_{2S} = 0 \Leftrightarrow$  the metric function  $g_{uu}$  is *at most quadratic* in the coordinate  $r$ ,

$$g_{uu} = a(u, x) r^2 + b(u, x) r + c(u, x), \quad (60)$$

and  $a$  is uniquely given as

$$a = \frac{1}{4} f^p f_p - \frac{1}{D-2} \left( \frac{{}^S R}{D-3} + f \right), \quad (61)$$

where  ${}^S R$  is the Ricci scalar of the spatial metric  $g_{pq}$ , and  $f$  is defined by (39). Notice that these spacetimes form a subclass of the degenerate Kundt spacetimes studied in [52].

- The Kundt spacetime (20) is of *algebraic type II(b)*  $\Leftrightarrow \tilde{\Psi}_{2T^{(ij)}} = 0 \Leftrightarrow$

$${}^S R_{pq} - \frac{g_{pq}}{D-2} {}^S R = -\frac{1}{2}(D-4) \left( f_{pq} - \frac{g_{pq}}{D-2} f \right). \quad (62)$$

This is identically satisfied when  $D = 4$  since for any transverse 2-dimensional Riemannian space there is  ${}^S R_{pq} = \frac{1}{2} g_{pq} {}^S R$ .



- The Kundt spacetime (20) is of *algebraic type* II(c)  $\Leftrightarrow \tilde{\Psi}_{2^{ijkl}} = 0 \Leftrightarrow$

$${}^S C_{mpnq} = 0. \quad (63)$$

This is always satisfied when  $D = 4$  and  $D = 5$  since the Weyl tensor vanishes identically in dimensions 2 and 3, see also [60, 61].

- The Kundt spacetime (20) is of *algebraic type* II(d)  $\Leftrightarrow \Psi_{2T^{[ij]}} = 0 \Leftrightarrow$

$$F_{pq} = 0, \quad (64)$$

i.e.,  $f_{p,q} - f_{q,p} = 0$  for all  $p, q$ , see (40). This condition can be conveniently rewritten in a geometric form. Introducing the 1-form  $\phi \equiv f_p dx^p$  in the transverse  $(D - 2)$ -dimensional Riemannian space, (64) is equivalent to the condition that  $\phi$  is *closed*,  $d\phi = 0$ . On any *contractible domain*, every closed form is exact by the Poincaré lemma, so that there exists a *potential function*  $\mathcal{F}$  such that  $\phi = d\mathcal{F}$ ,

$$f_p \equiv \mathcal{F}_{,p}. \quad (65)$$

In a general case, such  $\mathcal{F}$  exists only *locally*.

#### 4. Type III Kundt spacetimes

The Kundt spacetime (20) is of *algebraic type* III with respect to the triple WAND  $\mathbf{k} = \partial_r$  if all these four independent conditions (60)–(64) are *satisfied simultaneously*, that is III $\equiv$ II(abcd). Recall that in four spacetime dimensions, the conditions (62) and (63) are identically valid. For type III Kundt spacetimes the general expressions (25), (26), (27) for the corresponding Weyl scalars of the boost weight  $-1$  thus reduce to expressions

$$\Psi_{3T^j} = -m_j^p \frac{D-3}{D-2} \left[ (a_{,p} + f_p a) r + \frac{1}{2} (b_{,p} - f_{p,u}) - \frac{1}{2} T_p \right], \quad (66)$$

$$\Psi_{3^{ijk}} = \frac{1}{D-3} (\delta_{ij} \Psi_{3T^k} - \delta_{ik} \Psi_{3T^j}) + \tilde{\Psi}_{3^{ijk}}, \quad (67)$$

$$\tilde{\Psi}_{3^{ijk}} = m_i^p m_j^m m_k^q \left[ X_{pmq} - \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m) \right], \quad (68)$$

where  $a$  is given by (61),  $f_p$  can locally be written as (65),

$$T_p \equiv \frac{2e_p}{D-2} \left( \frac{{}^S R}{D-3} + f \right) - \frac{1}{2} f^q e_q f_p + f^q E_{qp} + \frac{2}{D-3} X_p, \quad (69)$$

$X_q \equiv g^{pm} X_{pmq}$ , and  $X_{pmq}$  defined in (29) simplifies to

$$X_{pmq} = e_{[q||m]||p} + \frac{1}{2} (e_{pm} f_q - e_{pq} f_m) - \frac{1}{2} (f_{pm} e_q - f_{pq} e_m) + g_{p[m,u]||q}, \quad (70)$$

because for the subtype II(ad) there is

$$\frac{1}{2} (r f_p g_{uu,rr} + g_{uu,rp} - f_{p,u}) = (a_{,p} + f_p a) r + \frac{1}{2} (b_{,p} - f_{p,u}), \quad (71)$$

$$\left( \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p \right) = -\frac{1}{D-2} \left( \frac{{}^S R}{D-3} + f \right), \quad (72)$$

$$F_{pq} = 0. \quad (73)$$

It can be seen that  $\tilde{\Psi}_{3^{i^i k}} = 0$  which confirms relation  $\Psi_{3^{i^i k}} = \Psi_{3T^k}$ , see (5).

Following the definition in section 2.3 of [6] we can thus explicitly write the conditions for the subtypes III(a) and III(b) with triple WAND  $\mathbf{k}$ :

- The Kundt type III spacetimes are of *algebraic type* III(a) if, and only if,  $\Psi_{3T^j} = 0$ . Due to (66), which is linear in  $r$ , this yields the following two conditions that must be satisfied simultaneously:

$$a_{,p} + f_p a = 0, \quad (74)$$

$$b_{,p} - f_{p,u} = T_p. \quad (75)$$

These put specific constraints on the spatial derivatives of the functions  $a$  and  $b$  in the expression (60) for  $g_{uu}$ . Notice that, in view of (61) and (65), the condition (74) can be (locally) integrated as

$$a(u, x) = \frac{1}{4} f^p f_p - \frac{1}{D-2} \left( \frac{{}^S R}{D-3} + f \right) = \alpha \exp(-\mathcal{F}), \quad (76)$$

where  $\mathcal{F}(x, u)$  is the potential function of  $f_p = \mathcal{F}_{,p}$  while  $\alpha(u)$  is an arbitrary function of the retarded time  $u$ .

- The Kundt type III spacetimes are of *algebraic type* III(b) if, and only if,  $\tilde{\Psi}_{3^{ijk}} = 0$ , which is explicitly

$$X_{pmq} = \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m). \quad (77)$$

This is the set of constraints on the metric functions  $e_p, f_p, g_{pq}$ . In four spacetime dimensions,  $D = 4$ , the condition (77) is satisfied identically, so that III(b)  $\equiv$  III, while III(a)  $\equiv$  N. Indeed, in such a case the transverse metric is two-dimensional and thus conformally flat,  $g_{pm} = \Omega \delta_{pm}$ , which implies  $X_{pmq} = g_{pm} X_q - g_{pq} X_m$  with  $X_q = \Omega^{-1}(X_{22q} + X_{33q})$  for all combinations of the indices  $p, m, q = 2, 3$ .

## 5. Type N Kundt spacetimes

Kundt spacetime (20) is of *algebraic type* N with quadruple WAND  $\mathbf{k} = \partial_r$  if, and only if, the four independent conditions (60)–(64) and the three independent conditions (74), (75), (77) are all satisfied simultaneously for every combination of the spatial components  $m, n, p, q = 2, \dots, D-1$ , i.e., N  $\equiv$  III(ab).

