Gyratons in the Robinson-Trautman and Kundt classes

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In our previous paper [Phys. Rev. D **89**, 124029 (2014)], we attempted to find Robinson-Trautman-type solutions of Einstein's equations representing gyratonic sources (a matter field in the form of an aligned null fluid, or particles propagating with the speed of light, with an additional internal spin). Unfortunately, by making a mistake in our calculations, we came to the wrong conclusion that such solutions do not exist. We are now correcting this mistake. In fact, this allows us to explicitly find a new large family of gyratonic solutions in the Robinson-Trautman class of spacetimes in any dimension greater than (or equal to) 3. Gyratons thus exist in all twist-free and shear-free geometries, that is, both in the expanding Robinson-Trautman and in the nonexpanding Kundt classes of spacetimes. We derive, summarize, and compare explicit canonical metrics for all such spacetimes in arbitrary dimension.

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I. INTRODUCTION

The Robinson-Trautman class of spacetimes [1,2] and the closely related Kundt class [3] are important families of exact solutions to Einstein's field equations. They are geometrically defined by admitting a geodesic, shear-free, and twist-free null congruence. For the Robinson-Trautman class, such a congruence is expanding, while for the Kundt class it is nonexpanding.

In the usual dimension $D = 4$, these classes contain a great number of famous solutions, namely, Schwarzschild-like static black holes, accelerating black holes (C-metric), Vaidya metric, Kinnersley photon rockets, spacetimes with gravitational waves of various types (including well-known pp-waves) propagating on various backgrounds (Minkowski, de Sitter, anti–de Sitter, direct-product universes, etc.), and many other exact spacetimes. These are vacuum solutions with any value of the cosmological constant Λ, they admit pure radiation, electromagnetic fields (both null and non-null), and other forms of matter. More details and specific references can be found, e.g., in chapters 28 and 31 of [4] or chapters 18 and 19 of [5], respectively.

During the past decade, the large Robinson-Trautman class of solutions was extended to any higher dimension $D > 4$ for the case of an empty space with any Λ or aligned pure radiation [6], for aligned electromagnetic fields [7], and general p-form fields [8]. Similarly, extension of the Kundt class to higher dimensions was presented in [9]; see also [10–13]. Complementarily, all Robinson-Trautman and Kundt solutions to Einstein's equations for Λ-vacuum, aligned pure radiation and gyratonic matter in lower dimension $D = 3$ were recently found in [14].

Gyratonic matter is a null field with internal spin/helicity. It was first considered in 1970 by Bonnor [15] who studied both the interior and the exterior solution of a "spinning null fluid" in the class of axially symmetric pp-waves (see also Griffiths [16] who studied neutrino fields). Such matter is characterized not only by a specific energy density profile, but also by a nonzero angular momentum density profile. Spacetimes with localized spinning sources of this kind (spinning null particles accompanied by impulsive gravitational waves) moving at the speed of light were then independently rediscovered and investigated in 2005 by Frolov *et al.* [17,18]. These pp-wave-type gyratons in $D \geq 4$ were subsequently studied in greater detail, and also generalized to include $\Lambda < 0$ [19], electromagnetic field [20], and various other settings including nonflat backgrounds or supergravity models. Summary of these gyratonic solutions can be found, e.g., in [21,22].

All the so-far-known spacetimes with gyratonic matter sources belong to the Kundt class. Five years ago we asked ourselves a question: are there gyratons in other geometries as well? The most natural candidate to investigate was the Robinson-Trautman class because it shares the twist-free and shear-free properties. It differs only in having a nonvanishing expansion of the privileged null congruence. In our paper [23] we attempted to systematically study the possible existence of Robinson-Trautman gyratonic solutions (in any dimension) which would be analogous to those known in the Kundt class. Unfortunately, by making a mistake in evaluating the gyratonic energy-momentum conservation equation, we came to the wrong conclusion that such solutions do not exist. Here we are correcting this specific mistake, and we explicitly derive a new large family of gyratonic solutions in the Robinson-Trautman class. Gyratons thus exist in all twist-free and shear-free $D \geq 3$ geometries.

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In Sec. II we summarize the general form of nontwisting shear-free geometries and Einstein's field equations, including the correct form of the gyratonic matter. Complete integration of the field equations is presented in Sec. III. The obtained Robinson-Trautman spacetimes are summarized and discussed in concluding Sec. IV. In particular, we compare the $D > 4$, $D = 4$, and $D = 3$ cases. Moreover, in a compact and explicit form we present the entire class of Kundt solutions with aligned gyratonic matter in any dimension D, and we compare it with the newly obtained Robinson-Trautman class.

II. GENERAL ROBINSON-TRAUTMAN AND KUNDT GEOMETRIES AND EINSTEIN'S EQUATIONS FOR ALIGNED GYRATONIC MATTER

The *metric* of the most general D-dimensional Robinson-Trautman or Kundt geometry can be written as

$$\mathrm{d}s^2 = g_{pq}(r,u,x)\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\mathrm{d}u\,\mathrm{d}x^p - 2\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2 \tag{1}$$

[see Eq. (1) in [23]], where x is a shorthand for $(D-2)$ spatial coordinates x^p. Recall also that the nonvanishing contravariant metric components are g^{pq} (an inverse matrix to g_{pq}), $g^{ru} = -1$, $g^{rp} = g^{pq}g_{uq}$, and $g^{rr} = -g_{uu} + g^{pq}g_{up}g_{uq}$ (so that $g_{up} = g_{pq}g^{rq}$ and $g_{uu} = -g^{rr} + g_{pq}g^{rp}g^{rq}$). The null vector field $\mathbf{k} = \partial_r$ generates a geodesic and affinely parametrized null congruence which is twist-free and shear-free, provided $g_{pq,r} = 2\Theta g_{pq}$. In the Robinson-Trautman class of geometries, this congruence has a nonvanishing expansion $\Theta \neq 0$, while $\Theta = 0$ defines the Kundt class.

