GEODESICS IN NONEXPANDING IMPULSIVE GRAVITATIONAL WAVES WITH Λ, PART I

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Abstract. We investigate the geodesics in the entire class of nonexpanding impulsive gravitational waves propagating in an (anti-)de Sitter universe using the distributional form of the metric. Employing a 5-dimensional embedding formalism and a general regularisation technique we prove existence and uniqueness of geodesics crossing the wave impulse leading to a completeness result. We also derive the explicit form of the geodesics thereby confirming previous results derived in a heuristic approach.

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1. Introduction

Impulsive *pp*-waves have now been studied for several decades and have become textbook examples of exact radiative spacetimes modelling short but intense bursts of gravitational radiation propagating in a Minkowski background (see e.g. [16, Sec. 20.2] and the references therein). Such geometries have been introduced by Roger Penrose using his 'scissors and paste method' (see e.g. [25]) leading to the *distributional Brinkmann form* of the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\mathrm{d}u\mathrm{d}v + f(x, y)\delta(u)\mathrm{d}u^2, \tag{1.1}$$

i.e., impulsive limits of sandwich *pp*-waves [5]. Alternatively, almost at the same time, Aichelburg and Sexl in [1] have obtained a special impulsive *pp*-wave as the ultrarelativistic limit of the Schwarzschild geometry and several authors have applied the same approach to other solutions of the Kerr-Newman family (see e.g. [4, Ch. 4] and [27, Sec. 3.5.1] for an overview). This procedure of boosting static sources to the speed of light was later generalised to the case of a non-vanishing cosmological constant Λ in the pioneering work [19] by Hotta and Tanaka (see also [28, 29]) which lead to an increased interest in nonexpanding impulsive waves in cosmological de Sitter and anti-de Sitter backgrounds. The Penrose 'scissors and paste method' for non-vanishing Λ was described in [38, 30] while impulsive limits in the Kundt class were considered in [26] and elsewhere, see [16, Sec. 20.3] for an overview.

Generally, nonexpanding impulsive waves in all backgrounds of constant curvature can be described by a continuous as well as by a distributional form of the metric tensor. To give a brief discussion of these we start with the conformally flat form of Minkowski ($\Lambda = 0$) and (anti-)de Sitter ($\Lambda \neq 0$) background spacetimes

$$\mathrm{d}s_0^2 = \frac{2\,\mathrm{d}\eta\,\mathrm{d}\bar{\eta} - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V}}{[\,1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\,]^2}\,. \tag{1.2}$$

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As in [30], here $\mathcal{U}, \mathcal{V}$ are the usual null and $\eta, \bar{\eta}$ the usual complex spatial coordinates. Now for $\mathcal{U} > 0$ we perform the transformation

$$\mathcal{U} = U\,, \quad \mathcal{V} = V + H + UH_{,Z}H_{,\bar{Z}}\,, \quad \eta = Z + UH_{,\bar{Z}}\,, \tag{1.3}$$

where $H(Z, \bar{Z})$ is an arbitrary real-valued function. Combining this with the background line element (1.2) for $U < 0$ in which we set $\mathcal{U} = U$, $\mathcal{V} = V$, $\eta = Z$ we obtain the *continuous* form of the metric

$$\mathrm{d}s^2 = \frac{2\,|\mathrm{d}Z + U_+(H_{,Z\bar{Z}}\mathrm{d}Z + H_{,\bar{Z}\bar{Z}}\mathrm{d}\bar{Z})|^2 - 2\,\mathrm{d}U\mathrm{d}V}{[\,1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV + U_+G)\,]^2}\,, \tag{1.4}$$

where† $G(Z, \bar{Z}) = ZH_{,Z} + \bar{Z}H_{,\bar{Z}} - H$ and $U_+ \equiv U_+(U) = 0$ for $U \leq 0$ and $U_+(U) = U$ for $U \geq 0$. This *'kink-function'* U_+ is Lipschitz continuous, hence the spacetime (apart from possible poles of H, which indeed occur in physically realistic models, see e.g. [1, 19], and Section 2, below) is locally Lipschitz. Recall that a locally Lipschitz metric possesses a locally bounded connection and hence a distributional curvature, which, however, in general is unbounded. In fact the discontinuity in the derivatives of the metric introduces impulsive components in the Weyl and curvature tensors (see [30]), and the metric (1.4) explicitly describes impulsive waves in de Sitter, anti-de Sitter or Minkowski backgrounds.

For $\Lambda = 0$, (1.4) reduces to the classic Rosen form of impulsive *pp*-waves (cf. [16, Sec. 17.5]). In this special case the geodesic equation has been rigorously solved in [23] using Carathéodory's solution concept (see e.g. [12, Ch. 1]), which allows to deal with the locally bounded but discontinuous right hand side of the equation. The geodesics thereby obtained coincide with the limits of the geodesics for the distributional form (1.1) calculated in [21] which have been previously derived heuristically (e.g. in [11, 2, 35]).

To deal, however, with the geodesic equation for the full class of nonexpanding impulsive waves for arbitrary Λ, that is the complete metric (1.4), the more sophisticated solution concept of Filippov ([12, Ch. 2]) has been applied recently in [32]. Building on a general result for all locally Lipschitz spacetimes ([40]), existence, uniqueness and global $\mathcal{C}^1$-regularity of the geodesics has been established. This, in particular, justifies the $\mathcal{C}^1$-matching procedure which has been used before to explicitly derive the geodesics in this and similar situations ([35, 41, 30, 31, 33, 34]).

On the other hand, the *distributional form* of the impulsive metric arises by writing the transformation relating (1.2) and (1.4) in a combined way for all U using the Heaviside function Θ as

$$\mathcal{U} = U\,, \quad \mathcal{V} = V + \Theta\, H + U_+\, H_{,Z}H_{,\bar{Z}}\,, \quad \eta = Z + U_+\, H_{,\bar{Z}}\,. \tag{1.5}$$

Of course, (1.5) is discontinuous in the coordinate $\mathcal{V}$ and merely Lipschitz continuous in η across $\{\mathcal{U} = 0\}$ but applying it *formally* to (1.4) we arrive at

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\eta\,\mathrm{d}\bar{\eta} - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2H(\eta, \bar{\eta})\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2}{[\,1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\,]^2}\,, \tag{1.6}$$

which has the striking advantage of coinciding with the background metric $\mathrm{d}s_0^2$ (1.2) off the impulse located at $\mathcal{U} = 0$. This, however, comes at the price of introducing a distributional coefficient in the metric which leads us out of the Geroch–Traschen class ([15]) of metrics (of

†This choice of sign of G is in accordance with [31], which is our main point of reference, but different from more recent papers, e.g. [32].

regularity $W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$), which guarantees existence and stability of the curvature in distributions (see also [24, 42]). However, due to its simple geometrical structure the metric (1.6) nevertheless allows to calculate the curvature as a distribution, again leading to impulsive components in the Weyl and curvature tensors ([30]). Also, in Minkowski background the metric (1.6) reduces to the Brinkmann form of impulsive *pp*-waves (1.1).

Clearly, a mathematically sound treatment of the transformation (1.5) is a delicate matter. In the special case of *pp*-waves this has been achieved in [22] using nonlinear distributional geometry ([17, Ch. 3]) based on algebras of generalised functions ([8]). More precisely, the 'discontinuous coordinate change' was shown to be the distributional limit of a 'generalised diffeomorphism', a concept further studied in [9, 10]. The key to these results was provided by a nonlinear distributional analysis of the geodesics in the metric (1.1), and, in particular, an existence, uniqueness and completeness result for the geodesic equation in nonlinear generalised functions ([39, 21]).
This suggests that a first step towards the long-term goal of understanding the transformation (1.5) for $\Lambda \neq 0$ is to reach a mathematically sound understanding of the geodesics in the distributional metric (1.6). The geodesic equation for (1.6), however, displays a very singular behaviour including terms proportional to the square of the Dirac-δ (cf. [38]). For this reason the authors of [31] have employed a five-dimensional formalism ([29, 30]) of embedding (anti-)de Sitter space into a 5-dimensional *pp*-wave spacetime (see Section 2 below). In this approach the geodesic equation takes a form that is distributionally accessible at all, however, not mathematically rigorously. In the absence of a valid solution concept for this nonlinear distributional equations a natural ansatz was used to derive the geodesics and to study them in detail in [31, Sec. 4–5]. Nevertheless, a desirable nonlinear distributional analysis of the geodesic equation in the $\Lambda \neq 0$-cases, which will eventually lead to a mathematical understanding of the transformation (1.5), has been missing to date.
In this paper we provide such an analysis. Thereby, we follow [31] in employing the five-dimensional formalism. We will, however, not use any theory of nonlinear distributions leaving a detailed study of nonexpanding impulsive waves in (anti-)de Sitter universe as (nonlinear) distributional geometries to a subsequent paper. Instead, we will employ a regularisation approach and view (1.6) as a spacetime with a short but finitely extended impulse (i.e., a generic sandwich wave with support in a regularisation strip which we will also call the 'wave zone') and employ an analysis in the spirit of [36] where impulsive limits in a class of *pp*-wave type spacetimes with a more general wave surface but vanishing Λ ([6, 13, 7, 14]) have been considered.

We will detail this regularisation approach in the next section after introducing the 5-dimensional formalism.In particular, we will replace the Dirac-δ in the metric (2.1) below by a very general regularisation δ_ε, thereby ensuring that our results are *regularisation independent* (within the class of so called a *model delta nets*). Then in Section 3 we will employ a fixed point argument to show that the regularised equations have unique smooth solutions which cross the regularisation strip. This will lead to our main result on completeness of nonexpanding impulsive gravitational waves in a cosmological background. The technical proofs allowing for the application of the fixed point theorem are deferred to Appendix A. In Section 4 we study boundedness properties of the regularised geodesics which are essential when dealing with their limits in Section 5. There we show that the solutions of the regularised geodesic equation converge, as the regularisation parameter goes to zero, to geodesics of the background (anti-)de Sitter spacetime which have to be matched appropriately across the impulse and have been derived previously in [31]. Overly technical calculations are collected in Appendix B.

2. The geodesic equation for $\Lambda \neq 0$

In this section we detail our regularisation approach and derive the respective geodesic equations. To begin with, however, we recall the 5-dimensional formalism of [29, 30] for the full class of nonexpanding impulsive waves with non-vanishing Λ. To this end we start with the 5-dimensional impulsive *pp*-wave spacetime $\overline{M}$ with metric (extending (1.1))

$$\mathrm{d}s^2 = \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \sigma \mathrm{d}Z_4^2 - 2\mathrm{d}U\mathrm{d}V + H(Z_2, Z_3, Z_4)\delta(U)\mathrm{d}U^2 \tag{2.1}$$

and consider the four-dimensional (anti-)de Sitter hyperboloid (M, g) given by the constraint

$$Z_2^2 + Z_3^2 + \sigma Z_4^2 - 2UV = \sigma a^2, \tag{2.2}$$

where $a := \sqrt{3/(\sigma\Lambda)}$, $\sigma := \mathrm{sign}(\Lambda) = \pm 1$, and $U = \frac{1}{\sqrt{2}}(Z_0 + Z_1)$, $V = \frac{1}{\sqrt{2}}(Z_0 - Z_1)$ are null-coordinates.[†] Here $(Z_0, \ldots, Z_4)$ are global Cartesian coordinates of $\mathbb{R}^5$. The impulse is located on the null hypersurface $\{U = 0\}$ given by

$$ {Z_2}^2 + {Z_3}^2 + \sigma {Z_4}^2 = \sigma a^2 \,, \tag{2.3}$$

which is a *nonexpanding* 2-sphere in the de Sitter universe ($\Lambda > 0$) and a hyperboloidal 2-surface in the anti-de Sitter universe ($\Lambda < 0$), respectively. Various 4-dimensional coordinate parametrizations of these spacetimes have been considered e.g. in [28]. Physically the spacetime (2.1), (2.2) describes impulsive gravitational waves as well as impulses of null matter. Purely gravitational waves occur in case the vacuum field equations are satisfied. It was demonstrated in [29, 30] that such solutions can be explicitly written as

$$H(z, \phi) = b_0\, Q_1(z) + \textstyle\sum_{m=1}^{\infty} b_m\, Q_1^m(z) \cos[m(\phi - \phi_m)] \,, \tag{2.4}$$

where $z = Z_4/a$, $\tan\phi = Z_3/Z_2$ and $Q_1^m(z)$ are associated Legendre functions of the second kind generated by the relation $Q_1^m(z) = (-\sigma)^m |1 - z^2|^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}z^m} Q_1(z)$. The first term for $m = 0$, i.e., $Q_1(z) = \frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1$, represents the simplest axisymmetric Hotta–Tanaka solution ([19]). The components with $m \geq 1$ describe nonexpanding impulsive gravitational waves in (anti-)de Sitter universe generated by null point sources with an m-pole structure, localized on the wave-front at the singularities $z = \pm 1$. See [30, 29, 27] for more details.

Now, the geodesics γ of M with tangent T are characterized by the condition that their $\overline{M}$-acceleration $A = \overline{\nabla}_T T$ is everywhere normal to M. Denoting by N the normal vector to M in $\overline{M}$ with $g(N, N) = \sigma$, we hence obtain

$$\overline{\nabla}_T T = -\sigma\, g(T, \overline{\nabla}_T N) N. \tag{2.5}$$

[†]These coordinates are different from those used in the metric (1.4). Since in this paper we will not use the continuous form (1.4) we simplify the notation by not distinguishing them by a bar (as we did in [32]).

Using this identity and the constraint (2.2) the explicit form of the geodesic equation was given in [31, eq. (28)] as

$$
\begin{aligned}
\ddot{U} &= -\frac{1}{3}\,\Lambda\, U\, e\,, \\
\ddot{V} - \frac{1}{2}\, H\, \delta'\, \dot{U}^2 - \delta^{pq}\, H_{,p}\, \delta\, \dot{Z}_q\, \dot{U} &= -\frac{1}{3}\,\Lambda\, V\,\big(e + \frac{1}{2}\, G\, \delta\, \dot{U}^2\big)\,, \\
\ddot{Z}_i - \frac{1}{2}\, H_{,i}\, \delta\, \dot{U}^2 &= -\frac{1}{3}\,\Lambda\, Z_i\,\big(e + \frac{1}{2}\, G\, \delta\, \dot{U}^2\big)\,, \\
\ddot{Z}_4 - \frac{\sigma}{2}\, H_{,4}\, \delta\, \dot{U}^2 &= -\frac{1}{3}\,\Lambda\, Z_4\,\big(e + \frac{1}{2}\, G\, \delta\, \dot{U}^2\big)\,,
\end{aligned}
\tag{2.6}
$$

where

$$
G := \delta^{pq}\, Z_p\, H_{,q} - H, \quad \text{and} \quad e := g(T,T) = \pm 1, 0
\tag{2.7}
$$

denotes the normalisation constant for spacelike ($e = 1$), timelike ($e = -1$) and null ($e = 0$) geodesics respectively. Observe, that $\dot{}$ denotes the derivative with respect to an affine parameter t which we have suppressed in the equations. Here and in the following we also adopt the convention that Greek indices α, β take all values $0, 1, 2, 3, 4$ while the indices p, q, r are restricted to the values $2, 3, 4$ and i, j run from 2 to 3 only.
Obviously, these equations reduce to the geodesic equations of the (anti-)de Sitter background off the impulse located at $\{U = 0\}$. Also observe that the first equation decouples from the rest of the system and can be easily integrated. Consequently U can be used as a parameter of the remaining equations, a fact which is essential for the analysis of the system (2.6) presented in [31, Sec. 4]. In fact, there the geodesics of the (anti-)de Sitter background in front and behind the wave impulsive are matched using a natural ansatz for solutions in the entire spacetime. However, this procedure has to be viewed as being only heuristic since the solution's Z_p-components are (assumed to be) continuous but not $\mathcal{C}^1$, while the V-component is (assumed to be) even discontinuous. Consequently the solutions cannot be plugged back into the equations due to the occurrence of undefined products of distributions, and so the question arises in which sense they actually solve the equations, see the discussion at the end of Sec. 4 of [31]. The situation is similar to the one encountered for impulsive *pp*-waves with $\Lambda = 0$ and we refer to the discussion in [39, Sec. II] as well as to the general discussion in [18].