Therefore, the only non-vanishing Weyl tensor component is  $\Psi_{4^{ij}}$  given by (28). For all type N Kundt geometries, this scalar of the lowest boost weight  $-2$  is

$$\Psi_{4^{ij}} = m_i^p m_j^q \left[ W_{pq} - \frac{1}{D-2} g_{pq} W \right], \quad (78)$$

where the symmetric matrix  $W_{pq}$  given by (31) simplifies, using (60), (61), (64), (74), (75), to

$$W_{pq} \equiv r \left[ \frac{1}{2} a g_{pq,u} + U_{(p||q)} + U_{(p} f_{q)} \right] - \frac{1}{4} \left[ (c_{,p} - c f_p)_{||q} + (c_{,q} - c f_q)_{||p} \right] - \frac{1}{2} b e_{pq} + \left( a - \frac{1}{4} f^m f_m \right) e_p e_q + Z_{(pq)}, \quad (79)$$

in which  $a$  is given by (61),  $U_p = \frac{1}{2}(f_{p,u} - T_p) - a e_p$  is

$$U_p \equiv \frac{1}{2} f_{p,u} - \frac{1}{4} f^q f_q e_p + \frac{1}{4} f^q e_q f_p - \frac{1}{2} f^q E_{qp} - \frac{1}{D-3} X_p, \quad (80)$$

$$Z_{pq} \equiv \frac{1}{4} e^m e_m f_p f_q + e_{p,u||q} - \frac{1}{2} g_{pq,uu} - e^m E_{mp} f_q + g^{mn} E_{mp} E_{nq} - \frac{2}{D-3} X_p e_q, \quad (81)$$

and the trace of  $W_{pq}$  is  $W = g^{pq} W_{pq}$ . The symmetric  $(D-2) \times (D-2)$  matrix  $\Psi_{4^{ij}}$  represents the specific amplitudes of Kundt gravitational waves, which in any dimension  $D$  are thus always transverse and traceless. This result applies to all Kundt type N geometries, irrespective of any field equations.

## 6. Type O Kundt spacetimes

Type O Kundt geometries arise when *all* components of the Weyl tensor vanish. From (78) it follows that this occurs for type N Kundt spacetimes if, and only if,

$$W_{pq} = \frac{1}{D-2} g_{pq} W. \quad (82)$$

This condition, which is symmetric in the spatial components  $p$  and  $q$ , obviously splits into the part linear in the coordinate  $r$ , and the part independent of it.

Due to (82), the matrix  $W_{pq}$  identically vanishes if, and only if, its trace vanishes,  $W_{pq} = 0 \Leftrightarrow W = 0$ . Therefore, there exists a *specific subtype* of conformally flat Kundt geometries, which we may denote as type O', given by the condition  $W_{pq} = 0$ . Using (79) this is equivalent to

$$U_{(p||q)} + U_{(p}f_{q)} = -\frac{1}{2} a g_{pq,u}, \quad (83)$$

$$(c_{,p} - c f_p)_{||q} + (c_{,q} - c f_q)_{||p} = -2b e_{pq} + (4a - f^m f_m) e_p e_q + 4Z_{(pq)}, \quad (84)$$

where the functions  $a, b, c$ , introduced in (60), must satisfy the constraints (61), (74), (75), while the functions  $U_p, Z_{pq}$  were defined in (80), (81).

## 7. Type D Kundt spacetimes

Finally, we may complete the explicit classification of Kundt geometries by identifying those algebraically special spacetimes for which the Weyl scalars of the lowest boost weights  $-2$  and  $-1$  vanish.

Let us consider a generic Kundt spacetime of type II with respect to the double WAND  $\mathbf{k} = \partial_r$ , described in section 3. If its boost weight  $-2$  scalar (28) (in which the functions  $W_{pq}$  and  $W$  are given by (31) and (34), respectively) vanishes,  $\Psi_{4^{ij}} = 0$ , the vector  $\mathbf{l}$  given by (6) is WAND and the spacetime is of the *algebraic type* II<sub>*i*</sub>.

If, in addition, all the conditions (60)–(64) are satisfied, the boost weight 0 Weyl scalars also vanish and the spacetime is thus of *algebraic type* III<sub>*i*</sub> with respect to triple WAND  $\mathbf{k}$  and WAND  $\mathbf{l}$ .

Complementary, the Kundt geometries of *algebraic type* D are represented by those metrics (20) for which all Weyl scalars of the *two* lowest boost weights, namely  $\Psi_{3T^j}$ ,  $\Psi_{3^{ijk}}$  and  $\Psi_{4^{ij}}$ , are zero. These scalars are explicitly given by expressions (25)–(28). All type D Kundt spacetimes for which *both*  $\mathbf{k}$  and  $\mathbf{l}$  given by (6) are *double WANDs* thus can be written in the form (20) in which the metric functions satisfy the conditions

$$\begin{aligned} \frac{1}{2} \left( r f_p g_{uu,rr} + g_{uu,rp} - f_{p,u} \right) + e_p \left( \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^q f_q \right) = \\ r \left( \frac{1}{2} f^q F_{qp} + \frac{1}{D-3} Y_p \right) - \frac{1}{4} f^q e_q f_p + \frac{1}{2} f^q E_{qp} + \frac{1}{D-3} X_p, \end{aligned} \quad (85)$$

$$X_{pmq} = \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m), \quad (86)$$

$$Y_{pmq} = \frac{1}{D-3} (g_{pm} Y_q - g_{pq} Y_m), \quad (87)$$

$$W_{pq} = \frac{1}{D-2} g_{pq} W, \quad (88)$$

where the corresponding functions are defined in (29)–(34).

Moreover, if the relations (60) with (61), (62), (63), (64) are also valid, we obtain the particular *subtypes* D(a), D(d), D(c), D(d), respectively. The subtype D(abcd) is equivalent to (conformally flat) type O Kundt spacetimes, described in section 6.

In the case of *four-dimensional spacetimes* ( $D = 4$ ), the conditions (86) and (87) are identically satisfied (due to the arguments analogous to those given at the end of section 4), and the type D spacetimes are automatically of type D(bc).

## 8. Summary of the results

Let us summarize the classification scheme of the Kundt spacetimes of type II with respect to double WAND  $\mathbf{k}$  in an arbitrary dimension  $D$ . Such algebraically special Kundt geometries (20) are classified into specific distinct subclasses listed in table 2.

type	necessary and sufficient conditions	equations
II(a)	$g_{uu} = a(u, x) r^2 + b(u, x) r + c(u, x)$ where $a = \frac{1}{4} f^p f_p - \frac{1}{D-2} \left( \frac{S_R}{D-3} + f \right)$	(60) (61)
II(b)	${}^S R_{pq} - \frac{1}{D-2} g_{pq} {}^S R = -\frac{1}{2} (D-4) \left( f_{pq} - \frac{1}{D-2} g_{pq} f \right)$	(62)
II(c)	${}^S C_{mpnq} = 0$	(63)
II(d)	$F_{pq} = 0$	(64)
III	II(abcd)	
III(a)	$a_{,p} + f_p a = 0$ where $a$ is given by (61) $b_{,p} - f_{p,u} = T_p$	(74), (61) (75), (69)
III(b)	$X_{pmq} = \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m)$	(77), (70)
N	III(ab)	
O	N with $W_{pq} = \frac{1}{D-2} g_{pq} W$	(82), (79)
O'	$W_{pq} = 0$	(83), (84)
D	$\frac{1}{2} (r f_p g_{uu,rr} + g_{uu,rp} - f_{p,u}) + e_p \left( \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^q f_q \right)$ $= r \left( \frac{1}{2} f^q F_{qp} + \frac{1}{D-3} Y_p \right) - \frac{1}{4} f^q e_q f_p + \frac{1}{2} f^q E_{qp} + \frac{1}{D-3} X_p$ $X_{pmq} = \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m)$ $Y_{pmq} = \frac{1}{D-3} (g_{pm} Y_q - g_{pq} Y_m)$ $W_{pq} = \frac{1}{D-2} g_{pq} W$	(85), (32), (33) (86), (29), (32) (87), (30), (33) (88), (31), (34)
D(a)	D with II(a)	
D(b)	D with II(b)	
D(c)	D with II(c)	
D(d)	D with II(d)	

**Table 2.** The complete classification scheme of algebraically special Kundt geometries (20) in any dimension  $D$  (in the classic  $D = 4$  case, the conditions for II(b), II(c), III(b) are automatically satisfied). The vector  $\mathbf{k} = \partial_r$  is multiple WAND, and for type D subclass the vector  $\mathbf{l} = \frac{1}{2} g_{uu} \partial_r + \partial_u$  is double WAND.