Einstein's equations for the metric g_{ab} read $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, where Λ is any cosmological constant. We study spacetimes with a gyratonic matter aligned with $\mathbf{k}$ [15,17,21]. In the coordinates of (1), the nonvanishing components of the energy-momentum tensor T_{ab} are

$$T_{uu}(r,u,x), \qquad T_{up}(r,u,x), \tag{2}$$

where T_{uu} corresponds to the classical pure radiation component, while T_{up} encodes inner gyratonic angular momentum. Since its trace $T \equiv g^{ab}T_{ab}$ vanishes, Einstein's equations simplify to

$$R_{ab} = \frac{2}{D-2}\Lambda g_{ab} + 8\pi T_{ab}. \tag{3}$$

In our previous paper [23], we explicitly calculated all complicated components of the Ricci tensor R_{ab}, namely, Eqs. (32)–(37). While these are correct, we made an unfortunate mistake in evaluating the conditions $T^{ab}_{\ ;b} = 0$ following from the Bianchi identities. Indeed, Eqs. (54) and (55) in [23] are wrong. Their correct form is

$$T_{up,r} + (D-2)\Theta T_{up} = 0, \tag{4}$$

$$T_{uu,r} + (D-2)\Theta T_{uu} = g^{pq}T_{up||q} + g^{rp}{}_{,r}T_{up}, \tag{5}$$

where the symbol $_{||}$ denotes the covariant derivative with respect to the spatial metric g_{pq}, that is, $T_{up||q} \equiv T_{up,q} - T_{um}{}^{S}\Gamma^{m}_{pq}$, in which ${}^{S}\Gamma^{m}_{pq} \equiv \frac{1}{2}g^{mn}(2g_{n(p,q)} - g_{pq,n})$ are the Christoffel symbols with respect to the spatial coordinates only.

III. COMPLETE INTEGRATION OF THE FIELD EQUATIONS

As in [23], we will now perform a step-by-step integration of the Einstein field equations (3) for $\Theta \neq 0$. Some results will remain the same, but due to the corrected constraints (4) and (5), gyratonic solutions are actually found to exist.

A. The equation $R_{rr} = 0$

This field equation remains unchanged, providing us with the expansion scalar

$$\Theta = \frac{1}{r}, \tag{6}$$

and thus the $(D-2)$-dimensional spatial metric

$$g_{pq} = r^2 h_{pq}(u,x), \tag{7}$$

which are the same expressions as Eqs. (57) and (58) of [23].

B. The equation $R_{rp} = 0$

Also this equation has a correct solution given by Eqs. (61) and (62) of [23], that is,

$$g^{rq} = e^q(u,x) + r^{1-D}f^q(u,x), \tag{8}$$

and

$$g_{up} = r^2 e_p(u,x) + r^{3-D}f_p(u,x), \tag{9}$$

respectively. Here $e_p \equiv h_{pq}e^q$ and $f_p \equiv h_{pq}f^q$ are arbitrary functions of u and x.

Using (6)–(8), we can fully integrate the corrected energy-momentum conservation equations (4) and (5), yielding

$$T_{up} = \mathcal{J}_p r^{2-D}, \tag{10}$$

$$T_{uu} = \mathcal{N}r^{2-D} - \mathcal{J}^p{}_{||p}r^{1-D} + f^p\mathcal{J}_p r^{3-2D}, \tag{11}$$

where $\mathcal{J}_p(u,x)$ and $\mathcal{N}(u,x)$ are arbitrary integration functions of u and x, and $\mathcal{J}^p{}_{||p} \equiv h^{pq}\mathcal{J}_{p||q}$. These expressions rectify wrong Eqs. (63) and (64) of [23].

C. The equation $R_{ru} = -\frac{2}{D-2}\Lambda$

Since this field equation is unaffected by the above-mentioned mistakes, Eq. (67) of [23] is correct, so that the corresponding metric function is

$$g^{rr} = a + br^{3-D} + cr - \frac{2\Lambda}{(D-1)(D-2)}r^2 + \frac{D-3}{D-2}f^p{}_{||p}r^{2-D} + \frac{D-1}{2(D-2)}f^p f_p r^{2(2-D)}, \tag{12}$$

where

$$c \equiv -\frac{2}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right), \tag{13}$$

which leads to

$$g_{uu} = -g^{rr} + r^2 e^p e_p + 2r^{3-D}e^p f_p + r^{2(2-D)}f^p f_p. \tag{14}$$

