Explicit black hole solutions in higher-derivative gravity

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We present, in an explicit form, the metric for all spherically symmetric Schwarzschild–Bach black holes in Einstein–Weyl theory. In addition to the black hole mass, this complete family of spacetimes involves a parameter that encodes the value of the Bach tensor on the horizon. When this additional "non-Schwarzschild parameter" is set to zero the Bach tensor vanishes everywhere and the "Schwa–Bach" solution reduces to the standard Schwarzschild metric of general relativity. Compared with previous studies, which were mainly based on numerical integration of a complicated form of field equations, the new form of the metric enables us to easily investigate geometrical and physical properties of these black holes, such as specific tidal effects on test particles, caused by the presence of the Bach tensor, as well as fundamental thermodynamical quantities.

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I. INTRODUCTION

Einstein's general relativity, formulated about a century ago [1], is the most successful theory of gravity. By predicting and correctly describing new fundamental phenomena such as black holes [2], gravitational waves, and cosmic expansion, it has become a cornerstone of modern theoretical physics and astronomy. Most recently, its predictions have been confirmed by the first direct detection of gravitational waves from a merger of two black holes at cosmological distance.

Despite such enormous successes, it has its limitations. As a classical field theory, it does not take quantum effects into account. In order to understand them, with an ultimate vision to unify general relativity with quantum theory, it is necessary to go beyond the Einstein theory. In string and other effective theories, Einstein's gravity is extended by higher-order terms in curvature that represent quantum corrections at high energies. In particular, in *quadratic gravity* theory, the usual Einstein–Hilbert action is generalized to include the square of the Ricci scalar R and a contraction of the Weyl tensor C_{abcd} [3, 4]. In absence of matter, such an action reads

$$S = \int \mathrm{d}^4x \sqrt{-g}\left(\gamma\, R + \beta R^2 - \alpha\, C_{abcd}\, C^{abcd}\right), \qquad (1.1)$$

where $\gamma = 1/G$ (G is the Newtonian constant) and α, β are additional parameters. The *Einstein–Weyl* theory is obtained by setting $\beta = 0$. In this case, the field equations are $\gamma\,(R_{ab} - \frac{1}{2}R\, g_{ab}) = 4\alpha\, B_{ab}$, where B_{ab} is the *Bach tensor*

$$B_{ab} \equiv \big(\nabla^c\nabla^d + \tfrac{1}{2}R^{cd}\big)C_{acbd}\,, \qquad (1.2)$$

which is traceless, symmetric and conserved ($g^{ab}B_{ab} = 0$, $B_{ab} = B_{ba}$, ${B_{ab}}^{;b} = 0$). Taking the trace of the field equations we obtain $R = 0$, so that they reduce to

$$R_{ab} = 4k\, B_{ab}\,, \qquad (1.3)$$

where $k \equiv \alpha\, G$. For $k = 0$, vacuum Einstein's equations are immediately recovered. Interestingly, in the case of *general quadratic gravity* ($\beta \neq 0$), it can be observed from the corresponding field equations (see e.g. Eq. (2) in [5]) that all solutions to (1.3) are also solutions of (1.1) since the trace of (1.3) implies $R = 0$.

The field equations (1.3) form a highly complicated system of fourth-order non-linear PDEs. Only few non-trivial exact solutions are known. Surprisingly, a static spherically symmetric *non-Schwarzschild black hole* has been recently identified and discussed in [6]. Its metric functions in standard coordinates are determined by involved system of ODEs which was analyzed, e.g., in [7–9], mainly by numerical approaches.