In table 2 the necessary and sufficient conditions and the corresponding equation references for all the algebraic types and subtypes are given. Recall that  $g_{pq}$ , where  $p, q = 2, 3, \dots, D-1$ , denotes the metric of the transverse Riemannian space, the corresponding Ricci tensor and Ricci scalar are  ${}^S R_{pq}$  and  ${}^S R$ , respectively, and its Weyl tensor is  ${}^S C_{mpnq}$ . The remaining quantities are defined in (35)–(53).

To illustrate the usefulness of this scheme, in the remaining sections we apply it to three most important subfamilies of the Kundt geometry, namely the pp-waves, the VSI spacetimes, and generalization of the Bertotti–Robinson, Nariai, and Plebański–Hacyan direct-product spacetimes.

## 9. pp-waves

The class of pp-waves is defined geometrically by the property that the spacetimes admit a *covariantly constant null vector field*  $\mathbf{k}$ , see [3, 4]. All the optical scalars (1) thus vanish, so that the pp-waves necessarily belong to the Kundt family (2) with  $\mathbf{k} = \partial_r$ . In view of (A.10), (A.11), the defining condition  $0 = k_{a;b} = \frac{1}{2}g_{ab,r}$  immediately implies that for pp-waves all the metric functions must be independent of the coordinate  $r$ , i.e., the line element can be written in the Brinkmann form [7] as

$$ds^2 = g_{pq} dx^p dx^q + 2 e_p du dx^p - 2 du dr + c du^2, \quad (89)$$

where  $g_{pq}(u, x)$ ,  $e_p(u, x)$ ,  $c(u, x)$  are function of  $u$  and  $x \equiv (x^2, x^3, \dots, x^{D-1})$  only. This is the particular case of the metric (20), (60) in the case when

$$f_p = 0, \quad a = 0 = b. \quad (90)$$

All pp-waves are thus algebraically special, that is of type II(d) or more special with multiple WAND  $\mathbf{k}$ , in agreement with [5, 6]. The relevant quantities which enter in table 2 are

$$f = 0, \quad f_{pq} = 0, \quad F_{pq} = 0, \quad (91)$$

$$Y_{pmq} = 0 = Y_q, \quad g_{uu,rr} = 0 = g_{uu,rp}, \quad (92)$$

$$T_p = \frac{2}{D-3} \left( X_p + \frac{1}{D-2} {}^S R e_p \right), \quad (93)$$

$$E_{pq} = e_{[p||q]} + \frac{1}{2} g_{pq,u}, \quad (94)$$

$$X_{pmq} = e_{[q||m]||p} + g_{p[m,u||q]}, \quad (95)$$

$$W_{pq} = -\frac{1}{2} c_{||p||q} + e_{(p,u||q)} - \frac{1}{2} g_{pq,uu} + g^{mn} E_{mp} E_{nq}, \quad (96)$$

and their contractions read

$$X_q = g^{mn} (e_{[q||m]||n} + g_{m[n,u||q]}), \quad (97)$$

$$W = -\frac{1}{2} \Delta c + g^{pq} (e_{(p,u||q)} - \frac{1}{2} g_{pq,uu}) + g^{mn} g^{pq} E_{mp} E_{nq}. \quad (98)$$

We thus obtain the complete algebraic classification of all pp-wave geometries, as summarized in table 3.

Notice that the classification into the subtypes II(a), II(b), II(c) is directly determined by vanishing of the three distinct components of the unique geometric decomposition of the Riemann curvature tensor  ${}^S R_{mpnq}$  of the  $(D-2)$ -dimensional transverse space, namely the Ricci scalar  ${}^S R$ , the traceless part of the Ricci tensor  ${}^S R_{pq} - \frac{1}{D-2} g_{pq} {}^S R$ , and the Weyl tensor  ${}^S C_{mpnq}$ , respectively.

Consequently, the pp-waves are of type III $\equiv$ II(abc) $\equiv$ II(abcd) and more special types N and O if, and only if, their *transverse space is flat*,  $g_{pq} = \delta_{pq}$ , i.e.,  ${}^S R_{mpnq} = 0$ .

type	necessary and sufficient conditions
II(a)	${}^S R = 0$
II(b)	${}^S R_{pq} = \frac{1}{D-2} g_{pq} {}^S R$
II(c)	${}^S C_{mpnq} = 0$
II(d)	always
III	$g_{pq} = \delta_{pq}$
III(a)	$g_{pq} = \delta_{pq}$ and $X_p = 0$
III(b)	$g_{pq} = \delta_{pq}$ and $X_{pmq} = \frac{1}{D-3} (\delta_{pm} X_q - \delta_{pq} X_m)$
N	$g_{pq} = \delta_{pq}$ and $X_p = 0 = X_{pmq}$
O	$g_{pq} = \delta_{pq}$ and $X_p = 0 = X_{pmq}$ and $W_{pq} = \frac{1}{D-2} \delta_{pq} W$
O'	$g_{pq} = \delta_{pq}$ and $X_p = 0 = X_{pmq}$ and $W_{pq} = 0 = W$
D	$X_p = 0 = X_{pmq}$ and $W_{pq} = \frac{1}{D-2} g_{pq} W$
D(a)	$X_p = 0 = X_{pmq}$ and $W_{pq} = \frac{1}{D-2} g_{pq} W$ and ${}^S R = 0$
D(b)	$X_p = 0 = X_{pmq}$ and $W_{pq} = \frac{1}{D-2} g_{pq} W$ and ${}^S R_{pq} = \frac{1}{D-2} g_{pq} {}^S R$
D(c)	$X_p = 0 = X_{pmq}$ and $W_{pq} = \frac{1}{D-2} g_{pq} W$ and ${}^S C_{mpnq} = 0$
D(d)	identical to D

**Table 3.** The complete classification scheme of all pp-wave geometries (89) in any dimension  $D$ .

Further subclassification of the type III, N and O pp-waves is then given by mutual relations and possible vanishing of the functions  $X_{pmq}$ ,  $X_p$  and  $W_{pq}$ ,  $W$ , as given by expressions (95)–(98). Due to  $g_{pq} = \delta_{pq}$ , these simplify to

$$X_{pmq} = e_{[q,m],p}, \quad (99)$$

$$X_q = \delta^{mn} e_{[q,m],n}, \quad (100)$$

$$W_{pq} = -\frac{1}{2} c_{,pq} + e_{(p,uq)} + \delta^{mn} e_{[m,p]} e_{[n,q]}, \quad (101)$$

$$W = -\frac{1}{2} \Delta c + g^{pq} e_{(p,uq)} + \delta^{mn} \delta^{pq} e_{[m,p]} e_{[n,q]}. \quad (102)$$

In particular, when  $e_p = 0$  (in the absence of a gyratonic matter) this further simplifies to  $X_{pmq} = 0 = X_q$ ,  $W_{pq} = -\frac{1}{2} c_{,pq}$ , and  $W = -\frac{1}{2} \Delta c$ . Therefore, the class of pp-wave metrics

$$ds^2 = \delta_{pq} dx^p dx^q - 2 du dr + c du^2, \quad (103)$$

is necessarily of type N, unless  $c_{,pq} = \frac{1}{D-2} \delta_{pq} \Delta c$ , in which case the spacetimes are type O, cf. [11].

For pp-wave geometries of type D, the transverse space with metric  $g_{pq}$  can not be flat (otherwise, they would become of type O). In such a case, the general expressions (95)–(98) must be used.

Recall that the conditions for types II(b), II(c), III(b) are identically satisfied in the  $D = 4$  subcase.

Our results extend those presented in Proposition 7.3 of [6].