D. The equation $R_{pq} = \frac{2}{D-2}\Lambda g_{pq}$

This Einstein field equation was also correctly evaluated and integrated in [23]. It turns out that in any dimension $D \geq 4$, necessarily

$$f_p = 0 \tag{15}$$

for all $(D-2)$ spatial indices p (interestingly, in *lower* dimension $D = 3$, the single function f remains arbitrary; see [14] and Sec. IV B below). Consequently, the most general Robinson-Trautman line element takes the form

$$\mathrm{d}s^2 = r^2 h_{pq}\mathrm{d}x^p\mathrm{d}x^q + 2r^2 e_p \mathrm{d}u\mathrm{d}x^p - 2\mathrm{d}u\mathrm{d}r + (r^2 e^p e_p - g^{rr})\mathrm{d}u^2, \tag{16}$$

where

$$g^{rr} = a + br^{3-D} + cr - \frac{2\Lambda}{(D-1)(D-2)}r^2. \tag{17}$$

The functions h_{pq} and e_p are constrained by the equations

$$\mathcal{R}_{pq} = \frac{\mathcal{R}}{D-2}h_{pq}, \tag{18}$$

$$\frac{1}{2}h_{pq,u} = e_{(p||q)} + \frac{1}{2}ch_{pq}, \tag{19}$$

that are also imposed by the field equation $R_{pq} = \frac{2}{D-2}\Lambda g_{pq}$, together with the relation

$$a = \frac{\mathcal{R}}{(D-2)(D-3)}. \tag{20}$$

Here, $\mathcal{R} \equiv h^{pq}\mathcal{R}_{pq}$ is the Ricci scalar curvature of the spatial metric h_{pq}, which is the r-independent part of g_{pq}. Notice that due to (7), the corresponding Ricci tensor is $\mathcal{R}_{pq} \equiv {}^S R_{pq}$, while $\mathcal{R} \equiv {}^S R r^2$. Due to (18), the transverse $(D-2)$-dimensional Riemannian space must be an Einstein space.

E. The equation $R_{up} = \frac{2}{D-2}\Lambda g_{up} + 8\pi T_{up}$

This Einstein equation now takes the form

$$-\frac{1}{D-2}\mathcal{R}e_p - \frac{D-3}{D-2}\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)_{,p} + h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) + \frac{(D-4)}{2(D-2)(D-3)}\mathcal{R}_{,p}r^{-1} - \frac{1}{2}b_{,p}r^{2-D} + \left[(D-2)\left(e^n e_{[n,p]} - \frac{1}{2}(e^n e_n)_{,p} + \frac{1}{2}e^n h_{np,u}\right) + e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)\right]r = 8\pi T_{up}. \tag{21}$$

The gyratonic term T_{up} on the right-hand side is given by the corrected expression (10), namely, $T_{up} = \mathcal{J}_p r^{2-D}$. This gives us four conditions:

$$\mathcal{R}e_p + (D-3)\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right)_{,p} - (D-2)h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) = 0, \tag{22}$$

$$(D-4)\mathcal{R}_{,p} = 0, \tag{23}$$

$$b_{,p} = -16\pi\mathcal{J}_p, \tag{24}$$

$$(D-2)\left(e^n e_{[n,p]} - \frac{1}{2}(e^n e_n)_{,p} + \frac{1}{2}e^n h_{np,u}\right) + e_p\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right) = 0. \tag{25}$$

In our previous paper we used the wrong expression $T_{up} = \mathcal{J}_p r$, which led us to the wrong relations $b_{,p} = 0$ and subsequently $\mathcal{J}_p = 0$; cf. Eqs. (86) and (92) in [23]. Thus, we were misled to the incorrect conclusion that there are no gyratonic solutions in the Robinson-Trautman class of geometries. But such solutions do exist since nonzero $\mathcal{J}_p$ is obviously allowed by admitting a spatial dependence of the function $b(u,x)$ in (24).

Moreover, as shown in our paper [23], complicated Eqs. (22) and (25) are *identically satisfied*. Equation (23)

clearly restricts the dependence of the spatial Ricci scalar $\mathcal{R}$ on the spatial coordinates x^p, namely,

$$\mathcal{R} = \mathcal{R}(u) \quad \text{for } D > 4, \tag{26}$$

$$\mathcal{R} = \mathcal{R}(u, x) \quad \text{for } D = 4. \tag{27}$$

There is thus a significant difference between the $D = 4$ case of classical relativity and the extension of Robinson-Trautman spacetimes to higher dimensions. The remaining Eq. (24) gives

$$\mathcal{J}_p = -\frac{1}{16\pi} b_{,p}. \tag{28}$$

Therefore, in *any* dimension $D \geq 4$ we obtain the gyratonic matter component

$$T_{up} = -\frac{1}{16\pi} b_{,p} r^{2-D}. \tag{29}$$

F. The equation $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi T_{uu}$

This final equation determines the relation between the Robinson-Trautman geometry and the pure radiation matter field represented by the profile $\mathcal{N}(u, x)$ in (11).

