Classical and Quantum Gravity

PAPER • OPEN ACCESS

Algebraic classification of 2+1 geometries: a new approach

To cite this article: Matúš Papajčík and Jiří Podolský 2024 *Class. Quantum Grav.* **41** 115008

OPEN ACCESS
IOP Publishing

Classical and Quantum Gravity

Class. Quantum Grav. **41** (2024) 115008 (39pp)

https://doi.org/10.1088/1361-6382/ad3ffd

Algebraic classification of 2+1 geometries: a new approach

Matúš Papajčík and Jiří Podolský*

Charles University, Faculty of Mathematics and Physics, Institute of Theoretical Physics, V Holešovičkách 2, 18000 Prague 8, Czech Republic

E-mail: podolsky@mbox.troja.mff.cuni.cz and matus.papajcik@matfyz.cuni.cz

Received 25 November 2023; revised 6 April 2024
Accepted for publication 17 April 2024
Published 7 May 2024

Abstract

We present a convenient method of algebraic classification of 2+1 spacetimes into the types I, II, D, III, N and O, without using any field equations. It is based on the 2+1 analogue of the Newman–Penrose curvature scalars Ψ_{A} of distinct boost weights, which are specific projections of the Cotton tensor onto a suitable null triad. The algebraic types are then simply determined by the gradual vanishing of such Cotton scalars, starting with those of the highest boost weight. This classification is directly related to the specific multiplicity of the Cotton-aligned null directions and to the corresponding Bel–Debever criteria. Using a bivector (that is 2-form) decomposition, we demonstrate that our method is fully equivalent to the usual Petrov-type classification of 2+1 spacetimes based on the eigenvalue problem and determining the respective canonical Jordan form of the Cotton–York tensor. We also derive a simple synoptic algorithm of algebraic classification based on the key polynomial curvature invariants. To show the practical usefulness of our approach, we perform the classification of several explicit examples, namely the general class of Robinson–Trautman spacetimes with an aligned electromagnetic field and a cosmological constant, and other metrics of various algebraic types.

Keywords: algebraic classification, 3D Lorentzian manifolds, Cotton tensor, Cotton–York tensor, Newman–Penrose scalars, Bel–Debever criteria, multiplicity of the Cotton-aligned null directions

* Author to whom any correspondence should be addressed.

Contents

1. Introduction

Algebraic classification of spacetimes is an important tool for investigation and understanding of exact solutions of Einstein's field equations and other theories of gravity. In the context of $D=4$ *general relativity* (that is for $3+1$ geometries) this was developed at the end of the 1950s by Petrov, Géhéniau, Pirani, Bell, Debever and Penrose [1–7] using various equivalent approaches. In its most convenient formulation, related to the study of gravitational radiation (spacetimes of type N) and also stationary black holes (of type D), this is based on finding the multiplicity of four possible principal null directions (PNDs) of the Weyl curvature tensor, encoded in its null-frame components which are denoted as the complex Newman–Penrose scalars Ψ_A, where $A=0,1,2,3,4$, see [8]. Comprehensive reviews of this topic can be found in the monographs [9, 10].

In 2004, this key concept of algebraic classification was extended to *higher dimensions* $D>4$ by Coley *et al* [11, 12]. In such a case, there are many more components of the Weyl tensor, but all their null-frame projections can again be sorted into just five groups with distinct boost weights. This fact enables one to perform the classification of the Weyl tensor in an analogous way as in the $D = 4$ case, i.e. by the multiplicity of four Weyl-aligned null directions (WANDs), see the reviews [13, 14]. To keep the closest possible analogy with the standard Newman–Penrose formalism, Krtouš and Podolský [15] introduced the familiar notation $\Psi_{\rm A}$ to represent all the relevant *real* Weyl scalars in any $D > 4$.

In fact, it should be emphasized that the classification scheme developed in [12] applies to *any tensor* in *arbitrary Lorentzian geometry*. Although not explicitly mentioned in this seminal work, it can be immediately observed that the scheme is valid also in the lower-dimensional case $D = 3$ admitting two independent null directions and just one additional spatial direction.

From this general point of view, our classification method is an application of the scheme presented in [12, 14] to 2+1 Lorentzian geometries in which we take the rank-3 *Cotton tensor* [16] (instead of the *identically vanishing* rank-4 Weyl tensor) as the key geometric quantity. The Cotton algebraic types correspond to the general classification into (primary) *principal* and *secondary alignment types* (PAT and SAT), as introduced for an arbitrary tensor by definitions 4.1 and 4.2 in [12], and 2.5 in [14]).

Classification of spacetimes in lower dimension $D = 3$ was introduced many years ago. Neither the Petrov approach (based on the eigenvalue problem of the Weyl tensor) nor the Debever–Penrose analysis (based on the multiplicity of the Weyl tensor PNDs) could be directly applied because in 2+1 geometries the rank-4 Weyl tensor vanishes. Instead, it was found that the fundamental role for the algebraic classification plays the rank-3 Cotton tensor. The number of its independent components in 2+1 gravity is *five*, so that it can be mapped onto the rank-2 symmetric and traceless *Cotton–York tensor*. This tensor can be represented by a 3×3 matrix, and thus its algebraic classification can be performed analogously to the original Petrov approach. This was done in 1986 by Barrow *et al* [17].

Such a classification in 2+1 gravity is, nevertheless, different from its $D = 4$ counterpart. In the *actual* formulation of the eigenvalue problem, the symmetry of the Cotton–York tensor is no longer manifest. The eigenvalues and also the corresponding eigenvectors can thus generally be complex. This feature was pointed out and remedied by García, Hehl, Heinicke and Macías in 2004. In their paper [18], it was proposed to classify the spacetimes according to the possible *Jordan forms* of the Cotton–York tensor in a suitable orthonormal basis. By this method, the spacetimes were divided into the types I, II, D, III, N and O. To deal with the possible complex eigenvalues, an additional type I′ was proposed which restricts the solutions to only real numbers.

Alternative approaches to classification of 2+1 spacetimes were also presented. The formalism of *null basis* was developed in [19], while in [20] a *spinor algebra* was established and used for the Ricci and Cotton–York tensors. An invariant *Karlhede classification* method was developed in [21] employing the Ricci and Cotton–York real spinors. Interestingly, in *topologically massive gravity* (TMG), whose action involves a gravitational Chern–Simons term, the field equations imply that the Cotton–York tensor is proportional to the traceless Ricci tensor. Therefore, the Petrov-type classification of 2+1 spacetimes in TMG is equivalent to the Segre classification of the simpler traceless Ricci tensor, see [22, 23].

Actually, the Segre–Pirani–Plebański classification of the *energy-momentum tensor* of matter, related to the traceless Ricci tensor, is another important way of characterizing the spacetime. It takes advantage of its symmetry property, so that the eigenvalues and eigenvectors can be directly determined by a standard procedure, and classified using the nomenclature of Plebański [24]. More details on these schemes, and their application to many important classes of exact solutions to 2+1 gravity, are given in the monograph [25], see in particular sections 1.2 and 20.5 therein.

In our work, we now propose a *simpler and general method* of algebraic classification of spacetimes in 2+1 gravity which does not assume any field equations. It is based *directly* on the Cotton tensor, namely on *five Cotton scalars* Ψ_{A} obtained by specific projections onto a null triad. In fact, this is a lower-dimensional analogue of the standard Newman–Penrose method of $D=4$ general relativity which uses the Weyl tensor. It is naturally related to the multiplicity of the Cotton-aligned null directions (CANDs), in full analogy to the multiplicity of PNDs and WANDs. We show that this approach is equivalent to the classification developed in [18] which relies on the canonical Jordan forms of the Cotton–York tensor. We also identify key scalar polynomial invariants constructed from the Cotton scalars Ψ_{A}, which conveniently assist with the algebraic classification.

We begin in section 2 by establishing the notation and introducing the Cotton tensor C_{abc}. In subsequent section 3 we define a null triad onto which the Cotton tensor is projected, obtaining thus the key Newman–Penrose-type Cotton scalars Ψ_{A}. This allows us to present a very simple classification scheme in section 4. Then in section 5 we define a bivector basis and prove that the corresponding components of the Cotton tensor are just the scalars Ψ_{A}. Relation to the Bel–Debever criteria for the privileged aligned null vector $\boldsymbol{k}$ is demonstrated in section 6. All Lorentz transformations are investigated in section 7, in particular their effect on the key Cotton scalars Ψ_{A}. It is then demonstrated that a suitable null rotation can always be performed in which $\Psi_0=0$, identifying thus the principle null triad and the CAND, see section 8. In fact, as shown in section 9, the specific multiplicities of CANDs $\boldsymbol{k}$ uniquely determine the algebraic types of spacetimes. In section 10 we present the related symmetric traceless Cotton–York tensor, and we write it in terms of the Cotton scalars Ψ_{A}. Expressing it in the orthonormal basis, in section 11 we are able to prove a full equivalence with the previous method of classification of 2+1 geometries based on the eigenvalues and the canonical Jordan forms of the Cotton–York tensor. In section 12 we investigate scalar curvature polynomial invariants constructed from the Cotton and Cotton–York tensors, and their relation to various algebraic types. In fact, we derive a simple practical classification algorithm based on these invariants. Section 13 introduces the refinement to subtypes $\mathrm{I_r}$, $\mathrm{II_r}$, $\mathrm{D_r}$ for which all four (possibly multiple) CANDs are real, and subtypes $\mathrm{I_c}$, $\mathrm{II_c}$, $\mathrm{D_c}$ for which some of the CANDs are complex. This is indirectly related to complex eigenvalues of the Cotton–York tensor. In final section 14, we explicitly apply this procedure on an interesting class of Robinson–Trautman spacetimes with a cosmological constant and an electromagnetic field, demonstrating that it is algebraically general (of type I), but with only Ψ_1 and Ψ_3 scalars non-vanishing. Similarly, we analyze several other examples of metrics of various algebraic types and subtypes.

2. Cotton tensor

Let $(\mathcal{M},\mathbf{g})$ be a general three-dimensional Lorentzian manifold with the metric signature $(-,+,+)$. On such a manifold, at any point we construct the *basis* of the tangent space consisting of three vectors $\boldsymbol{e}_a$, and the cotangent space *dual basis* given by three 1-forms $\boldsymbol{\omega}^a$. In local coordinates x^α, these are

$$\boldsymbol{e}_a = e_a^\alpha\,\partial_\alpha\,, \qquad \boldsymbol{\omega}^a = e^a_\alpha\,\mathrm{d}x^\alpha\,. \tag{1}$$

By the Latin letters $a,b,\ldots$ we denote the frame (anholonomic) indices, while by the Greek letters $\alpha,\beta,\ldots$ we denote the coordinate (holonomic) indices. In terms of the dual basis, the line element corresponding to the *metric* g_{ab} is

$$\mathrm{d}s^2 = g_{ab}\,\boldsymbol{\omega}^a\boldsymbol{\omega}^b\,. \tag{2}$$

We also assume that the manifold is equipped with the symmetric Levi-Civita connection ∇.