To resolve this open problem we now employ a regularisation approach and detail the setting we are working with: To begin with we replace the Dirac-δ in the line element (2.1) by a fairly general class of smooth approximations called *model delta nets*. Chose an arbitrary smooth function ρ on $\mathbb{R}$ with unit integral and its support contained in $[-1, 1]$. Then for $0 < \varepsilon \leq 1$ set

$$
\delta_\varepsilon(x) := \frac{1}{\varepsilon}\,\rho\Big(\frac{x}{\varepsilon}\Big)\,.
\tag{2.8}
$$

We now consider for fixed $\varepsilon \in (0, 1]$ the five-dimensional sandwich wave

$$
\mathrm{d}s_\varepsilon^2 = \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \sigma \mathrm{d}Z_4^2 - 2\mathrm{d}U\mathrm{d}V + H(Z_2, Z_3, Z_4)\delta_\varepsilon(U)\mathrm{d}U^2\,,
\tag{2.9}
$$

and define the spacetime of our interest as (M, g_ε) given by the constraint (2.2), i.e.,

$$
F(U, V, Z_2, Z_3, Z_4) := -2UV + Z_2^2 + Z_3^2 + \sigma Z_4^2 - \sigma a^2 = 0\,.
\tag{2.10}
$$

Observe that while the differential $\mathrm{d}F = 2(-V, -U, Z_2, Z_3, \sigma Z_4)$ is independent of ε, the normal vector $N_\varepsilon^\alpha := g_\varepsilon^{\alpha\beta}\mathrm{d}F_\beta$ depends on ε. Indeed we choose to work with the *non-normalised* normal vector N_ε to M given by

$$N_\varepsilon = (U, V + HU\delta_\varepsilon(U), Z_p) \quad \text{with} \quad g_\varepsilon(N_\varepsilon, N_\varepsilon) = \sigma a^2 - U^2 H\delta_\varepsilon(U)\,. \tag{2.11}$$

The non-zero Christoffel symbols of (2.9) are given by

$$\Gamma^V_{\varepsilon\, UU} = -\frac{1}{2}H\delta_\varepsilon'(U)\,, \quad \Gamma^V_{\varepsilon\, Up} = -\frac{1}{2}H_{,p}\delta_\varepsilon(U)\,, \tag{2.12}$$

$$\Gamma^i_{\varepsilon\, UU} = -\frac{1}{2}H_{,i}\delta_\varepsilon(U)\,, \quad \Gamma^4_{\varepsilon\, UU} = -\frac{1}{2}\sigma H_{,4}\delta_\varepsilon(U)\,, \tag{2.13}$$

and the geodesics of (M, g_ε) are now characterised by

$$\overline{\nabla}^\varepsilon_{T_\varepsilon} T_\varepsilon = -g_\varepsilon(T_\varepsilon, \overline{\nabla}^\varepsilon_{T_\varepsilon} N_\varepsilon)\ \frac{N_\varepsilon}{g_\varepsilon(N_\varepsilon, N_\varepsilon)}\,, \tag{2.14}$$

where (suppressing the parameter) we write $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ for the geodesics with tangent $T_\varepsilon = (\dot U_\varepsilon, \dot V_\varepsilon, \dot Z_{p\varepsilon})$ and $\overline{\nabla}^\varepsilon$ denotes the Levi-Civita connection of (2.9). By a straightforward calculation we now obtain

$$g_\varepsilon(T_\varepsilon, \overline{\nabla}^\varepsilon_{T_\varepsilon} N_\varepsilon) = e + \frac{1}{2}\,\dot U_\varepsilon^2\, \tilde G_\varepsilon(U_\varepsilon, Z_{p\varepsilon}) - \dot U_\varepsilon \Big(H(Z_{p\varepsilon})\, \delta_\varepsilon(U_\varepsilon)\, U_\varepsilon\Big)^{\dot{}}, \tag{2.15}$$

where we have used the abbreviations

$$\tilde G_\varepsilon(U, Z_r) := \delta^{pq}\, H_{,p}(Z_r)\, \delta_\varepsilon(U)\, Z_q + H(Z_r)\, \delta_\varepsilon'(U)\, U\,, \quad \text{and} \quad e := g_\varepsilon(T_\varepsilon, T_\varepsilon) = \pm 1, 0\,. \tag{2.16}$$

Observe that since the g_ε-norm of the tangent vector T_ε is constant along the geodesic γ_ε, we have chosen it also to be constant in ε, which means that we have fixed the normalisation independently of ε. Finally we obtain the following explicit form of the geodesic equations

$$\begin{aligned}
\ddot U_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\, \tilde G_\varepsilon - \dot U_\varepsilon \left(H\, \delta_\varepsilon\, U_\varepsilon\right)^{\dot{}}\Big)\ \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon}\,, \\
\ddot V_\varepsilon - \frac{1}{2}\, H\, \delta_\varepsilon'\, \dot U_\varepsilon^2 - \delta^{pq} H_{,p}\, \delta_\varepsilon\, \dot Z^\varepsilon_q\, \dot U_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\, \tilde G_\varepsilon - \dot U_\varepsilon \left(H\, \delta_\varepsilon\, U_\varepsilon\right)^{\dot{}}\Big)\ \frac{V_\varepsilon + H\,\delta_\varepsilon U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon}\,, \\
\ddot Z_{i\varepsilon} - \frac{1}{2} H_{,i}\, \delta_\varepsilon \dot U_\varepsilon^2 &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\, \tilde G_\varepsilon - \dot U_\varepsilon \left(H\, \delta_\varepsilon\, U_\varepsilon\right)^{\dot{}}\Big)\ \frac{Z_{i\varepsilon}}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon}\,, \\
\ddot Z_{4\varepsilon} - \frac{\sigma}{2}\, H_{,4}\, \delta_\varepsilon \dot U_\varepsilon^2 &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\, \tilde G_\varepsilon - \dot U_\varepsilon \left(H\, \delta_\varepsilon\, U_\varepsilon\right)^{\dot{}}\Big)\ \frac{Z_{4\varepsilon}}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon}\,,
\end{aligned} \tag{2.17}$$

where we again have suppressed the parameter t as well as the dependencies on the variables. However, note that always

$$\begin{aligned}
&\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,, \quad \delta_\varepsilon' = \delta_\varepsilon'(U_\varepsilon(t))\,, \\
&\tilde G_\varepsilon = \tilde G_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t))\,, \quad H = H(Z_{p\varepsilon}(t))\,, \quad \text{and} \quad H_{,p} = H_{,p}(Z_{q\varepsilon}(t))\,.
\end{aligned} \tag{2.18}$$

The right hand sides of these equations are considerably more complicated than their 'distributional counterparts' in (2.6), the reason being that in the regularised equations the distributional identities $\delta(U)U = 0$ and $\delta'(U)U = -\delta(U)$ do not apply. Indeed the lack of the first one leads to the more complicated form of the normal vector (see (2.11)) and is reflected in the second summand in the denominator of (2.17). On the other hand the lack of the second one is responsible for the different form of the terms involving $\tilde G_\varepsilon$ as compared to the ones involving G (cf. (2.7)) in (2.6), to which they reduce in the limit $\varepsilon \to 0$. Finally the

terms proportional $(H\delta_\varepsilon U_\varepsilon)^\cdot$ which occur in all four equations vanish in the limit $\varepsilon \to 0$ again due to the first identity. The same holds true for the term proportional to $\tilde{G}_\varepsilon U_\varepsilon$ contained in the first equation. In this sense the equations (2.17) converge weakly to (2.6) as $\varepsilon \to 0$.
The more complicated form of the system (2.17), in particular, results in the fact that the U_ε-equation does *not decouple* from the rest of the system and consequently U cannot be used as a parameter along the geodesics. This issue greatly complicates our analysis. However, the V_ε-equation still is linear and decoupled, hence can be simply integrated once the rest of the system is solved.

3. Existence, uniqueness and completeness of geodesics

In this section we prove an existence and uniqueness result for solutions of the initial value problem for (2.17) that additionally guarantees that the geodesics that enter the sandwich region $U \in [-\varepsilon, \varepsilon]$ at one side exist long enough to leave it on the other side. This allows us to obtain global solutions of the geodesic equations, since outside the strip $\{-\varepsilon \le U \le \varepsilon\}$ the spacetime coincides with the background (anti-)de Sitter universe.
Observe that for fixed ε the equations are smooth and hence a local solution is guaranteed to exist. However, the time of existence might depend on ε and in principle could even shrink to zero as $\varepsilon \to 0$. So the main objective here is to provide a result which guarantees that the time of existence is independent of ε, and large enough such that the solutions pass through the regularisation sandwich (at least for all small ε). To this end we employ a fixed point argument in the spirit of [36, Appendix A] based on Weissinger's fixed point theorem ([43]). However, the significant increase in the complexity of the equations forces the use of new ideas to derive the required estimates. In particular, since it is not possible to use the U-coordinate as a parameter along the geodesics, the 'singular terms' such as δ_ε are composed with the U-coordinate of the solution U_ε, see (2.18). We have separated the technical proofs preparing the grounds for the application of the fixed point theorem from the main line of arguments and have deferred them to Appendix A.

Let us start by giving the general setup. Consider any geodesic

$$\gamma = (U, V, Z_p) \tag{3.1}$$

on the background (anti-)de Sitter universe *without* impulsive wave but reaching $U = 0$. All other geodesics are not of interest to the present analysis and will be dealt with separately. We choose an affine parameter t in such a way that $U(t = 0) = 0$ and assume $\dot{\gamma}$ to be normalised by $e = \pm 1, 0$. Such geodesics can explicitly be written as

$$U = t\,, \qquad U = a\dot{U}^0 \sinh(t/a)\,, \qquad U = a\dot{U}^0 \sin(t/a)\,, \tag{3.2}$$

in the cases $\sigma e = 0$, $\sigma e < 0$, and $\sigma e > 0$, respectively, see [31, eq. (33)]. Recall that the case $\sigma e = 0$ corresponds to null geodesics in both de Sitter and anti-de Sitter space, while the case $\sigma e < 0$ corresponds to timelike geodesics in de Sitter as well as to spacelike geodesics in anti-de Sitter spacetime, and finally $\sigma e > 0$ describes spacelike geodesics in de Sitter and timelike geodesics in anti-de Sitter space. Without loss of generality we assume the constant $\dot{U}^0$ to be positive[†] so that in all three cases U is increasing (at least for $t \in [-a\pi/2, a\pi/2]$). It is thus most convenient to prescribe initial data at $t = 0$, that is we set

$$\gamma(t = 0) =: (0, V^0, Z_p^0)\,, \qquad \dot{\gamma}(t = 0) =: (\dot{U}^0 > 0, \dot{V}^0, \dot{Z}_p^0)\,, \tag{3.3}$$

[†]The time reversed case with $\dot{U}^0 < 0$ can be treated in complete analogy.

where the constants satisfy the constraints

$$(Z_2^0)^2+(Z_3^0)^2+\sigma(Z_4^0)^2-2U^0V^0=\sigma a^2, \quad Z_2^0\dot{Z}_2^0+Z_3^0\dot{Z}_3^0+\sigma Z_4^0\dot{Z}_4^0-V^0\dot{U}^0-U^0\dot{V}^0=0\,, \tag{3.4}$$

which are simply consequences of the fact that we are dealing with γ on the (anti-)de Sitter manifold, see (2.2). Note however that $U^0=0$, so the last term on the left hand side of either condition actually vanishes. In addition the normalisation condition

$$-2\dot{U}^0\dot{V}^0+(\dot{Z}_2^0)^2+(\dot{Z}_3^0)^2+\sigma(\dot{Z}_4^0)^2=e \tag{3.5}$$

holds.
Now we start to think of γ as a geodesic in the impulsive wave spacetime (2.1), (2.2) *'in front' of the impulse* that is for $U<0$. Also, γ is a geodesic in the regularised spacetime (2.9), (2.10) *'in front' of the sandwich wave*, that is for $U\leq-\varepsilon$. We will call it *'seed geodesic'* and denote the affine parameter time when γ enters this regularisation wave region by α_ε,

$$U(t=\alpha_\varepsilon)=-\varepsilon\,. \tag{3.6}$$

Observe that, by continuity of γ, $\alpha_\varepsilon\to0$ from below as $\varepsilon\to0$. More precisely, we have $\alpha_\varepsilon=-\varepsilon$, $\alpha_\varepsilon=-a\mathrm{Arcsinh}(\varepsilon/a\dot{U}^0)$, and $\alpha_\varepsilon=-a\arcsin(\varepsilon/a\dot{U}^0)$, respectively for the three cases in (3.2), leading to $\alpha_\varepsilon=-\varepsilon/\dot{U}^0+O(\varepsilon^3)$ in the latter cases and hence overall

$$-C\varepsilon\leq\alpha_\varepsilon<0\,, \tag{3.7}$$

for some positive constant C.
To investigate the geodesics in the regularised spacetime (2.9), (2.10), which is the main topic of this paper, we follow γ up to the beginning of wave zone, i.e., up to $t=\alpha_\varepsilon$, and then extend it (smoothly) to a geodesic

$$\gamma_\varepsilon=(U_\varepsilon,V_\varepsilon,Z_{p\varepsilon}) \tag{3.8}$$

solving the regularised geodesic equations (2.17). This means γ_ε at α_ε assumes the data (see Figure 1)

$$\gamma_\varepsilon(\alpha_\varepsilon):=\gamma(\alpha_\varepsilon)=:(-\varepsilon,V_\varepsilon^0,Z_{p\varepsilon}^0)\,,\quad \dot{\gamma}_\varepsilon(\alpha_\varepsilon):=\dot{\gamma}(\alpha_\varepsilon)=:(\dot{U}_\varepsilon^0,\dot{V}_\varepsilon^0,\dot{Z}_{p\varepsilon}^0)\,. \tag{3.9}$$

Observe that by smoothness of γ the data $\gamma_\varepsilon(\alpha_\varepsilon)$ and $\dot{\gamma}_\varepsilon(\alpha_\varepsilon)$ converge to $\gamma(0)$ and $\dot{\gamma}(0)$, respectively, as $\varepsilon\to0$. In fact, by a mean value argument and (3.7) we even have

$$\begin{aligned}|(-\varepsilon,V_\varepsilon^0,Z_{p\varepsilon}^0)-(0,V^0,Z_p^0)|&\leq\sup_{\alpha_\varepsilon\leq t\leq0}\|\dot{\gamma}(t)\|_h\,|\alpha_\varepsilon|\;\leq\;C\varepsilon\,,\\ |(\dot{U}_\varepsilon^0,\dot{V}_\varepsilon^0,\dot{Z}_{p\varepsilon}^0)-(\dot{U}^0,\dot{V}^0,\dot{Z}_p^0)|&\leq\sup_{\alpha_\varepsilon\leq t\leq0}\|\ddot{\gamma}(t)\|_h\,|\alpha_\varepsilon|\;\leq\;C\varepsilon\,,\end{aligned} \tag{3.10}$$

where h is any Riemannian background metric and C again is a generic constant.

Based on Theorem A.6 proved in Appendix A we may now state and prove our main results on the existence, uniqueness and completeness of the geodesics in the *regularised* spacetime (2.9), (2.10):

Theorem 3.1 (Existence and uniqueness)**.** *Consider the geodesic equations* (2.17) *with initial data* (3.9)*. Then for all ε small enough (more precisely for all $\varepsilon\leq\varepsilon_0$, where ε_0 is constrained by* (A.14)*), there exists a unique smooth solution $\gamma_\varepsilon=(U_\varepsilon,V_\varepsilon,Z_{p\varepsilon})$ on $[\alpha_\varepsilon,\alpha_\varepsilon+\eta]$, where η is independent of ε (and explicitly given by* (A.12)*).*

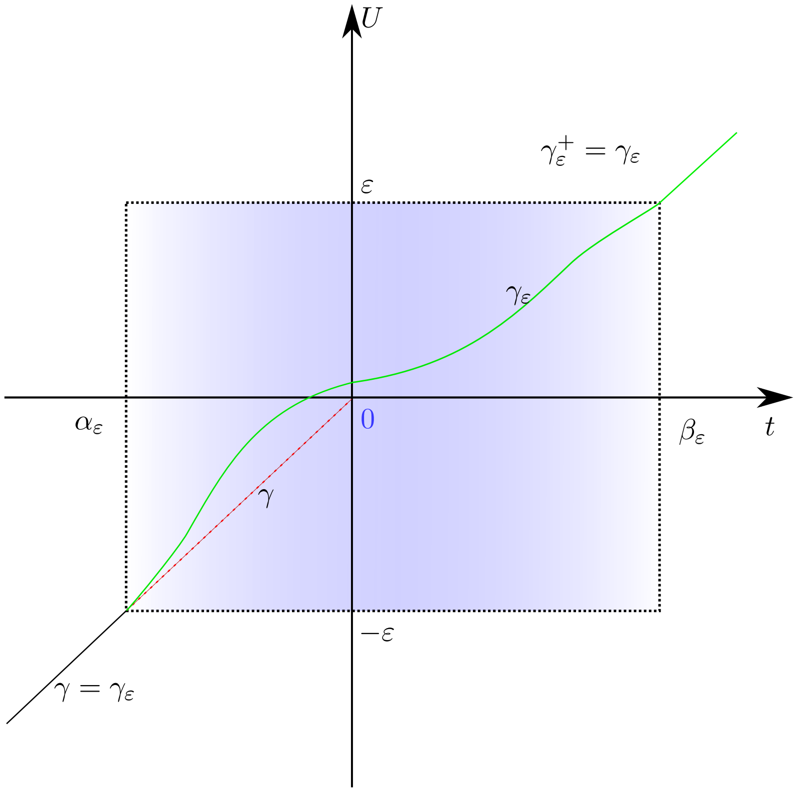

FIGURE 1. The U-component of the 'seed geodesic' γ is depicted in black until it reaches the regularisation sandwich at parameter time $t = \alpha_\varepsilon$, i.e., $U(\alpha_\varepsilon) = -\varepsilon$. While in the background spacetime it would continue as the dotted red line to $U = 0$ at $t = 0$, in the regularised spacetime it continues as γ_ε of (3.8) (depicted in green) solving the equations (2.17) with data (3.9). Theorem 3.2 guarantees that γ_ε (for ε small) leaves the regularisation sandwich at $t = \beta_\varepsilon$ and continues as background geodesic γ_ε^+ of (3.14) with data (3.13).