In our contribution, we present an *exact solution* for such black holes in the form of *explicit infinite series*. Instead of using usual coordinates, we express the metric in a more convenient form conformal to type D direct-product Kundt geometries [5]. Higher-order corrections to the Einstein theory are represented here by the conformally well-behaved Bach tensor. This leads to a remarkable simplification, providing us with two compact field equations whose solutions can be found in terms of power series to *any* order around *any value* of a radial coordinate. In addition to mass, these black holes contain a further parameter determining the components of the Bach tensor. By setting this additional parameter to zero, the Schwarzschild metric is recovered. These solutions in higher-derivative gravity can thus be called *Schwarzschild–Bach* (or Schwa–Bach) *black holes.*

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II. NEW CONVENIENT FORM OF A BLACK HOLE METRIC

For static spherically symmetric black holes, the metric

$$\mathrm{d}s^2 = -h(\bar r)\,\mathrm{d}t^2 + \frac{\mathrm{d}\bar r^2}{f(\bar r)} + \bar r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \quad (2.1)$$

is commonly employed. The *Schwarzschild* solution [2] is given by $f = h = 1 - 2m/\bar r$. The metric (2.1) was also used in [6] to investigate black holes in quadratic gravity. It was demonstrated that such a class contains further *non-Schwarzschild* black hole for which $f \neq h$.

However, in this paper we are going to use an *alternative metric form*, namely

$$\mathrm{d}s^2 = \Omega^2(r)\big[\,\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2-2\,\mathrm{d}u\,\mathrm{d}r+\mathcal{H}(r)\,\mathrm{d}u^2\,\big]. \quad (2.2)$$

This is related to the metric (2.1) via the transformation

$$\bar r = \Omega(r)\,, \qquad t = u - \textstyle\int \mathcal{H}(r)^{-1}\mathrm{d}r\,, \quad (2.3)$$

and the new metric functions Ω, $\mathcal{H}$ are related to f, h as

$$h(\bar r) = -\Omega^2\,\mathcal{H}\,, \qquad f(\bar r) = -(\Omega'/\Omega)^2\,\mathcal{H}\,, \quad (2.4)$$

where Ω' denotes the derivative of Ω with respect to r.

The *Killing horizon* associated with $\partial_t = \partial_u$ is located at r_h such that

$$\mathcal{H}\big|_{r=r_h} = 0\,, \quad (2.5)$$

and, due to (2.4), also $h(\bar r_h) = 0 = f(\bar r_h)$. This is unchanged under the *time-scaling freedom* $t \to \sigma^{-1}\,t$ implying $h \to \sigma^2\,h$, which can be used, e.g., to set $h = 1$ at spatial infinity for asymptotically flat solutions.

The metric (2.2), writtten as $\mathrm{d}s^2 = \Omega^2(r)\,\mathrm{d}s^2_{\text{Kundt}}$, is *conformal to* $\mathrm{d}s^2_{\text{Kundt}}$ which belongs to the class of *Kundt geometries* [10, 11] (in fact, to a subclass that is the direct-product of two 2-spaces [11, Chap. 7]).

III. THE FIELD EQUATIONS

The conformal approach to investigating black holes, based on the metric (2.2), is very convenient since it enables us to evaluate the Ricci and Bach tensors from the corresponding tensors of the simpler metric $\mathrm{d}s^2_{\text{Kundt}}$. In particular, the Bach tensor is given by $B_{ab} = \Omega^{-2}\,B^{\text{Kundt}}_{ab}$. A direct calculation yields three non-trivial components of the field equations (1.3) for the metric functions $\Omega(r)$ and $\mathcal{H}(r)$. By employing the Bianchi identities, it can be shown [12] that they reduce to two ODEs

$$\Omega\Omega'' - 2\Omega'^2 = \tfrac{1}{3}k\,\mathcal{B}_1\mathcal{H}^{-1}\,, \quad (3.1)$$

$$\Omega\Omega'\mathcal{H}' + 3\Omega'^2\mathcal{H} + \Omega^2 = \tfrac{1}{3}k\,\mathcal{B}_2\,, \quad (3.2)$$

where *2 independent components of the Bach tensor* are

$$\mathcal{B}_1 \equiv \mathcal{H}\mathcal{H}''''\,, \qquad \mathcal{B}_2 \equiv \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\,. \quad (3.3)$$

This system is considerably simpler than the previously used equations for the metric (2.1), see e.g. [9]. Moreover, Eqs. (3.1), (3.2) form an *autonomous system* (they do not explicitly depend on the variable r) which is essential for finding their solution in the form (5.1) below.