The role of the key geometrical object in 2+1 spaces plays the (conformally invariant) *Cotton tensor*, first investigated by Cotton [16] already in 1899, and later by Schouten [26]. It is the best analogue for the Weyl tensor which identically vanishes in 2+1 geometries. The Cotton tensor is defined as

$$C_{abc} \equiv 2\left(\nabla_{[a}R_{b]c} - \frac{1}{4}\nabla_{[a}R\,g_{b]c}\right), \tag{3}$$

where R_{ab} is the *Ricci tensor* of the metric g_{ab}, see equation (20.39) in [25]. From the definition (3) it follows that the Cotton tensor is *antisymmetric in the first two indices*[1],

$$C_{abc} = -C_{bac}\,, \tag{4}$$

and that it also satisfies the constraints

$$C_{[abc]} = 0\,, \tag{5}$$

$$C_{ab}{}^{a} = 0\,. \tag{6}$$

For a detailed exposition of the Cotton tensor see [18] or chapter 20 in [25]. These constraints restrict the Cotton tensor in 2+1 geometries to have *only 5 independent components*. Indeed, due to (4), the Cotton tensor has $3\times 3 = 9$ independent components which are constrained by 1 condition (5) and 3 independent conditions (6).

3. Null triad and the Cotton scalars $\Psi_{\rm A}$

The next step is to project the Cotton tensor onto a suitable basis on the tangent space. We choose the *null triad* $\{\boldsymbol{e}_a\} \equiv \{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$, such that $\boldsymbol{k}\cdot\boldsymbol{k} = 0 = \boldsymbol{l}\cdot\boldsymbol{l}$, $\boldsymbol{k}\cdot\boldsymbol{m} = 0 = \boldsymbol{l}\cdot\boldsymbol{m}$, and

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1\,, \qquad \boldsymbol{m}\cdot\boldsymbol{m} = 1\,, \tag{7}$$

or written explicitly in the components

$$k_a\, l^a = -1\,, \qquad m_a\, m^a = 1\,. \tag{8}$$

It means that both $\boldsymbol{k}$ and $\boldsymbol{l}$ are *null vectors* (future-oriented and mutually normalized to -1), while $\boldsymbol{m}$ is the *spatial unit vector* orthogonal to $\boldsymbol{k}$ and $\boldsymbol{l}$.

A dual basis $\{\boldsymbol{\omega}^b\}$ is given by the relation $e_a^\alpha\,\omega_\alpha^b = \delta_a^b$. In view of the scalar products (7), such a dual basis can be written as $\{\boldsymbol{\omega}^b\} \equiv \{-\boldsymbol{l}, -\boldsymbol{k}, \boldsymbol{m}\}$. By this notation we mean that the dual to the vector $\boldsymbol{e}_1 = \boldsymbol{k} = k^\alpha\,\partial_\alpha$ is the 1-form $\boldsymbol{\omega}^1 = -l_\alpha\,\mathrm{d}x^\alpha$, and similarly for the remaining two basis vectors.

Now we define the *Newman–Penrose-type* curvature *Cotton scalars* $\Psi_{\rm A}$ as

$$\begin{aligned}
\Psi_0 &\equiv C_{abc}\, k^a\, m^b\, k^c\,,\\
\Psi_1 &\equiv C_{abc}\, k^a\, l^b\, k^c\,,\\
\Psi_2 &\equiv C_{abc}\, k^a\, m^b\, l^c\,,\\
\Psi_3 &\equiv C_{abc}\, l^a\, k^b\, l^c\,,\\
\Psi_4 &\equiv C_{abc}\, l^a\, m^b\, l^c\,.
\end{aligned} \tag{9}$$

[1] Unfortunately, in mathematical literature a different convention is also used for the position of the antisymmetric indices, as for example in equation (3.89) in [10].

Table 1. The algebraic classification of 2+1 geometries.

Algebraic type	The conditions	
I	$\Psi_0 = 0\,,$	$\Psi_1 \neq 0$
II	$\Psi_0 = \Psi_1 = 0\,,$	$\Psi_2 \neq 0$
III	$\Psi_0 = \Psi_1 = \Psi_2 = 0\,,$	$\Psi_3 \neq 0$
N	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,,$	$\Psi_4 \neq 0$
D	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4\,,$	$\Psi_2 \neq 0$
O	all $\Psi_{\rm A} = 0$	

These are fully analogous to standard definition of the Newman–Penrose scalars constructed from the *Weyl* curvature tensor in $D = 4$ (see [10]) and in any $D > 4$ (see [15], equivalent to [11–14]). Notice that these scalars are *real*, and completely represent the *5 independent components* of the Cotton tensor.

4. Algebraic classification based on the $\Psi_{\rm A}$ scalars

We propose that the algebraic classification of 2+1 geometries can easily be made by using these curvature scalars $\Psi_{\rm A}$, which are the components of the Cotton tensor with respect to the null triad, defined in (9). The specific algebraic types are given by simple conditions, namely that *in a suitable triad* $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ the *specific Cotton scalars vanish*, as summarized in table 1.

In fact, this is a direct analogue of the Petrov–Penrose algebraic classification in standard general relativity based on the multiplicity of the PNDs of the Weyl tensor (see section 4.3 in [10] for the review), or of PAT/SAT and the multiplicity of the WANDs in higher dimensions (see [14]).

To justify the definition of algebraic types presented in table 1 and to demonstrate that it is equivalent to the previous definition based on the Jordan forms of the Cotton–York tensor, it is now necessary to introduce a convenient bivector basis of 2-forms, which effectively represent the first two (antisymmetric) indices of the Cotton tensor (3).

5. Cotton tensor in the bivector basis

The space of all *2-forms* (also called *bivectors*) in 2+1 geometries has dimension 3, and we now construct a basis

$$\left\{\boldsymbol{Z}^I\right\} = \{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\}\,, \tag{10}$$

where $I = 1, 2, 3$, to express them. In particular, employing the null triad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ normalized as (7), we define these base 2-forms as the wedge products

$$\begin{aligned} \boldsymbol{U} &\equiv 2\boldsymbol{m}\wedge\boldsymbol{l}\,, \\ \boldsymbol{V} &\equiv 2\boldsymbol{k}\wedge\boldsymbol{m}\,, \\ \boldsymbol{W} &\equiv 2\boldsymbol{l}\wedge\boldsymbol{k}\,. \end{aligned} \tag{11}$$

More explicitly, in the null triad frame such a bivector basis is $\{Z^I_{ab}\} = \{U_{ab}\,, V_{ab}, W_{ab}\}$, where

$$
\begin{aligned}
U_{ab} &= m_a\, l_b - l_a\, m_b\,,\\
V_{ab} &= k_a\, m_b - m_a\, k_b\,,\\
W_{ab} &= l_a\, k_b - k_a\, l_b\,.
\end{aligned}
\tag{12}
$$

It is the analogous definition as in $D=4$, see equation (3.40) in [10]. A direct calculation using (8) reveals that these bivectors satisfy the normalization relations

$$
U_{ab}\, V^{ab} = 2\,, \qquad W_{ab}\, W^{ab} = -2\,,
\tag{13}
$$

while all other contractions are zero.

The rank-3 Cotton tensor, antisymmetric in the first two indices, can be expressed in the basis given by (all combinations of) the tensor product of a basis bivector $\boldsymbol{Z}^I$ and a 1-form $\boldsymbol{\omega}^J$, that is

$$
C_{abc} = \sum_{I,J=1}^{3} C_{IJ}\, Z^I_{ab}\, \omega^J_c\,,
\tag{14}
$$

where C_{IJ} are the corresponding components[2]. Written explicitly, it has nine terms,

$$
\begin{aligned}
C_{abc} = &- C_{11}\, U_{ab}\, l_c - C_{12}\, U_{ab}\, k_c + C_{13}\, U_{ab}\, m_c\\
&- C_{21}\, V_{ab}\, l_c - C_{22}\, V_{ab}\, k_c + C_{23}\, V_{ab}\, m_c\\
&- C_{31}\, W_{ab}\, l_c - C_{32}\, W_{ab}\, k_c + C_{33}\, W_{ab}\, m_c\,.
\end{aligned}
\tag{15}
$$

Since the bivectors U_{ab}, V_{ab}, W_{ab} are antisymmetric, the condition (4) is trivially satisfied. Now we employ the vanishing trace condition (6). Using the relations

$$
\begin{aligned}
&U_{ab}\, l^a = 0\,, && U_{ab}\, k^a = m_b\,, && U_{ab}\, m^a = l_b\,,\\
&V_{ab}\, l^a = -m_b\,, && V_{ab}\, k^a = 0\,, && V_{ab}\, m^a = -k_b\,,\\
&W_{ab}\, l^a = l_b\,, && W_{ab}\, k^a = -k_b\,, && W_{ab}\, m^a = 0\,,
\end{aligned}
\tag{16}
$$

we obtain the constraint

$$
(C_{13} - C_{31})\, l_b + (C_{32} - C_{23})\, k_b + (C_{21} - C_{12})\, m_b = 0\,.
\tag{17}
$$

This 1-form must be identically zero, and so we obtain three conditions for the components, namely

$$
C_{13} = C_{31}\,, \qquad C_{23} = C_{32}\,, \qquad C_{12} = C_{21}\,.
\tag{18}
$$

The Cotton tensor thus can be written in the form

$$
\begin{aligned}
C_{abc} = &- C_{11}\, U_{ab}\, l_c - C_{12}\, (U_{ab}\, k_c + V_{ab}\, l_c) + C_{13}\, (U_{ab}\, m_c - W_{ab}\, l_c)\\
&- C_{22}\, V_{ab}\, k_c + C_{23}\, (V_{ab}\, m_c - W_{ab}\, k_c) + C_{33}\, W_{ab}\, m_c\,.
\end{aligned}
\tag{19}
$$

Finally, we have to apply the remaining condition (5). It is helpful first to calculate that

$$
3!\, \left(U_{[ab}\, k_{c]} + V_{[ab}\, l_{c]}\right) = 4\, \left(U_{ab}\, k_c + V_{ab}\, l_c + W_{ab}\, m_c\right)\,,
$$

[2] Here $Z^I_{ab}\, \omega^J_c$ is a shorthand for $Z^I_{ab} \otimes \omega^J_c$.

$$3!\,W_{[ab}\,m_{c]} = 2\,(U_{ab}\,k_c + V_{ab}\,l_c + W_{ab}\,m_c)\,, \tag{20}$$

where the factorial was included just to compensate the factor in the definition of the antisymmetrization. All other terms of the tensor-product basis are trivially zero under the complete antisymmetrization, namely

$$\begin{aligned} U_{[ab}\,l_{c]} &= 0\,, & U_{[ab}\,m_{c]} &= 0\,,\\ V_{[ab}\,k_{c]} &= 0\,, & V_{[ab}\,m_{c]} &= 0\,,\\ W_{[ab}\,k_{c]} &= 0\,, & W_{[ab}\,l_{c]} &= 0\,. \end{aligned} \tag{21}$$