Proof. As noted in the appendix it suffices to first solve the (simplified) model system (A.1) for $(u_\varepsilon, z_\varepsilon)$. Identifying $(U_\varepsilon, Z_{p\varepsilon})$ with $(u_\varepsilon, z_\varepsilon)$ the initial data (3.3) and (3.9) transfer to the data (A.3), (A.2). Then (3.10) implies (A.4) and Theorem A.6 applies to provide a unique smooth solution $(U_\varepsilon, Z_{p\varepsilon})$ of (2.17), (3.9) on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$, with η given by (A.12).
Finally we solve the linear equation for V_ε to obtain the claimed smooth solution $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$. □

Next we make sure that the solutions just obtained, which by construction enter the wave zone at $U_\varepsilon = -\varepsilon$ at parameter time $t = \alpha_\varepsilon$ with positive speed $\dot{U}_\varepsilon^0$, in fact *leave the sandwich region*, that is they reach $U_\varepsilon = \varepsilon$ within their time of existence η. Consequently, they naturally extend to the background *(anti-)de Sitter universe 'behind'* the sandwich region. Observe that here it is vital that η in (A.12) is independent of ε.

Theorem 3.2 (Extension of geodesics)**.** *The unique smooth geodesics γ_ε of Theorem 3.1 extend to geodesics of the background (anti-)de Sitter spacetime 'behind' the sandwich wave zone.*

Proof. Let $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ be the unique solution of (2.17), (3.9) given by Theorem 3.1. By the definition of the 'solution space' $\mathfrak{X}_\varepsilon$ (see (A.5)) we obtain

$$U_\varepsilon(\alpha_\varepsilon + \eta) = -\varepsilon + \int_{\alpha_\varepsilon}^{\alpha_\varepsilon+\eta} \dot{U}_\varepsilon(s)\,\mathrm{d}s \geq -\varepsilon + \frac{\eta}{2}\dot{U}^0 \geq -\varepsilon + 3\varepsilon \geq \varepsilon\,, \tag{3.11}$$

since $\varepsilon \leq \eta\,\dot{U}^0/6$ by (A.13).
So for such ε the geodesic γ_ε leaves the wave zone and extends to a geodesic of the background spacetime since the geodesic equations (2.17) coincide with the geodesic equations of the background (anti-)de Sitter spacetime for $U_\varepsilon \geq \varepsilon$. □

Recall that by construction the global geodesic $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ with data (3.9) of Theorem 3.2 for $t \leq \alpha_\varepsilon$, i.e. 'in front' of the sandwich, coincide for all (small) ε with the single 'seed geodesic' γ with data (3.3). However, 'behind' the sandwich the geodesics γ_ε for each ε will coincide with a *different* geodesic of the *background* spacetime. To make this observation more precise, define the affine parameter time when the geodesic γ_ε leaves the sandwich wave zone by β_ε,

$$U_\varepsilon(t = \beta_\varepsilon) = \varepsilon\,. \tag{3.12}$$

and denote the corresponding values of γ_ε at β_ε by

$$\gamma_\varepsilon(\beta_\varepsilon) =: (\varepsilon, V_\varepsilon^{0+}, Z_{p\varepsilon}^{0+})\,, \quad \dot{\gamma}_\varepsilon(\beta_\varepsilon) =: (\dot{U}_\varepsilon^{0+}, \dot{V}_\varepsilon^{0+}, \dot{Z}_{p\varepsilon}^{0+})\,. \tag{3.13}$$

Then for $t \geq \beta_\varepsilon$ the geodesic γ_ε will coincide with the geodesic

$$\gamma_\varepsilon^+ = (U_\varepsilon^+, V_\varepsilon^+, Z_{p\varepsilon}^+) \tag{3.14}$$

of the *background* (anti-)de Sitter space with the data (3.13), see Figure 1 and also Figure 2. Observe that the data (3.13) is normalised and constrained, more precisely we have:

Remark 3.3 (Preservation of constraints and normalisation). The fact that the data (3.3) of the 'seed geodesic' γ in (3.1) is constrained and normalised, i.e., it satisfies (3.4) and (3.5), implies that also the data (3.9) of γ_ε is constrained and normalised. Clearly these conditions are propagated by γ_ε being a solution to (2.17). Moreover, at $t = \beta_\varepsilon$ the regularised metric g_ε and the background metric g coincide and so the data (3.13) of γ_ε^+ is constrained and normalised with respect to the background spacetime.
While the preservation of the constraints confirms the consistency of our construction, the preservation of the normalisation, in particular, implies that the causal character of γ_ε (and γ_ε^+) is the same as the one of the 'seed' γ.

Note that the geodesic γ_ε^+ 'behind' the regularisation sandwich depends on ε (only) via this initial data. Interestingly, as we will detail in Section 5, for $t > 0$ the family of geodesics γ_ε of the regularised spacetime converges for $\varepsilon \to 0$ to a unique geodesic γ^+ in the background with data given by the limits of (3.13). This will explicitly describe the effect of the *impulsive* gravitational wave on the geodesics in (anti-)de Sitter universe.

In the remainder of this section we will formulate completeness results for the regularised spacetimes. First we remark that actually our results allow us to make the smallness assumption on ε precise: ε has to be smaller than ε_0 constrained by (A.14). This, however, means that the specific ε from which on a certain geodesic γ_ε becomes complete depends on its data (3.3), i.e., on the 'seed geodesic', and there is in general *no global* ε from which on *all* geodesics hence the spacetime is complete. A 'global' completeness result in the spirit of

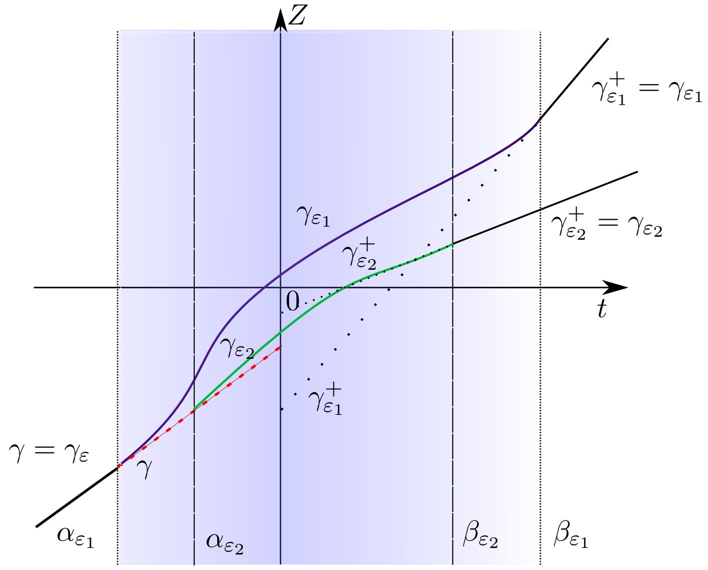

FIGURE 2. The Z-components of two solutions γ_{ε_1} (purple) and γ_{ε_2} (green) of the regularised equations (2.17) with the same 'seed geodesic' γ of (3.1) are depicted for $\varepsilon_1 > \varepsilon_2$. The regularisation sandwich is given by $[\alpha_{\varepsilon_1}, \beta_{\varepsilon_1}]$ and $[\alpha_{\varepsilon_2}, \beta_{\varepsilon_2}]$, respectively. The dotted red line represents the Z-components of γ, while the black dotted lines are said components of $\gamma^+_{\varepsilon_i}$.

[37], however, can be obtained using the geometric theory of generalised functions ([17, Ch. 3]) and we reserve its detailed presentation for future work.
To formulate completeness results in our current setting, the dependence of ε on the data discussed above also makes it necessary to be careful about global effects. Indeed, geodesics in the background spacetimes with $\sigma e > 0$, that is spacelike geodesics in de Sitter space and timelike geodesics in anti-de Sitter space are periodic. Consequently geodesics γ_ε in the regularised spacetimes constructed from such 'seed geodesics' γ will share their causal character (see Remark 3.3) and hence cross the wave zone infinitely often and we (have to) (re)apply Theorems 3.1, 3.2 again and again. However, note that the geodesics may enter the regularisation sandwich region each time with different data. So the ε from which on Theorem 3.2 applies may in principal become smaller and smaller on successive crossings with no positive infimum. In such a case, given an initial geodesic γ as in (3.1) and given any finite number N of crossings we can specify an ε from which on the geodesic γ_ε extends to cross the wave zone N-times. However, we cannot give a (global) ε for which the geodesic γ_ε extends to all (positive) values of its affine parameter. Consequently we prefer to avoid multiple crossings of the impulse in the formulation of our results by restricting to causal geodesics in the de Sitter-case (neglecting unphysical tachyons only) and working with the universal covering spacetime in case of anti-de Sitter space.
Note also that in our discussion so far (see the specification of the 'seed geodesics' γ at the beginning of this section) we have exclusively dealt with geodesics with non-constant U-component. Hence it remains to deal with geodesics travelling parallel to the surface $\{U = 0\}$. In case $U = \text{const} \neq 0$ the geodesic will never enter the sandwich region of the regularised spacetime, provided ε is small enough. Staying entirely on the constant curvature background

such a geodesic clearly is complete. To discuss the geodesics with $U = 0$, observe that the surface $\{U = 0\}$ is totally geodesic (not only in the background but also) in the regularised spacetime, which can be seen from the U-component of the geodesic equations (2.17) (see also [32], the discussion prior to Thm. 3.6). Hence such geodesics have trivial U-components and consequently the system (2.17) reduces to the background geodesic equations which again leads to completeness. So we finally arrive at:

Theorem 3.4 (Causal completeness for positive Λ)**.** *Every causal geodesic in the entire class of regularised nonexpanding impulsive gravitational waves propagating in de Sitter universe (i.e.,* (2.9)*,* (2.10) *with $\Lambda > 0$ and a smooth profile function H) is complete, provided the regularisation parameter ε is chosen small enough.*

Theorem 3.5 (Completeness for negative Λ)**.** *Every geodesic in the entire class of regularised nonexpanding impulsive gravitational waves propagating in the universal cover of anti-de Sitter universe (i.e.,* (2.9)*,* (2.10)*, with $\Lambda < 0$ and a smooth profile function H) is complete, provided the regularisation parameter ε is chosen small enough.*

Remark 3.6 (Non-smooth profiles H)*.* In case the profile function H in the metric (2.9) is non-smooth — which, in fact, occurs in physically interesting models where H possesses poles on the wave front at $z = \pm 1$, see (2.4) — our method still applies but some care is needed. Indeed, if a 'seed geodesic' γ hits the surface at $U = 0$ away from the poles we may work on an open subset of the spacetime with the poles of H removed. We only have to choose the constant C_1 in (A.5) so small that the curves in the 'solution space' $\mathfrak{X}_\varepsilon$ stay away from the poles, and then Theorems 3.1 and 3.2 still apply. The only 'seed geodesics' γ which do *not* allow for such a treatment are those which *directly head* at the poles (i.e., γ of (3.1) hits the pole at $\gamma(0)$), which is in complete agreement with physical expectations.

4. Boundedness properties of geodesics

In this section we establish boundedness properties of the global geodesics in the regularised spacetimes obtained in the previous section. In particular, we will prove local boundedness of γ_ε and of some components of its velocity *uniformly in* ε. These properties will be essential in the next section where we derive the limits of γ_ε.

To begin with observe that the fixed point argument of the appendix already gives uniform boundedness of the U- and Z_p-components together with their first order derivatives. On the other hand, the V-component was not involved in the fixed point argument and we have to establish its boundedness properties using the V-component of the geodesic equation (2.17). Since this equation involves a δ'-term which is *not* multiplied by U_ε, the V-component of $\dot{\gamma}_\varepsilon$ is not uniformly bounded in the regularisation sandwich. However, we will show that $\dot{V}_\varepsilon(\beta_\varepsilon)$, i.e., the V-speed when the geodesic *leaves* the regularisation sandwich is uniformly bounded.

Proposition 4.1 (Uniform boundedness of geodesics)**.** *The global geodesics $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ of Theorem 3.2 satisfy*

(i) *U_ε and $\dot{U}_\varepsilon$ are locally uniformly bounded in ε,*
(ii) *$Z_{p\varepsilon}$ and $\dot{Z}_{p\varepsilon}$ are locally uniformly bounded in ε,*
(iii) *V_ε is locally uniformly bounded in ε, and*
(iv) *$\dot{V}_\varepsilon(\beta_\varepsilon)$ is uniformly bounded in ε.*

Observe that by Lemma A.2 the time β_ε when the geodesic leaves the regularisation strip satisfies $\beta_\varepsilon \leq \alpha_\varepsilon + 4\varepsilon/\dot{U}^0 \to 0$.

Proof. Items (i) and (ii) are immediate from Theorem A.6.
To deal with the V-component we write

$$|V_\varepsilon(t)| \le |V^0| + |\dot V^0|\,|t-\alpha_\varepsilon| + \int\limits_{\alpha_\varepsilon}^{t}\int\limits_{\alpha_\varepsilon}^{s} |\ddot V_\varepsilon(r)|\,\mathrm{d}r\mathrm{d}s \tag{4.1}$$

and estimate each term in the differential equation (2.17) for V_ε. To begin with we claim that

$$V_\varepsilon \text{ is bounded on } [\alpha_\varepsilon, \beta_\varepsilon]\,. \tag{4.2}$$

Indeed, proceeding as in the proof of Proposition A.3 and, in particular, using (A.5) and Lemmata A.1,A.2 we obtain

$$\begin{aligned}
\int\limits_{\alpha_\varepsilon}^{t}\int\limits_{\alpha_\varepsilon}^{s} |\ddot V_\varepsilon(r)|\,\mathrm{d}r\mathrm{d}s \le{}& \Big(\frac{4\varepsilon}{\dot U^0}\Big)^2 \Big(\frac12\|H\|_\infty \frac{1}{\varepsilon^2}\|\rho'\|_\infty \frac94 (\dot U^0)^2 + 3\|DH\|_\infty \frac1\varepsilon \|\rho\|_\infty \big(|\dot Z^0| + C_2\big)\frac32 \dot U^0\Big)\\
&+ \frac{2}{a^2}\Big(\frac{4\varepsilon}{\dot U^0}\Big)^2 \Big(1 + \frac98 (\dot U^0)^2 \|\tilde G_\varepsilon\|_\infty + \frac32 \dot U^0 \|(H\delta_\varepsilon U_\varepsilon)^{\dot{}}\|_\infty\Big)\,\|H\|_\infty \|\rho\|_\infty\\
&+ \frac{2}{a^2}\Big(1 + \frac98 (\dot U^0)^2 \|\tilde G_\varepsilon\|_\infty + \frac32 \dot U^0 \|(H\delta_\varepsilon U_\varepsilon)^{\dot{}}\|_\infty\Big) \int\limits_{\alpha_\varepsilon}^{t}\int\limits_{\alpha_\varepsilon}^{s} |V_\varepsilon(r)|\,\mathrm{d}r\mathrm{d}s\\
\le{}& C + C\int\limits_{\alpha_\varepsilon}^{t}\int\limits_{\alpha_\varepsilon}^{s} |V_\varepsilon(r)| \Big(1 + \frac{\chi_\varepsilon(r)}{\varepsilon}\Big)\,\mathrm{d}r\mathrm{d}s\,,
\end{aligned} \tag{4.3}$$

where C is some generic constant and χ_ε is the characteristic function of $[\alpha_\varepsilon, \beta_\varepsilon]$. Moreover, we have used that $\|\tilde G_\varepsilon\|_\infty = O(1/\varepsilon) = \|(H\delta_\varepsilon U_\varepsilon)^{\dot{}}\|_\infty$. So overall we obtain by a generalization of Gronwall's inequality due to Bykov [3, Thm.11.1]†

$$|V_\varepsilon| \le C(1+\varepsilon)\exp\Big(\int\limits_{\alpha_\varepsilon}^{\beta_\varepsilon}\int\limits_{\alpha_\varepsilon}^{s} C\big(1+\frac1\varepsilon\big)\mathrm{d}r\mathrm{d}s\Big) \le C\,e^C\,, \tag{4.4}$$

establishing the claim (4.2).