Recall that the trace of (1.3) gives $R = 0$, which reads

$$\mathcal{H}\Omega'' + \mathcal{H}'\Omega' + \tfrac{1}{6}(\mathcal{H}'' + 2)\Omega = 0\,. \quad (3.4)$$

In fact, this equation is obtained by subtracting (3.1) multiplied by $\mathcal{H}'$ from the derivative of (3.2).

For a geometrical/physical interpretation, let us evaluate the Bach and Weyl *scalar curvature invariants*:

$$B_{ab}\,B^{ab} = \tfrac{1}{72}\,\Omega^{-8}\big[(\mathcal{B}_1)^2 + 2(\mathcal{B}_1 + \mathcal{B}_2)^2\big]\,, \quad (3.5)$$

$$C_{abcd}\,C^{abcd} = \tfrac{1}{3}\,\Omega^{-4}\,(\mathcal{H}'' + 2)^2\,. \quad (3.6)$$

In fact, $B_{ab} = 0$ if (and only if) $B_{ab}\,B^{ab} = 0$. Moreover, $C_{abcd}\,C^{abcd} = 0$ implies $B_{ab} = 0$. Notice also from (3.3), (2.5) that $\mathcal{B}_1$ *always vanishes on the horizon.* Based on the invariant (3.5), there are thus *two geometrically distinct classes of solutions* to (3.1), (3.2), depending on the Bach tensor. The first corresponds to $B_{ab} = 0$, while the involved second case arises when $B_{ab} \neq 0$.

IV. VANISHING BACH TENSOR: UNIQUENESS OF SCHWARZSCHILD

In the case $\mathcal{B}_1 = 0 = \mathcal{B}_2$, using a coordinate freedom $r \to \lambda\,r + \nu$, $u \to \lambda^{-1}u$ of the metric (2.2), the complete solution of Eqs. (3.1)–(3.3) is

$$\Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = -r^2 - 2m\,r^3\,. \quad (4.1)$$

This is the Schwarzschild solution, since (2.3), (2.4) give $r = -1/\bar r$, $f(\bar r) = 1 - 2m/\bar r = h(\bar r)$, where $\bar r > 0$ corresponds to $r < 0$ (r increases with $\bar r$). The *Schwarzschild black hole is thus the only possible solution with vanishing Bach tensor*, in accordance with Birkhoff's theorem.

V. NON-VANISHING BACH TENSOR: GENERAL SCHWARZSCHILD–BACH

With $\mathcal{B}_1, \mathcal{B}_2 \neq 0$, the system (3.1), (3.2) of non-linear field equations is coupled in a non-trivial way. However, it is autonomous, so that its solutions can be found as *expansions in the powers of* r *around any fixed value* r_0,

$$\Omega(r) = \Delta^n \sum_{i=0}^{\infty} a_i\,\Delta^i\,, \qquad \mathcal{H}(r) = \Delta^p \sum_{i=0}^{\infty} c_i\,\Delta^i\,, \quad (5.1)$$

where $\Delta \equiv r - r_0$. Inserting the series (5.1) with $n, p \in \mathbb{R}$ into Eqs. (3.1), (3.2), (3.4), it can be shown [12] that the dominant powers of Δ imply specific restrictions such that *only four classes of solutions of the form* (5.1) *are allowed*, namely $[n,p] = [-1,2]$, $[0,1]$, $[0,0]$, $[1,0]$. We have proved [12] that the only solution in the class $[-1,2]$ is the Schwarzchild black hole (4.1), while the class $[1,0]$ is equivalent to the peculiar $(s,t)=(2,2)$ class of [7, 13]. The Schwarzschild–Bach black hole is contained in the classes $[0,1]$ and $[0,0]$.