For (6)–(9) and (14) with (15), the Ricci tensor component R_{uu} becomes[1]

$$\begin{aligned} R_{uu} = {} & \frac{1}{2} g^{rr} g^{rr}{}_{,rr} + \frac{1}{2}\left[e^n{}_{||n} - \frac{1}{2} h^{mn} h_{mn,u} + (D-2) g^{rr} r^{-1} - 2 e^n e_n r \right] g^{rr}{}_{,r} \\ & + e^n \left[g^{rr}{}_{,r} + \frac{1}{2}(D-6) g^{rr} r^{-1} \right]_{,n} + \frac{1}{2} h^{mn} g^{rr}{}_{||m||n} r^{-2} + \frac{1}{2}(D-2) g^{rr}{}_{,u} r^{-1} \\ & - (D-3) e^n e_n g^{rr} + h^{mn} \left[e_{m,u||n} - \frac{1}{2}(e^p e_p)_{||m||n} - \frac{1}{2} h_{mn,uu} \right] \\ & + h^{mn} h^{pq} \left(e_{[p,m]} + \frac{1}{2} h_{pm,u} \right) \left(e_{[q,n]} + \frac{1}{2} h_{qn,u} \right) \\ & + \left[\frac{1}{2}(D-2)(e^m e^n h_{mn,u} - e^n (e^p e_p)_{,n}) - e^p e_p \left(e^n{}_{||n} - \frac{1}{2} h^{mn} h_{mn,u} \right) \right] r. \end{aligned} \tag{30}$$

Employing the explicit form (17) of g^{rr} with the help of (19) we obtain

$$\begin{aligned} R_{uu} = {} & \frac{2}{D-2}\Lambda g_{uu} + \frac{1}{2}\left[(D-2) b_{,u} + \frac{1}{2}(D-2)(D-1) bc - D e^n b_{,n} \right] r^{2-D} + \frac{1}{2}\Delta b r^{1-D} + \frac{1}{2}\Delta a r^{-2} \\ & + \frac{1}{2}[(D-2)(a_{,u} + ac) + (D-6) e^n a_{,n} + \Delta c] r^{-1} \\ & + \frac{1}{2}(D-2)(c_{,u} + c^2) + e^n{}_{||n} c + \frac{1}{2}(D-4) e^n c_{,n} - (D-3) e^p e_p a \\ & + h^{mn}\left[e_{m,u||n} - \frac{1}{2} h_{mn,uu} - \frac{1}{2}(e^p e_p)_{||m||n} + h^{pq} e_{p||m} e_{q||n} \right] + \frac{1}{2}(D-2)[e^m e^n h_{mn,u} - e^n (e^p e_p)_{,n} - e^n e_n c] r, \end{aligned} \tag{31}$$

where a is given by (20), c is given by (13), and $\Delta a \equiv h^{mn} a_{||m||n}$ denotes the covariant Laplace operator on the $(D-2)$-dimensional transverse Riemannian space.

Now, in the Appendix of our previous work [23] we proved the nontrivial identities

$$e^m e^n h_{mn,u} - e^n (e^p e_p)_{,n} - e^n e_n c = 0, \tag{32}$$

$$\frac{1}{2}(D-2)(c_{,u} + c^2) + e^n{}_{||n} c + \frac{1}{2}(D-4) e^n c_{,n} - (D-3) e^p e_p a + h^{mn}\left[e_{m,u||n} - \frac{1}{2} h_{mn,uu} - \frac{1}{2}(e^p e_p)_{||m||n} + h^{pq} e_{p||m} e_{q||n} \right] = 0, \tag{33}$$

[1]Recall that $e^n{}_{||n} \equiv h^{nm} e_{m||n}$, $e_{p||q} \equiv e_{p,q} - e_m {}^S\Gamma^m_{pq}$, $a_{||p||q} \equiv a_{,pq} - a_{,n} {}^S\Gamma^n_{pq}$, etc.; see [23] for more details.

$$(D-2)(a_{,u}+ac)+(D-6)e^n a_{,n}+\Delta c=(D-4)e^n a_{,n}, \tag{34}$$

which are valid in any dimension $D \geq 4$. These appear in the terms in (31) proportional to r, r^0, and r^{-1}, respectively. Einstein's equation $R_{uu}=\frac{2}{D-2}\Lambda g_{uu}+8\pi T_{uu}$ with (11) thus simplifies to[2]

$$\left[(D-2)b_{,u}+\frac{1}{2}(D-2)(D-1)bc-De^n b_{,n}\right]r^{2-D}+\Delta b r^{1-D} +\Delta a r^{-2}+(D-4)e^n a_{,n}r^{-1}=16\pi[\mathcal{N}r^{2-D}-\mathcal{J}^p{}_{||p}r^{1-D}]. \tag{35}$$

Moreover, due to (28) the gyratonic matter functions $\mathcal{J}_p$ always obey the "divergence relation"

$$-16\pi\mathcal{J}^p{}_{||p}=\Delta b, \tag{36}$$

so that the r^{1-D} part of Eq. (35) is identically valid. Also, $(D-4)a_{,n}=0$ in any dimension $D \geq 4$; see Eqs. (23) and (20). Consequently, the field equation (35) reduces to

$$\left[(D-2)b_{,u}+\frac{1}{2}(D-2)(D-1)bc-De^n b_{,n}\right]r^{2-D} +\Delta a\, r^{-2}=16\pi\mathcal{N}r^{2-D}. \tag{37}$$

The factor Δa proportional to r^{-2} is always zero in any $D>4$ due to (26), while in the $D=4$ case it is combined with the terms proportional to $r^{2-D}=r^{-2}$. The last Einstein's field equation thus reads

$$(D-2)b_{,u}+\frac{1}{2}(D-2)(D-1)bc-De^n b_{,n}=16\pi\mathcal{N} \text{ for } D>4, \tag{38}$$

$$\Delta\left(\frac{1}{2}\mathcal{R}\right)+2b_{,u}+3bc-4e^n b_{,n}=16\pi\mathcal{N} \quad \text{for } D=4. \tag{39}$$

This is a complete and explicit solution for gyratons with aligned pure radiation in the Robinson-Trautman class of geometries (16) in four and any higher dimension D.