Now we may prove (iv): Writing $\dot V_\varepsilon(\beta_\varepsilon)$ also as an integral and using boundedness of V_ε on $[\alpha_\varepsilon, \beta_\varepsilon]$ we may proceed as in (4.3). Then again we obtain uniform boundedness of all the terms but the first one, which involves the δ'_ε-term. To estimate this one we use integration by parts to obtain

$$\begin{aligned}
\int_{\alpha_\varepsilon}^{\beta_\varepsilon} H(Z_\varepsilon(s))\,\delta'_\varepsilon(U_\varepsilon(s))\,\dot U_\varepsilon^2(s)\,\mathrm{d}s ={}& \int\limits_{-\varepsilon}^{\varepsilon} H\big(Z_\varepsilon(U_\varepsilon^{-1}(r))\big)\,\delta'_\varepsilon(r)\dot U_\varepsilon(U_\varepsilon^{-1}(r))\,\mathrm{d}r\\
={}& \delta_\varepsilon(r)\,H\big(Z_\varepsilon(U_\varepsilon^{-1}(r))\big)\,\dot U_\varepsilon(U_\varepsilon^{-1}(r))\Big|_{-\varepsilon}^{\varepsilon} - \int\limits_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(r)\Big(H\big(Z_\varepsilon(U_\varepsilon^{-1}(r))\big)\,\dot U_\varepsilon\big(U_\varepsilon^{-1}(r)\big)\Big)^{\dot{}}\,\mathrm{d}r\,.
\end{aligned} \tag{4.5}$$

---

†Note that it suffices that χ_ε is integrable rather than continuous.

Now the first term vanishes and the second one is bounded independently of ε by (i), (ii), the fact that $\dot{U}_\varepsilon^{-1}$ is uniformly bounded away from zero by (A.5), and since $\ddot{U}_\varepsilon$ is uniformly bounded by (2.17). This establishes (iv).

To prove (iii) it remains to show that V_ε is bounded on any compact interval disjoint from $(\alpha_\varepsilon, \beta_\varepsilon)$. But this follows from the fact that γ_ε outside of $(\alpha_\varepsilon, \beta_\varepsilon)$ solves the geodesic equation of the background spacetime of constant curvature, and that $\gamma_\varepsilon(\beta_\varepsilon)$ and $\dot{\gamma}_\varepsilon(\beta_\varepsilon)$ are uniformly bounded by (i), (ii), (iv) and (4.2). □

5. Limiting geodesics

In this final section we consider the limit $\varepsilon \to 0$ of the unique global smooth geodesics γ_ε of the regularised spacetime (2.9), (2.10) obtained in Section 3 (Thm. 3.1, Thm. 3.2). This physically amounts to *explicitly determine the geodesics of the distributional form* (2.1), (2.2) of all nonexpanding impulse gravitational waves propagating in (anti-)de Sitter universe. In particular, we will prove that the geodesics γ_ε converge to geodesics of the background (anti-)de Sitter spacetime with appropriate *but different* data on either side of the impulse ($U < 0$ and $U > 0$ respectively), which from a global point of view amounts to convergence of γ_ε to geodesics of the background which have to be *matched appropriately* across the impulse. The technical calculation of the limits is given in Appendix B.

To make our claims on the convergence of $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ precise we introduce the following notation for the geodesics of the background (anti-)de Sitter universe: Let $\gamma = (U, V, Z_p)$ be a 'seed geodesic' as in (3.1), that is $U(t = 0) = 0$ and γ assumes the data (3.3), i.e.,

$$\gamma(0) = (0, V^0, Z_p^0)\,, \quad \text{and} \quad \dot{\gamma}(0) = (\dot{U}^0 > 0, \dot{V}^0, \dot{Z}_p^0)\,, \tag{5.1}$$

where the constants satisfy the constraints (3.4) and are normalised as in (3.5). Furthermore let $\gamma^+ = (U^+, V^+, Z_p^+)$ be a geodesic of the background again crossing $U = 0$ at $t = 0$, i.e., with $U^+(t = 0) = 0$ and with data

$$\gamma^+(0) = (0, \mathsf{B}, Z_p^0)\,, \quad \text{and} \quad \dot{\gamma}^+(0) = (\dot{U}^0, \mathsf{C}, \mathsf{A_p})\,, \tag{5.2}$$

where we define

$$\mathsf{A_p} := \lim_{\varepsilon\to 0} \dot{Z}_{p\varepsilon}(\beta_\varepsilon)\,, \quad \mathsf{B} = \lim_{\varepsilon\to 0} V_\varepsilon(\beta_\varepsilon)\,, \quad \mathsf{C} = \lim_{\varepsilon\to 0} \dot{V}_\varepsilon(\beta_\varepsilon)\,. \tag{5.3}$$

Recall that $\beta_\varepsilon \leq \alpha_\varepsilon + 4\varepsilon/\dot{U}^0 \to 0$ is defined to be the time when the regularised geodesic γ_ε leaves the regularisation strip, i.e., $U_\varepsilon(\beta_\varepsilon) = \varepsilon$. Finally define $\tilde{\gamma} = (\tilde{U}, \tilde{V}, \tilde{Z}_p)$ by

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t)\,, & t \leq 0 \\ \gamma^+(t)\,, & t > 0\,. \end{cases} \tag{5.4}$$

We will show that γ_ε *converge to the 'matched geodesics'* $\tilde{\gamma}$ of the impulsive spacetime, which from now on we will also call 'limiting geodesics' with 'past branch' γ and 'future branch' γ^+, see also Figure 3. Note that the respective notion of convergence of the individual components of γ_ε will differ, subject to the regularity of the respective components of the 'limiting geodesics'. Indeed, $\tilde{\gamma} = (\tilde{U}, \tilde{V}, \tilde{Z}_p)$ has a smooth first component U, while Z_p is continuous with a finite jump in $\dot{Z}_p$ (determined by A_p) across the impulse at $t = 0$, and V is even discontinuous across $t = 0$ with a finite jump in V and $\dot{V}$ (determined by the coefficients B and C, respectively).

Observe that at the moment we only know the limits in (5.3) to exist for subsequences (by uniform boundedness of $\dot{Z}_{p\varepsilon}$, V_ε and $\dot{V}_\varepsilon(\beta_\varepsilon)$, cf. Proposition 4.1) and hence $\mathsf{A_p}$, B, and C need not be uniquely defined. We will, however, prove convergence and we will derive an explicit expressions for $\mathsf{A_p}$, B and C in Proposition 5.3, below. But first we state and prove the main assertion on the limits of the geodesics in the regularised spacetime:

Theorem 5.1. *The geodesics $\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon})$ of the regularised spacetime derived in Theorem 3.2 converge to the 'limiting geodesics' $\tilde\gamma$ of* (5.4) *in the following sense:*

(i) $U_\varepsilon \to \tilde U$ *in* $\mathcal{C}^1$,
(ii) $Z_{p\varepsilon} \to \tilde Z_p$ *locally uniformly,*
$\dot Z_{p\varepsilon} \to \dot{\tilde Z}_p$ *in distributions and uniformly on compact intervals not containing* $t = 0$,
(iii) $V_\varepsilon \to \tilde V$ *in distributions and in* $\mathcal{C}^1$ *on compact intervals not containing* $t = 0$.

Observe that $\dot{\tilde Z}_p$ is discontinuous across $t = 0$, hence convergence of $\dot Z_{p\varepsilon}$ cannot be uniform on any interval containing $t = 0^\dagger$, and the same holds true for V_ε and $\dot V_\varepsilon$.

Proof. First we consider the U-component of γ_ε on the interval $[\alpha_\varepsilon, \beta_\varepsilon]$, where we have from the geodesic equations (2.17) resp. (2.6)

$$|U_\varepsilon(t) - U(t)| \le \varepsilon + |e| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big| \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon} - \frac{U}{\sigma a^2} \Big| \,\mathrm{d}r\,\mathrm{d}s$$

$$+ \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big| \frac{\frac12 U_\varepsilon \dot U_\varepsilon^{\,2} \tilde G_\varepsilon - U_\varepsilon \dot U_\varepsilon \,(H\delta_\varepsilon U_\varepsilon)^{\boldsymbol\cdot}}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon} \Big| \,\mathrm{d}r\,\mathrm{d}s =: \varepsilon + |e| I + II\,. \tag{5.5}$$

To estimate I observe that (cf. the proof of Lemma A.1)

$$\Big| \frac{1}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon} - \frac{1}{\sigma a^2} \Big| \le \frac{2}{a^4} \varepsilon \|H\|_\infty \|\rho\|_\infty \le C\varepsilon\,, \tag{5.6}$$

with C a generic constant, and consequently by (A.5)

$$\begin{aligned} I &\le \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big(\Big| \frac{U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H\delta_\varepsilon} - \frac{U_\varepsilon}{\sigma a^2} \Big| + \Big| \frac{U_\varepsilon}{\sigma a^2} - \frac{U}{\sigma a^2} \Big| \Big) \,\mathrm{d}r\,\mathrm{d}s \\ &\le \eta^2(\varepsilon + C_1) C\varepsilon + \frac{1}{a^2} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big| U_\varepsilon - U \Big| \,\mathrm{d}r\,\mathrm{d}s\,. \end{aligned} \tag{5.7}$$

For the term II we obtain, as in the proof of Proposition A.3 (cf. (A.16), (A.18)), that $II \le C\varepsilon$. Hence, overall

$$|U_\varepsilon(t) - U(t)| \le C\varepsilon + \frac{|e|}{a^2} \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big| U_\varepsilon - U \Big| \,\mathrm{d}r\,\mathrm{d}s\,, \tag{5.8}$$

and so again by Bykov's inequality $|U_\varepsilon(t) - U(t)| = O(\varepsilon)$. In the same way we see that also $|\dot U_\varepsilon(t) - \dot U(t)| = O(\varepsilon)$ and so

$$\sup_{\alpha_\varepsilon \le t \le \beta_\varepsilon} |U_\varepsilon(t) - U(t)| + |\dot U_\varepsilon(t) - \dot U(t)| \to 0\,. \tag{5.9}$$

We now turn to the Z_p-components on the interval $[\alpha_\varepsilon, \beta_\varepsilon]$. We have

$$\sup_{\alpha_\varepsilon \le t \le \beta_\varepsilon} |Z_{p\varepsilon}(t) - \tilde Z_p(t)| \le \sup_{\alpha_\varepsilon \le t \le \beta_\varepsilon} |Z_{p\varepsilon}(t) - Z^0_{p\varepsilon}| + \sup_{\alpha_\varepsilon \le t \le \beta_\varepsilon} |Z^0_{p\varepsilon} - \tilde Z_p(t)| \to 0 \qquad (\varepsilon \to 0)\,, \tag{5.10}$$

---

†Unless $t = 0$ is the right endpoint of the interval. This, however, is rather an artefact due to our choice setting $\tilde\gamma(0) = \gamma(0)$, cf. (5.4).

where $Z^0_{p\varepsilon} = Z_p(\alpha_\varepsilon)$ was defined in (3.9). Indeed by continuity of $\tilde{Z}_p$ the second term converges to zero since α_ε, $\beta_\varepsilon \to 0$ and the first term can be estimated using the differential equation by

$$
\begin{aligned}
|Z_{p\varepsilon}(t) - Z^0_{p\varepsilon}| \leq \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \int_{\alpha_\varepsilon}^{s} \left| \frac{DH\delta_\varepsilon \dot{U}_\varepsilon^2}{2} - \frac{eZ_p^\varepsilon + \frac{1}{2} Z_p^\varepsilon \dot{U_\varepsilon}^2 \tilde{G}_\varepsilon - Z_p^\varepsilon \dot{U}_\varepsilon \,(H\delta_\varepsilon U_\varepsilon)^{\cdot}}{\sigma a^2 - U^2 H\delta_\varepsilon} \right| \mathrm{d}r\,\mathrm{d}s \\
+ |\dot{Z}^0_{p\varepsilon}|(\alpha_\varepsilon + \beta_\varepsilon) \leq C\varepsilon\,.
\end{aligned} \tag{5.11}
$$

Here we have used that the inner integral is bounded by (A.27)-(A.29).

Now we finish the proof of (i) and establish the claim on uniform convergence in (ii) and (iii). First note that by construction there is nothing to show for $t \leq 0$. For $t \geq \beta_\varepsilon$ we use continuous dependence of solutions to ODEs on the data. Indeed for such t both γ_ε and $\tilde{\gamma} = \gamma^+$ are solutions to the same differential equation, however, with different data which is given for γ_ε at $t = \beta_\varepsilon$ by (5.2) and for γ^+ at $t = 0$ by (3.13). More precisely, for all $T > 0$ (which implies $T > \beta_\varepsilon$ for ε small) we have

$$
\begin{aligned}
\sup_{\beta_\varepsilon \leq t \leq T} \big(|\gamma_\varepsilon(t) - \gamma^+(t)|, |\dot{\gamma}_\varepsilon(t) - \dot{\gamma}^+(t)|\big) \\
\leq \max\big(|\gamma_\varepsilon(\beta_\varepsilon) - \gamma^+(\beta_\varepsilon)|, |\dot{\gamma}_\varepsilon(\beta_\varepsilon) - \dot{\gamma}^+(\beta_\varepsilon)|\big)\, e^{TL}\,,
\end{aligned} \tag{5.12}
$$

where L is a Lipschitz constant for the right hand side of the geodesic equation of the background on a suitable compact set. Note that such a set exists by the boundedness properties of γ_ε established in Proposition 4.1, i.e., $\dot{\gamma}_\varepsilon(\beta_\varepsilon)$ is uniformly bounded. Finally, for the terms in the maximum in (5.12) we have

$$
\begin{aligned}
&|U_\varepsilon(\beta_\varepsilon) - U^+(\beta_\varepsilon)| \to 0\,, \quad |\dot{U}_\varepsilon(\beta_\varepsilon) - \dot{U}^+(\beta_\varepsilon)| \to 0 \quad \text{by (5.9), and} \\
&|Z_{p\varepsilon}(\beta_\varepsilon) - Z_p^+(\beta_\varepsilon)| \to 0 \quad \text{by (5.10)}\,,
\end{aligned} \tag{5.13}
$$

whereas for the remaining terms we write

$$
\begin{aligned}
|V_\varepsilon(\beta_\varepsilon) - V^+(\beta_\varepsilon)| &\leq |V_\varepsilon(\beta_\varepsilon) - V^+(0)| + |V^+(0) - V^+(\beta_\varepsilon)|\,, \\
|\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(\beta_\varepsilon)| &\leq |\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(0)| + |\dot{V}^+(0) - \dot{V}^+(\beta_\varepsilon)|\,, \\
|\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(\beta_\varepsilon)| &\leq |\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(0)| + |\dot{Z}_p^+(0) - \dot{Z}_p^+(\beta_\varepsilon)|\,.
\end{aligned} \tag{5.14}
$$

Now in each line the last term on the right hand side goes to zero by smoothness of γ^+, while for the respective first terms we have by our choice of data (5.2), (5.3)

$$
\begin{aligned}
|V_\varepsilon(\beta_\varepsilon) - V^+(0)| &= |V_\varepsilon(\beta_\varepsilon) - \mathsf{B}| \to 0\,, \\
|\dot{V}_\varepsilon(\beta_\varepsilon) - \dot{V}^+(0)| &= |\dot{V}_\varepsilon(\beta_\varepsilon) - \mathsf{C}| \to 0\,, \\
|\dot{Z}_{p\varepsilon}(\beta_\varepsilon) - \dot{Z}_p^+(0)| &= |\dot{Z}_{p\varepsilon}(t_\varepsilon) - \mathsf{A_p}| \to 0\,.
\end{aligned} \tag{5.15}
$$

Finally to prove the distributional convergence in (iii) thanks to the uniform convergence of V_ε established above we only have to consider the integral

$$
\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \big(V_\varepsilon(s) - V(s)\big)\varphi(s)\,\mathrm{d}s \tag{5.16}
$$

for a test function φ on $\mathbb{R}$. This, however, converges to zero by the local uniform boundedness of V_ε established in Proposition 4.1(iii). In case of the distributional convergence in (ii) we argue precisely in the same manner, now using Proposition 4.1(ii). □

Remark 5.2 (Normalisation and constraints in the limit). The convergence result provided by Theorem 5.1 also guarantees the preservation of the normalisation, i.e., the normalisation of the 'seed geodesic' γ, which by Remark 3.3 carries over to the regularised geodesics γ_ε (and hence to γ_ε^+), also carries over to the 'future branch' of the 'limiting geodesic' γ^+. To see this we just write

$$\begin{aligned} e &= g_\varepsilon(\gamma_\varepsilon(\beta_\varepsilon))\big(\dot\gamma_\varepsilon(\beta_\varepsilon),\dot\gamma_\varepsilon(\beta_\varepsilon)\big) = g\big(\dot\gamma_\varepsilon(\beta_\varepsilon),\dot\gamma_\varepsilon(\beta_\varepsilon)\big) \\ &= -2\dot U_\varepsilon(\beta_\varepsilon)\dot V_\varepsilon(\beta_\varepsilon) + (\dot Z_{2\varepsilon}(\beta_\varepsilon))^2 + (\dot Z_{3\varepsilon}(\beta_\varepsilon))^2 + \sigma(\dot Z_{4\varepsilon}(\beta_\varepsilon))^2 \\ &\to -2\dot U^0\mathsf{C} + \mathsf{A_2}^2 + \mathsf{A_3}^2 + \sigma\mathsf{A_4}^2 = g\big(\dot\gamma^+(0),\dot\gamma^+(0)\big). \end{aligned} \tag{5.17}$$

Here the first equality follows from Remark 3.3 and the second one follows from the fact, that at $\gamma_\varepsilon(\beta_\varepsilon)$ the regularised metric agrees with the (constant) background metric. Finally, convergence is due to Thm. 5.1(i) and our choice of the data (5.2), (5.3).
Also by a similar (actually simpler) argument the constraints carry over from γ_ε to γ^+, which again confirms consistency of our construction.