A. Class $[0,1]$: Schwa–Bach black hole expressed around the horizon r_h

In general, r_0 in Δ of expansions (5.1) can be any constant. However, in the $[0,1]$ class, r_0 is the root of $\mathcal{H}$, and thus the horizon r_h, see Eq. (2.5). A lengthy analysis shows [12] that this class of solutions of the Einstein–Weyl/quadratic gravity includes *non-Schwarzchild* black holes with $B_{ab}\neq 0$. Their explicit form (5.1) is

$$\Omega(r) = -\frac{1}{r} - \frac{b}{r_h}\sum_{i=1}^{\infty}\alpha_i\Big(1-\frac{r}{r_h}\Big)^i, \tag{5.2}$$

$$\mathcal{H}(r) = (r-r_h)\left[\frac{r^2}{r_h} + 3b\,r_h\sum_{i=1}^{\infty}\gamma_i\Big(\frac{r}{r_h}-1\Big)^i\right], \tag{5.3}$$

where the initial coefficients are

$$\alpha_1 = 1\,,\quad \gamma_1 = 1\,,\quad \gamma_2 = \frac{1}{3}\Big(4-\frac{1}{2kr_h^2}+3b\Big)\,, \tag{5.4}$$

and α_l, γ_{l+1} for $l\geq 2$ are given by the recurrent relations

$$\begin{aligned}\alpha_l &= \frac{1}{l^2}\Big[\alpha_{l-1}\big(2l^2-2l+1\big)-\alpha_{l-2}(l-1)^2\\ &\quad -3\sum_{i=1}^{l}(-1)^i\,\gamma_i\,(1+b\,\alpha_{l-i})\big(l(l-i)+\tfrac{1}{6}i(i+1)\big)\Big],\\ \gamma_{l+1} &= \frac{(-1)^l}{kr_h^2\,(l+2)(l+1)l(l-1)}\\ &\quad\times\sum_{i=0}^{l-1}\big(\alpha_i+\alpha_{l-i}(1+b\,\alpha_i)\big)(l-i)(l-1-3i)\,,\end{aligned} \tag{5.5}$$

(with $\alpha_0\equiv 0$) so that $\alpha_2 = 2+\frac{1}{8kr_h^2}+b$, $\gamma_3=\frac{1}{96k^2r_h^4}$ etc.

This family of spherically symmetric black holes depends on *two parameters* with a clear interpretation:

- The parameter r_h identifies the *horizon position.* Clearly, $r=r_h$ is the root of $\mathcal{H}$ given by (5.3).
- The dimensionless *Bach parameter b distinguishes* the Schwarzschild solution ($b=0$) from the more general non-Schwarzschild (Schwa–Bach) black hole with non-zero Bach tensor ($b\neq 0$).

Indeed, setting $b=0$, the solution (5.2), (5.3) reduces to (4.1), the Schwarzschild solution (its horizon is given by $r_h=-\frac{1}{2m}$, where m is the black hole mass). Moreover, we have chosen the new parameter b to *determine the value of the Bach tensor* (3.3) *on the horizon* r_h, namely

$$\mathcal{B}_1(r_h) = 0\,,\qquad \mathcal{B}_2(r_h) = -\frac{3}{kr_h^2}\,b\,. \tag{5.6}$$

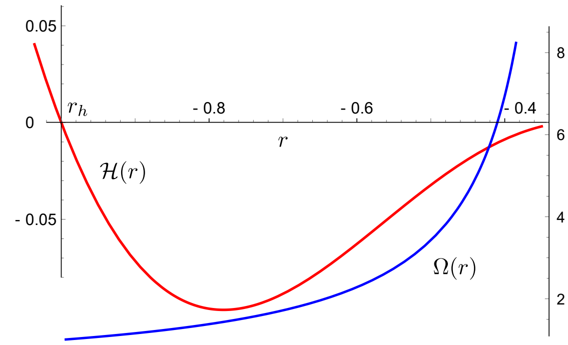

FIG. 1. The functions $\mathcal{H}(r)$ and $\Omega(r)$ for the Schwa–Bach black hole in the form (2.2). The first 20 terms in (5.3) for $\mathcal{H}$ agree with a numerical solution with precision 10^{-4}, and the first 40 terms in (5.2) for Ω agree with precision 10^{-5} on $[-1,-0.5]$. The horizon is at $r_h=-1$, and $k=0.5$, $b=0.3633018769168$, which are the same values as in [6].