According to (28), specific properties of the corresponding gyraton are encoded in the metric function $b(u,x)$ and in the related off-diagonal functions $e_p(u,x)$. The gyratonic matter is absent when $\mathcal{J}_p=0$, which is equivalent to $b_{,p}=0$. In other words, there are no gyratons if (and only if) the function $b(u)$ is independent of any spatial coordinates.

[2]Recall that necessarily $f^p=0$; see (15).

IV. SUMMARY AND DISCUSSION

By fully integrating all Einstein's equations we explicitly proved that there are gyratons in the Robinson-Trautman class, as they are in the Kundt class. A null matter field in these geometries can thus have its "internal spin"/angular momentum.

A. Robinson-Trautman gyratons in $D \geq 4$

The most general D-dimensional ($D \geq 4$) Robinson-Trautman line element in vacuum, with a cosmological constant Λ, and possibly the pure radiation matter field with an additional gyratonic component, characterized by

$$T_{up}=\mathcal{J}_p r^{2-D}, \tag{40}$$

$$T_{uu}=\mathcal{N}r^{2-D}-\mathcal{J}^p{}_{||p}r^{1-D}, \tag{41}$$

can be written as

$$\mathrm{d}s^2=r^2 h_{pq}\mathrm{d}x^p\mathrm{d}x^q+2r^2 e_p \mathrm{d}u\mathrm{d}x^p-2\mathrm{d}u\mathrm{d}r+g_{uu}\mathrm{d}u^2, \tag{42}$$

where

$$g_{uu}=-\frac{\mathcal{R}}{(D-2)(D-3)}-\frac{b}{r^{D-3}} +\frac{2}{D-2}\left(e^n{}_{||n}-\frac{1}{2}h^{mn}h_{mn,u}\right)r +\left(\frac{2\Lambda}{(D-1)(D-2)}+e^n e_n\right)r^2, \tag{43}$$

with the functions $h_{pq}(u,x)$, $e_p(u,x)$, and $b(u,x)$ constrained by the field equations (18), (19), (24), and (37), that is

$$\mathcal{R}_{pq}=\frac{h_{pq}}{D-2}\mathcal{R}, \tag{44}$$

$$e_{(p||q)}-\frac{1}{2}h_{pq,u}=\frac{h_{pq}}{D-2}\left(e^n{}_{||n}-\frac{1}{2}h^{mn}h_{mn,u}\right), \tag{45}$$

$$-b_{,p}=16\pi\mathcal{J}_p, \tag{46}$$

$$\frac{\Delta\mathcal{R}}{(D-2)(D-3)}-(D-1)\left(e^n{}_{||n}-\frac{1}{2}h^{mn}h_{mn,u}\right)b +(D-2)b_{,u}-De^n b_{,n}=16\pi\mathcal{N}. \tag{47}$$

The first equation (44) restricts the Riemannian metric h_{pq} of the transverse $(D-2)$-dimensional space covered by

the coordinates x^p (with $\mathcal{R}_{pq}$ and $\mathcal{R}$ being its Ricci tensor and Ricci scalar). Any Einstein space metric h_{pq} is admitted. The second constraint (45) imposes a specific coupling between this spatial metric h_{pq} and the off-diagonal metric components represented by $(D-2)$ functions e^p.

Equation (46) directly expresses the gyratonic matter profile functions $\mathcal{J}_p(u,x)$ in (40) in terms of the spatial derivatives of $b(u,x)$ [recall also the relation (36) which enables us to express the function $\mathcal{J}^p{}_{||p}$ in (41) as $-\frac{1}{16\pi}\Delta b$], while Eq. (47) effectively relates these functions to the pure radiation profile $\mathcal{N}(u,x)$.

In particular, in any higher dimension $D>4$, the field equation (47) simplifies to (38), while in the usual $D=4$ case it takes the form (39). In the no-gyraton ($\mathcal{J}_p=0$) case, that is, for $b_{,p}=0$, Eq. (39) reduces exactly to the classical Robinson-Trautman equation [see [4,5] with the identification $a=\frac{1}{2}\mathcal{R}=\Delta(\log P)=K$, $b=-2m(u)$, $c=-2(\log P)_{,u}$, where K is the Gaussian curvature of the spatial metric $h_{pq}=P^{-2}\delta_{pq}$]. Equation (38) generalizes the field equation previously derived in [6] to admit the gyratonic matter in $D>4$.

Vacuum spacetimes are obtained when $\mathcal{J}_p=0=\mathcal{N}$. First of all, this arises when $b=0$ [and $\mathcal{R}$ is constant, which is true in any $D>4$ due to (23)].