To end this section and the entire paper we now explicitly evaluate the limits in (5.3), thereby showing that the 'limiting geodesics' (5.4) of the smooth global geodesics γ_ε of Theorem 3.2 in the regularised spacetime (2.9), (2.10) coincide with the geodesics (2.6) of the distributional spacetime (2.1), (2.2) derived previously in [31].

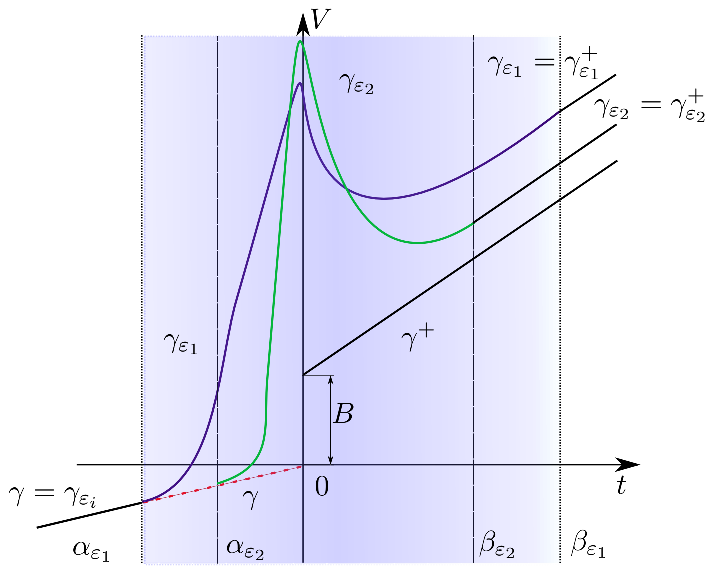

FIGURE 3. The V-components of γ_{ε_1} (purple) and γ_{ε_2} (green) for $\varepsilon_1 > \varepsilon_2$ are depicted. They converge to the the 'limiting geodesic' $\tilde\gamma$ whose 'future branch' γ^+ is separated from its 'past branch' γ (black outside and dotted red inside the regularisation sandwich) by the 'jump' B calculated in Proposition 5.3. The 'jump' C in $\dot V$ is indicated by the different V-slopes of the 'past branch' γ and the 'future branch' γ^+.

Proposition 5.3. *The constants* $\mathsf{A_p}$*,* B*, and* C *determining the data for the 'future branch'* γ^+ *of the 'limiting geodesics'* $\tilde{\gamma}$ *are explicitly given by*

$$
\begin{aligned}
\mathsf{A_i} = \lim_{\varepsilon\to 0} \dot{Z}_{i\varepsilon}(\beta_\varepsilon) &= \frac{1}{2}\dot{U}^0 \Big(H_{,i}(Z_r^0) + \frac{Z_i^0}{\sigma a^2}\big(H(Z_r^0) - \delta^{pq} Z_p^0 H_{,q}(Z_r^0)\big)\Big) + \dot{Z}_i^0 \,, \\
\mathsf{A_4} = \lim_{\varepsilon\to 0} \dot{Z}_{4\varepsilon}(\beta_\varepsilon) &= \frac{1}{2}\dot{U}^0 \Big(\sigma H_{,4}(Z_r^0) + \frac{Z_4^0}{\sigma a^2}\big(H(Z_r^0) - \delta^{pq} Z_p^0 H_{,q}(Z_r^0)\big)\Big) + \dot{Z}_4^0 \,, \\
\mathsf{B} = \lim_{\varepsilon\to 0} V_\varepsilon(\beta_\varepsilon) &= \frac{1}{2} H(Z_p^0) + V^0 \,, \\
\mathsf{C} := \lim_{\varepsilon\to 0} \dot{V}_\varepsilon(\beta_\varepsilon) &= \dot{V}^0 + \frac{\dot{U}^0}{8}\Big(H_{,2}(Z_r^0)^2 + H_{,3}(Z_r^0)^2 + \sigma H_{,4}(Z_r^0)^2 \\
&\qquad + \frac{1}{\sigma a^2} H(Z_r^0)^2 - \frac{1}{\sigma a^2}\left(\delta^{pq} Z_p^0 H_{,q}(Z_r^0)\right)^2 \Big) \\
&\qquad - \frac{\dot{U}^0}{2\sigma a^2}\left(\delta^{pq} Z_p^0 H_{,q}(Z_r^0) - H(Z_r^0)\right) V^0 + \frac{1}{2}\delta^{pq} H_{,p}(Z_r^0) \dot{Z}_q^0 \,.
\end{aligned}
\tag{5.18}
$$

The calculation is rather technical and we sketch the main points in Appendix B. Finally we remark that our results are fully compatible with the ones in [31]:

Remark 5.4. To simplify the comparison of the results on the 'limiting geodesics' of Proposition 5.3 with the heuristically derived geodesics of the impulsive wave spacetime of [31, eqs. (38),(39)] we remark that there the geodesics were restricted to $V^0 = \dot{Z}^0 = 0$, and $\dot{U}^0 = 1$. Moreover recalling that $1/(\sigma a^2) = \Lambda/3$ and using the notations $G(0) = G(Z_p^0) = \delta^{pq} Z_p^0 H_{,q}(Z_r^0) - H(Z_r^0)$ and $H(0) = H(Z_p^0)$ of [31], equations (5.18) take the form

$$
\begin{aligned}
\mathsf{A_i} &= \frac{1}{2}\Big(H_{,i}(0) - \frac{\Lambda}{3} Z_i^0\, G(0)\Big), \quad \mathsf{A_4} = \frac{1}{2}\Big(\sigma H_{,4}(0) - \frac{\Lambda}{3} Z_4^0\, G(0)\Big), \quad \mathsf{B} = \frac{1}{2} H(0), \\
\mathsf{C} &= \frac{1}{8}\Big(H_{,2}(0)^2 + H_{,3}(0)^2 + \sigma H_{,4}(0)^2 + \frac{\Lambda}{3} H(0)^2 - \frac{\Lambda}{3}\left(\delta^{pq} Z_p^0 H_{,q}(0)\right)^2\Big) \;+\; \dot{V}^0.
\end{aligned}
\tag{5.19}
$$

Finally taking into account that $\epsilon = \mathrm{sign}(\Lambda)$ in [31] as well as the slightly different definition of C in [31, eqs. (37)] we see that (5.19) indeed agrees with eqs. (38) of [31].

Summary

In this paper we have rigorously investigated all geodesics in the entire class of nonexpanding impulsive gravitational waves propagating in (anti-)de Sitter universe, extending thus many previous studies of geodesic motion in the class of impulsive *pp*-wave spacetimes with vanishing cosmological constant. Following [31] we employed the *distributional form of the metric in the context of a* 5*-dimensional embedding formalism*. We have applied a regularisation technique, replacing the Dirac-δ by a general class of smooth functions—the model delta nets (2.8). Since we have never used any special property of the regularising net, our results are completely *regularisation independent* within this class. In physical terms this means that the formal distributional form of the impulsive metric (2.1), (2.2) is understood as a limit of a family of spacetimes with ever shorter but ever stronger sandwich gravitational waves (2.9), (2.10) of an *arbitrary smooth profile* δ_ε.

Although the resulting regularized geodesic equations (2.17) form a highly complicated coupled system, we were able to prove in Section 3 the existence and uniqueness of geodesics crossing the wave impulse, leading to completeness results, see Theorems 3.1, 3.2 and 3.4, 3.5. Observe that, in particular, we prove that the geodesics of the regularised spacetime hitting the wave zone and hence interacting *nonlinearly* with the impulse actually cross it. This is a physical information not provided by the approach of [31] in which this feature is built into the heuristic ansatz.

Our proof is based on the application of a fixed point theorem, the technical details of which can be found in Appendix A. There we have extended the range of applicability of this kind of fixed point techniques, originating in [22] and generalised in [36], to a far more involved situation where the higher order 'contraction estimate' contains a term proportional to $1/\varepsilon$, cf. (A.32). In this way we have pushed on the crucial point of the method to the estimate (A.29), cf. the remark in the middle of page 25 below the proof of Proposition A.3. This raises hopes that these techniques can be extended to the even wilder '4-dimensional' distributional form of the metric (1.6) in the future.

In Section 4 we studied boundedness properties of the global geodesics in the regularized spacetimes. These technical results, summarized in Proposition 4.1, were essential for performing their limits in the final Section 5. Due to the complexity of the system of geodesic equations (2.17) it seemed advisable to simplify the 'usual arguments' in this limiting procedure. We have done so by abstracting from the concrete form of the (limiting) geodesics and repeatedly using continuous dependence of solutions of ODEs on its data. In this way we were able to show that, as the regularisation parameter goes to zero, the solutions of the regularised geodesic equation converge to unique geodesics of the background (anti-)de Sitter spacetime (Theorem 5.1) which have to be matched appropriately across the impulse. In fact, we rigorously derived the explicit form of these matching conditions (some overly technical related calculations are contained in Appendix B). The resulting coefficients (5.18) of Proposition 5.3 fully agree with previous results derived by a heuristic approach in [31]. Remarkably, the impulsive limit is completely independent of the specific regularization, i.e., in the limit $\varepsilon \to 0$ it is *the same for any smooth profile* of the sandwich gravitational waves.

Finally, as mentioned in the introduction in [32], we have recently investigated the complete family of nonexpanding impulsive gravitational waves propagating in spaces of constant curvature (Minkowski, de Sitter and anti-de Sitter universes) employing the (Lipschitz) *continuous form of the metric* (1.4). Using Filippov's solution concept for differential equations with discontinuous right hand side we proved existence and uniqueness of continuously differentiable geodesics. In section 4 of [32] we explicitly derived such geodesics using a $\mathcal{C}^1$-matching procedure resulting in specific matching conditions, namely equations (4.4)–(4.10) of [32].

A natural question thus arises about the mutual consistency of the two results, both obtained in a rigorous way but starting from two different forms of the metric, namely the continuous form of the metric (employed in [32]) and the distributional form of the metric in the context of the 5-dimensional embedding formalism (employed here and in [31]). In fact, it was shown in the recent work [20] that the *matching conditions* of [32] and [31] *are fully equivalent* when appropriate coordinate transformations are applied. This result confirms that both our approaches are consistent. It follows that the understanding of geodesics in the complete family of spacetimes with nonexpanding impulsive gravitational waves and any cosmological constant now rests on firm mathematical grounds.

These results set the stage for a sound mathematical analysis of the 'discontinuous coordinate transformations' between the continuous and the distributional forms of the metric. Together with the results of [20] is seems now feasible to rigorously relate the continuous form of the metric (1.4) to the '5-dimensional' distributional form (2.1), (2.2). On the other hand the technical advances of the fixed point techniques made here might eventually bring into reach a direct approach on the mathematical intricacies of the transformation (1.5).

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Appendix A. The fixed point argument

In this appendix we detail the fixed point argument used to prove a suitable existence and uniqueness result for solutions of the regularised geodesic equations (2.17) with data (3.9) that additionally guarantees the solutions to live long enough to leave the regularisation sandwich. To do so, we only have to prove existence of the U_ε and $Z_{p\varepsilon}$-components, since the equation for V_ε decouples and is linear, hence can then be solved on the domain of existence of (U_ε, $Z_{p\varepsilon}$). Moreover, the sign-difference between the $Z_{i\varepsilon}$-equations and the $Z_{4\varepsilon}$-equation can safely be ignored in the estimates leading to the fixed point argument. Therefore, in this appendix, we only (have to) deal with the following *simplified model system*

$$\begin{aligned} \ddot{u}_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\big)^{\dot{}}\Big)\,\frac{u_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\,, \\ \ddot{z}_\varepsilon - \frac{1}{2}DH\,\delta_\varepsilon\dot{u}_\varepsilon^2 &= -\Big(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\big)^{\dot{}}\Big)\,\frac{z_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\,, \end{aligned} \tag{A.1}$$

where $H = H(z_\varepsilon)$ is a smooth function on $\mathbb{R}^3$, DH denotes its gradient, and $\tilde{G}_\varepsilon(u_\varepsilon, z_\varepsilon) := DH(z_\varepsilon)\,\delta_\varepsilon(u_\varepsilon)\,z_\varepsilon + H(z_\varepsilon)\,\delta'_\varepsilon(u_\varepsilon)\,u_\varepsilon$. We will also frequently use the notation $x_\varepsilon = (u_\varepsilon, z_\varepsilon)$.

We begin by setting up the *initial data*. Let $\eta > 0$ and let $J_\varepsilon = [\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ be the parameter interval where we look for solutions. In accordance with the strategy employed in Section 3, we will pose initial data at $t = \alpha_\varepsilon$ and compare it to fixed data (corresponding to the initial data of the 'seed geodesic' at $t = 0$). So let

$$\begin{aligned} &x_\varepsilon^0 = (u_\varepsilon^0, z_\varepsilon^0) \in \mathbb{R}\times\mathbb{R}^3 \quad \text{and} \quad \dot{x}_\varepsilon^0 := (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \in \mathbb{R}\times\mathbb{R}^3 \text{ be given and set} \\ &x_\varepsilon(\alpha_\varepsilon) = (u_\varepsilon(\alpha_\varepsilon), z_\varepsilon(\alpha_\varepsilon)) = (u_\varepsilon^0, z_\varepsilon^0) \quad \text{and} \quad \dot{x}_\varepsilon(\alpha_\varepsilon) = (\dot{u}_\varepsilon(\alpha_\varepsilon), \dot{z}_\varepsilon(\alpha_\varepsilon)) = (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \end{aligned} \tag{A.2}$$

and let additionally

$$u^0, \dot{u}^0 \in \mathbb{R},\ z^0, \dot{z}^0 \in \mathbb{R}^3 \text{ be given and write } x^0 := (u^0, z^0),\ \dot{x}^0 := (\dot{u}^0, \dot{z}^0)\,. \tag{A.3}$$

As detailed in Section 3 we exclusively deal with data satisfying

$$\begin{aligned} &u_\varepsilon^0 = -\varepsilon,\ \dot{u}_\varepsilon^0 > 0 \text{ and } u^0 = 0,\ \dot{u}^0 > 0\,, \text{ with the additional assumption (cf. (3.10))} \\ &x_\varepsilon^0 \to x^0 \text{ and } \dot{x}_\varepsilon^0 \to \dot{x}^0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{A.4}$$

We will apply our fixed point argument on a *complete metric space* which we will call the 'solution space' and which is given as the closed subset of $\mathcal{C}^1(J_\varepsilon, \mathbb{R}^4)$ defined by

$$\mathfrak{X}_\varepsilon := \Big\{ x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{C}^1(J_\varepsilon, \mathbb{R}^4) : x_\varepsilon(\alpha_\varepsilon) = x^0_\varepsilon,\ \dot{x}_\varepsilon(\alpha_\varepsilon) = \dot{x}^0_\varepsilon \text{ and}$$

$$\|x_\varepsilon - x^0\|_\infty \le C_1,\ \|\dot{z}_\varepsilon - \dot{z}^0\|_\infty \le C_2,\ \dot{u}_\varepsilon \in \Big[\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0\Big] \Big\}. \tag{A.5}$$

Observe that we have 'centred' the functions in $\mathfrak{X}_\varepsilon$ around the 'fixed' initial data (A.3), while the prospective solutions are required to assume the ε-dependent data (A.2) at $t = \alpha_\varepsilon$. Also note that the final condition forces $\dot{u}_\varepsilon$ to stay positive, which is the essential ingredient that forces the solutions to leave the regularisation sandwich. Now we arrange the constants as follows: First let $C_1 > 0$ and set

$$C_2 := 1 + \max\Big\{ 9\dot{u}^0 \|DH\|_\infty \|\rho\|_1,\ \frac{36}{a^2}\dot{u}^0(|z^0| + C_1)\Big(3\|DH\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|H\|_\infty\|\rho'\|_\infty\Big),$$

$$\frac{48}{a^2}(|z^0| + C_1)\left(1 + \frac{3}{2}\dot{u}^0\|H\|_\infty\Big(\|\rho'\|_\infty + \|\rho\|_\infty\Big)\right) \Big\}, \tag{A.6}$$

where $\|H\|_\infty$ and $\|DH\|_\infty$ are taken over the closed Euclidean ball $B_{C_1}(z^0)$. Also ρ is as in (2.8). Observe that the space $\mathfrak{X}_\varepsilon$ only depends on ε via the domain J_ε and the initial data (A.2).