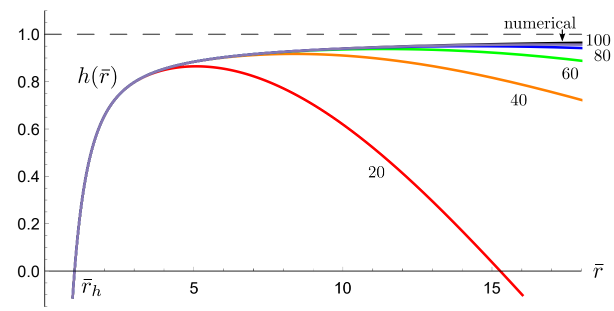

FIG. 2. To demonstrate the rapid convergence in the near-horizon region, we plot the function $h(\bar r)$ of the metric (2.1) expressed using (2.4). The first 20 (red), 40 (orange), 60 (green), 80 (blue), and 100 (violet) terms in the series are compared with the numerical solution of [6] (black). The horizon is located at $\bar r_h = 1$ (that is $r_h=-1$). The scaling freedom with $\sigma^2\approx 2.18$ has been used to obtain $h\to 1$ asymptotically.

The invariants (3.5) and (3.6) are $B_{ab}\,B^{ab}(r_h) = \frac{r_h^4}{4k^2}\,b^2$ and $C_{abcd}\,C^{abcd}(r_h) = 12\,r_h^4\,(1+b)^2$, respectively.

The behavior of the metric functions $\mathcal{H}$ and Ω given by (5.2)–(5.5) is shown on Fig. 1 for a special value of b when the Bach tensor approaches zero for large $\bar r\equiv\Omega(r)$, see Fig. 3. Close to the horizon, the series rapidly converge to the numerical solution of [6]. This can be seen in Fig. 2, where, using the parametric plot and (2.4), the function $h(\bar r)$ of the metric (2.1) is expressed from Ω and $\mathcal{H}$.

B. Class $[0,0]$: Schwa–Bach black hole expressed around any point $r_0\neq r_h$

In this case, the solution to Eqs. (3.1), (3.2) of the form (5.1) with $n=0=p$ is given by the Taylor expansions, where a_0, a_1, c_0, c_1, c_2 are five free parameters,

$$c_3 = \frac{1}{6kc_1}\left[3a_0(a_0+a_1c_1)+9a_1^2c_0+2k(c_2^2-1)\right]\,,$$

$$a_{l+1} = \frac{-1}{l(l+1)\,c_0}\Big[\tfrac{1}{3}\,a_{l-1} + \sum_{i=1}^{l+1} c_i\,a_{l+1-i}\left(l(l+1-i)+\tfrac{1}{6}i(i-1)\right)\Big],$$

$$c_{l+3} = \frac{3}{k\,(l+3)(l+2)(l+1)l} \times \sum_{i=0}^{l} a_i\,a_{l+1-i}(l+1-i)(l-3i)\,, \tag{5.7}$$

for any $l \geq 1$, see [12]. This is a large class of solutions with non-trivial Bach tensor. To identify the Schwa–Bach black hole (5.2), (5.3), previously expressed around the horizon r_h in the class $[0,1]$, we have to uniquely determine the five free parameters by evaluating the functions (5.2), (5.3) and their derivatives at $r = r_0$. Interestingly, for $b = 0$, the coefficients a_i form a geometrical series, r_0 disappears, and the metric functions simplify to the Schwarzschild solution in the form (4.1) with $2m = -1/r_h$. For $B_{ab} = 0$, both classes $[0,0]$ and $[0,1]$ thus reduce to the Schwarzchild black hole. Recall that the parameter r_0 in the class $[0,1]$ *equals* r_h, while $r_0 \neq r_h$ can be chosen *arbitrarily* in the class $[0,0]$.