B. Comparison to Robinson-Trautman gyratons in $D=3$

In our recent work [14], we integrated Einstein's field equations for a general three-dimensional Robinson-Trautman metric in vacuum, with a cosmological constant Λ, and possibly a pure radiation field and gyratons. The matter field takes the form

$$T_{ux}=\frac{\mathcal{J}}{r}, \tag{48}$$

$$T_{uu}=\frac{\mathcal{N}}{r}-\frac{P(P\mathcal{J})_{,x}}{r^2}+\frac{fP^2\mathcal{J}}{r^3}, \tag{49}$$

where $\mathcal{N}(u,x)$ and $\mathcal{J}(u,x)$ are functions determining the (density of) energy and angular momentum. The corresponding generic metric can be written in the form

$$\begin{aligned}\mathrm{d}s^2=&\frac{r^2}{P^2}\mathrm{d}x^2+2(er^2+f)\mathrm{d}u\mathrm{d}x-2\mathrm{d}u\mathrm{d}r\\&+(-a+2[P(Pe)_{,x}+(\ln P)_{,u}]r+(\Lambda+P^2e^2)r^2)\mathrm{d}u^2.\end{aligned} \tag{50}$$

The functions $P(u,x)$, $e(u,x)$, $f(u,x)$, and $a(u,x)$ are constrained just by two equations, namely,

$$a_{,x}=cf-2f_{,u}-16\pi\mathcal{J}, \tag{51}$$

$$\begin{aligned}a_{,u}=&ac+\Delta c+2(\Lambda+P^2e^2)P(Pf)_{,x}+3P^2f(P^2e^2)_{,x}\\&-2P^2fe_{,u}-P^2e(4f_{,u}-cf+48\pi\mathcal{J})+16\pi\mathcal{N},\end{aligned} \tag{52}$$

where $\Delta c\equiv P(Pc_{,x})_{,x}$ is the transverse-space Laplace operator applied on the function c, defined by $c\equiv 2[P(Pe)_{,x}+(\ln P)_{,u}]$.

Generically, by prescribing an arbitrary gyratonic function $\mathcal{J}$ (as well as *any* metric functions P, e, f) we can always integrate (51) to obtain $a(u,x)$. Subsequently, its partial derivative $a_{,u}$ (and other given functions) uniquely determines the pure radiation energy profile $\mathcal{N}$ via the field equation (52).

It is remarkable that in $D=3$ the function $f(u,x)$ in the metric (50) remains arbitrary and, in general, nonvanishing. This is an entirely new feature which does not occur in dimensions $D\geq 4$. Indeed, it was demonstrated in [6–8] that for the Robinson-Trautman class of spacetimes in four and any higher dimensions necessarily $f_p=0$ for all $(D-2)$ spatial components. In this sense, the $D=3$ case is surprisingly richer than the $D\geq 4$ cases.

In the specific subcase $f=0$, the metric (50) basically reduces to the form (42) and (43) (where, of course, $\mathcal{R}=0$) with the two remaining field equations (51) and (52) simplifying considerably to

$$a_{,x}=-16\pi\mathcal{J}, \tag{53}$$

$$a_{,u}=ac+\Delta c-48\pi P^2e\mathcal{J}+16\pi\mathcal{N}. \tag{54}$$

Since a here corresponds to b in (43), these two equations are very similar to Eqs. (46) and (47). The only difference is the additional term Δc in (54). In fact, it is not possible to set $D=3$ in (47) because in this number of dimensions the terms in (31) proportional to r^{2-D} and r^{-1} combine together, introducing thus the term Δc into the correct field equation (54).

C. Comparison to Kundt gyratons in $D\geq 3$

Finally, it is useful to compare the newly found complete class of Robinson-Trautman-type ($\Theta\neq 0$) gyratons in any dimension D with the most general gyratonic solutions in the closely related Kundt family ($\Theta=0$) of spacetimes, completing thus the derivation of all solutions with aligned gyratonic matter in any nontwisting and shear-free geometry.

We obtain the most general Kundt gyratons by a direct integration of the field equations, using the explicit form of the Ricci tensor components which we presented in [23]. By setting $\Theta=0$, they simplify considerably. First, from the geometric relation $g_{pq,r}=2\Theta g_{pq}$ we immediately obtain $g_{pq}=h_{pq}(u,x)$ independent of r, instead of (7) in the Robinson-Trautman case. The second field equation $R_{rp}=0$ for $\Theta=0$ yields $g_{up}=e_p+f_pr$, so that $g^{rp}=e^p+f^pr$ (recall that $e^p\equiv h^{pq}e_q$, $f^p\equiv h^{pq}f_q$). The gyratonic/pure radiation matter field is then obtained by integrating (4) and (5) as

$$T_{up} = \mathcal{J}_p, \tag{55}$$

$$T_{uu} = \mathcal{N} + (\mathcal{J}^p{}_{||p} + f^p \mathcal{J}_p) r, \tag{56}$$

where $\mathcal{J}_p$ and $\mathcal{N}$ are arbitrary functions of u and x. The Einstein's equation $R_{ru} = -\frac{2}{D-2}\Lambda$ gives $g_{uu} = ar^2 + br + c$, with[3]

$$a = \frac{2\Lambda}{D-2} + \frac{1}{2}(f^p{}_{||p} + f^p f_p), \tag{57}$$

so that the Kundt metric takes the form

$$\begin{aligned} \mathrm{d}s^2 &= h_{pq}\mathrm{d}x^p\mathrm{d}x^q + 2(e_p + f_p r)\mathrm{d}u\mathrm{d}x^p - 2\mathrm{d}u\mathrm{d}r \\ &\quad + (ar^2 + br + c)\mathrm{d}u^2. \end{aligned} \tag{58}$$