## 10. VSI spacetimes

The VSI spacetimes have the property that their *scalar curvature invariants of all orders vanish* identically. They necessarily belong to the Kundt class and, relative to  $\mathbf{k}$ , their Riemann tensor is of type III or more special (Proposition 5.2 of [6]). As shown in [11], see equations (20)–(23) therein, these spacetimes must be of the form (20) with  $g_{uu}$  quadratic in  $r$  and flat transverse space  $g_{pq} = \delta_{pq}$ ,

$$ds^2 = \delta_{pq} dx^p dx^q + 2(e_p + f_p r) du dx^p - 2 du dr + (a r^2 + b r + c) du^2, \quad (104)$$

where

$$a = \frac{1}{4} f^p f_p, \quad f_{pq} = 0, \quad f = 0, \quad F_{pq} = 0, \quad (105)$$

$${}^S R = 0, \quad {}^S R_{pq} = 0, \quad {}^S C_{mpnq} = 0. \quad (106)$$

In full agreement with [6, 11], all such spacetimes are of Weyl type III or more special, since the conditions in table 2 for II(a), II(b), II(c) and II(d) are automatically satisfied.

The possible subtypes are determined by the conditions listed in table 4, in which

$$T_p = -\frac{1}{2} f^q e_q f_p + f^q e_{[q,p]} + \frac{2}{D-3} X_p, \quad (107)$$

$$U_p = \frac{1}{2} (f_{p,u} - T_p) - \frac{1}{4} f^q f_q e_p, \quad (108)$$

$$Z_{pq} = \frac{1}{4} e^m e_m f_p f_q + e_{p,uq} - e^m e_{[m,p]} f_q + g^{mn} e_{[m,p]} e_{[n,q]} - \frac{2}{D-3} X_p e_q, \quad (109)$$

$$X_{pmq} = e_{[q,m],p} + \frac{1}{2} (e_{(p,m)} f_q - e_{(p,q)} f_m), \quad (110)$$

$$W_{pq} = r [U_{(p,q)} + U_{(p} f_q)] - \frac{1}{4} [(c_{,p} - c f_p)_{,q} + (c_{,q} - c f_q)_{,p}] - \frac{1}{2} b e_{(p,q)} + Z_{(pq)}, \quad (111)$$

and

$$X_q = \delta^{mn} e_{[q,m],n} + \frac{1}{2} \delta^{mn} e_{m,n} f_q - \frac{1}{2} f^p e_{(p,q)}, \quad (112)$$

$$W = r \delta^{pq} (U_{p,q} + U_p f_q) - \frac{1}{2} \delta^{pq} (c_{,p} - c f_p)_{,q} - \frac{1}{2} b \delta^{pq} e_{p,q} + \delta^{pq} Z_{pq}. \quad (113)$$

type	necessary and sufficient conditions
III(a)	$a_{,p} + f_p a = 0$ where $a = \frac{1}{4} f^p f_p$ $b_{,p} - f_{p,u} = T_p$
III(b)	$X_{pmq} = \frac{1}{D-3} (\delta_{pm} X_q - \delta_{pq} X_m)$
N	III(ab)
O	N with $W_{pq} = \frac{1}{D-2} \delta_{pq} W$
O'	$W_{pq} = 0 = W$

**Table 4.** The complete classification scheme of all VSI geometries (104) in any dimension  $D$ .

This agrees with the results presented in [11], and extends them because no field equations and gauge fixing have been employed.

## 11. Generalized Bertotti–Robinson, Nariai, and other type D and O spacetimes

Finally, let us consider a subclass of the Kundt geometries of the form

$$ds^2 = g_{pq} dx^p dx^q - 2 du dr + a r^2 du^2, \quad (114)$$

which is the special case of (20) with

$$e_p = 0, \quad f_p = 0, \quad b = 0, \quad c = 0. \quad (115)$$

As recently discussed in [18], such spacetimes include interesting CSI backgrounds on which the Kundt waves and gyratons in any (higher) dimension propagate.

Algebraic classification of these metrics depends on the following functions:

$$f = 0, \quad f_{pq} = 0, \quad F_{pq} = 0, \quad (116)$$

$$Y_{pmq} = 0 = Y_q, \quad (117)$$

$$g_{uu,rp} = 2a_{,p} r, \quad (118)$$

$$T_p = \frac{2}{D-3} X_p, \quad (119)$$

$$X_{pmq} = g_{p[m,u][q]}, \quad (120)$$

$$X_q = g^{mn} g_{m[n,u][q]}, \quad (121)$$

$$W_{pq} = -\frac{1}{2} a_{||p||q} r^2 + \frac{1}{2} a r g_{pq,u} - \frac{1}{2} g_{pq,uu} + \frac{1}{4} g^{mn} g_{mp,u} g_{nq,u}, \quad (122)$$

$$W = -\frac{1}{2} \Delta a r^2 + \frac{1}{2} a r g^{pq} g_{pq,u} - \frac{1}{2} g^{pq} g_{pq,uu} + \frac{1}{4} g^{mn} g^{pq} g_{mp,u} g_{nq,u}. \quad (123)$$

The corresponding subtypes, as determined generally in table 2, can thus be summarized in table 5. Notice that the condition for subtype O' splits, using (122), into  $a g_{pq,u} = 0$  and  $g_{pq,uu} = \frac{1}{2} g^{mn} g_{mp,u} g_{nq,u}$ . There are thus two distinct subclasses, namely  $a = 0$  and  $g_{pq,u} = 0$ .

An important subfamily of (114) arises when  $a$  is *constant* and  $g_{pq,u} = 0$ :

$$ds^2 = g_{pq}(x) dx^p dx^q - 2 du dr + a r^2 du^2. \quad (124)$$

In such a case,  $X_{pmq} = 0 = X_q$  and  $W_{pq} = 0 = W$ , so that the spacetimes can only be of algebraic type D or type O, as described in table 6. In fact, these are the *direct-product spacetimes*, where the first part is a  $(D-2)$ -dimensional Riemannian space with metric  $g_{pq}$ , while the second part is a 2-dimensional Lorentzian spacetime which has a constant curvature. By performing the transformation

$$U = \frac{1}{a u}, \quad V = \frac{4}{a r} + 2 u, \quad (125)$$

the metric (124) is put to the canonical form

$$ds^2 = g_{pq}(x) dx^p dx^q - \frac{2 dU dV}{(1 - \frac{1}{2} a UV)^2}. \quad (126)$$

According to the sign of the constant  $a$  there are three possibilities: for  $a = 0$  the temporal surface  $x = \text{const.}$  is the 2-dimensional Minkowski space  $M_2$ , for  $a > 0$  it is the 2-dimensional de Sitter space  $dS_2$ , and for  $a < 0$  we get the 2-dimensional anti-de Sitter space  $AdS_2$ . The scale of the (anti-)de Sitter space is given by  $\ell = 1/\sqrt{|a|}$ .