Since, in general, $\mathcal{B}_1(r_0), \mathcal{B}_2(r_0)$ are independent, the $[0,0]$ class admits one more parameter than the Schwa-Bach black hole and thus it is a *larger family* of solutions. Moreover, the power series (5.1) with integer exponents transforms in some cases to series with *non-integer* exponents in the usual coordinate $\bar{r}$. For example, a new class $(w,t) = (4/3, 0)$ in the notation of [7] also belongs to our $[0,0]$ class, see [12].

VI. OBSERVABLE EFFECTS CAUSED BY THE SCHWA–BACH BLACK HOLE

The two independent parts $\mathcal{B}_1, \mathcal{B}_2$ of the Bach invariant (3.5) can be observed via a *specific influence on test particles*, namely their *relative motion* described by the equation of geodesic deviation [14]. To obtain measurable information, we project it onto an orthonormal frame associated with an *initially static observer* ($\dot{r} = 0$, $\dot{\theta} = 0 = \dot{\phi}$), namely $\boldsymbol{e}_{(0)} = \boldsymbol{u} = \dot{u}\,\partial_u$, $\boldsymbol{e}_{(1)} = -\dot{u}\,(\partial_u + \mathcal{H}\,\partial_r) = -\mathcal{H}\,\Omega'\,\dot{u}\,\partial_{\bar{r}}$, $\boldsymbol{e}_{(2)} = \Omega^{-1}\partial_\theta$, $\boldsymbol{e}_{(3)} = (\Omega\sin\theta)^{-1}\partial_\phi$. Indeed, $\boldsymbol{e}_{(a)} \cdot \boldsymbol{e}_{(b)} = \eta_{ab}$, and the normalization of the observer's velocity $\boldsymbol{u}$ implies $\Omega^2\mathcal{H}\,\dot{u}^2 = -1$. Denoting the *relative position* of two particles as $Z^{(a)} \equiv e^{(a)}{}_{\mu}\,Z^{\mu}$, and their *mutual acceleration* as $\ddot{Z}^{(a)} \equiv e^{(a)}{}_{\mu}\,\frac{\mathrm{D}^2 Z^{\mu}}{\mathrm{d}\,\tau^2}$, we obtain

$$\ddot{Z}^{(1)} = \frac{1}{6}\frac{\mathcal{H}''+2}{\Omega^2}\,Z^{(1)} - \frac{k}{3}\,\frac{\mathcal{B}_1+\mathcal{B}_2}{\Omega^4}\,Z^{(1)}\,, \tag{6.1}$$

$$\ddot{Z}^{(i)} = -\frac{1}{12}\frac{\mathcal{H}''+2}{\Omega^2}\,Z^{(i)} - \frac{k}{6}\,\frac{\mathcal{B}_1}{\Omega^4}\,Z^{(i)}\,, \tag{6.2}$$

where $i = 2,3$. There is the classical *Newtonian tidal deformation* caused by the *Weyl curvature* proportional to $(\mathcal{H}''+2)\,\Omega^{-2}$, i.e., square root of the invariant (3.6). The Schwa–Bach black hole causes *two additional efects* due to the *Bach tensor*. The first is observed in the *transverse* components of the acceleration (6.2) along $\partial_\theta, \partial_\phi$, while the second occurs in the *radial* component (6.1) along $\partial_{\bar{r}}$. Their amplitudes are given by $\mathcal{B}_1, \mathcal{B}_2$ defined in (3.3). Interestingly, *on the horizon* there is *only the radial* effect caused by $\mathcal{B}_2$ since $\mathcal{B}_1(r_h) = 0$, see (5.6). It can also be proven [12] that $\mathcal{B}_1, \mathcal{B}_2$ *cannot mimic* the Newtonian tidal effect, i.e., cannot be "incorporated" into the first terms in (6.1), (6.2). Therefore, by detecting free fall of a set of test particles *it is possible to distinguish* the pure Schwarzschild from the general Schwa–Bach geometry.