The next field equation $R_{pq} = \frac{2}{D-2}\Lambda g_{pq}$ yields just one constraint, namely,

$$\mathcal{R}_{pq} = \frac{2\Lambda}{D-2} h_{pq} + f_{pq}, \quad \text{where } f_{pq} \equiv f_{(p||q)} + \frac{1}{2} f_p f_q. \tag{59}$$

It couples the Ricci curvature $\mathcal{R}_{pq}$ of the $(D-2)$-dimensional spatial metric h_{pq} to the tensor f_{pq} constructed from the functions f_p determining the metric components g_{up}. The trace of (59) is $\mathcal{R} = 2\Lambda + f^p{}_{||p} + \frac{1}{2} f^p f_p$, which enables us to rewrite a as

$$a = \frac{1}{2}\mathcal{R} - \frac{D-4}{D-2}\Lambda + \frac{1}{4} f^p f_p. \tag{60}$$

Evaluating the field equation $R_{up} = \frac{2}{D-2}\Lambda g_{up} + 8\pi T_{up}$, we obtain the following two conditions:

$$\begin{aligned} &a_{,p} + \frac{1}{2} f_p (f^n{}_{||n} + f^n f_n) - 2 f^n f_{[n,p]} - h^{mn} f_{[m,p]||n} \\ &\quad + \frac{2\Lambda}{D-2} f_p = 0, \end{aligned} \tag{61}$$

$$\begin{aligned} &b_{,p} - f_{p,u} - e^n (f_{n||p} - 2 f_{p||n} - f_p f_n) \\ &\quad + f_p \left(e^n{}_{||n} - \frac{1}{2} h^{mn} h_{mn,u} \right) \\ &\quad - f^n e_{n||p} - 2 h^{mn} (h_{m[p,u||n]} + e_{[m,p]||n}) \\ &\quad + \frac{4\Lambda}{D-2} e_p = -16\pi \mathcal{J}_p. \end{aligned} \tag{62}$$

Effectively, they determine the spatial derivatives of the metric functions a and b, respectively. The last Einstein equation $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi T_{uu}$ contains terms proportional to r^2, r^1, and r^0. Separately, they form three constraints, namely,

$$\Delta a + f^n{}_{||n} a + 3 f^n a_{,n} + 2 f^n f_n a - 2 h^{mn} h^{pq} f_{[p,m]} f_{[q,n]} = 0, \tag{63}$$

$$\begin{aligned} &\Delta b + f^n b_{,n} + 4 e^n a_{,n} + 2a \left(e^n{}_{||n} - \frac{1}{2} h^{mn} h_{mn,u} \right) \\ &\quad + 4 f^n e_n a - 2 f^n f_{n,u} - 4 f^n e^m f_{[n,m]} \\ &\quad - 2 h^{mn} f_{m,u||n} - 2 h^{mn} h^{pq} f_{[p,m]} (2 e_{[q,n]} + h_{qn,u}) \\ &\quad = -16\pi (\mathcal{J}^p{}_{||p} + f^p \mathcal{J}_p), \end{aligned} \tag{64}$$

$$\begin{aligned} &\Delta c - f^n{}_{||n} c - f^n c_{,n} + 2 e^n b_{,n} + b \left(e^n{}_{||n} - \frac{1}{2} h^{mn} h_{mn,u} \right) \\ &\quad + h^{mn} h_{mn,uu} + 2 e^n e_n a - e^n e_n f^m f_m + e^n f_n e^m f_m \\ &\quad - 2 e^n f_{n,u} - 4 f^n e^m e_{[n,m]} - 2 h^{mn} e_{m,u||n} \\ &\quad - 2 h^{mn} h^{pq} \left(e_{[p,m]} + \frac{1}{2} h_{pm,u} \right) \left(e_{[q,n]} + \frac{1}{2} h_{qn,u} \right) \\ &\quad = -16\pi \mathcal{N}. \end{aligned} \tag{65}$$

Surprisingly, a lengthy calculation [using (57) and (59), standard properties of covariant derivatives, the identity (A.15) from [23], and also the Bianchi identities] reveals that Eqs. (63) and (64) are, in fact, identically satisfied as a consequence of previous Eqs. (61) and (62) (As shown in [24], see footnote 8, the same is true for the Kundt spacetimes with aligned electromagnetic field.). We thus conclude that the most general Kundt metric with aligned gyratonic matter can be written in the form (58) with (59), in which the metric function a given by (60) is constrained by (61), the function b is determined by (62), and the function c satisfies Eq. (65). The particular subcase $D = 3$ is presented and discussed in more detail in [14].

There is a great simplification in the case when $f_p = 0$ for all p. In fact, it was shown in our previous work [9] that this is a geometrically distinct subclass of the Kundt class. The complete family of such gyratonic solutions reads

$$\begin{aligned} \mathrm{d}s^2 &= h_{pq}\mathrm{d}x^p\mathrm{d}x^q + 2 e_p \mathrm{d}u\mathrm{d}x^p - 2\mathrm{d}u\mathrm{d}r \\ &\quad + \left(\frac{2\Lambda}{D-2} r^2 + br + c \right) \mathrm{d}u^2, \end{aligned} \tag{66}$$

where, as in the Robinson-Trautman case [cf. (18)], h_{pq} is the spatial metric of any Einstein space,

$$\mathcal{R}_{pq} = \frac{2\Lambda}{D-2} h_{pq}, \qquad \mathcal{R} = 2\Lambda, \tag{67}$$

[3]The meanings of a, b, c, e_p, and f_p are here, of course, different from those in the Robinson-Trautman case.