From the condition (5) for (19) we now obtain

$$(-2C_{12} + C_{33})\,(U_{ab}\,k_c + V_{ab}\,l_c + W_{ab}\,m_c) = 0\,, \tag{22}$$

which implies the last constraint

$$C_{33} = 2\,C_{12}\,. \tag{23}$$

The *generic Cotton tensor* in the bivector-null basis thus takes the form

$$\begin{aligned} C_{abc} = &-C_{11}\,U_{ab}\,l_c - C_{12}\,(U_{ab}\,k_c + V_{ab}\,l_c - 2W_{ab}\,m_c)\\ &+ C_{13}\,(U_{ab}\,m_c - W_{ab}\,l_c) - C_{22}\,V_{ab}\,k_c + C_{23}\,(V_{ab}\,m_c - W_{ab}\,k_c)\,. \end{aligned} \tag{24}$$

It has five independent components, namely $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}$. They can be *uniquely* expressed in terms of the Newman–Penrose-type curvature *Cotton scalars* Ψ_{A} defined in (9). Indeed, using the normalization relations (8) and (13), the coefficients in (24) can be expressed as

$$\begin{aligned} C_{11} &= \tfrac{1}{2}C_{abc}\,V^{ab}\,k^c\,,\\ C_{12} &= \tfrac{1}{2}C_{abc}\,V^{ab}\,l^c \;= \tfrac{1}{2}C_{abc}\,U^{ab}\,k^c \;= -\tfrac{1}{4}C_{abc}\,W^{ab}\,m^c\,,\\ C_{13} &= \tfrac{1}{2}C_{abc}\,V^{ab}\,m^c = -\tfrac{1}{2}C_{abc}\,W^{ab}\,k^c\,,\\ C_{22} &= \tfrac{1}{2}C_{abc}\,U^{ab}\,l^c\,,\\ C_{23} &= \tfrac{1}{2}C_{abc}\,U^{ab}\,m^c = -\tfrac{1}{2}C_{abc}\,W^{ab}\,l^c\,. \end{aligned} \tag{25}$$

After explicitly putting the bivectors (12) into the first terms on the right-hand side of (25), using the definition (9) of the scalars Ψ_{A} and the antisymmetry of the Cotton tensor (4), we arrive at a very simple expressions for the five independent components of the Cotton tensor, namely

$$C_{11} = \Psi_0\,, \quad C_{12} = \Psi_2\,, \quad C_{13} = \Psi_1\,, \quad C_{22} = -\Psi_4\,, \quad C_{23} = -\Psi_3\,. \tag{26}$$

Moreover, the four other basis components of the Cotton tensor in the expansion (14) are *not* independent because from the remaining four expressions on the right-hand side of (25) we get

$$\begin{aligned} C_{12} &= C_{abc}\,m^a\,l^b\,k^c = \tfrac{1}{2}C_{abc}\,k^a\,l^b\,m^c\,,\\ C_{13} &= C_{abc}\,k^a\,m^b\,m^c\,,\\ C_{23} &= C_{abc}\,m^a\,l^b\,m^c\,. \end{aligned} \tag{27}$$

Using (26) we can thus write that

$$\Psi_1 = C_{abc}\,k^a\,m^b\,m^c\,,$$

$$\Psi_2 = C_{abc}\, m^a\, l^b\, k^c = \tfrac{1}{2} C_{abc}\, k^a\, l^b\, m^c\,,$$
$$\Psi_3 = C_{abc}\, l^a\, m^b\, m^c\,, \tag{28}$$

which are the four *alternative* expressions for the three scalars Ψ_1, Ψ_2, Ψ_3, equivalent to their definitions given in (9).

We can thus conclude that the *most general Cotton tensor* in the bivector-null basis (24) takes the form

$$\begin{aligned} C_{abc} = &-\Psi_0\, U_{ab}\, l_c + \Psi_1\, (U_{ab}\, m_c - W_{ab}\, l_c) \\ &- \Psi_2\, (U_{ab}\, k_c + V_{ab}\, l_c - 2W_{ab}\, m_c) \\ &- \Psi_3\, (V_{ab}\, m_c - W_{ab}\, k_c) + \Psi_4\, V_{ab}\, k_c\,. \end{aligned} \tag{29}$$

This is an important expression of the Cotton tensor in terms of the five key scalars $\Psi_{\rm A}$, which we will employ in proving many further properties and relations. In fact, it is obviously an analogue of the standard expression valid in $D=4$ general relativity, see equation (3.58) in [10].

6. Bel–Debever criteria

It is now possible to explicitly connect the algebraic classification of 2+1 gravity fields, summarized in table 1, to another property, namely to the *Bel–Debever criteria* which involve the Cotton tensor and the related *(aligned) null vectors* $\boldsymbol{k}$.

These criteria were presented in 1959 by Bel and Debever [4–6] in the context of Einstein's general relativity in $D=4$, employing the Riemann or Weyl tensors and the corresponding Debever–Penrose null vectors $\boldsymbol{k}$. Relatively recently, they were also generalized to geometries of any higher dimension $D>4$ by Ortaggio [27] (using the principal directions of the Weyl tensor) and equivalently by Senovilla [28] (using the Bel–Robinson tensor).

We claim that in $D=3$ spacetimes the Bel–Debever criteria involve the Cotton scalars, and they have the following form:

$$k_{[d}\, C_{a]bc}\, k^b\, k^c = 0 \quad \Leftrightarrow \quad \Psi_0 = 0\,, \tag{30}$$
$$C_{abc}\, k^b\, k^c = 0 \quad \Leftrightarrow \quad \Psi_0 = \Psi_1 = 0\,, \tag{31}$$
$$k_{[d}\, C_{a]bc}\, k^b = 0 \quad \Leftrightarrow \quad \Psi_0 = \Psi_1 = \Psi_2 = 0\,, \tag{32}$$
$$C_{abc}\, k^b = 0 \quad \Leftrightarrow \quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,. \tag{33}$$

The proof is not difficult. Using (29) we get

$$C_{abc}\, k^b = \Psi_0\, m_a\, l_c - \Psi_1\, (m_a\, m_c + k_a\, l_c) + \Psi_2\, (m_a\, k_c + 2k_a\, m_c) + \Psi_3\, k_a\, k_c\,, \tag{34}$$

and then

$$C_{abc}\, k^b\, k^c = -\Psi_0\, m_a + \Psi_1\, k_a\,. \tag{35}$$

After multiplying this expression by k_d, the antisymmetrization yields

$$k_{[d}\, C_{a]bc}\, k^b\, k^c = \tfrac{1}{2}\, \Psi_0\, V_{ad}\,, \tag{36}$$

from which we obtain the equivalence (30). The equivalence (31) follows immediately from (35). As for (32), we employ (34) which implies

$$k_{[d}\, C_{a]bc}\, k^b = \tfrac{1}{2}\, V_{da}\, (\Psi_0\, l_c - \Psi_1\, m_c + \Psi_2\, k_c)\,, \tag{37}$$

Table 2. For all algebraic types, the Bel–Debever criteria in 2+1 geometries involving the Cotton tensor C_{abc} (left) and in 3+1 geometries involving the Weyl tensor C_{abcd} (right) are fully analogous.

Algebraic type	2+1 geometries	3+1 geometries
I	$k_{[d}\,C_{a]bc}\,k^b\,k^c = 0$	$k_{[f}\,C_{a]bc[d}\,k_{e]}\,k^b\,k^c = 0$
II	$C_{abc}\,k^b\,k^c = 0$	$C_{abc[d}\,k_{e]}\,k^b\,k^c = 0$
III	$k_{[d}\,C_{a]bc}\,k^b = 0$	$k_{[f}\,C_{a]bcd}\,k^b = 0$
N	$C_{abc}\,k^b = 0$	$C_{abcd}\,k^b = 0$
D	$C_{abc}\,k^b\,k^c = 0$ and $C_{abc}\,l^b\,l^c = 0$	$C_{abc[d}\,k_{e]}\,k^b\,k^c = 0$ and $C_{abc[d}\,l_{e]}\,l^b\,l^c = 0$
O	$C_{abc} = 0$	$C_{abcd} = 0$

from which the equivalence (32) is clear. The last relation (33) is obvious from (34).

For type D spacetimes, not only $\Psi_0 = \Psi_1 = 0$ but also $\Psi_4 = \Psi_3 = 0$. Because

$$C_{abc}\,l^b\,l^c = \Psi_3\,l_a - \Psi_4\,m_a\,, \tag{38}$$

it follows that

$$C_{abc}\,l^b\,l^c = 0 \quad\Leftrightarrow\quad \Psi_3 = \Psi_4 = 0\,. \tag{39}$$

Therefore, for type D geometries *both* $C_{abc}\,k^b\,k^c = 0$ and $C_{abc}\,l^b\,l^c = 0$. This concludes the proof of the Bel–Debever criteria in 2+1 gravity.

The results for all algebraic types are summarized in table 2. The second column contains the Bel–Debever criteria in 2+1 geometries, while the last column contains the classic Bel–Debever criteria in 3+1 geometries, see equations (4.21)–(4.24) and (4.27) in [10]. It is obvious that there is a perfect analogy when the Weyl tensor C_{abcd} is replaced by the Cotton tensor C_{abc} in lower dimension $D = 3$.

In fact, the privileged (aligned) null vectors $\boldsymbol{k}$ which enter the Bel–Debever criteria in table 2 are the (possibly multiple) *principal null directions* of the Cotton and the Weyl tensor, respectively. Now we will demonstrate that these can be systematically investigated and easily found also by using the Newman–Penrose-type Cotton scalars $\Psi_{\rm A}$ defined in (9).

7. Lorentz transformations of the Cotton scalars $\Psi_{\rm A}$

The key curvature scalars $\Psi_{\rm A}$, which conveniently represent five independent components of the Cotton tensor (9), are not unique in the sense that they *depend on the choice of the null triad* $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$. However, as in the case $D \geqslant 4$ this freedom is simple, given just by the local Lorentz transformations between various triads at a given point of the spacetime manifold. These are the only admitted changes of the null basis of the tangent space which keep the normalization conditions (7).

In particular, there are three subgroups of such Lorentz transformations, namely:

$$\boldsymbol{k}' = B\boldsymbol{k}\,, \qquad \boldsymbol{l}' = B^{-1}\boldsymbol{l}\,, \qquad \boldsymbol{m}' = \boldsymbol{m}\,, \tag{40}$$

$$\boldsymbol{k}' = \boldsymbol{k}\,, \qquad \boldsymbol{l}' = \boldsymbol{l} + \sqrt{2}L\boldsymbol{m} + L^2\boldsymbol{k}\,, \qquad \boldsymbol{m}' = \boldsymbol{m} + \sqrt{2}L\boldsymbol{k}\,, \tag{41}$$

$$\boldsymbol{k}' = \boldsymbol{k} + \sqrt{2}K\boldsymbol{m} + K^2\boldsymbol{l}\,, \qquad \boldsymbol{l}' = \boldsymbol{l}\,, \qquad \boldsymbol{m}' = \boldsymbol{m} + \sqrt{2}K\boldsymbol{l}\,. \tag{42}$$

The *boost* (40) in the $\boldsymbol{k}-\boldsymbol{l}$ subspace is parameterized by B, the *null rotation (41) with* $\boldsymbol{k}$ *fixed* (changing $\boldsymbol{l}$ and $\boldsymbol{m}$) is parameterized by L, while the complementary *null rotation (42) with* $\boldsymbol{l}$ *fixed* (changing $\boldsymbol{k}$ and $\boldsymbol{m}$) is parameterized by K. All these three parameters B, K, L are *real*.