Next we define the *solution operator* A_ε acting on $\mathfrak{X}_\varepsilon$ for all $t \in J_\varepsilon$ via $A_\varepsilon(x_\varepsilon)(t) := (A^1_\varepsilon(x_\varepsilon)(t), A^2_\varepsilon(x_\varepsilon)(t))$ with

(A.7)

$$A^1_\varepsilon(x_\varepsilon)(t) := -\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{e u_\varepsilon + \frac{1}{2} u_\varepsilon \dot{u}^2_\varepsilon \tilde{G}_\varepsilon - u_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^{\cdot}}{\sigma a^2 - u^2_\varepsilon H \delta_\varepsilon}\, \mathrm{d}r\, \mathrm{d}s\ + \dot{u}^0_\varepsilon(t - \alpha_\varepsilon) - \varepsilon\,,$$

$$A^2_\varepsilon(x_\varepsilon)(t) := \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left(\frac{1}{2} DH\delta_\varepsilon \dot{u}^2_\varepsilon - \frac{e z_\varepsilon + \frac{1}{2} z_\varepsilon \dot{u}^2_\varepsilon \tilde{G}_\varepsilon - z_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^{\cdot}}{\sigma a^2 - u^2_\varepsilon H\delta_\varepsilon} \right) \mathrm{d}r\, \mathrm{d}s + \dot{z}^0_\varepsilon(t - \alpha_\varepsilon) + z^0_\varepsilon\,,$$

where again we have suppressed the dependence of δ_ε, $\tilde{G}_\varepsilon$ and H as well as their derivatives on the variables.

Our first step will be to show that the operator A_ε takes $\mathfrak{X}_\varepsilon$ to $\mathfrak{X}_\varepsilon$, see Proposition A.3 below. We begin with two preliminary results. First we bound the term in the denominator of A_ε from below.

Lemma A.1. *Suppose $z \in B_{C_1}(z^0)$ then for all $u \in \mathbb{R}$*

$$\left| \frac{1}{\sigma a^2 - u^2\, H(z)\, \delta_\varepsilon(u)} \right| \le \frac{2}{a^2}\,, \tag{A.8}$$

provided $\varepsilon \le a^2/(2\|\rho\|_\infty \|H\|_\infty)$.

Proof. First, in case $|u| > \varepsilon$ we have $u \notin \operatorname{supp}(\delta_\varepsilon)$ and consequently $\dfrac{1}{|\sigma a^2 - u^2 H\delta_\varepsilon|} = \dfrac{1}{a^2}$. Second, in case $|u| \le \varepsilon$ we have $|u^2 H(z)\delta_\varepsilon(u)| \le \varepsilon\|H\|_\infty\|\rho\|_\infty \le a^2/2$ and therefore in both cases

$$\frac{1}{|\sigma a^2 - u^2 H\delta_\varepsilon|} \le \frac{2}{a^2}\,.$$

□

The second preliminary result shows that the conditions on $\dot u_\varepsilon$ imposed in (A.5), i.e., that $\dot u_\varepsilon \geq \dot u^0/2$, prevents the u-component from slowing down too much in the sense that $u_\varepsilon(t)$ leaves the sandwich region early enough. To state the result in a precise way we define for $x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathfrak{X}_\varepsilon$ the set

$$\Gamma_\varepsilon(x_\varepsilon) \equiv \Gamma_\varepsilon(u_\varepsilon) := \{t \in J_\varepsilon \ : \ |u_\varepsilon(t)| \leq \varepsilon\} \subseteq J_\varepsilon \,, \tag{A.9}$$

which is the maximal set where the terms in (A.1), (A.7) involving δ_ε or δ'_ε are non vanishing. We now have:

Lemma A.2. *The diameter of* $\Gamma_\varepsilon(u_\varepsilon)$ *is bounded for all* $x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathfrak{X}_\varepsilon$ *by*

$$\operatorname{diam}(\Gamma_\varepsilon(u_\varepsilon)) \leq \frac{4\varepsilon}{\dot u^0} \,. \tag{A.10}$$

Proof. For $x_\varepsilon \in \mathfrak{X}_\varepsilon$ let $t \in \Gamma_\varepsilon(x_\varepsilon)$ which implies $|u_\varepsilon(t)| \leq \varepsilon$ and so

$$\varepsilon \geq u_\varepsilon(t) = u^0_\varepsilon + \int_{\alpha_\varepsilon}^t \dot u_\varepsilon(\tau)\,\mathrm{d}\tau \geq u^0_\varepsilon + \frac{1}{2}\dot u^0(t - \alpha_\varepsilon) \,. \tag{A.11}$$

But this implies $t \leq \alpha_\varepsilon + 2\,(\varepsilon - u^0_\varepsilon)/\dot u^0 = \alpha_\varepsilon + 4\varepsilon/\dot u^0$. □

Now we may state and prove that $A_\varepsilon(\mathfrak{X}_\varepsilon) \subseteq \mathfrak{X}_\varepsilon$ provided η is chosen appropriately and ε is small enough.

Proposition A.3. *Set*

$$\begin{aligned}\eta := \min\Bigg\{ &1,\ \frac{a^2}{24\dot u^0},\ \frac{C_1}{\frac{3}{2} + \dot u^0},\ \frac{2C_1}{54\|\rho\|_1\|DH\|_\infty \dot u^0},\ \frac{a^2\,C_1}{12(|z^0| + C_1)},\ \frac{a^2\,C_2}{8(|z^0| + C_1)},\\ &\frac{C_1 a^2}{54}\Big(\dot u^0(|z^0| + C_1)(3\|DH\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|H\|_\infty\|\rho'\|_\infty)\Big)^{-1},\ \frac{C_1}{6(1 + |\dot z^0|)},\\ &\frac{C_1 a^2}{72}\left((|z^0| + C_1)\Big(3\|DH\|_\infty\|\rho\|_\infty(|\dot z^0| + C_2) + \frac{3}{2}\dot u^0\|H\|_\infty\big(\|\rho'\|_\infty + \|\rho\|_\infty\big)\Big)\right)^{-1}\Bigg\}\end{aligned} \tag{A.12}$$

and

$$\begin{aligned}\varepsilon'_0 := \min\Bigg\{ &\frac{a^2}{2\|\rho\|_\infty\|H\|_\infty},\ \frac{a^2}{72\dot u^0}\Big(3\|DH\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|H\|_\infty\|\rho'\|_\infty\Big)^{-1},\\ &\frac{a^2}{96}\left(3\|DH\|_\infty\|\rho\|_\infty(|\dot z^0| + C_2) + \frac{3}{2}\dot u^0\|H\|_\infty\Big(\|\rho'\|_\infty + \|\rho\|_\infty\Big)\right)^{-1},\\ &\big(3\|DH\|_\infty\|\rho\|_\infty(|\dot z^0| + C_2)\big)^{-1},\ \frac{\eta\dot u^0}{6},\ \eta\Bigg\}.\end{aligned} \tag{A.13}$$

Now choose ε_0 *such that*

$$\begin{aligned}&0 < \varepsilon_0 \leq \varepsilon'_0\,, \quad \textit{and}\\ &|\dot u^0_\varepsilon - \dot u^0| \leq \frac{1}{8}\,, \quad |z^0_\varepsilon - z^0| \leq \frac{C_1}{6}\,, \quad \textit{and} \quad |\dot z^0_\varepsilon - \dot z^0| \leq 1 \quad \textit{for all} \quad 0 < \varepsilon \leq \varepsilon_0\,.\end{aligned} \tag{A.14}$$

Then for all $\varepsilon \leq \varepsilon_0$ *the operator* A_ε *maps* $\mathfrak{X}_\varepsilon$ *to* $\mathfrak{X}_\varepsilon$.

Observe that by (A.4) there exists ε_0, that guarantees the estimates in (A.14) to hold.

Proof. We begin by estimating

$$\frac{d}{dt}A^1_\varepsilon(x_\varepsilon)(t) = -\int_{\alpha_\varepsilon}^t \frac{eu_\varepsilon + \frac{1}{2}u_\varepsilon\dot{u}_\varepsilon^2\tilde{G}_\varepsilon - u_\varepsilon\dot{u}_\varepsilon\,(H\delta_\varepsilon u_\varepsilon)^{\cdot}}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\,\mathrm{d}s + \dot{u}^0_\varepsilon \tag{A.15}$$

and proceed term by term beginning with the latter two under the integral which we will see to vanish as $\varepsilon \to 0$. Indeed we have for all $\varepsilon \le \varepsilon_0$ by the definition of $\mathfrak{X}_\varepsilon$ (A.5) and by Lemma A.1

$$\begin{aligned} &\Big|\int_{\alpha_\varepsilon}^t \frac{u_\varepsilon\dot{u}_\varepsilon^2\big(DH\delta_\varepsilon z_\varepsilon + H\delta'_\varepsilon u_\varepsilon\big)}{2\,(\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon)}\mathrm{d}s\Big| \\ &\le \frac{1}{a^2}\,\mathrm{diam}\,(\Gamma_\varepsilon(u_\varepsilon))\;\varepsilon\left(\frac{3}{2}\,\dot{u}^0\right)^2\left(3\|DH\|_\infty\|\rho\|_\infty\,\frac{1}{\varepsilon}(|z^0|+C_1) + \|H\|_\infty\frac{1}{\varepsilon^2}\|\rho'\|_\infty\,\varepsilon\right) \\ &\le \frac{9\dot{u}^0}{a^2}\,\varepsilon\;\big(3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1) + \|H\|_\infty\|\rho'\|_\infty\big) \le \frac{1}{8}\,, \end{aligned} \tag{A.16}$$

where for the second inequality we have used Lemma A.2 and for the third that

$$\varepsilon_0 \le \frac{a^2}{72\dot{u}^0}\,\big(3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1) + \|H\|_\infty\|\rho'\|_\infty\big)^{-1}\,. \tag{A.17}$$

Similarly, we have

$$\begin{aligned} &\Big|\int_{\alpha_\varepsilon}^t \frac{u_\varepsilon\dot{u}_\varepsilon\,(H\delta_\varepsilon u_\varepsilon)^{\cdot}}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\mathrm{d}s\Big| \\ &\le \frac{12}{a^2}\,\varepsilon\,\left(3\varepsilon\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0|+C_2) + \frac{3}{2}\dot{u}^0\|H\|_\infty\Big(\|\rho'\|_\infty + \|\rho\|_\infty\Big)\right) \le \frac{1}{8}\,, \end{aligned} \tag{A.18}$$

where the final estimate again follows from our assumptions on ε_0.
Finally to estimate the first term under the integral in (A.15) we write for $u_\varepsilon \in \mathfrak{X}_\varepsilon$

$$u_\varepsilon(t) = u^0_\varepsilon + \int_{\alpha_\varepsilon}^{t} \dot{u}_\varepsilon(s)\,\mathrm{d}s \le -\varepsilon + \eta\,\frac{3}{2}\dot{u}^0 \le \frac{3}{2}\dot{u}^0\eta\,. \tag{A.19}$$

Since $-\varepsilon \le u_\varepsilon(t)$ and by the last condition on ε_0 in (A.14) we obtain $|u_\varepsilon(t)| \le \frac{3}{2}\dot{u}^0\eta$ and hence

$$\Big|\int_{\alpha_\varepsilon}^t \frac{eu_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\,\mathrm{d}s\Big| \le \frac{2}{a^2}\int_{\alpha_\varepsilon}^t |u_\varepsilon(s)|\;\mathrm{d}s \le \frac{3}{a^2}\dot{u}^0\eta^2 \le \frac{3}{a^2}\dot{u}^0\eta \le \frac{1}{8}\,, \tag{A.20}$$

where we have used that $\eta \le 1$ and $\eta \le a^2/(24\dot{u}^0)$, cf. (A.12). Thus, by $|\dot{u}^0_\varepsilon - \dot{u}^0| \le \frac{1}{8}$ we obtain overall $\|\frac{\mathrm{d}}{\mathrm{d}t}A^1_\varepsilon(x_\varepsilon) - \dot{u}^0\|_\infty \le \frac{1}{2}$, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}A^1_\varepsilon(x_\varepsilon)(t) \in [\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0]$ for all $t \in J_\varepsilon$.

Moreover, using the above estimates, integrating once more and using $\varepsilon \le \eta$ we find that

$$\|A^1_\varepsilon(x_\varepsilon) - u^0\|_\infty \le \varepsilon + \eta\frac{3}{8} + \eta\dot{u}^0_\varepsilon \le \eta(\frac{3}{2} + \dot{u}^0) \le C_1\,, \tag{A.21}$$

due to the assumption that $\eta \le C_1/(\frac{3}{2} + \dot{u}^0)$.

Now we turn to the 'spatial component' A^2_ε of the solution operator. We have to show that

$$\begin{aligned}
&\|A^2_\varepsilon(x_\varepsilon) - z^0\|_\infty \\
&= \Big\| \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \Big(\frac{1}{2} DH\delta_\varepsilon \dot u_\varepsilon^2 - \frac{ez_\varepsilon + \frac{1}{2} z_\varepsilon \dot u_\varepsilon^2 \tilde G_\varepsilon - z_\varepsilon \dot u_\varepsilon \big(H\delta_\varepsilon u_\varepsilon\big)^{\cdot}}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\Big) \mathrm{d}r\, \mathrm{d}s \ + \dot z^0_\varepsilon (t-\alpha_\varepsilon)\Big\|_\infty \qquad \text{(A.22)}\\
&\le C_1
\end{aligned}$$

and again proceed term by term. To begin with we note the following auxiliary estimate

$$\begin{aligned}
\int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| \mathrm{d}s &= \frac{2}{\dot u^0} \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| \frac{\dot u^0}{2} \mathrm{d}s \qquad \text{(A.23)}\\
&\le \frac{2}{\dot u^0} \int_{\alpha_\varepsilon}^t |\delta_\varepsilon(u_\varepsilon(s))| \dot u_\varepsilon(s) \mathrm{d}s = \frac{2}{\dot u^0} \int_{-\varepsilon}^{u_\varepsilon(t)} |\delta_\varepsilon(r)| \mathrm{d}r \le \frac{2}{\dot u^0} \|\rho\|_1 .
\end{aligned}$$

Now we have once more using the definition of $\mathfrak{X}_\varepsilon$

$$\frac{1}{2}\Big|\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s DH\delta_\varepsilon \dot u_\varepsilon^2 \, \mathrm{d}r \mathrm{d}s\Big| \le \frac{2}{\dot u^0} \|\rho\|_1 \, \|DH\|_\infty \Big(\frac{3}{2}\dot u^0\Big)^2 \eta = \frac{9}{2}\, \|\rho\|_1 \, \|DH\|_\infty \dot u^0\, \eta \, \le \frac{C_1}{6}\,, \qquad \text{(A.24)}$$

where we have made use of $\eta \le C_1/((2/54)\|\rho\|_1 \|DH\|_\infty \dot u^0)$. Similarly since $\eta \le (a^2\, C_1)/(12(|z^0| + C_1))$ we obtain

$$\Big|\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{ez_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} \, \mathrm{d}r\, \mathrm{d}s\Big| \le \frac{2}{a^2}\, \big(|z^0| + C_1\big)\, \eta^2 \le \frac{2}{a^2}\, \big(|z^0| + C_1\big)\, \eta \le \frac{C_1}{6}\,. \qquad \text{(A.25)}$$

Furthermore, we estimate

$$\begin{aligned}
&\Big|\frac{1}{2}\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{z_\varepsilon \dot u_\varepsilon^2 \tilde G_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\, \mathrm{d}r\, \mathrm{d}s\Big| \\
&\qquad \le \frac{9}{a^2} \dot u^0 \, \big(|z^0| + C_1\big) \, \big(3\|DH\|_\infty \|\rho\|_\infty (|z^0| + C_1) + \|H\|_\infty \|\rho'\|_\infty\big)\, \eta \le \frac{C_1}{6}\,,
\end{aligned}$$

where we have used $\eta \le \frac{C_1 a^2}{54}\Big(\dot u^0(|z^0|+C_1)(3\|DH\|_\infty \|\rho\|_\infty(|z^0|+C_1) + \|H\|_\infty \|\rho'\|_\infty)\Big)^{-1}$, and finally

$$\begin{aligned}
&\Big|\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{z_\varepsilon \dot u_\varepsilon\, (H\delta_\varepsilon u_\varepsilon)^{\cdot}}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\, \mathrm{d}r\, \mathrm{d}s\Big| \qquad \text{(A.26)}\\
&\quad \le \frac{12}{a^2}\, \big(|z^0| + C_1\big) \, \Big(3\varepsilon \|DH\|_\infty \|\rho\|_\infty (|\dot z^0| + C_2) + \frac{3}{2} \dot u^0 \|H\|_\infty \Big(\|\rho'\|_\infty + \|\rho\|_\infty\Big)\Big)\, \eta \le \frac{C_1}{6}\,,
\end{aligned}$$

where we have used the final condition on η in (A.12). This establishes (A.22) using the one before last condition on η in (A.12) together with $|z^0_\varepsilon - z^0| \le C_1/6$.