For generic type D direct-product spacetimes (124), or equivalently (126), the transverse metric  $g_{pq}$  *need not* be of constant curvature. However, for the subtype D(a), the Gaussian curvature  $a$  is *uniquely related* to the *constant Ricci scalar*  ${}^S R$  of the transverse  $(D-2)$ -dimensional Riemannian space. The metrics



type	necessary and sufficient conditions
II(a)	$a = -\frac{1}{(D-2)(D-3)} {}^S R$
II(b)	${}^S R_{pq} = \frac{1}{D-2} g_{pq} {}^S R$
II(c)	${}^S C_{mpnq} = 0$
II(d)	always
III	II(abcd)
III(a)	${}^S R_{,p} = 0, \quad X_q = g^{mn} g_{m[n,u][q]} = 0$
III(b)	$X_{pmq} = \frac{1}{D-3} (g_{pm} X_q - g_{pq} X_m)$
N	III(ab)
O	N with $W_{pq} = \frac{1}{D-2} g_{pq} W$
O'	$W_{pq} = 0$
D	$a_{,p} = 0$ $X_q = 0 \Leftrightarrow X_{pmq} = 0$ $W_{pq} = \frac{1}{D-2} g_{pq} W$
D(a)	D with II(a)
D(b)	D with II(b)
D(c)	D with II(c)
D(d)	D with II(d)

**Table 5.** The complete classification scheme of all Kundt geometries (114) in any dimension  $D$  with respect to multiple WAND  $\mathbf{k} = \partial_r$  and (possibly) double WAND  $\mathbf{l} = \frac{1}{2} ar^2 \partial_r + \partial_u$ .

type	necessary and sufficient conditions
D=D(d)	always
D(a)	$a = -\frac{1}{(D-2)(D-3)} {}^S R = \text{const.}$
D(b)	${}^S R_{pq} = \frac{1}{D-2} g_{pq} {}^S R$
D(c)	${}^S C_{mpnq} = 0$
O	D(abcd)= D(abc)

**Table 6.** Algebraic classification of the direct-product Kundt geometries (124) in any dimension  $D$ . These involve generalizations of the Bertotti–Robinson, (anti-)Nariai, and Plebański–Hacyan spacetimes of type D or O.

then represent higher-dimensional generalizations of the well-known  $D = 4$  Bertotti–Robinson, (anti-)Nariai, and Plebański–Hacyan spacetimes of types D or O, for which the two-dimensional transverse Riemannian spaces are either  $S^2$ ,  $E^2$ , or  $H^2$  of *constant* positive, zero, or negative curvatures, see chapter 7 of [4] for more details.

General conditions for direct-product spacetimes to be of type O were worked out a long time ago in the pioneering work [62], and the fact that such spacetimes must be of type D(d) or O was noticed in [56], see Proposition 4 therein. In our recent work [18] we analyzed in detail exact type II Kundt gravitational waves and gyratons propagating on such direct-product backgrounds.

## 12. Conclusions

We derived the complete and explicit algebraic classification, based on the null alignment properties of the Weyl tensor, of the general Kundt class of spacetimes (2) in an arbitrary dimension  $D$  for the case when the generator  $\mathbf{k}$  of optically privileged null congruence with vanishing expansion, twist, and shear is a (multiple) WAND. The results apply to any metric theory of gravity which admits the Kundt geometries. Classification of the Kundt spacetimes in standard general relativity is obtained by setting  $D = 4$  (and applying the Einstein field equations).

We calculated all components of the Riemann and Ricci curvature tensors, see (A.13)–(A.28), and projecting the corresponding Weyl tensor onto the natural null frame (6) we showed that the Kundt geometries are of type I(b) (or more special) with  $\mathbf{k} = \partial_r$  being WAND aligned with the optically privileged null congruence.

The spacetimes become algebraically special with respect to multiple WAND  $\mathbf{k}$  if, and only if, the metric functions  $g_{up}$  are (at most) linear in the affine parameter  $r$  along the privileged null congruence with vanishing optical scalars, see the line element (20). For such a generic metric we explicitly evaluated all (rather complicated) coordinate components of the Riemann, Ricci and Weyl tensors, as summarized in Appendix B. The corresponding Weyl scalars (of various boost weights and spins) with respect to the most natural null frame (6) have surprisingly simple structure (21)–(28). This enabled us to determine all possible algebraic types with respect to multiple WAND  $\mathbf{k}$  (and potential additional WAND  $\mathbf{l}$ ), including the refinement to various subtypes, namely II(a), II(b), II(c), II(d), III(a), III(b), N, O, II<sub>*i*</sub>, III<sub>*i*</sub>, D(a), D(b), D(c), and D(d), see sections 3–7 respectively. The explicit conditions, given in table 2 of section 8, are expressed in an invariant geometric form. Some of them are identically satisfied in the  $D = 4$  case.

In the final sections 9–11 we applied our classification scheme to the most important subclasses of the Kundt geometries, namely the pp-waves, the VSI spacetimes, and the generalization of direct-product spacetimes of any dimension. The main results are contained in tables 3–6.

We hope that these results will be useful for further studies of the Kundt family of geometries and its various interesting representatives. For example, in a subsequent paper [63] extending the preliminary analysis in [64], we will relate the algebraic classification of these spacetimes to specific local motion of test particles, as given by the deviation of geodesics. This will further clarify their geometrical properties and physical interpretation.

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**Appendix A. Curvature tensors for a general Kundt geometry**

For the most general Kundt line element (2) the Christoffel symbols read

$$\Gamma_{ru}^r = -\frac{1}{2}g_{uu,r} + \frac{1}{2}g^{rp}g_{up,r}, \quad (\text{A.1})$$

$$\Gamma_{rp}^r = -\frac{1}{2}g_{up,r}, \quad (\text{A.2})$$

$$\Gamma_{uu}^r = \frac{1}{2}[-g^{rr}g_{uu,r} - g_{uu,u} + g^{rp}(2g_{up,u} - g_{uu,p})], \quad (\text{A.3})$$

$$\Gamma_{uq}^r = \frac{1}{2}[-g^{rr}g_{uq,r} - g_{uu,q} + g^{rp}(2g_{p(u,q)} - g_{uq,p})], \quad (\text{A.4})$$

$$\Gamma_{pq}^r = \frac{1}{2}[g_{pq,u} - 2g_{u(p,q)} + g^{rm}(2g_{m(p,q)} - g_{pq,m})], \quad (\text{A.5})$$

$$\Gamma_{ru}^p = \frac{1}{2}g^{pq}g_{uq,r}, \quad (\text{A.6})$$

$$\Gamma_{uu}^p = \frac{1}{2}[-g^{rp}g_{uu,r} + g^{pq}(2g_{uq,u} - g_{uu,q})], \quad (\text{A.7})$$

$$\Gamma_{um}^p = \frac{1}{2}[-g^{rp}g_{um,r} + g^{pq}(2g_{q(u,m)} - g_{um,q})], \quad (\text{A.8})$$

$$\Gamma_{pq}^m = \frac{1}{2}g^{mn}(2g_{n(p,q)} - g_{pq,n}), \quad (\text{A.9})$$

$$\Gamma_{uu}^u = \frac{1}{2}g_{uu,r}, \quad (\text{A.10})$$

$$\Gamma_{up}^u = \frac{1}{2}g_{up,r}, \quad (\text{A.11})$$

where the contravariant metric coefficients of (2) are

$$g^{rp} = g^{pq}g_{uq}, \quad g^{rr} = -g_{uu} + g^{pq}g_{up}g_{uq}, \quad (\text{A.12})$$

with  $g^{pq}$  denoting the inverse of  $g_{pq}$  (inversely,  $g_{up} = g_{pq}g^{rq}$ ,  $g_{uu} = -g^{rr} + g_{pq}g^{rp}g^{rq}$ ). A straightforward but lengthy calculation gives the following coordinate components of the Riemann curvature tensor in a fully explicit and compact form

$$R_{rprq} = 0, \quad (\text{A.13})$$

$$R_{rpru} = -\frac{1}{2}g_{up,rr}, \quad (\text{A.14})$$

$$R_{ruur} = -\frac{1}{2}g_{uu,rr} + \frac{1}{4}g^{pq}g_{up,r}g_{uq,r}, \quad (\text{A.15})$$

$$R_{rpmq} = 0, \quad (\text{A.16})$$

$$R_{rpuq} = \frac{1}{2}g_{up,rq} + \frac{1}{4}g_{up,r}g_{uq,r} - \frac{1}{4}g^{mn}g_{um,r}(2g_{n(p,q)} - g_{pq,n}), \quad (\text{A.17})$$

$$R_{rupq} = g_{u[p,q],r}, \quad (\text{A.18})$$

$$R_{ruup} = g_{u[u,p],r} + \frac{1}{4}g^{rq}g_{up,r}g_{uq,r} - \frac{1}{4}g^{mn}g_{um,r}(2g_{n(u,p)} - g_{up,n}), \quad (\text{A.19})$$