It is now straightforward to determine the transformation properties of the Cotton scalars (9). In particular, under the *boost* (40) they transform as a *rescaling*

$$\Psi'_{\mathrm{A}} = B^{2-\mathrm{A}}\,\Psi_{\mathrm{A}}\,. \tag{43}$$

It means that they are naturally *ordered* in the definition (9) according to their specific *boost weight*, which is the corresponding *power* $(2-\mathrm{A})$ of the boost parameter B. This fact is fundamental for the algebraic classification of the Cotton tensor, as an application of a general scheme developed in [12, 14] for any tensor.

Under the null rotation (41) the Cotton scalars transform as

$$\begin{aligned}
\Psi'_0 &= \Psi_0\,,\\
\Psi'_1 &= \Psi_1 + \sqrt{2}L\,\Psi_0\,,\\
\Psi'_2 &= \Psi_2 + \sqrt{2}L\,\Psi_1 + L^2\,\Psi_0\,,\\
\Psi'_3 &= \Psi_3 - 3\sqrt{2}L\,\Psi_2 - 3L^2\,\Psi_1 - \sqrt{2}L^3\,\Psi_0\,,\\
\Psi'_4 &= \Psi_4 + 2\sqrt{2}L\,\Psi_3 - 6L^2\,\Psi_2 - 2\sqrt{2}L^3\,\Psi_1 - L^4\,\Psi_0\,.
\end{aligned} \tag{44}$$

It follows that the classification of 2+1 geometries summarized in table 1 is *invariant with respect to both types of the Lorentz transformations (40) and (41)*. Indeed, if the corresponding condition for a certain algebraic type is satisfied for Ψ_{A}, it remains satisfied for Ψ'_{A}.

Finally—and more importantly—it remains to investigate the effect of the null rotation (42) *with fixed* $\boldsymbol{l}' = \boldsymbol{l}$ which changes the vectors $\boldsymbol{k}$ and $\boldsymbol{m}$ of the null triad to $\boldsymbol{k}'$ and $\boldsymbol{m}'$. In such a case the Cotton scalars (9) transform as

$$\begin{aligned}
\Psi'_0 &= \Psi_0 + 2\sqrt{2}K\,\Psi_1 + 6K^2\,\Psi_2 - 2\sqrt{2}K^3\,\Psi_3 - K^4\,\Psi_4\,,\\
\Psi'_1 &= \Psi_1 + 3\sqrt{2}K\,\Psi_2 - 3K^2\,\Psi_3 - \sqrt{2}K^3\,\Psi_4\,,\\
\Psi'_2 &= \Psi_2 - \sqrt{2}K\,\Psi_3 - K^2\,\Psi_4\,,\\
\Psi'_3 &= \Psi_3 + \sqrt{2}K\,\Psi_4\,,\\
\Psi'_4 &= \Psi_4\,.
\end{aligned} \tag{45}$$

Notice that these expressions are complementary to (44) under the swap of the null vectors $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$ and $K \leftrightarrow L$, which implies $\Psi_0 \leftrightarrow \Psi_4$, $\Psi_1 \leftrightarrow \Psi_3$ and $\Psi_2 \leftrightarrow -\Psi_2$, see (9) and (28).

8. Principle null triad and the CAND

Now we come to a crucial observation, namely that the null rotation (45) *always* allows us to *achieve* $\Psi'_0 = 0$ by a *suitable choice* of the (complex) parameter K. Consequently, in the new null triad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}'\}$ the condition for algebraic type I given in table 1 is satisfied. Such a special frame is called the *principle null triad*, and its special null vector $\boldsymbol{k}'$ is said to be *aligned with the Cotton curvature tensor* C_{abc}. The existence of the principal null triad demonstrates that *all 2+1 geometries are (at least) of algebraic type I*. Recall that the same is true for all $3+1$ geometries, considering the Weyl tensor instead of the Cotton tensor, but in higher-dimensional spacetimes such a principal null frame need not exist at all (see [10] and [14], respectively).

For practical reasons, however, we will consider the equivalent opposite procedure, in which one starts with the Cotton scalars Ψ'_{A} calculated with respect to an *arbitrarily chosen* null triad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}'\}$. It is then *possible to achieve* $\Psi_0 = 0$ by performing the *inverse* of the null rotation (42), that is

$$\boldsymbol{k} = \boldsymbol{k}' - \sqrt{2}K\boldsymbol{m}' + K^2\boldsymbol{l}' , \qquad \boldsymbol{l} = \boldsymbol{l}' , \qquad \boldsymbol{m} = \boldsymbol{m}' - \sqrt{2}K\boldsymbol{l}' . \tag{46}$$

The resulting special triad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ becomes the *principle null triad*, and its null vector $\boldsymbol{k}$ is the *Cotton-aligned null direction*, which we can abbreviate as CAND. It is the 2+1 analogue of the usual concept of PND of the Weyl tensor in $D = 4$ general relativity, and of WAND in $D \geqslant 4$ gravity, as introduced in [11, 12].

In fact, such an algebraically privileged triad with the CAND can be *explicitly found*. Under the null rotation (46) the Cotton scalar Ψ_0 (having the highest boost weight +2) transforms as

$$\Psi_0 = \Psi'_0 - 2\sqrt{2}K\Psi'_1 + 6K^2\Psi'_2 + 2\sqrt{2}K^3\Psi'_3 - K^4\Psi'_4 . \tag{47}$$

Actually, it is obtained from (45) by the simple swap $\Psi_{\mathrm{A}} \leftrightarrow \Psi'_{\mathrm{A}}$ and $K \leftrightarrow -K$. The condition $\Psi_0 = 0$ thus takes the form

$$\Psi'_4 K^4 - 2\sqrt{2}\Psi'_3 K^3 - 6\Psi'_2 K^2 + 2\sqrt{2}\Psi'_1 K - \Psi'_0 = 0 . \tag{48}$$

It is an *algebraic* equation of the *fourth order* in the parameter K which, in general, admits *four complex solutions* (not necessarily distinct). It thus follows that, *at any event* of the 2+1 spacetime there exist, in general, *four CANDs* determined by the local algebraic structure of the (non-vanishing) Cotton tensor.

Each of these CANDs $\boldsymbol{k}$ is obtained using the relation (46), in which the parameter K is the corresponding root of the equation (48). Moreover, any multiplicity of these roots K implies the same *multiplicity of the CANDs*. We will now demonstrate that such multiplicities are uniquely related to the algebraic types.

9. Algebraic types and the CANDs multiplicity

A 2+1 spacetime is said to be *algebraically general* if its CANDs, i.e. the four roots of (48), are *all distinct*. Such a spacetime is of algebraic *type I*.

A spacetime is *algebraically special* if at least two its CANDs *coincide*. If *just two* CANDs $\boldsymbol{k}$ coincide, it is of *type II*. Analogously, higher multiplicity defines *type III* (triple CAND/root) and the most special *type N* (quadruple CAND/root) geometries.

In addition, there exists another degenerate case of *type D*. It is a subtype of type II such that there are two distinct CANDs $\boldsymbol{k}$ and $\boldsymbol{l}$, both of multiplicity 2 (*two pairs* of coinciding roots). For completeness, the algebraic type O denotes a spacetime with everywhere vanishing Cotton tensor (a *conformally flat* 2+1 spacetime). The complete scheme is summarized in table 3.

More specifically, if the vector $\boldsymbol{k}$ of the principal null triad is the CAND then $\Psi_0 = 0$, and the key equation (48) in such a triad becomes

$$\left(\Psi_4 K^3 - 2\sqrt{2}\Psi_3 K^2 - 6\Psi_2 K + 2\sqrt{2}\Psi_1\right) K = 0 . \tag{49}$$

The root $K = 0$ corresponds to the CAND $\boldsymbol{k}$. The special algebraic types arise when also the cubic expression in the bracket has another root(s) $K = 0$. It is now obvious that type II arises when $\Psi_1 = 0$, and type III arises when $\Psi_1 = \Psi_2 = 0$. Type N occurs when $\Psi_1 = \Psi_2 = \Psi_3 = 0$, in which case (49) reduces to $\Psi_4 K^4 = 0$. The quadruple root $K = 0$ corresponds to the unique and privileged quadruple CAND $\boldsymbol{k}$. For type D spacetimes with the Cotton scalars having the

Table 3. Possible algebraic types of 2+1 geometries. The classification is uniquely related to the multiplicity of the Cotton-aligned null directions (CANDs), that is to the multiplicity of the four (complex) roots of the key equation (48). The canonical forms of the five real Cotton scalars Ψ_A for each algebraic type are also included.

Algebraic type	CANDs	Multiplicity	Canonical Cotton scalars	
I		$1+1+1+1$	$\Psi_0 = 0\,,$	$\Psi_1 \neq 0$
II		$1+1+2$	$\Psi_0 = \Psi_1 = 0\,,$	$\Psi_2 \neq 0$
D		$2+2$	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4\,,$	$\Psi_2 \neq 0$
III		$1+3$	$\Psi_0 = \Psi_1 = \Psi_2 = 0\,,$	$\Psi_3 \neq 0$
N		4	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,,$	$\Psi_4 \neq 0$
O		N/A	all $\Psi_A = 0$	

form $\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4$ the equation (49) reduces to quadratic equation $\Psi_2 K^2 = 0$, so that $\boldsymbol{k}$ is the double CAND, as in type II. To be more specific:

- **Type I** geometries with the CAND $\boldsymbol{k}$ satisfy the equation (49) in which the simple root $K = 0$ corresponds to $\boldsymbol{k}$. Because $\Psi_1 \neq 0$ the remaining part of the equation is of the third order, admitting *in general* three distinct (complex) roots different from zero. In such a generic case, there exist *four different* CANDs with no multiplicities, symbolically denoted as $1+1+1+1$.
- **Type II** geometries have the canonical form $\Psi_0 = 0 = \Psi_1$ and $\Psi_2 \neq 0$, in which case (49) reduces to

$$\left(\Psi_4 K^2 - 2\sqrt{2}\,\Psi_3 K - 6\,\Psi_2\right) K^2 = 0\,. \tag{50}$$

  It has the solution $K = 0$ with the *multiplicity 2* (which means that $\boldsymbol{k}$ is a *double* CAND), and other $K \neq 0$ solutions are given by the roots of the quadratic equation in the bracket. In general, there exist two different (complex) roots, meaning that the multiplicities of the *three different* CANDs are $1+1+2$.
- **Type III** geometries have the canonical form of the Cotton scalars $\Psi_0 = \Psi_1 = \Psi_2 = 0$ with $\Psi_3 \neq 0$. The key equation (49) thus takes the form

$$\left(\Psi_4 K - 2\sqrt{2}\,\Psi_3\right) K^3 = 0\,. \tag{51}$$

  The trivial solution $K = 0$ has the *multiplicity 3* (which means that $\boldsymbol{k}$ is a *triple* CAND), and there exists another (real) root $K = 2\sqrt{2}\,\Psi_3/\Psi_4 \neq 0$. The multiplicities of the *two different* CANDs are thus $1+3$.
- **Type N** geometries are defined by the canonical condition $\Psi_4 \neq 0$ with all remaining Cotton scalars zero, so that the equation (49) simplifies to

$$K^4 = 0\,. \tag{52}$$

  Therefore, $\boldsymbol{k}$ is a *quadruple* CAND, corresponding to the *multiplicity 4* of the solution $K = 0$.