It remains to show $\|\frac{\mathrm{d}}{\mathrm{d}\,t} A^2_\varepsilon(x_\varepsilon) - \dot z^0\|_\infty \le C_2$. As in (A.24), (A.25) we estimate

$$\begin{aligned}
\frac{1}{2}\Big|\int_{\alpha_\varepsilon}^t DH\delta_\varepsilon(u_\varepsilon) \dot u_\varepsilon^2 \, \mathrm{d}s\Big| &\le \frac{9}{4}\, \|\rho\|_1 \, \|DH\|_\infty \dot u^0 \le \frac{C_2}{4}\,, \qquad \text{(A.27)}\\
\Big|\int_{\alpha_\varepsilon}^t \frac{ez_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon}\, \mathrm{d}s\Big| &\le \frac{2}{a^2}\, \big(|z^0| + C_1\big)\, \eta \le \frac{C_2}{4}\,,
\end{aligned}$$

where we have used the first condition on C_2 in (A.6) and the sixth one on η in (A.12).

For the remaining two terms we have

$$\begin{aligned}&\left|\frac{1}{2}\int_{\alpha_\varepsilon}^{t}\frac{z_\varepsilon \dot u_\varepsilon^2\tilde G_\varepsilon}{\sigma a^2-u_\varepsilon^2 H\delta_\varepsilon}\,\mathrm{d}s\right|\\ &\le \frac{36}{4a^2}\dot u^0\left(|z^0|+C_1\right)\left(3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1)+\|H\|_\infty\|\rho'\|_\infty\right)\le\frac{C_2}{4}\,,\end{aligned}\tag{A.28}$$

where we have used the second condition on C_2 in (A.6), and

$$\begin{aligned}&\left|\int_{\alpha_\varepsilon}^{t}\frac{z_\varepsilon\dot u_\varepsilon\left(H\delta_\varepsilon u_\varepsilon\right)^{\cdot}}{\sigma a^2-u_\varepsilon^2H\delta_\varepsilon}\,\mathrm{d}s\right|\\ &\le\frac{12}{a^2}\left(|z^0|+C_1\right)\left(3\varepsilon\|DH\|_\infty\|\rho\|_\infty(|\dot z^0|+C_2)+\frac{3}{2}\dot u^0\|H\|_\infty\Big(\|\rho'\|_\infty+\|\rho\|_\infty\Big)\right)\\ &\le\frac{12}{a^2}\left(|z^0|+C_1\right)\left(1+\frac{3}{2}\dot u^0\|H\|_\infty\Big(\|\rho'\|_\infty+\|\rho\|_\infty\Big)\right)\le\frac{C_2}{4}\,.\end{aligned}\tag{A.29}$$

Here we have used the fourth condition on ε_0 in (A.14) as well as the final condition on C_2 in (A.6). ☐

Observe that in the estimate (A.29) it is absolutely *vital* that the term in the second line involving C_2 is proportional to ε — otherwise we would end up in a circle and our method would fail.

Our next step is to prove that the solution operator A_ε has a fixed point on $\mathfrak{X}_\varepsilon$. To this end we need the following technical preparation.

Lemma A.4. *There exist constants $\tilde C$ and $\tilde C'$ (independent of ε) such that for all $x_\varepsilon, x_\varepsilon^* \in \mathfrak{X}_\varepsilon$ we have*

(i) $\left|\int_{\alpha_\varepsilon}^{t}\left(\delta_\varepsilon(u_\varepsilon)u_\varepsilon-\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*\right)\mathrm{d}s\right|\le\tilde C\|u_\varepsilon-u_\varepsilon^*\|_\infty$, *and*

(ii) $\left|\int_{\alpha_\varepsilon}^{t}\left(\delta_\varepsilon'(u_\varepsilon)u_\varepsilon^2-\delta_\varepsilon'(u_\varepsilon^*)(u_\varepsilon^*)^2\right)\mathrm{d}s\right|\le\tilde C'\|u_\varepsilon-u_\varepsilon^*\|_\infty$.

Proof. To prove (i) we first consider the case $\|u_\varepsilon-u_\varepsilon^*\|_\infty\le\varepsilon$. We have by (A.23)

$$\begin{aligned}\left|\int_{\alpha_\varepsilon}^{t}\left(\delta_\varepsilon(u_\varepsilon)u_\varepsilon-\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*\right)\mathrm{d}s\right|&\le\int_{\alpha_\varepsilon}^{t}|\delta_\varepsilon(u_\varepsilon)u_\varepsilon-\delta_\varepsilon(u_\varepsilon)u_\varepsilon^*|\,\mathrm{d}s+\int_{\alpha_\varepsilon}^{t}|\delta_\varepsilon(u_\varepsilon)u_\varepsilon^*-\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*|\,\mathrm{d}s\\ &\le\frac{2}{\dot u^0}\|\rho\|_1\|u_\varepsilon-u_\varepsilon^*\|_\infty+\int_{\Gamma_\varepsilon(x_\varepsilon)\cup\Gamma_\varepsilon(x_\varepsilon^*)}|\delta_\varepsilon(u_\varepsilon)-\delta_\varepsilon(u_\varepsilon^*)|\,|u_\varepsilon^*|\,\mathrm{d}s\,.\end{aligned}\tag{A.30}$$

Now the last integral is non-vanishing only if $|u_\varepsilon|\le\varepsilon$ or $|u_\varepsilon^*|\le\varepsilon$ hence we have in any case $|u_\varepsilon^*|\le2\varepsilon$. Since both x_ε, $x_\varepsilon^*\in\mathfrak{X}_\varepsilon$, Lemma A.2 applies so that using a mean value argument we may bound the integral by $\frac{8}{\dot u^0}\|\rho'\|_\infty\|u_\varepsilon-u_\varepsilon^*\|_\infty$.

In case $\|u_\varepsilon-u_\varepsilon^*\|_\infty>\varepsilon$ we obtain again from Lemma A.2

$$\begin{aligned}\left|\int_{\alpha_\varepsilon}^{t}\left(\delta_\varepsilon(u_\varepsilon)u_\varepsilon-\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*\right)\mathrm{d}s\right|&\le\int_{\Gamma_\varepsilon(x_\varepsilon)}|\delta_\varepsilon(u_\varepsilon)u_\varepsilon|\,\mathrm{d}s+\int_{\Gamma_\varepsilon(x_\varepsilon^*)}|\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*|\,\mathrm{d}s\\ &\le\frac{8}{\dot u^0}\|\rho\|_\infty\,\varepsilon<\frac{8}{\dot u^0}\|\rho\|_\infty\|u_\varepsilon-u_\varepsilon^*\|_\infty\,.\end{aligned}\tag{A.31}$$

So we may chose $\tilde C=\frac{2}{\dot u^0}\max\left(\|\rho\|_1+4\|\rho'\|_\infty,4\|\rho\|_\infty\right)$ and (i) is proved.

(ii) is proved analogously with the choice $\tilde{C}' = \frac{4}{\dot{u}^0} \max\left(4\|\rho'\|_\infty + \|\rho''\|_\infty, 2\|\rho'\|_\infty\right)$. $\square$

We finally prove the key estimates which will allow the application of Weissinger's fixed point theorem.

Proposition A.5. *There exists a sequence of positive real numbers $(\alpha_n)_n$ (depending on ρ, ρ', ρ'', H, DH, D^2H, and $\dot{u}^0$ but independent of ε) with $\sum_{n\in\mathbb{N}} \alpha_n < \infty$ such that for all $x_\varepsilon, x_\varepsilon^* \in \mathfrak{X}_\varepsilon$ with $\varepsilon \le \varepsilon_0$ of* (A.14) *and η as in* (A.12) *and all $n \in \mathbb{N}$ we have*

$$\|(A_\varepsilon)^n(x_\varepsilon) - (A_\varepsilon)^n(x_\varepsilon^*)\|_{\mathcal{C}^1} \le \frac{1}{\varepsilon}\alpha_n \|x_\varepsilon - x_\varepsilon^*\|_{\mathcal{C}^1}\,. \tag{A.32}$$

Proof. It suffices to show $\|A_\varepsilon(x_\varepsilon) - A_\varepsilon(x_\varepsilon^*)\|_{\mathcal{C}^1} \le (C/\varepsilon)\|x_\varepsilon - x_\varepsilon^*\|_{\mathcal{C}^1}$ for some appropriate constant C, since for higher powers we then may use

$$\int_{\alpha_\varepsilon}^{t_{2n}} \dots \int_{\alpha_\varepsilon}^{t_1} 1\,\mathrm{d}t\mathrm{d}t_1 \dots \mathrm{d}t_{2n-1} \le \frac{\eta^{2n}}{(2n)!} \tag{A.33}$$

to obtain a converging series.
We again proceed term by term, skipping some of the details of the (by now) routine estimates and only stress the technical key points.
We start with the first term in $\|A_\varepsilon^1(x_\varepsilon) - A_\varepsilon^1(x_\varepsilon^*)\|_{\mathcal{C}^1}$. By writing the two summands on a common denominator we obtain

$$\begin{aligned}
&\left|\int_{\alpha_\varepsilon}^{t} \frac{e u_\varepsilon}{\sigma a^2 - u_\varepsilon^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)}\mathrm{d}s - \frac{e u_\varepsilon^*}{\sigma a^2 - (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)}\,\mathrm{d}s\right| \\
&\quad\le \frac{4}{a^4}\int_{\alpha_\varepsilon}^{t} a^2|u_\varepsilon - u_\varepsilon^*|\mathrm{d}s + \frac{4}{a^4}\int_{\alpha_\varepsilon}^{t} |u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*) - u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon^*)|\mathrm{d}s \\
&\quad\quad + \frac{4}{a^4}\int_{\alpha_\varepsilon}^{t} |u_\varepsilon (u_\varepsilon^*)^2 H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon^*) - u_\varepsilon^2 u_\varepsilon^* H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)|\mathrm{d}s \\
&\quad\le \frac{4}{a^2}\eta\|u_\varepsilon - u_\varepsilon^*\|_\infty + \frac{4}{a^4}(|u^0| + C_1)^2\left(\mathrm{Lip}(H)\|z_\varepsilon - z_\varepsilon^*\|_\infty \frac{4}{\dot{u}^0}\|\rho\|_\infty + \|H\|_\infty \tilde{C}\|u_\varepsilon - u_\varepsilon^*\|_\infty\right) \\
&\quad\le \frac{4}{a^2}\left(\eta + \frac{1}{a^2}(1 + C_1)^2\Big(\mathrm{Lip}(H)\frac{4}{\dot{u}^0}\|\rho\|_\infty + \|H\|_\infty \tilde{C}\Big)\right)\|x_\varepsilon - x_\varepsilon^*\|_{\mathcal{C}^1}\,,
\end{aligned} \tag{A.34}$$

where $\mathrm{Lip}(H)$ denotes the Lipschitz constant of H on $B_{C_1}(z^0)$ and $\tilde{C}$ is the constant given by Lemma A.4.
For the second term we need the following auxiliary estimate which is proven by a combination of (i) and (ii) in Lemma A.4:

$$\int_{\alpha_\varepsilon}^{t} |\tilde{G}u_\varepsilon - \tilde{G}^* u_\varepsilon^*|\mathrm{d}s \le C'\|x_\varepsilon - x_\varepsilon^*\|_{\mathcal{C}^1}\,, \tag{A.35}$$

where $C' = \|D^2H\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|DH\|_\infty\big((|z^0| + C_1)\tilde{C} + \|\rho\|_\infty + \|\rho'\|_\infty\big) + \|H\|_\infty\tilde{C}'$.

Abbreviating $C_{\tilde G} := 3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1)+\|H\|_\infty\|\rho'\|_\infty$ we are able to estimate

$$\begin{aligned}
&\left|\int_{\alpha_\varepsilon}^t\left(\frac{\frac12\,u_\varepsilon\dot u_\varepsilon^2\tilde G}{\sigma a^2-u_\varepsilon^2H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)}-\frac{\frac12\,u_\varepsilon^*(\dot u_\varepsilon^*)^2\tilde G^*}{\sigma a^2-(u_\varepsilon^*)^2H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)}\right)\mathrm{d}s\right|\\
&\le\frac{2}{a^4}\int_{\alpha_\varepsilon}^t a^2\left|u_\varepsilon\dot u_\varepsilon^2\tilde G-u_\varepsilon^*(\dot u_\varepsilon^*)^2\tilde G^*\right|+\left|u_\varepsilon\dot u_\varepsilon^2\tilde G(u_\varepsilon^*)^2H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)-u_\varepsilon^*(\dot u_\varepsilon^*)^2\tilde G^*u_\varepsilon^2H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)\right|\,\mathrm{d}s\\
&\le\frac{18(\dot u^0)}{a^4}\left(\frac{a^2C'}{4}+\frac{a^2C_{\tilde G}}{3\dot u^0}+\frac{C'}{4}\|H\|_\infty\|\rho\|_\infty+\frac{\tilde C C_{\tilde G}}{4}\|H\|_\infty+\frac{C_{\tilde G}}{4}\mathrm{Lip}(H)\|\rho\|_\infty+\frac{C_{\tilde G}}{3\dot u^0}\|H\|_\infty\|\rho\|_\infty\right)\\
&\quad\times\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}\,,
\end{aligned}$$

by using (A.35) and Lemma A.4.

The final term in $A_\varepsilon^1(x_\varepsilon)$, i.e.,

$$\left|\int_{\alpha_\varepsilon}^t\left(\frac{u_\varepsilon\dot u_\varepsilon\Big(H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)u_\varepsilon\Big)^{\cdot}}{\sigma a^2-u_\varepsilon^2H(z_\varepsilon)\delta_\varepsilon(u_\varepsilon)}-\frac{u_\varepsilon^*\dot u_\varepsilon^*\Big(H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)u_\varepsilon^*\Big)^{\cdot}}{\sigma a^2-(u_\varepsilon^*)^2H(z_\varepsilon^*)\delta_\varepsilon(u_\varepsilon^*)}\right)\mathrm{d}s\right| \tag{A.36}$$

can be estimated in perfect analogy to the previous terms inserting and subtracting appropriate terms wherever necessary to arrive at an estimate proportional to $\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}$.