$$R_{mpnq} = {}^S R_{mpnq}, \quad (\text{A.20})$$

$$\begin{aligned} R_{upmq} &= g_{p[m,q],u} - g_{u[m,q],p} \\ &\quad + \frac{1}{4}[g_{um,r}(g_{pq,u} - 2g_{u(p,q)}) - g_{uq,r}(g_{pm,u} - 2g_{u(p,m)})] \\ &\quad + \frac{1}{4}g^{rn}[g_{um,r}(2g_{n(p,q)} - g_{pq,n}) - g_{uq,r}(2g_{n(p,m)} - g_{pm,n})] \\ &\quad + \frac{1}{4}g^{ns}(2g_{s(u,q)} - g_{uq,s})(2g_{n(p,m)} - g_{pm,n}) \\ &\quad - \frac{1}{4}g^{ns}(2g_{s(u,m)} - g_{um,s})(2g_{n(p,q)} - g_{pq,n}), \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned} R_{rupq} &= g_{u(p,q),u} - \frac{1}{2}(g_{pq,uu} + g_{uu,pq}) + \frac{1}{4}g^{rr}g_{up,r}g_{uq,r} \\ &\quad - \frac{1}{4}g_{uu,r}[2g_{u(p,q)} - g_{pq,u} - g^{rm}(2g_{m(p,q)} - g_{pq,m})] \\ &\quad + \frac{1}{4}g_{up,r}[g_{uu,q} - g^{rm}(2g_{m(u,q)} - g_{uq,m})] \\ &\quad + \frac{1}{4}g_{uq,r}[g_{uu,p} - g^{rm}(2g_{m(u,p)} - g_{up,m})] \\ &\quad + \frac{1}{4}g^{mn}(2g_{n(u,p)} - g_{up,n})(2g_{m(u,q)} - g_{uq,m}) \\ &\quad - \frac{1}{4}g^{mn}(2g_{un,u} - g_{uu,n})(2g_{m(p,q)} - g_{pq,m}), \end{aligned} \quad (\text{A.22})$$

where  $p, q, m, n, s$  denote the spatial components (and derivatives with respect to)  $x$ . The superscript “ $S$ ” labels tensor quantities corresponding to the spatial metric  $g_{pq}$ , with derivatives taken only with respect to the coordinates  $x$ . The corresponding coordinate components of the Ricci tensor are||

$$R_{rr} = 0, \quad (\text{A.23})$$

$$R_{rp} = -\frac{1}{2}g_{up,rr}, \quad (\text{A.24})$$

$$R_{ru} = -\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{rp}g_{up,rr} + \frac{1}{2}g^{pq}g_{up,rq} + \frac{1}{2}g^{pq}g_{up,r}g_{uq,r} - \frac{1}{4}g^{pq}g^{mn}g_{um,r}(2g_{np,q} - g_{pq,n}), \quad (\text{A.25})$$

$$R_{pq} = {}^S R_{pq} - g_{u(p,q),r} - \frac{1}{2}g_{up,r}g_{uq,r} + \frac{1}{2}g^{mn}g_{um,r}(2g_{n(p,q)} - g_{pq,n}), \quad (\text{A.26})$$

$$R_{us} = -\frac{1}{2}g^{rr}g_{us,rr} - g_{u[s],r} + g^{rp}\left(\frac{1}{2}g_{up,sr} - g_{us,pr}\right) + g^{pq}(g_{p[s,q],u} - g_{u[s,q],p}) - \frac{1}{2}g^{rp}g_{us,r}g_{up,r} + \frac{1}{4}g^{pq}g^{rm}\left[4g_{uq,r}g_{s[p,m]} + g_{us,r}(2g_{m(p,q)} - g_{pq,m})\right] + \frac{1}{4}g^{pq}\left[2g_{up,r}g_{uq,s} - g_{us,r}(2g_{up,q} - g_{pq,u})\right] + \frac{1}{4}g^{pq}g^{mn}\left(2g_{n(u,q)} - g_{uq,n}\right)\left(2g_{m(p,s)} - g_{ps,m}\right) - \frac{1}{4}g^{pq}g^{mn}\left(2g_{n(u,s)} - g_{us,n}\right)\left(2g_{mp,q} - g_{pq,m}\right), \quad (\text{A.27})$$

$$R_{uu} = -\frac{1}{2}g^{rr}g_{uu,rr} - 2g^{rp}g_{u[p],r} + \frac{1}{2}g^{pq}(2g_{up,uq} - g_{pq,uu} - g_{uu,pq}) - \frac{1}{2}g^{rp}g^{rq}g_{up,r}g_{uq,r} + \frac{1}{2}g^{rr}g^{pq}g_{up,r}g_{uq,r} + \frac{1}{2}g^{pq}g^{rm}g_{up,r}(2g_{q(u,m)} - g_{um,q}) - \frac{1}{4}g^{pq}g_{uu,r}\left[2g_{up,q} - g_{pq,u} - g^{rm}(2g_{mp,q} - g_{pq,m})\right] + \frac{1}{2}g^{pq}g_{up,r}\left[g_{uu,q} - g^{rm}(2g_{m(u,q)} - g_{uq,m})\right] + \frac{1}{4}g^{pq}g^{mn}\left(2g_{n(u,p)} - g_{up,n}\right)\left(2g_{m(u,q)} - g_{uq,m}\right) - \frac{1}{4}g^{pq}g^{mn}\left(2g_{un,u} - g_{uu,n}\right)\left(2g_{mp,q} - g_{pq,m}\right), \quad (\text{A.28})$$

and the Ricci scalar curvature of the general Kundt spacetime (2) is given by

$$R = {}^S R + g_{uu,rr} - 2g^{rp}g_{up,rr} - 2g^{pq}g_{up,rq} - \frac{3}{2}g^{pq}g_{up,r}g_{uq,r} + g^{pq}g^{mn}g_{um,r}(2g_{np,q} - g_{pq,n}). \quad (\text{A.29})$$

The Weyl tensor components are defined as

$$C_{abcd} = R_{abcd} - \frac{1}{D-2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}) + \frac{1}{(D-1)(D-2)}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \quad (\text{A.30})$$

Notice that  $C_{rprq} = 0$ .

## Appendix B. Curvature tensors for Kundt geometries of type II

In this appendix we explicitly present the Riemann, Ricci, and Weyl tensors for an arbitrary Kundt spacetime of dimension  $D \geq 4$  which is *algebraically special* (of type II or more special) *with respect to*  $\mathbf{k}$ , the generator of a privileged null congruence with vanishing optical scalars. As in the main text, we do not apply any field equations.

|| Let us remark that the first term in  $R_{us}$  is missing in the corresponding component (25) presented in [51]. Other components are equivalent using the relations  $(\ln \sqrt{g})_{,u} = \Gamma_{ua}^a = \frac{1}{2}g^{pq}(2g_{q(u,p)} - g_{up,q})$  and  $(\ln \sqrt{g})_{,p} = \Gamma_{pa}^a = \Gamma_{pq}^q = \frac{1}{2}g^{qm}(2g_{m(p,q)} - g_{pq,m})$ , where  $g \equiv \det g_{pq} = -\det g_{ab}$ .