Eq. (61) is satisfied identically, and Eqs. (62) and (65) for the functions b, c reduce to

$$b_{,p} - 2h^{mn}(h_{m[p,u||n]} + e_{[m,p]||n}) + \frac{4\Lambda}{D-2}e_p = -16\pi\mathcal{J}_p, \tag{68}$$

$$\begin{aligned}&\Delta c + 2e^n b_{,n} + b\left(e^n{}_{||n} - \frac{1}{2}h^{mn}h_{mn,u}\right) + h^{mn}h_{mn,uu} \\ &+ \frac{4\Lambda}{D-2}e^n e_n - 2h^{mn}e_{m,u||n} \\ &- 2h^{mn}h^{pq}\left(e_{[p,m]} + \frac{1}{2}h_{pm,u}\right)\left(e_{[q,n]} + \frac{1}{2}h_{qn,u}\right) = -16\pi\mathcal{N},\end{aligned} \tag{69}$$

respectively. Equation (68) relating $b_{,p}$ to $\mathcal{J}_p$ is similar to Eq. (24) in the Robinson-Trautman case, while Eq. (69) relates the metric function c to $\mathcal{N}$. The corresponding gyratonic matter takes the form

$$T_{up} = \mathcal{J}_p, \tag{70}$$

$$T_{uu} = \mathcal{N} + \mathcal{J}^p{}_{||p}r. \tag{71}$$

In fact, this $f_p = 0$ subclass of Kundt spacetimes (66)–(71) contains *all* particular gyratonic solutions discussed in the literature so far; see [21,22] for a review and a list of references.

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[1] I. Robinson and A. Trautman, Spherical Gravitational Waves, Phys. Rev. Lett. **4**, 431 (1960).
[2] I. Robinson and A. Trautman, Some spherical gravitational waves in general relativity, Proc. R. Soc. A **265**, 463 (1962).
[3] W. Kundt, The plane-fronted gravitational waves, Z. Phys. **163**, 77 (1961).
[4] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[5] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).
[6] J. Podolský and M. Ortaggio, Robinson–Trautman spacetimes in higher dimensions, Classical Quantum Gravity **23**, 5785 (2006).
[7] M. Ortaggio, J. Podolský, and M. Žofka, Robinson–Trautman spacetimes with an electromagnetic field in higher dimensions, Classical Quantum Gravity **25**, 025006 (2008).
[8] M. Ortaggio, J. Podolský, and M. Žofka, Static and radiating p-form black holes in the higher dimensional Robinson–Trautman class, J. High Energy Phys. 02 (2015) 045.
[9] J. Podolský and M. Žofka, General Kundt spacetimes in higher dimensions, Classical Quantum Gravity **26**, 105008 (2009).
[10] A. Coley, A. Fuster, S. Hervik, and N. Pelavas, Higher dimensional VSI spacetimes, Classical Quantum Gravity **23**, 7431 (2006).
[11] A. Coley, S. Hervik, G. Papadopoulos, and N. Pelavas, Kundt spacetimes, Classical Quantum Gravity **26**, 105016 (2009).
[12] M. Ortaggio, V. Pravda, and A. Pravdová, Algebraic classification of higher dimensional spacetimes based on null alignment, Classical Quantum Gravity **30**, 013001 (2013).
[13] J. Podolský and R. Švarc, Algebraic structure of Robinson–Trautman and Kundt geometries in arbitrary dimension, Classical Quantum Gravity **32**, 015001 (2015).
[14] J. Podolský, R. Švarc, and H. Maeda, All solutions of Einstein's equations in $2+1$ dimensions: Λ-vacuum, pure radiation, or gyratons, Classical Quantum Gravity **36**, 015009 (2019).
[15] W. B. Bonnor, Spinning null fluid in general relativity, Int. J. Theor. Phys. **3**, 257 (1970).
[16] J. B. Griffiths, Some physical properties of neutrino-gravitational fields, Int. J. Theor. Phys. **5**, 141 (1972).
[17] V. P. Frolov and D. V. Fursaev, Gravitational field of a spinning radiation beam pulse in higher dimensions, Phys. Rev. D **71**, 104034 (2005).
[18] V. P. Frolov, W. Israel, and A. Zelnikov, Gravitational field of relativistic gyratons, Phys. Rev. D **72**, 084031 (2005).
[19] V. P. Frolov and A. Zelnikov, Relativistic gyratons in asymptotically AdS spacetime, Phys. Rev. D **72**, 104005 (2005).
[20] V. P. Frolov and A. Zelnikov, Gravitational field of charged gyratons, Classical Quantum Gravity **23**, 2119 (2006).

[21] P. Krtouš, J. Podolský, A. Zelnikov, and H. Kadlecová, Higher-dimensional Kundt waves and gyratons, Phys. Rev. D **86**, 044039 (2012).

[22] J. Podolský, R. Steinbauer, and R. Švarc, Gyratonic *pp*-waves and their impulsive limit, Phys. Rev. D **90**, 044050 (2014).

[23] R. Švarc and J. Podolský, Absence of gyratons in the Robinson–Trautman class, Phys. Rev. D **89**, 124029 (2014).

[24] M. Ortaggio and V. Pravda, Electromagnetic fields with vanishing scalar invariants, Classical Quantum Gravity **33**, 115010 (2016).