The 'spatial component' A_ε^2 of the solution operator can be treated in a similar way. The only new aspect when estimating $\|A_\varepsilon^2(x_\varepsilon)-A_\varepsilon^2(x_\varepsilon^*)\|_{\mathcal C^1}$ is the following. When bounding terms like $|\tilde G-\tilde G^*|$ by multiples of $\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}$ we find that they are no longer multiplied by u_ε and u_ε^*, respectively. Thus we cannot use the auxiliary result (A.35) and consequently terms proportional to $1/\varepsilon$ remain. (Note, however, that the occurrence of $1/\varepsilon$-terms at this stage causes no problem at all in the application of the fixed point theorem, see below.) Summing up we arrive at

$$\|\frac{\mathrm d}{\mathrm d\,t}A_\varepsilon(x_\varepsilon)-\frac{\mathrm d}{\mathrm d\,t}A_\varepsilon(x_\varepsilon^*)\|_\infty\le\frac1\varepsilon C\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}\,,$$

where C is some constant (as above depending on H, ρ, etc.). Furthermore, since $\eta\le1$ we obtain the same estimate for the zeroth order, i.e., $\|A_\varepsilon(x_\varepsilon)-A_\varepsilon(x_\varepsilon^*)\|_\infty\le\frac1\varepsilon C\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}$, and hence

$$\|A_\varepsilon(x_\varepsilon)-A_\varepsilon(x_\varepsilon^*)\|_{\mathcal C^1}\le\frac1\varepsilon C\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}.$$

Finally, for higher powers of A_ε we obtain (using (A.33))

$$\|(A_\varepsilon)^n(x_\varepsilon)-(A_\varepsilon)^n(x_\varepsilon^*)\|_{\mathcal C^1}\le\frac1\varepsilon\alpha_n\|x_\varepsilon-x_\varepsilon^*\|_{\mathcal C^1}\,,$$

where $\alpha_n:=C\frac{\eta^{2n}}{(2n)!}$ $(n\in\mathbb N)$. □

At this point we finally obtain the existence of a unique solution to (A.1) in $\mathfrak X_\varepsilon$ for all fixed small ε by applying Weissinger's fixed point theorem ([43]). Note that the factor $1/\varepsilon$ in the estimate (A.32) provided by Proposition A.5 does not cause any trouble. Its only effect is that the approximating sequence $(A_\varepsilon)^n(x_\varepsilon)$ converges to the fixed point slower as ε gets smaller. Nevertheless we obtain a fixed point for every fixed (small) ε:

Theorem A.6 (Existence and uniqueness)**.** *Consider the system* (A.1) *with initial data* (A.2)*, satisfying* (A.3)*,* (A.4)*. Then for all $\varepsilon \le \varepsilon_0$ where ε_0 is constrained by* (A.14) *and for η given by* (A.12) *we have a unique smooth solution $(u_\varepsilon, z_\varepsilon)$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$. Moreover, u_ε and z_ε as well as their first order derivatives are uniformly bounded in ε.*

Proof. Propositions A.3 and A.5 allow the application of Weissinger's fixed point theorem ([43]) for fixed $\varepsilon \le \varepsilon_0$ and suitable η, providing thus a unique fixed point for the operator A_ε on the space $\mathfrak{X}_\varepsilon$ which in turn gives a unique $\mathcal{C}^1$-solution $x_\varepsilon = (u_\varepsilon, z_\varepsilon)$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ to the system (A.1) with data (A.2). Moreover, since the right hand sides of (A.1) are smooth the solution is smooth as well.
The solution obtained via the fixed point argument is unique in the space $\mathfrak{X}_\varepsilon$ and thereby unique among all smooth solutions assuming this data by the usual argument from ODE-theory.
Finally, u_ε, $\dot u_\varepsilon$, z_ε, and $\dot z_\varepsilon$ are bounded uniformly in ε on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ by the very definition of $\mathfrak{X}_\varepsilon$. □

Appendix B. Limits

In this appendix we deal with the explicit form of the limits $\mathsf{A_p} = \lim_{\varepsilon\to 0} \dot Z_{p\varepsilon}(\beta_\varepsilon)$, $\mathsf{B} = \lim_{\varepsilon\to 0} V_\varepsilon(\beta_\varepsilon)$, and $\mathsf{C} = \lim_{\varepsilon\to 0} \dot V_\varepsilon(\beta_\varepsilon)$ as stated in Proposition 5.3. Since the actual calculations are overly technical we only sketch the main points.
Again the sign-difference between the Z_i-components and Z_4 is minor and to simplify the notation we will use a similar convention as in Appendix A and write Z_ε and Z instead of $Z_{p\varepsilon}$ and Z_p and analogously for their derivatives. Also we will write DH instead of $H_{,p}$.
Starting with $\mathsf{A_p}$ we use the differential equation (2.17) for $Z_{p\varepsilon}$ and the uniform converge of $Z_{p\varepsilon}$ and $\dot U_\varepsilon$ established in Theorem 5.1 to show that

$$\mathsf{A} = \lim_{\varepsilon\to 0} \dot Z_\varepsilon(\beta_\varepsilon) \;=\; \frac{1}{2}\dot U^0 \Big(DH(Z^0) + \frac{Z^0}{\sigma a^2}\big(H(Z^0) - DH(Z^0)Z^0\big)\Big) + \dot Z^0. \tag{B.1}$$

To begin with we express $\dot Z_\varepsilon(\beta_\varepsilon)$ according to (2.17)

$$\begin{aligned}
\dot Z_\varepsilon(\beta_\varepsilon) &= \dot Z^0_\varepsilon + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \ddot Z_\varepsilon(r)\,\mathrm{d}r \\
&= \dot Z^0_\varepsilon + \frac{1}{2}\int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH\delta_\varepsilon \dot U^2\,\mathrm{d}r - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{eZ_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon)\delta_\varepsilon}\,\mathrm{d}r \\
&\quad - \frac{1}{2}\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot U_\varepsilon^2 DH\delta_\varepsilon Z_\varepsilon^2}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon)\delta_\varepsilon}\,\mathrm{d}r + \frac{1}{2}\int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot U_\varepsilon^2 H\delta_\varepsilon' U_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon)\delta_\varepsilon}\,\mathrm{d}r \\
&\quad + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot U_\varepsilon DH\dot Z_\varepsilon\delta_\varepsilon U_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon)\delta_\varepsilon}\,\mathrm{d}r + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\dot U_\varepsilon^2 H\delta_\varepsilon Z_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H(Z_\varepsilon)\delta_\varepsilon}\,\mathrm{d}r \\
&=: \dot Z^0_\varepsilon + \mathsf{I}_\varepsilon + \mathsf{II}_\varepsilon + \mathsf{III}_\varepsilon + \mathsf{IV}_\varepsilon + \mathsf{V}_\varepsilon + \mathsf{VI}_\varepsilon\,,
\end{aligned} \tag{B.2}$$

where we have used that

$$\frac{1}{2}\dot U_\varepsilon^2 \tilde G_\varepsilon - \dot U_\varepsilon (H(Z_\varepsilon)\delta_\varepsilon U_\varepsilon)^{\cdot} = \frac{1}{2}\dot U_\varepsilon^2 DH\delta_\varepsilon Z_\varepsilon - \frac{1}{2}\dot U_\varepsilon^2 H\delta_\varepsilon' U_\varepsilon - \dot U_\varepsilon DH\dot Z_\varepsilon\delta_\varepsilon U_\varepsilon - \dot U_\varepsilon^2 H\delta_\varepsilon\,. \tag{B.3}$$

Proceeding term by term we have

$$\begin{aligned}\Big|\mathsf{I}_\varepsilon - \frac{1}{2}DH(Z^0)\dot{U}^0\Big| &= \frac{1}{2}\Big|\int_{-\varepsilon}^{\varepsilon}\Big(DH(Z_\varepsilon(U_\varepsilon^{-1}(s)))\delta_\varepsilon(s)\dot{U}_\varepsilon(U_\varepsilon^{-1}(s)) - DH(Z^0)\delta_\varepsilon(s)\dot{U}^0\Big)\,\mathrm{d}s\Big|\\ &\le \frac{1}{2}\sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])}\Big|DH(Z_\varepsilon(w))\dot{U}_\varepsilon(w) - DH(Z^0)\dot{U}^0\Big|\ \|\rho\|_{L^1}\ \to 0\,,\end{aligned}$$

where we have used that $U_\varepsilon^{-1}([-\varepsilon,\varepsilon]) = [\alpha_\varepsilon,\beta_\varepsilon]$ together with Lemma A.2. The next term, II_ε, vanishes in the limit by the uniform boundedness of the integrand, the same holds true for V_ε. Now III_ε can be treated as I_ε, additionally using (5.6) to conclude

$$\mathsf{III}_\varepsilon \to -\frac{1}{2}\frac{\dot{U}^0 DH(Z^0)(Z^0)^2}{\sigma a^2}\,. \tag{B.4}$$

We treat IV_ε using $\int \delta'_\varepsilon(s)s\,\mathrm{d}s = -1$ to obtain

$$\begin{aligned}&\Big|\mathsf{IV}_\varepsilon + \frac{\dot{U}^0 H(Z^0)Z^0}{2\sigma a^2}\Big|\\ &\le \frac{1}{2}\sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])}\Big|\frac{\dot{U}_\varepsilon(w)H(Z_\varepsilon(w))Z_\varepsilon(w)}{\sigma a^2 - U_\varepsilon^2(w)H(Z_\varepsilon(w))\delta_\varepsilon(U_\varepsilon(w))} - \frac{\dot{U}^0H(Z^0)Z^0}{\sigma a^2}\Big|\ \|\delta'_\varepsilon(s)s\|_{L^1}\ \to 0\,.\end{aligned} \tag{B.5}$$

Finally, the limit of VI_ε is proportional to the limit of IV_ε,

$$\begin{aligned}&\Big|\mathsf{VI}_\varepsilon - \frac{\dot{U}^0 H(Z^0)Z^0}{\sigma a^2}\Big|\\ &\le \sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])}\Big|\frac{\dot{U}_\varepsilon(w)H(Z_\varepsilon(w))Z_\varepsilon(w)}{\sigma a^2 - U_\varepsilon^2(w)H(Z_\varepsilon(w))\delta_\varepsilon(U_\varepsilon(w))} - \frac{\dot{U}^0H(Z^0)Z^0}{\sigma a^2}\Big|\ \|\rho\|_{L^1} \to 0\,.\end{aligned} \tag{B.6}$$

By adding up the terms and using (3.10) we establish (B.1).

The calculations for B are relatively simple. Using equation (2.17) for $\ddot{V}_\varepsilon$ we write (cf. (4.3))

$$V_\varepsilon(\beta_\varepsilon) = V_\varepsilon^0 + \frac{1}{2}\int_{\alpha_\varepsilon}^{\beta_\varepsilon}\int_{\alpha_\varepsilon}^{s} H(Z_\varepsilon(r))\delta'_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon(r)^2\,\mathrm{d}r\,\mathrm{d}s + O(\varepsilon)\,. \tag{B.7}$$

We substitute twice, use $\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{s}\delta'_\varepsilon(r)\,\mathrm{d}r\,\mathrm{d}s = 1$ and insert appropriate terms to obtain

$$\begin{aligned}&\frac{1}{2}\Big|\int_{\alpha_\varepsilon}^{\beta_\varepsilon}\int_{\alpha_\varepsilon}^{s} H(Z_\varepsilon(r))\delta'_\varepsilon(U_\varepsilon(r))\dot{U}_\varepsilon(r)^2\,\mathrm{d}r\,\mathrm{d}s - H(Z^0)\Big|\\ &\quad= \frac{1}{2}\Big|\int_{-\varepsilon}^{\varepsilon}\frac{1}{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))}\int_{-\varepsilon}^{l} H(Z_\varepsilon(U_\varepsilon^{-1}(\tau)))\delta'_\varepsilon(\tau)\dot{U}_\varepsilon(U_\varepsilon^{-1}(\tau))\,\mathrm{d}\tau\,\mathrm{d}l\\ &\qquad\qquad - \int_{-\varepsilon}^{\varepsilon}\frac{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))}{\dot{U}_\varepsilon(U_\varepsilon^{-1}(l))}\int_{-\varepsilon}^{l} H(Z^0)\delta'_\varepsilon(\tau)\,\mathrm{d}\tau\,\mathrm{d}l\Big|\\ &\quad\le \frac{4\|\rho'\|_\infty}{\dot{U}^0}\Bigg(\sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])}|H(Z_\varepsilon(w))\dot{U}_\varepsilon(w) - H(Z^0)\dot{U}^0|\\ &\qquad\qquad + |H(Z^0)|\sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])}|\dot{U}_\varepsilon(w) - \dot{U}^0|\Bigg)\,,\end{aligned} \tag{B.8}$$

which goes to zero by the uniform convergence of Z_ε and $\dot{U}_\varepsilon$, establishing the claimed form of B.

Finally we turn to the calculation of C which is the most demanding one. As above we express $\dot V_\varepsilon(\beta_\varepsilon)$ using the geodesic equation (2.17) to obtain

$$\begin{aligned}
\dot V_\varepsilon(\beta_\varepsilon) &= \dot V_\varepsilon^0 + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{1}{2} H(Z_\varepsilon(r))\delta_\varepsilon'(U_\varepsilon(r))\dot U_\varepsilon^2(r)\,\mathrm{d}r + \int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))\dot U_\varepsilon(r)\dot Z_\varepsilon(r)\,\mathrm{d}r \\
&\quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{e\left(V_\varepsilon(r) + H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon(r)\right)}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)}\,\mathrm{d}r \\
&\quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\left(\frac12 \dot U_\varepsilon^2(r)\tilde G_\varepsilon(r) - \dot U_\varepsilon(r)\big(H(Z_\varepsilon(r))U_\varepsilon(r)\delta_\varepsilon(U_\varepsilon(r))\dot{\big)}\right) V_\varepsilon(r)}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)}\,\mathrm{d}r \\
&\quad - \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \frac{\left(\frac12 \dot U_\varepsilon^2(r)\tilde G_\varepsilon(r) - \dot U_\varepsilon(r)\big(H(Z_\varepsilon(r))U_\varepsilon(r)\delta_\varepsilon(U_\varepsilon(r))\dot{\big)}\right) H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon(r)}{\sigma a^2 - H(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))U_\varepsilon^2(r)}\,\mathrm{d}r \\
&=: \dot V_\varepsilon^0 + \mathsf{I}_\varepsilon + \mathsf{II}_\varepsilon + \mathsf{III}_\varepsilon + \mathsf{IV}_\varepsilon + \mathsf{V}_\varepsilon\,.
\end{aligned} \tag{B.9}$$

Note that $\mathsf{III}_\varepsilon \to 0$ because the integrand is uniformly bounded. Now we rewrite I_ε substituting $s = U_\varepsilon(r)$, abbreviating $w := U_\varepsilon^{-1}(s)$, and using equation (2.17) for $\ddot U_\varepsilon$

$$\begin{aligned}
\mathsf{I}_\varepsilon &= \frac12 \int_{-\varepsilon}^{\varepsilon} H(Z_\varepsilon(w))\delta_\varepsilon'(s)\dot U_\varepsilon(w)\,\mathrm{d}s = 0 - \frac12 \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(s)\big(H(Z_\varepsilon(w))\dot U_\varepsilon(w)\dot{\big)}\,\mathrm{d}s \\
&= -\frac12 \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(s) DH(Z_\varepsilon(w))\dot Z_\varepsilon(w)\,\mathrm{d}s \\
&\quad + \frac12 \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(s) H(Z_\varepsilon(w)) \frac{\left(\frac12 \dot U_\varepsilon(w)\tilde G_\varepsilon(w) - \big(H(Z_\varepsilon)\delta_\varepsilon U_\varepsilon\dot{\big)}(w)\right) s}{\sigma a^2 - s^2 H(Z_\varepsilon(w))\delta_\varepsilon(s)}\,\mathrm{d}s + O(\varepsilon)\,.
\end{aligned} \tag{B.10}$$

Now the integrals on the right-hand-side of (B.10) combine with II_ε and V_ε to give

$$\dot V_\varepsilon(\beta_\varepsilon) = \dot V_\varepsilon^0 + \frac12 \mathsf{II}_\varepsilon + \mathsf{III}_\varepsilon + \mathsf{IV}_\varepsilon + \frac12 \mathsf{V}_\varepsilon\,. \tag{B.11}$$

Now we insert equation (2.17) for $\dot Z_\varepsilon$ into II_ε and follow the same pattern as before. The only remarkable new point is the occurrence of the regularisation-dependent term $\int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(s)^2 s\,\mathrm{d}s$, whose prefactors cancel after a long and tedious calculation, where we repeatedly use identities as e.g. $\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{s} \delta_\varepsilon(r)\,\mathrm{d}r\,\mathrm{d}s = \frac12$. For example we obtain for the term in (B.11) related to $\dot Z_\varepsilon^0$

$$\begin{aligned}
&\Big|\frac12 \int_{\alpha_\varepsilon}^{\beta_\varepsilon} DH(Z_\varepsilon(r))\delta_\varepsilon(U_\varepsilon(r))\dot U_\varepsilon(r)\dot Z_\varepsilon(r)\,\mathrm{d}r - \frac12 DH(Z^0)\dot Z^0\Big| \\
&\quad = \frac12 \Big| \int_{-\varepsilon}^{\varepsilon} \left(DH(Z_\varepsilon(U_\varepsilon^{-1}(s)))\delta_\varepsilon(s)\dot Z_\varepsilon(U_\varepsilon^{-1}(s)) - DH(Z^0)\delta_\varepsilon(s)\dot Z^0\right)\,\mathrm{d}s\Big| \\
&\qquad \le \sup_{w\in U_\varepsilon^{-1}([-\varepsilon,\varepsilon])} \frac12 \|\rho\|_{L^1} \left(|DH(Z_\varepsilon(w))| \left|\dot Z_\varepsilon^0 - \dot Z^0\right| + \left|\dot Z^0\right| \left|DH(Z_\varepsilon(w)) - DH(Z^0)\right| \right) \to 0\,.
\end{aligned}$$

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