Table 4. Algebraic classification of 2+1 geometries for the special case $\Psi_4 = 0 = \Psi_0$.

$\Psi_1 = 0$	$\Psi_2 = 0$	$\Psi_3 = 0$		type O
		$\Psi_3 \neq 0$		type III
	$\Psi_2 \neq 0$	$\Psi_3 = 0$		type D
		$\Psi_3 \neq 0$		type II
$\Psi_1 \neq 0$	$\Psi_2 = 0$	$\Psi_3 = 0$		type III
		$\Psi_3 \neq 0$		type I
	$\Psi_2 \neq 0$	$\Psi_3 = 0$		type II
		$\Psi_3 \neq 0$	$9\Psi_2^2 = -8\Psi_1\Psi_3$	type II
			$9\Psi_2^2 \neq -8\Psi_1\Psi_3$	type I

- **Type D** geometries have the canonical form $\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4$ and $\Psi_2 \neq 0$. The key equation (49) thus reduces to

$$K^2 = 0\,, \tag{53}$$

from which it follows that $\boldsymbol{k}$ is a *double* CAND, corresponding to the *multiplicity 2* of the solution $K = 0$. In section 8 we have defined the CAND $\boldsymbol{k}'$ as the null vector of the principle null triad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}'\}$ in which $\Psi_0' = 0$. Analogously[3], the null vector $\boldsymbol{l}'$ of the principle null triad is CAND if $\Psi_4' = 0$. In view of the transformation property (44) of the Cotton scalars under the null rotation (41) with $\boldsymbol{k}$ fixed, that is

$$\boldsymbol{k}' = \boldsymbol{k}\,, \qquad \boldsymbol{l}' = \boldsymbol{l} + \sqrt{2}L\boldsymbol{m} + L^2\boldsymbol{k}\,, \qquad \boldsymbol{m}' = \boldsymbol{m} + \sqrt{2}L\boldsymbol{k}\,, \tag{54}$$

for the canonical form of type D geometries we obtain

$$\Psi_0' = 0\,, \qquad \Psi_1' = 0\,, \qquad \Psi_2' = \Psi_2\,, \qquad \Psi_3' = -3\sqrt{2}L\Psi_2\,, \qquad \Psi_4' = -6L^2\,\Psi_2\,. \tag{55}$$

Therefore, the condition $\Psi_4' = 0$ for $\boldsymbol{l}'$ being the Cotton-aligned null direction is simply

$$L^2 = 0\,. \tag{56}$$

It means that the null vector $\boldsymbol{l} = \boldsymbol{l}'$ is, in fact, a *double* CAND, corresponding to the *multiplicity 2* of the solution $L = 0$. To summarize, geometries of algebraic type D admit *two distinct* CANDs $\boldsymbol{k}$ and $\boldsymbol{l}$, both of multiplicity 2.

This completes the proof of the relations contained in table 3.

A special situation $\Psi_4 = 0 = \Psi_0$ has to be treated separately. In such a case the quartic equation (49) reduces to the cubic

$$\left(\sqrt{2}\,\Psi_3\,K^2 + 3\,\Psi_2\,K - \sqrt{2}\,\Psi_1\right)K = 0\,, \tag{57}$$

with the CAND $\boldsymbol{k}$ (corresponding to $K = 0$) and the distinct CAND $\boldsymbol{l}$ (corresponding to $L = 0$). The respective multiplicities of the remaining roots of (57) are given by the nature of the quadratic polynomial in the bracket, depending on the (non-)vanishing of the scalars Ψ_1, Ψ_2, Ψ_3,

[3] Recall that the swap of the null vectors $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$ results in $\Psi_0 \leftrightarrow \Psi_4$, $\Psi_1 \leftrightarrow \Psi_3$, $\Psi_2 \leftrightarrow -\Psi_2$.

and also on the discriminant $D = 9\,\Psi_2^2 + 8\,\Psi_1\Psi_3$. Full discussion of all possible algebraic types in this case is presented in table 4.

We should emphasize that the algebraic classification in 2+1 dimensions (presented here) is actually more subtle than in 3+1 geometries. The complication arises from the fact that the key *real* equation (48) can in general have some *complex roots* K. This implies that some of the null vectors $\boldsymbol{k}$ representing the CANDs may *formally be complex*. This somewhat unwelcome consequence is closely related to the property that the important Cotton–York matrix $Y_a{}^b$, see (79) for its explicit form, *is not symmetric* and there is thus *no guarantee that its eigenvalues are real*. Nevertheless, it is a common practice in the field of algebraic classification of 2+1 spacetimes to *formally admit* the complex classification. A geometrically more justified (sub)classification based on the *real roots* can be introduced. By restricting the eigenvalues to only real numbers, a new algebraic subtype of spacetimes denoted as Class I′ can be added, see for example section 20.5.2 in the García-Díaz monograph [25]. We will return to this issue later in section 13, after presenting the usual approach to algebraic classification based on the Jordan form of the Cotton–York tensor in section 11.

10. Cotton–York tensor

The number of independent components of the Cotton tensor C_{abc} (3) in three dimensions, being five, is exactly equal to the number of components of a symmetric and traceless rank-2 tensor. This can be obtained as the *Hodge dual*.

More specifically, following the conventions given in [25], with only slight modifications, the *Cotton–York tensor* (sometimes also called the Schouten–Cotton–York tensor) is geometrically defined by equation (20.111) in [25] (but denoted as $C_{\alpha\beta}$ therein) as

$$Y_{ab} \equiv \boldsymbol{e}_a \rfloor {}^*\mathbf{C}_b = {}^*\left(\mathbf{C}_b \wedge \boldsymbol{\omega}_a\right)\,, \tag{58}$$

where $\boldsymbol{\omega}_a = g_{ab}\,\boldsymbol{\omega}^b$ is the linear combination of the basis 1-forms (1) and $\mathbf{C}_b$ is the Cotton ('vector valued') 2-form

$$\text{2-form:} \qquad \mathbf{C}_b \equiv \tfrac{1}{2}\,C_{mnb}\,\boldsymbol{\omega}^m \wedge \boldsymbol{\omega}^n\,. \tag{59}$$

The symbol $\rfloor$ in (58) stands for the *interior product* defined on a general p-form $\boldsymbol{\sigma}$ as[4]

$$\boldsymbol{e}_a \rfloor \boldsymbol{\sigma} \equiv \frac{1}{(p-1)!}\,\sigma_{ab_2\ldots b_p}\,\boldsymbol{\omega}^{b_2} \wedge \ldots \wedge \boldsymbol{\omega}^{b_p}\,. \tag{60}$$

Another common notation for this operation is $\iota_{\boldsymbol{e}}\,\boldsymbol{\sigma}$.

Recall that a metric-independent *Hodge dual operator* can be defined by employing the so called ϵ-basis. This is constructed by subsequent interior products of the Levi-Civita tensor

$$\boldsymbol{\omega} \equiv -3!\,\sqrt{-g}\;\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2\,, \tag{61}$$

in which g is the determinant of the metric g_{ab}. In components, this tensor explicitly reads

$$\omega_{abc} = -\sqrt{-g}\;\varepsilon_{abc}\,, \qquad \text{or} \qquad \omega^{abc} = \frac{1}{\sqrt{-g}}\;\varepsilon^{abc}\,, \tag{62}$$

[4] The exterior calculus notation and definitions used here are mainly taken from appendix A in [29].

where $\varepsilon^{abc} = \varepsilon_{abc}$ is the Levi-Civita symbol. Without loss of generality we assume that the null triad has the following orientation

$$\omega_{abc}\, k^a\, l^b\, m^c = 1\,. \tag{63}$$

It is equivalent to defining the Levi-Civita symbol in the null basis $\{\boldsymbol{\omega}^b\}$ as $\varepsilon^{123} = \varepsilon_{123} = -1$. Such an orientation ensures that, in an orthonormal frame, the spatial part will have the right-handed orientation. The correspondence between the Hodge dual operation and the ϵ-basis is

$$\begin{aligned}
&\text{2-form:} \quad && \boldsymbol{\epsilon}_a \equiv {}^*\boldsymbol{\omega}_a && = \boldsymbol{e}_a \rfloor \boldsymbol{\omega} = \tfrac{1}{2}\omega_{abc}\,\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c\,,\\
&\text{1-form:} \quad && \boldsymbol{\epsilon}_{ab} \equiv {}^*\left(\boldsymbol{\omega}_a \wedge \boldsymbol{\omega}_b\right) && = \boldsymbol{e}_b \rfloor \boldsymbol{\epsilon}_a = \omega_{abc}\,\boldsymbol{\omega}^c\,,\\
&\text{0-form:} \quad && \epsilon_{abc} \equiv {}^*\left(\boldsymbol{\omega}_a \wedge \boldsymbol{\omega}_b \wedge \boldsymbol{\omega}_c\right) && = \boldsymbol{e}_c \rfloor \boldsymbol{\epsilon}_{ab} = \omega_{abc}\,.
\end{aligned} \tag{64}$$

Applying this construction of the Hodge dual to the (vector valued) Cotton 2-form (59) we obtain ${}^*\mathbf{C}_b = \frac{1}{2} C_{mnb}\,\boldsymbol{\epsilon}^{mn}$, that gives the following expression for the dual

$$\text{1-form:} \qquad {}^*\mathbf{C}_b = \tfrac{1}{2}\,\omega^{mnk}\,C_{mnb}\,g_{kc}\,\boldsymbol{\omega}^c\,. \tag{65}$$

By performing the contractions $\boldsymbol{e}_a \rfloor \boldsymbol{\omega}^c = \delta_a{}^c$ we get from the definition (58) an explicit prescription for the Cotton–York tensor, namely

$$\begin{aligned}
Y_{ab} &= \tfrac{1}{2}\,g_{ak}\,\omega^{kmn}\,C_{mnb}\\
&= -\sqrt{-g}\,\varepsilon_{amn}\left(\nabla^m R^n{}_b - \frac{1}{4}\,\delta^n{}_b\,\partial^m R\right).
\end{aligned} \tag{66}$$

This alternative form of the Cotton tensor appeared in York's work [30], but was already discussed before by Arnowitt *et al* [31]. It encodes the same information as the Cotton tensor, but it is a rank-lower tensor. One of its major advantages is that it is *symmetric*

$$Y_{ab} = Y_{ba}\,, \tag{67}$$

and also *traceless*

$$Y_a{}^a = 0\,. \tag{68}$$

Moreover, 2+1 spacetime is locally *conformally flat* if and only if $Y_{ab} = 0$. This Cotton–York tensor is the key tensor in the context of 2+1 gravity, whose algebraic classification has already been introduced and successfully employed. Since it is a rank-2 tensor, the eigenvalue problem can be formulated exactly as a standard eigenvalue problem for matrices, see [18, 25].