From section 3 it follows that the most general Kundt metric of algebraic type II with double WAND  $\mathbf{k} = \partial_r$  can be written in the form (20) with

$$g_{up} = e_p(u, x) + f_p(u, x) r, \quad (\text{B.1})$$

for any  $p = 2, \dots, D-1$ . The contravariant terms (A.12) are thus

$$g^{rp} = e^p + f^p r, \quad g^{rr} = -g_{uu} + e^p e_p + 2 e^p f_p r + f^p f_p r^2, \quad (\text{B.2})$$

where  $e^p(u, x) \equiv g^{pq} e_q$ ,  $f^p(u, x) \equiv g^{pq} f_q$ , whereas  $g_{uu}(r, u, x)$  is an arbitrary function of all the coordinates. The coordinate components of the Riemann curvature tensor (A.13)–(A.22) then reduce to

$$R_{rprq} = 0, \quad (\text{B.3})$$

$$R_{rpru} = 0, \quad (\text{B.4})$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} f^p f_p, \quad (\text{B.5})$$

$$R_{rpmq} = 0, \quad (\text{B.6})$$

$$R_{rpuq} = \frac{1}{2} f_{pq} + \frac{1}{2} F_{pq}, \quad (\text{B.7})$$

$$R_{rupq} = F_{pq}, \quad (\text{B.8})$$

$$R_{ruup} = \frac{1}{2} g_{uu,rp} - \frac{1}{2} f_{p,u} + \frac{1}{4} r f^q (f_q f_p - 2F_{qp}) + \frac{1}{4} f^q (e_q f_p - 2E_{qp}), \quad (\text{B.9})$$

$$R_{mpnq} = {}^S R_{mpnq}, \quad (\text{B.10})$$

$$R_{upmq} = r \left[ f_{[q||m]||p} + \frac{1}{2} (f_{pm} f_q - f_{pq} f_m) \right] + e_{[q||m]||p} + \frac{1}{2} (e_{pm} f_q - e_{pq} f_m) + g_{p[m,u||q]}, \quad (\text{B.11})$$

$$R_{upuq} = -\frac{1}{2} (g_{uu})_{||p||q} - \frac{1}{4} g_{uu} f_p f_q - \frac{1}{2} g_{uu,r} (r f_{(p||q)} + e_{pq}) + \frac{1}{4} (g_{uu,p} f_q + g_{uu,q} f_p) + r^2 \left[ \frac{1}{4} f_p f_q f^m f_m - \frac{1}{2} f^m (F_{mp} f_q + F_{mq} f_p) + g^{mn} F_{mp} F_{nq} \right] + r \left[ f_{(p,u||q)} + \frac{1}{2} f_p f_q f^m e_m - \frac{1}{2} f^m (E_{mp} f_q + E_{mq} f_p) - \frac{1}{2} e^m (F_{mp} f_q + F_{mq} f_p) + g^{mn} (E_{mp} F_{nq} + E_{mq} F_{np}) \right] \quad (\text{B.12})$$

$$+ e_{(p,u||q)} - \frac{1}{2} g_{pq,uu} + \frac{1}{4} e^m e_m f_p f_q - \frac{1}{2} e^m (E_{mp} f_q + E_{mq} f_p) + g^{mn} E_{mp} E_{nq},$$

where the useful geometric quantities  $f_{p||q}$ ,  $f^p_{||p}$ ,  $f_{pq}$ ,  $f$ ,  $F_{pq}$ , etc. are defined in (36)–(53). The symbol  $||$  indicates the covariant derivative with respect to the spatial metric  $g_{pq}$  in the transverse  $(D-2)$ -dimensional Riemannian space.

Similarly, from (A.23)–(A.28) and (A.29), using the identity

$$(f^p e^q - f^q e^p) g_{qm,p} \equiv \Gamma_{pm}^q (e_q f^p - f_q e^p), \quad (\text{B.13})$$

we obtain the following coordinate components of the Ricci tensor and the Ricci scalar:

$$R_{rr} = 0, \quad (\text{B.14})$$

$$R_{rp} = 0, \quad (\text{B.15})$$

$$R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} f + \frac{1}{4} f^p f_p, \quad (\text{B.16})$$

$$R_{pq} = {}^S R_{pq} - f_{pq}, \quad (\text{B.17})$$

$$R_{up} = -\frac{1}{2} g_{uu,rp} + \frac{1}{2} f_{p,u} + r \left[ g^{mn} f_{[m||p]||n} + 2f^q F_{qp} - \frac{1}{2} (f + \frac{1}{2} f^q f_q) f_p \right] + \left[ g^{mn} e_{[m||p]||n} + e^q F_{qp} - \frac{1}{2} g^{mn} e_{mn} f_p + \frac{1}{2} (f^q e_{q||p} - e^q f_{p||q} - e^q f_q f_p) + g^{mn} g_{m[p,u||n]} \right], \quad (\text{B.18})$$

$$\begin{aligned}
R_{uu} = & -\frac{1}{2}\Delta g_{uu} + \frac{1}{2}g_{uu}(g_{uu,rr} - f^p f_p) \\
& + \frac{1}{2}g_{uu,p} f^p + (f_{p,u} - g_{uu,rp})(r f^p + e^p) \\
& - \frac{1}{2}g_{uu,r} g^{pq}(r f_{[p|q]} + e_{pq}) - \frac{1}{2}g_{uu,rr}(r^2 f^p f_p + 2r f^p e_p + e^p e_p) \\
& + r^2 g^{mn} g^{pq} F_{mp} F_{nq} \\
& + r [g^{pq}(f_{p,u})_{|q} + 2f^p e^q F_{pq} + 2g^{mn} g^{pq} E_{mp} F_{nq}] \\
& + [g^{pq}(e_{p,u})_{|q} - \frac{1}{2}g^{pq} g_{pq,uu} + \frac{1}{2}(f^p f_p)(e^q e_q) - \frac{1}{2}(f^p e_p)^2 \\
& + 2f^p e^q e_{[p|q]} + g^{mn} g^{pq} E_{mp} E_{nq}], \tag{B.19}
\end{aligned}$$

and

$$R = g_{uu,rr} - \frac{1}{2}f^p f_p + {}^S R - 2f. \tag{B.20}$$

Finally, after straightforward but very lengthy calculation, the explicit Weyl tensor coordinate components (A.30) become

$$C_{rprq} = 0, \tag{B.21}$$

$$C_{rpru} = 0, \tag{B.22}$$

$$C_{ruru} = \frac{D-3}{D-1} \left[ -\frac{1}{2}g_{uu,rr} + \frac{1}{4}f^p f_p - \frac{1}{D-2} \left( \frac{{}^S R}{D-3} + f \right) \right], \tag{B.23}$$

$$C_{rpmq} = 0, \tag{B.24}$$

$$\begin{aligned}
C_{rupq} = & \frac{g_{pq}}{(D-1)(D-2)} \left[ \frac{1}{2}(D-3)g_{uu,rr} - \frac{1}{4}(D-3)f^m f_m - {}^S R - \frac{1}{2}(D-5)f \right] \\
& + \frac{1}{D-2} \left[ {}^S R_{pq} + \frac{1}{2}(D-4)f_{pq} \right] + \frac{1}{2}F_{pq}, \tag{B.25}
\end{aligned}$$

$$C_{rupq} = F_{pq}, \tag{B.26}$$

$$\begin{aligned}
C_{ruup} = & \frac{D-3}{D-2} \left\{ \frac{1}{2}g_{uu,rp} + r \left[ \frac{f_p}{D-1} \left( \frac{1}{2}g_{uu,rr} + \frac{1}{4}(D-2)f^m f_m - \left( \frac{{}^S R}{D-3} + f \right) \right) \right. \right. \\
& + \left. \frac{1}{D-3} g^{mn} f_{[m|p]||n} - \frac{1}{2} \frac{D-6}{D-3} f^q F_{qp} \right] \\
& + \left[ \frac{e_p}{D-1} \left( \frac{1}{2}g_{uu,rr} - \frac{1}{4}f^m f_m - \left( \frac{{}^S R}{D-3} + f \right) \right) \right. \\
& \left. \left. - \frac{1}{2}f_{p,u} + \frac{1}{4}f^q e_q f_p - \frac{1}{2}f^q E_{qp} - \frac{1}{D-3} X_p \right] \right\}, \tag{B.27}
\end{aligned}$$