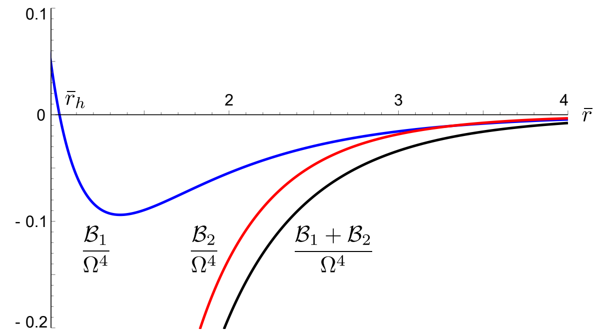

FIG. 3. The Bach tensor components (3.3), entering (6.1) and (6.2), as functions of $\bar{r}$. For the special value of b as in Fig. 1, they approach zero for large $\bar{r}$.

In Fig. 3, the dependence of the physically relevant functions $(\mathcal{B}_1+\mathcal{B}_2)\,\Omega^{-4}$ and $\mathcal{B}_1\Omega^{-4}$ on $\bar{r}$ is plotted. Interestingly, while the former approaches zero monotonously, the latter has its extreme at a specific distance outside the horizon $\bar{r}_h$. At this radius, the transverse Bachian tidal effect (6.2) is maximal.

VII. THERMODYNAMICAL PROPERTIES: HORIZON AREA, TEMPERATURE, ENTROPY

It is also important to determine main physical properties of the family of Schwarzschild–Bach black holes. The *horizon* in these spherically symmetric spacetimes is generated by the (rescaled) null Killing vector $\ell \equiv \sigma\partial_t = \sigma\partial_u$ and thus is located at $\mathcal{H} = 0$, i.e., at $r = r_h$, see (2.5), (5.3). Its *area* is, using (5.2),

$$\mathcal{A} = 4\pi\,\bar{r}_h^2 = 4\pi\,\Omega^2(r_h) = 4\pi\,r_h^{-2}\,. \tag{7.1}$$

Non-zero derivatives of ℓ are $\ell_{u;r} = -\ell_{r;u} = \frac{1}{2}\sigma(\Omega^2\mathcal{H})'$. The *surface gravity*, given by $\kappa^2 \equiv -\frac{1}{2}\,\ell_{\mu;\nu}\,\ell^{\,\mu;\nu}$ on the horizon [15], is thus

$$\kappa/\sigma = -\tfrac{1}{2}\,\mathcal{H}'(r_h) = -\tfrac{1}{2}\,r_h = \tfrac{1}{2}\,\bar{r}_h^{-1}\,. \tag{7.2}$$

It is *the same expression as in the Schwarzschild case* ($\sigma = 1$, $\kappa = \frac{1}{4m}$), independent of the Bach parameter b. The value of the scaling factor σ is fixed by the condition that $h = -\Omega^2\mathcal{H} \to 1$ asymptotically as $\bar{r} = \Omega(r) \to \infty$.

The black-hole horizon *temperature* is thus

$$T/\sigma = \tfrac{1}{2\pi}\,\kappa/\sigma = -\tfrac{1}{4\pi}\,r_h = \tfrac{1}{4\pi}\,\bar{r}_h^{\,-1}\,. \qquad (7.3)$$

However, in higher-derivative theories we have to apply the generalized definition of *entropy* $S = (2\pi/\kappa)\oint \mathbf{Q}$, see [16], where the Noether charge 2-form on the horizon is

$$\mathbf{Q} = -\frac{\Omega^2\,\mathcal{H}'}{16\pi}\Big[1+\tfrac{4}{3}k^2\,\frac{\mathcal{B}_1+\mathcal{B}_2}{\Omega^4}\Big]\Big|_{r=r_h}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \quad (7.4)$$

Evaluating the integral, using (7.1), (7.2), (5.6), we get

$$S = \tfrac{1}{4}\mathcal{A}\left(1-4k\,r_h^2\,b\right) = \tfrac{1}{4}\mathcal{A}\left(1-4k\,\bar{r}_h^{\,-2}\,b\right). \qquad (7.5)$$