To find an explicit relation to our new method of classification, we first express Y_{ab} in the null triad (7) as

$$Y_{ab} = \sum_{I,J=1}^{3} Y_{IJ}\,\omega^I_a\,\omega^J_b\,, \tag{69}$$

where Y_{IJ} are the corresponding components (recall that the dual basis is $\{\boldsymbol{\omega}^I\} \equiv \{-\boldsymbol{l}, -\boldsymbol{k}, \boldsymbol{m}\}$). We can uniquely relate them to the Newman–Penrose-like Cotton scalars (9). Writing the sum (69) explicitly, using just the symmetry property (67) of the Cotton–York tensor, we get

$$Y_{ab} = Y_{11}\, l_a\, l_b + Y_{12}\,(l_a\, k_b + k_a\, l_b) - Y_{13}\,(l_a\, m_b + m_a\, l_b) \\ + Y_{22}\, k_a\, k_b - Y_{23}\,(k_a\, m_b + m_a\, k_b) + Y_{33}\, m_a\, m_b\,. \tag{70}$$

Applying the normalization condition (7), from (70) the coefficients can be expressed as[5]

$$\begin{aligned} Y_{11} &= Y_{ab}\, k^a\, k^b\,, \\ Y_{12} &= Y_{ab}\, k^a\, l^b\,, \\ Y_{13} &= Y_{ab}\, k^a\, m^b\,, \\ Y_{22} &= Y_{ab}\, l^a\, l^b\,, \\ Y_{23} &= Y_{ab}\, l^a\, m^b\,, \\ Y_{33} &= Y_{ab}\, m^a\, m^b\,. \end{aligned} \tag{71}$$

Using the definition (66) with the relation (63) and the full expression (29) of the general Cotton tensor, one arrives at the following result

$$\begin{aligned} Y_{11} &= -\Psi_0\,, \qquad & Y_{12} &= -\Psi_2\,, \qquad & Y_{13} &= -\Psi_1\,, \\ Y_{22} &= \Psi_4\,, & Y_{23} &= \Psi_3\,, & Y_{33} &= -2\,\Psi_2\,. \end{aligned} \tag{72}$$

The *general Cotton–York tensor* (70) in the null triad basis thus takes the form

$$\begin{aligned} Y_{ab} = &- \Psi_0\, l_a\, l_b + \Psi_1\,(l_a\, m_b + m_a\, l_b) \\ &- \Psi_2\,(l_a\, k_b + k_a\, l_b + 2\, m_a\, m_b) \\ &- \Psi_3\,(k_a\, m_b + m_a\, k_b) + \Psi_4\, k_a\, k_b\,. \end{aligned} \tag{73}$$

This is the key expression that will allow us now to relate the Cotton scalars Ψ_{A} to the algebraic classification of spacetimes in 2+1 gravity, and to demonstrate its equivalence with the previous classification scheme based on the Jordan form of the Cotton–York tensor [25].

To this end, it is important to express the general Cotton–York tensor in an *orthonormal basis* $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2\}$ corresponding to the null triad (7) via the usual relations

$$\boldsymbol{E}_0 \equiv \tfrac{1}{\sqrt{2}}\,(\boldsymbol{k} + \boldsymbol{l})\,, \qquad \boldsymbol{E}_1 \equiv \tfrac{1}{\sqrt{2}}\,(\boldsymbol{k} - \boldsymbol{l})\,, \qquad \boldsymbol{E}_2 \equiv \boldsymbol{m}\,. \tag{74}$$

Due to the normalization (7), such a basis satisfies the conditions

$$\boldsymbol{E}_0 \cdot \boldsymbol{E}_0 = -1\,, \qquad \boldsymbol{E}_1 \cdot \boldsymbol{E}_1 = 1\,, \qquad \boldsymbol{E}_2 \cdot \boldsymbol{E}_2 = 1\,, \tag{75}$$

with all other scalar products equal to zero, i.e. the metric in this basis reads

$$g_{ab} = \mathrm{diag}\,(-1, 1, 1)\,. \tag{76}$$

It means that $\boldsymbol{E}_0$ is the (future-oriented) timelike unit vector, while $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are perpendicular Cartesian spatial vectors. It also follows that

$$E_0^a\, k_a = -\tfrac{1}{\sqrt{2}}\,, \qquad E_0^a\, l_a = -\tfrac{1}{\sqrt{2}}\,, \qquad E_0^a\, m_a = 0\,,$$

[5] Basically, these are the real symmetric quantities Ψ_{AB} introduced in section 5 of [21].

$$E_1^a\,k_a = \tfrac{1}{\sqrt{2}}\,, \qquad E_1^a\,l_a = -\tfrac{1}{\sqrt{2}}\,, \qquad E_1^a\,m_a = 0\,,$$
$$E_2^a\,k_a = 0\,, \qquad E_2^a\,l_a = 0\,, \qquad E_2^a\,m_a = 1\,. \tag{77}$$

Using (73), we thus easily obtain *all orthonormal projections of the Cotton–York tensor*, such as $Y_{00} \equiv E_0^a E_0^b\, Y_{ab} = -\Psi_2 - \frac{1}{2}(\Psi_0 - \Psi_4)$, etc. The result is

$$Y_{ab} = \begin{pmatrix} -\Psi_2 - \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{2}(\Psi_0 + \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) \\ -\frac{1}{2}(\Psi_0 + \Psi_4) & \Psi_2 - \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) \\ -\frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix}, \tag{78}$$

which is clearly a *symmetric real matrix* ($Y_{ab} = Y_{ba}$, ${Y_a}^a = 0$). Actually, it is a direct 2+1 analogue of the symmetric complex 3×3 matrix $\boldsymbol{Q}$ (with zero trace) which is used for the algebraic classification of the Weyl tensor in 3+1 spacetimes in the original Petrov approach (see equation (3.65) in [10]).

11. Equivalence with the previous method of classification

To complete this work, it remains to prove the equivalence of our new convenient method of algebraic classification, based on the Cotton scalars $\Psi_{\rm A}$ and the multiplicity of CANDs, with the previous 'Petrov-type' classification scheme based on finding the specific Jordan forms of the Cotton–York tensor. As summarized in Introduction, this was first considered in [17] and refined in [18].

Let us repeat the main results, following sections 1.2.1 and 20.5.2 of the monograph [25] by García-Díaz. The key idea is to solve the ordinary *eigenvalue problem* ${Y_a}^b\, v_b = \lambda\, v_a$ for the Cotton–York 3×3 matrix ${Y_a}^b \equiv Y_{ac}\, g^{cb}$. In view of (78) and (76), in the orthonormal basis (74) we get its explicit expression (a denotes rows, while b denotes the columns)

$${Y_a}^b = \begin{pmatrix} \Psi_2 + \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{2}(\Psi_0 + \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) \\ \frac{1}{2}(\Psi_0 + \Psi_4) & \Psi_2 - \frac{1}{2}(\Psi_0 - \Psi_4) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) \\ \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix}. \tag{79}$$

It is important to emphasize that the matrix ${Y_a}^b$ is traceless but *not symmetric*. Therefore, the roots of the characteristic *cubic polynomial* $\det({Y_a}^b - \lambda\,{\delta_a}^b) = 0$ may be *complex*. Nevertheless, according to the possible eigenvalues λ_1, λ_2 and $\lambda_3 = -\lambda_1 - \lambda_2$, one can find the corresponding canonical *Jordan forms*, defining the algebraic 'Petrov' types of all 2+1 geometries. Such forms are presented in table 5, which is actually the copy of table 1.2.1 of [25].

Now, we would like to find a one-to-one correspondence between the canonical Jordan forms J of ${Y_a}^b$ presented in table 5 for each 'Petrov' type, and the canonical values of the Cotton scalars $\Psi_{\rm A}$. By comparing the Jordan form of type I with the explicit expression (79) we uniquely obtain the conditions $\Psi_1 \pm \Psi_3 = 0$ (so that $\Psi_1 = 0 = \Psi_3$), $\Psi_0 + \Psi_4 = 0$, $\frac{1}{2}(\Psi_0 - \Psi_4) + \Psi_2 = \lambda_1$ and $-\frac{1}{2}(\Psi_0 - \Psi_4) + \Psi_2 = \lambda_2$ (so that $2\Psi_2 = \lambda_1 + \lambda_2$ and $\Psi_0 - \Psi_4 = \lambda_1 - \lambda_2$). Similarly for type D we immediately obtain $\Psi_0 = 0 = \Psi_4$, $\Psi_1 = 0 = \Psi_3$ and $\Psi_2 = \lambda_1$. However, for types II, III and N such an identification is *not directly possible*. Instead, in these cases it is necessary to employ an equivalent (alternative) *normal forms of the Cotton–York matrix* ${Y_a}^b$.

Table 5. Traditional algebraic classification of the Cotton–York tensor ${Y_a}^b$ based on the possible Jordan forms and eigenvalues.

'Petrov' type	Jordan form J of ${Y_a}^b$	Eigenvalues relation
I	$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & -\lambda_1-\lambda_2\end{pmatrix}$	$\lambda_1 \neq \lambda_2, \quad \lambda_3 = -\lambda_1 - \lambda_2$
II	$\begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix}$	$\lambda_1 = \lambda_2 \neq 0, \quad \lambda_3 = -2\lambda_1$
D	$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix}$	$\lambda_1 = \lambda_2 \neq 0, \quad \lambda_3 = -2\lambda_1$
III	$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
N	$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
O	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$	

More precisely, we look for a *similarity transformation* between the Jordan form J and the specific *normal form* N, such that

$$N = AJA^{-1}\,, \tag{80}$$

where A is an invertable matrix and A^{-1} its inverse. In particular, a direct calculation shows that for type II geometries such a similarity transformation takes the form

$$N = \begin{pmatrix}\lambda_1 - 1 & -1 & 0\\ 1 & \lambda_1 + 1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix} = \begin{pmatrix}-1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{81}$$

The subcase $\lambda_1 = 0$ gives the transformation for type N geometries. And for type III we get

$$N = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0 & 1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 1 & 1 & 0\end{pmatrix}. \tag{82}$$

Such normal forms of the Cotton–York tensor ${Y_a}^b$ can be uniquely identified with the canonical values of the Cotton scalars Ψ_{A} for each algebraic type. The results are summarized in table 6.

We have thus proven that for each 'Petrov' algebraic type in $D = 3$ there exists a privileged orthonormal basis, which can be called the *principal Cotton–York basis*, in which ${Y_a}^b$ has the corresponding normal form N, and the associated canonical values of the Cotton scalars Ψ_{A}, as given in the last column of table 6. Actually, it is an analogue of table 4.2 in [10] which contains the normal forms of the Weyl tensor for all Petrov types in $D = 4$.

Moreover, the specific values of the Cotton scalars Ψ_{A} in table 6 are *fully consistent* with our new simpler method of algebraic classification of 2+1 geometries, as summarized in table 1

Table 6. Algebraic classification based on the possible normal forms of the Cotton–York tensor ${Y_a}^b$ and the specific values of the Cotton scalars Ψ_A.