$$\begin{aligned}
C_{mpnq} = & {}^S R_{mpnq} - \frac{1}{D-2} (g_{mn}({}^S R_{pq} - f_{pq}) - g_{mq}({}^S R_{pn} - f_{pn}) \\
& + g_{pq}({}^S R_{mn} - f_{mn}) - g_{pn}({}^S R_{mq} - f_{mq})) \\
& + \frac{1}{(D-1)(D-2)} (g_{uu,rr} - \frac{1}{2}f^s f_s + {}^S R - 2f) (g_{mn}g_{pq} - g_{mq}g_{pn}), \tag{B.28}
\end{aligned}$$

$$\begin{aligned}
C_{upmq} = & X_{pmq} + r f_{[q|m]||p} - F_{qm} e_p + \frac{1}{2} (F_{pq} e_m - F_{pm} e_q) \\
& - \frac{r f_m + e_m}{D-2} \left[ \left( {}^S R_{pq} - \frac{g_{pq}}{D-2} {}^S R \right) + \frac{1}{2}(D-4) \left( f_{pq} - \frac{g_{pq}}{D-2} f \right) \right] \\
& + \frac{g_{pq}}{D-2} \left\{ \frac{1}{2}g_{uu,rm} + r \left[ \frac{f_m}{D-1} \left( g_{uu,rr} + \frac{1}{4}(D-3)f^s f_s \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{D-3}{D-2} \left( \frac{{}^S R}{D-3} + f \right) - g^{ns} f_{[n||m]||s} - 2f^n F_{nm} \Big] \\
& + \left[ \frac{e_m}{D-1} \left( g_{uu,rr} - \frac{1}{2} f^s f_s - \frac{D-3}{D-2} \left( \frac{{}^S R}{D-3} + f \right) \right) \right. \\
& \quad \left. - \frac{1}{2} f_{m,u} + \frac{1}{4} f^n e_n f_m - \frac{1}{2} f^n E_{nm} + X_m \right] \Big\}, \\
& - \frac{r f_q + e_q}{D-2} \left[ \left( {}^S R_{pm} - \frac{g_{pm}}{D-2} {}^S R \right) + \frac{1}{2} (D-4) \left( f_{pm} - \frac{g_{pm}}{D-2} f \right) \right] \\
& + \frac{g_{pm}}{D-2} \left\{ \frac{1}{2} g_{uu,rq} + r \left[ \frac{f_q}{D-1} \left( g_{uu,rr} + \frac{1}{4} (D-3) f^s f_s \right) \right. \right. \\
& \quad \left. \left. - \frac{D-3}{D-2} \left( \frac{{}^S R}{D-3} + f \right) \right) - g^{ns} f_{[n||q]||s} - 2f^n F_{nq} \right] \\
& \quad + \left[ \frac{e_q}{D-1} \left( g_{uu,rr} - \frac{1}{2} f^s f_s - \frac{D-3}{D-2} \left( \frac{{}^S R}{D-3} + f \right) \right) \right. \\
& \quad \left. - \frac{1}{2} f_{q,u} + \frac{1}{4} f^n e_n f_q - \frac{1}{2} f^n E_{nq} + X_q \right] \Big\}, \quad (B.29)
\end{aligned}$$

$$\begin{aligned}
C_{upuq} &= \frac{1}{2} \left[ \frac{g_{pq}}{D-2} \Delta g_{uu} - (g_{uu})_{||p||q} \right] \\
& - \frac{1}{2} \left[ \frac{g_{pq}}{D-2} g_{uu,m} f^m - \frac{1}{2} (g_{uu,p} f_q + g_{uu,q} f_p) \right] \\
& \frac{1}{2} g_{uu,r} \left[ \frac{g_{pq}}{D-2} g^{mn} (r f_{(m||n)} + e_{mn}) - (r f_{(p||q)} + e_{pq}) \right] \\
& - \frac{g_{pq}}{D-2} (f_{m,u} - g_{uu,rm}) (r f^m + e^m) \\
& \quad + \frac{1}{D-2} \left[ \frac{1}{2} (f_{p,u} - g_{uu,rp}) (r f_q + e_q) + \frac{1}{2} (f_{q,u} - g_{uu,rq}) (r f_p + e_p) \right] \\
& + \frac{g_{pq} g_{uu}}{(D-1)(D-2)} \left[ -\frac{1}{2} (D-3) g_{uu,rr} + \frac{1}{2} (D-2) f^m f_m + {}^S R - 2f \right] \\
& \quad - \frac{g_{uu}}{D-2} \left[ \frac{1}{4} (D-2) f_p f_q + {}^S R_{pq} - f_{pq} \right] \\
& + \frac{g_{pq}}{D-2} g_{uu,rr} \frac{1}{2} (r^2 f^m f_m + 2r f^m e_m + e^m e_m) \\
& - \frac{1}{(D-1)(D-2)} \left[ g_{uu,rr} - \frac{1}{2} f^m f_m + {}^S R - 2f \right] \\
& \quad \times \left( r^2 f_p f_q + r (f_p e_q + f_q e_p) + e_p e_q \right) \\
& + r^2 \left[ g^{mn} F_{mp} F_{nq} - \frac{g_{pq}}{D-2} g^{mn} g^{st} F_{ms} F_{nt} \right. \\
& \quad - \frac{1}{2} \frac{D-6}{D-2} f^m (F_{mp} f_q + F_{mq} f_p) + \frac{g^{mn}}{D-2} (f_{[m||p]||n} f_q + f_{[m||q]||n} f_p) \\
& \quad \left. - \frac{1}{D-2} \left( \frac{1}{4} (D-4) f^m f_m - f \right) f_p f_q \right]
\end{aligned}$$

$$\begin{aligned}
& + r \left[ f_{(p,u||q)} - \frac{g_{pq}}{D-2} g^{mn} f_{(m,u||n)} - \frac{1}{D-2} (X_p f_q + X_q f_p) \right. \\
& \quad + g^{mn} (E_{mp} F_{nq} + E_{mq} F_{np}) - 2 \frac{g_{pq}}{D-2} g^{mn} g^{st} E_{ms} F_{nt} \\
& \quad - \frac{1}{2} \frac{D-3}{D-2} f^m (E_{mp} f_q + E_{mq} f_p) + \frac{g^{mn}}{D-2} (f_{[m||p]||n} e_q + f_{[m||q]||n} e_p) \\
& \quad + \frac{2}{D-2} f^m [(F_{mp} e_q + F_{mq} e_p) - g_{pq} F_{mn} e^n] \\
& \quad - \frac{1}{2} e^m (F_{mp} f_q + F_{mq} f_p) \\
& \quad \left. + \frac{1}{2} \frac{D-3}{D-2} f^m e_m f_p f_q - \frac{1}{D-2} \left( f + \frac{1}{4} f^m f_m \right) (f_p e_q + f_q e_p) \right] \\
& + \left[ (e_{(p,u||q)} - \frac{1}{2} g_{pq,uu}) - \frac{g_{pq}}{D-2} g^{mn} (e_{(m,u||n)} - \frac{1}{2} g_{mn,uu}) \right. \\
& \quad - \frac{1}{D-2} (X_p e_q + X_q e_p) + g^{mn} E_{mp} E_{nq} - \frac{g_{pq}}{D-2} g^{mn} g^{st} E_{ms} E_{nt} \\
& \quad - \frac{1}{2} e^m (E_{mp} f_q + E_{mq} f_p) + \frac{1}{2} \frac{1}{D-2} f^m (E_{mp} e_q + E_{mq} e_p) \\
& \quad - 2 \frac{g_{pq}}{D-2} f^m e^n e_{[m||n]} + \frac{1}{2} \frac{1}{D-2} f^m e_m \left( g_{pq} f^n e_n - \frac{1}{2} (f_p e_q + f_q e_p) \right) \\
& \quad \left. - \frac{1}{2} e^m e_m \left( \frac{g_{pq}}{D-2} f^n f_n - \frac{1}{2} f_p f_q \right) - \frac{1}{D-2} f e_p e_q \right], \tag{B.30}
\end{aligned}$$

in which  $X_{pmq}$  is given by (29), and  $X_q$  by (32).

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