This explicit formula for the Schwa–Bach black hole entropy agrees with the results of [6], with the identification $k=\alpha$, $b=\delta^*$. In fact, it gives a *geometric interpretation* of the "non-Schwarzschild parameter" δ as the parameter b determining the *value of the Bach tensor on the horizon*, see (5.6). For the Schwarzschild black hole ($b=0$) or in Einstein's theory ($k=0$), we recover the standard expression. For smaller Schwa–Bach black holes (smaller $\bar{r}_h$), the deviations from $S=\frac{1}{4}\mathcal{A}$ are larger, analogously to [17]. To retain $S>0$, it is necessary to have $4kb<\bar{r}_h^2$, restricting the theory parameters if $\bar{r}_h\to 0$.

Combining expressions (7.3), (7.5), (7.1), *exact relation between the temperature and the entropy* is obtained,

$$T = \tfrac{1}{4}\sigma\left(\pi\,S + 4\pi^2 k\,b\right)^{-1/2}, \qquad (7.6)$$

generalizing $T=\frac{1}{4}(\pi S)^{-1/2}$ of the Schwarzschild case.

For the parameters of Fig. 1, the values of S and T agree with those given by Eq. (11) in [6]. From the behavior of the metric functions for large $\bar{r}$ we were also able to estimate the *mass of this Schwa–Bach black hole* as $2M\approx 0.55$, also in full agreement with [6]. In fact, in [6] the mass for this whole family of black holes was studied numerically, and the first law of thermodynamics was confirmed.

Our current research topics are Schwarzschild–Bach black holes with a cosmological constant, and the study of specific astrophysical consequences (e.g., pericenter precession or gravitational lensing).

VIII. ACKNOWLEDGEMENTS

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[1] Einstein A 1916 Die Grundlage der allgemeinen Relativitätstheorie, *Ann. der Physik* **49** 769

[2] Schwarzschild K 1916 Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, *Sitz. Preuss. Akad. Wiss. Berlin* **7** 189

[3] Weyl H 1919 Eine neue Erweiterung der Relativitätstheorie, *Ann. der Physik* **59** 101

[4] Bach R 1921 Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs, *Math. Zeitschrift* **9** 110

[5] Pravda V, Pravdová A, Podolský J and Švarc R 2017 Exact solutions to quadratic gravity, *Phys. Rev. D* **95** 084025

[6] Lü H, Perkins A, Pope C N and Stelle K S 2015 Black holes in higher derivative gravity, *Phys. Rev. Lett.* **114** 171601

[7] Lü H, Perkins A, Pope C N and Stelle K S 2015 Spherically symmetric solutions in higher-derivative gravity, *Phys. Rev. D* **92** 124019

[8] Holdom B and Ren J 2017 Not quite a black hole, *Phys. Rev. D* **95** 084034

[9] Kokkotas K D, Konoplya R A and Zhidenko A 2017 Non-Schwarzschild black-hole metric in four dimensional higher derivative gravity: Analytical approximation, *Phys. Rev. D* **96** 064007

[10] Stephani H, Kramer D, MacCallum M, Hoenselaers C, and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press).

[11] Griffiths J and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge: Cambridge University Press)

[12] Podolský J, Švarc R, Pravda V and Pravdová A, 2018 (in preparation)

[13] Stelle K S 1978 Classical gravity with higher derivatives, *Gen. Relativ. Gravit.* **9** 353

[14] Podolský J and Švarc R 2012 Interpreting spacetimes of any dimension using geodesic deviation, *Phys. Rev. D* **85** 044057

[15] Wald R M 1984 *General Relativity* (Chicago: University of Chicago Press)

[16] Wald R M 1993 Black hole entropy is the Noether charge, *Phys. Rev. D* **48** R3427(R)

[17] Nozawa M and Maeda H 2008 Dynamical black holes with symmetry in Einstein-Gauss-Bonnet gravity, *Class. Quantum Grav.* **25** 055009