'Petrov' type	Normal form N of ${Y_a}^b$	Values of the Cotton scalars
I	$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & -\lambda_1-\lambda_2\end{pmatrix}$	$\Psi_1 = 0 = \Psi_3$ $\Psi_0 = \frac{1}{2}(\lambda_1-\lambda_2) = -\Psi_4$ $\Psi_2 = \frac{1}{2}(\lambda_1+\lambda_2)$
II	$\begin{pmatrix}\lambda_1-1 & -1 & 0\\ 1 & \lambda_1+1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix}$	$\Psi_0 = 0,\ \Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$ $\Psi_4 = 2$
D	$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & -2\lambda_1\end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$
III	$\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & -1 & 0\end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_2$ $\Psi_3 = \sqrt{2}$
N	$\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\Psi_0 = 0 = \Psi_2$ $\Psi_1 = 0 = \Psi_3$ $\Psi_4 = 2$
O	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$	all $\Psi_\mathrm{A} = 0$

and corroborated in table 3 to also include the related multiplicity of the CANDs. To be more specific:

- **Type I** geometries with CAND $\boldsymbol{k}'$ are defined by the existence of the principle null triad such that $\Psi_0' = 0$, see section 8. By inspecting the last column of table 6 we observe that such a condition is not satisfied in the principal Cotton–York basis because $\Psi_0 \neq 0$ (recall that $\lambda_1 \neq \lambda_2$ for type I spacetimes). However, we can employ a suitable Lorentz transformation (45) which for the canonical values of the Cotton scalars reduces to

$$\begin{aligned}
\Psi_0' &= \left(K^4+1\right)\Psi_0 + 6K^2\,\Psi_2\,,\\
\Psi_1' &= \sqrt{2}K\left(K^2\,\Psi_0 + 3\,\Psi_2\right)\,,\\
\Psi_2' &= K^2\,\Psi_0 + \Psi_2\,,\\
\Psi_3' &= -\sqrt{2}K\Psi_0\,,\\
\Psi_4' &= -\Psi_0\,.
\end{aligned} \tag{83}$$

  Obviously, we achieve $\Psi_0' = 0$ by taking K to be any root of the bi-quadratic equation

$$\Psi_0 K^4 + 6\Psi_2 K^2 + \Psi_0 = 0\,. \tag{84}$$

  Because $\Psi_0 = \frac{1}{2}(\lambda_1-\lambda_2)$ and $\Psi_2 = \frac{1}{2}(\lambda_1+\lambda_2)$, these *four distinct* explicit roots are

$$K^2 = \frac{-3\Psi_2 \pm \sqrt{9\Psi_2^2-\Psi_0^2}}{\Psi_0} = -3\,\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} \pm \frac{\sqrt{9\left(\lambda_1+\lambda_2\right)^2-\left(\lambda_1-\lambda_2\right)^2}}{\lambda_1-\lambda_2}\,. \tag{85}$$

Table 7. Multiplicities of the Cotton-aligned null directions (CANDs) for all algebraic types. These are obtained from the key condition $\Psi_0' = 0$ expressed in the principal Cotton–York basis, in which the Cotton scalars Ψ_A have the canonical form presented in table 6. Multiplicity of the root corresponds to the multiplicity of the related CAND. If the root is $K = 0$ then the null vector $\boldsymbol{k}$ is CAND. Similarly, the root $L = 0$ identifies that $\boldsymbol{l}$ is CAND. For type N geometries the only nontrivial Cotton scalar is Ψ_4, so that the vector $\boldsymbol{k}$ is a quadruple CAND corresponding to the multiplicity 4 of the root $K = 0$.

Type	Condition $\Psi_0' = 0$	Roots	CANDs	Multiplicity
I	$K^4 + 6bK^2 + 1 = 0$	$K = \pm\sqrt{-3b \pm 2\sqrt{D}}$, where $b = \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ and $\sqrt{D} = \frac{\sqrt{2\lambda_1^2+5\lambda_1\lambda_2+2\lambda_2^2}}{\lambda_1-\lambda_2}$		1+1+1+1
II	$(K^2 - 3\lambda_1)\,K^2 = 0$	$K = \pm\sqrt{3\lambda_1}$ and double $K = 0$		1+1+2
D	$L^2 = 0$ and $K^2 = 0$	double $L = 0$ and double $K = 0$		2+2
III	$L = 0$ and $K^3 = 0$	$L = 0$ and triple $K = 0$		1+3
N	$K^4 = 0$	quadruple $K = 0$		4

The corresponding null vectors $\boldsymbol{k}' = \boldsymbol{k} + \sqrt{2}K\boldsymbol{m} + K^2\boldsymbol{l}$ are then CANDs because $\Psi_0' = 0$. Moreover, $K^2\Psi_0 + 3\Psi_2 = \pm\frac{1}{2}\sqrt{9(\lambda_1+\lambda_2)^2 - (\lambda_1-\lambda_2)^2}$, so that generally $\Psi_1' \neq 0$.

- **Type II** geometries in the principal Cotton–York basis have $\Psi_0 = \Psi_1 = 0$, $\Psi_2 = \lambda_1 \neq 0$, see table 6. This fully corresponds to our definition presented in table 1. In fact, we can even achieve $\Psi_3 = 0$ by performing (the inverse of) the Lorentz transformation (45), namely $\Psi_3 = \Psi_3' - \sqrt{2}K\Psi_4'$ for the particular choice of the null rotation parameter $\sqrt{2}K = \Psi_3'/\Psi_4'$. Using (45), the condition $\Psi_0' = 0$ for CAND $\boldsymbol{k}'$ in the principal Cotton–York basis reduces to a special form

$$\left(K^2 - 3\lambda_1\right)K^2 = 0\,. \tag{86}$$

The factor K^2 shows that $\boldsymbol{k}$ is a *double CAND*, and other two distinct CANDs are obtained by applying the null rotation with the parameters $K = \pm\sqrt{3\lambda_1}$.

- **Type D** geometries in the principal Cotton–York basis have $\Psi_0 = 0 = \Psi_1$, $\Psi_3 = 0 = \Psi_4$ and $\Psi_2 = \lambda_1 \neq 0$. This is a complete agreement with our definition presented in table 1. There are *two distinct* CANDs $\boldsymbol{k}$ and $\boldsymbol{l}$, both of multiplicity 2 because $K^2 = 0$ and also $L^2 = 0$, see relations (53) and (56).
- **Type III** geometries in the principal Cotton–York basis have $\Psi_0 = \Psi_1 = \Psi_2 = 0$, $\Psi_3 = \sqrt{2}$ and $\Psi_4 = 0$, which exactly corresponds to our definition in table 1. We can achieve $\Psi_4 = 0$ by performing (the inverse of) the Lorentz transformation (44). Indeed, for the particular choice of the null rotation parameter $2\sqrt{2}L = \Psi_4'/\Psi_3'$ we get $\Psi_4 = \Psi_4' - 2\sqrt{2}L\Psi_3' = 0$. The key equation $\Psi_0' = 0$ given by (45) reduces to

$$K^3 = 0\,. \tag{87}$$

It demonstrates that $\boldsymbol{k}$ is a *triple CAND*, while the fourth distinct CAND is $\boldsymbol{l}$ corresponding to $L = 0$.

- **Type N** geometries in the principal Cotton–York basis have $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and $\Psi_4 = 2$, in full agreement with table 1. We can achieve the fixed canonical value $\Psi_4 = 2$ from any $\Psi_4' \neq 0$ by the boost (40), which implies a simple rescaling $\Psi_4 = B^2\Psi_4'$, see (43).

In this case the condition $\Psi_0' = 0$ reduces to

$$K^4 = 0\,, \tag{88}$$

which proves that $\boldsymbol{k}$ is a *quadruple CAND*.

These results are summarized in table 7. They show that our new simpler method of algebraic classification of 2+1 geometries, based on the direct evaluation (9) of the Cotton scalars $\Psi_{\rm A}$ and using the conditions in table 1, is fully equivalent to the previous (rather cumbersome) 'Petrov' approach based on determining the eigenvalues and the respective Jordan form J of the Cotton–York tensor $Y_a{}^b$ (employed, e.g. in [25]). Moreover, our approach shows the unique relation of the algebraic types to the corresponding multiplicity of the CANDs, in a complete analogy with the multiplicities of the PNDs in $D = 4$ gravity (see section 4.3 of [10]) and the WANDs in $D > 4$ gravity theories (see [14]).

12. Invariants assisting with the algebraic classification

To complete our new procedure of algebraic classification of 2+1 geometries, we now investigate an important concept of *scalar curvature polynomial invariants* which can be constructed from the Cotton tensor and the related Cotton–York tensor. In fact, it will turn out that these invariants play a crucial role in easily determining the algebraic type of the spacetime.

From the explicit expression (29) for the Cotton tensor, using the normalization relations (8) and (13), we can directly evaluate the quadratic scalar invariant

$$C_{abc}\,C^{abc} = 4\left(\Psi_0\Psi_4 - 2\,\Psi_1\Psi_3 - 3\,\Psi_2^2\right), \tag{89}$$

and similarly from the expression (73) for the Cotton–York tensor we similarly obtain

$$Y_{ab}\,Y^{ab} = -2\left(\Psi_0\Psi_4 - 2\,\Psi_1\Psi_3 - 3\,\Psi_2^2\right). \tag{90}$$

Another invariant can be constructed as their specific cubic combination

$$C_{abc}\,C^{abd}\,Y^c{}_d = 6\left(\Psi_0\Psi_3^2 - \Psi_1^2\Psi_4 + 2\,\Psi_0\Psi_2\Psi_4 + 2\,\Psi_1\Psi_2\Psi_3 + 2\,\Psi_2^3\right). \tag{91}$$

It can be immediately seen that for *type N spacetimes*, in which the only non-vanishing Cotton scalar is Ψ_4, one gets

$$C_{abc}\,C^{abc} = 0 = Y_{ab}\,Y^{ab} \qquad \text{and} \qquad C_{abc}\,C^{abd}\,Y^c{}_d = 0\,. \tag{92}$$

In fact, *all algebraic types* can be uniquely identified by using such invariants expressed in terms of the specific polynomials constructed from the Cotton scalars $\Psi_{\rm A}$. The two key invariants are

$$\begin{aligned} I &= \tfrac{1}{4}\,C_{abc}\,C^{abc} = -\tfrac{1}{2}\,Y_{ab}\,Y^{ab}\,,\\ J &= \tfrac{1}{6}\,C_{abc}\,C^{abd}\,Y^c{}_d\,. \end{aligned} \tag{93}$$

In view of (89)–(91) they can be defined as

$$\begin{aligned} I &\equiv \Psi_0\Psi_4 - 2\,\Psi_1\Psi_3 - 3\,\Psi_2^2\,,\\ J &\equiv 2\,\Psi_0\Psi_2\Psi_4 + 2\,\Psi_1\Psi_2\Psi_3 + 2\,\Psi_2^3 + \Psi_0\Psi_3^2 - \Psi_4\Psi_1^2\,. \end{aligned} \tag{94}$$

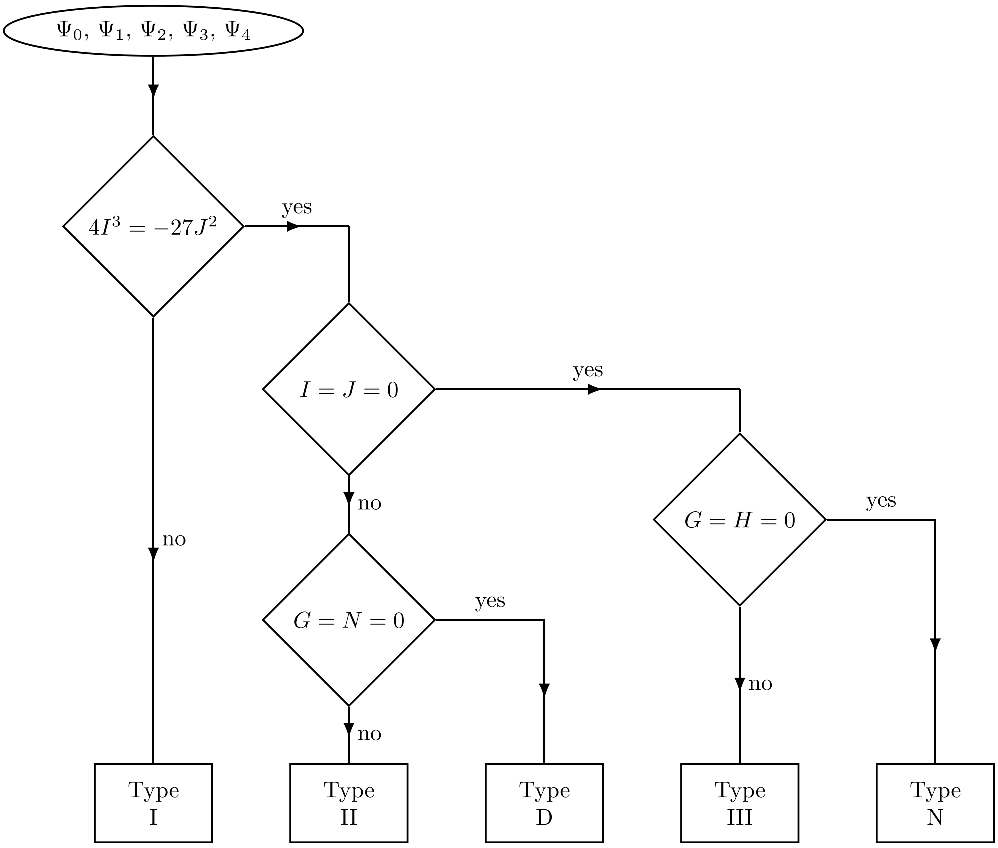

Figure 1. Flow diagram for determining the algebraic type of a 2+1 geometry using the invariants (94) and (97) constructed from the Cotton scalars $\Psi_{\rm A}$. In the special case $\Psi_4 = 0 = \Psi_0$ it is necessary to employ table 4.

These unique invariants naturally occur in the expression for the *discriminant Δ of the key quartic equation* (48), after dropping the primes. Indeed, a direct calculation shows that

$$-\Delta = 2^8\, I^3 + 2^6 3^3\, J^2\,. \tag{95}$$

It is well-known that the necessary and sufficient condition for the quartic equation to have a *multiple root* is $\Delta = 0$. In the present context it means that a spacetime is *algebraically special* (it admits at least one *multiple CAND*), i.e. it is at least of type II, if and only if

$$4\, I^3 = -27\, J^2\,. \tag{96}$$

Furthermore, if and only if $I = 0 = J$ the key equation (48) has at least a *triple root* and the corresponding spacetime is of type III or of type N. To distinguish them we define additional quantities

$$G \equiv \Psi_1 \Psi_4^2 - 3\,\Psi_2\Psi_3\Psi_4 - \Psi_3^3\,,$$
$$H \equiv 2\,\Psi_2\Psi_4 + \Psi_3^2\,,$$

Table 8. Consistency of the algebraic classification based on the normal forms of the Cotton–York tensor $Y_a{}^b$ with the invariants calculated from the corresponding specific values of the Cotton scalars Ψ_{A}.

Algebraic type	Special values of Ψ_{A}	Corresponding invariants
I	$\Psi_1 = 0 = \Psi_3$ $\Psi_0 = \frac{1}{2}(\lambda_1 - \lambda_2) = -\Psi_4$ $\Psi_2 = \frac{1}{2}(\lambda_1 + \lambda_2)$	$I = \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)^2$ $J = \lambda_1\lambda_2(\lambda_1 + \lambda_2)$
II	$\Psi_0 = 0,\ \Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$ $\Psi_4 = 2$	$I = -3\lambda_1^2$ $J = 2\lambda_1^3$ $G = 0,\ N = 36\lambda_1^2$
D	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$	$I = -3\lambda_1^2$ $J = 2\lambda_1^3$ $G = 0 = N$
III	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_2$ $\Psi_3 = \sqrt{2}$	$I = 0 = J$ $G = -2\sqrt{2},\ H = 2$
N	$\Psi_0 = 0 = \Psi_2$ $\Psi_1 = 0 = \Psi_3$ $\Psi_4 = 2$	$I = 0 = J$ $G = 0 = H$

$$N \equiv 3H^2 + \Psi_4^2 I. \tag{97}$$

A spacetime is of algebraic type N if $I = 0 = J$ and $G = 0 = H$. Finally, algebraically special spacetime with $I \neq 0 \neq J$ is of type D if and only if $G = 0 = N$ (otherwise it remains of type II).

These conditions follow from first eliminating the *cubic term* in the quartic equation (48), resulting in the so called *depressed quartic*. The quantity G is the coefficient in front of the *linear term* in this depressed quartic. Its vanishing reduces the equation to bi-quadratic equation, from which the subsequent analysis depending on $H = 0$ and $N = 0$ immediately follows.

The useful complete[6] algorithm of algebraic classification is synoptically summarized by the flow diagram in figure 1. Actually, it is a one-to-one analogue of the flow diagram for the algebraic classification of $D = 4$ spacetimes presented in the original work [32] by d'Inverno and Russell-Clark, and in figure 9.1 of [10].

With the help of the practical algorithm in figure 1, we can finally confirm the classification into the algebraic types contained in table 6. The invariants I, J, G, H, N for the corresponding special values of Ψ_{A} in the principal Cotton–York basis (contained in the last column of table 6) are shown in the last column of table 8. Their values and mutual relations are fully consistent with the flow diagram scheme in figure 1.

13. Complex CANDs and complex eigenvalues

We have already mentioned at the end of section 9 that in 2+1 gravity the algebraic classification suffers from the (somewhat unwelcome) property that the key *real* equation (48) can,

[6] The procedure is not applicable if $\Psi_4 = 0$ and $\Psi_0 \neq 0$. However, in such a case it is possible to perform the swap $\Psi_0 \leftrightarrow -\Psi_4$ and $\Psi_1 \leftrightarrow -\Psi_3$ in the expressions, after which the algorithm in figure 1 can be used.

in general, have some *complex roots* K. More specifically, an equation of the fourth order can have either four (possibly multiple) real roots, or four complex roots, or two real and two complex roots. This implies that some of the null vectors $\boldsymbol{k} = \boldsymbol{k}' - \sqrt{2}K\boldsymbol{m}' + K^2\boldsymbol{l}'$, representing the Cotton-aligned PNDs CANDs obtained by (46), *may be complex*. This cannot happen neither in 3+1 gravity (because the Newman–Penrose formalism in $D = 4$ with the Weyl scalars Ψ_A is complex, and thus complex roots K are allowed, leading to four real PNDs) nor in higher dimensional gravity $D > 4$ (because the corresponding real quartic equation involves more parameters K_i, leading to four real WANDs—except in type G spacetimes).

It is thus natural to suggest a *subclassification* which, for each algebraic type, *distinguishes the real and complex CANDs*. In particular, we may introduce the definition:

- Subtypes $\mathrm{I_r}$, $\mathrm{II_r}$ and $\mathrm{D_r}$: all four (possibly multiple) *CANDs are real*,
- Subtypes $\mathrm{I_c}$, $\mathrm{II_c}$ and $\mathrm{D_c}$: some of the *CANDs are (formally) complex*.

We need not distinguish such subtypes for geometries of algebraic type III and type N. Indeed, if a quartic equation (48) admits a root of multiplicity three or four, necessarily all roots must be real, and thus $\mathrm{III} \equiv \mathrm{III_r}$ and $\mathrm{N} \equiv \mathrm{N_r}$.

On the other hand, for type I and type II (and thus also for type D) geometries we have to investigate the complexity of the roots of the equation (48), which is best done using the invariants (94) and (97)[7].

- For *type I* (when $4I^3 \neq -27J^2$) the equation (48) has four distinct roots. If
$$4I^3 > -27J^2\,, \tag{98}$$
the discriminant Δ given by (95) is negative, and the equation has two distinct real roots and a pair of (conjugated) complex roots. It means that the geometry is of subtype $\mathrm{I_c}$ with two complex CANDs.

  If the relation (98) is *not* satisfied, there exist either four distinct real roots, or four distinct complex roots. If both the relations
$$H > 0 \qquad \text{and} \qquad N > 0 \tag{99}$$
hold then there are four real roots and the geometry is of subtype $\mathrm{I_r}$ with four real CANDs. Otherwise, all roots of the equation (48) are complex and the algebraic subtype is $\mathrm{I_c}$.
- For *type II* (when $4I^3 = -27J^2$) the discriminant Δ vanishes and there is at least one double root. The only possibility of the subtype $\mathrm{II_c}$ is when there is one double real root and a pair of complex conjugated roots and thus CANDs. This happens if and only if
$$N < 0\,. \tag{100}$$
For $G = 0 = N$ the geometry is of type D (see figure 1). The subtype $\mathrm{D_c}$ with two double complex CANDs occurs when the equation (48) admits two conjugated complex roots of multiplicity two. This can only happen in the case when
$$H < 0\,. \tag{101}$$
In particular, with the canonical form of the Cotton scalars $\Psi_0 = 0 = \Psi_1$ with $\Psi_2 \neq 0$, the subtype $\mathrm{II_r}$ occurs if and only if $3\Psi_2\Psi_4 + \Psi_3^2 > 0$. In the opposite case $3\Psi_2\Psi_4 + \Psi_3^2 < 0$ it is of subtype $\mathrm{II_c}$. The case $3\Psi_2\Psi_4 + \Psi_3^2 = 0$ gives the geometry of subtype $\mathrm{D_r}$ with two double real CANDs.

[7] We assume that $\Psi_4 \neq 0$, otherwise we perform the swap $\Psi_0 \leftrightarrow -\Psi_4$ and $\Psi_1 \leftrightarrow -\Psi_3$, provided $\Psi_0 \neq 0$.

Actually, the presence of the complex CANDs in the subtypes $\mathrm{I_c}$, $\mathrm{II_c}$ and $\mathrm{D_c}$ is (indirectly) related to the known property that the Cotton–York matrix ${Y_a}^b$ *is not symmetric* and thus its *eigenvalues can be complex*, see [18] and section 20.5.2 in the monograph [25]. Therefore, in these works a special algebraic subtype denoted as Class I$'$ was introduced. This represents the case when the Cotton–York matrix has three distinct eigenvalues—of which one is real and two are complex (necessarily complex conjugated).

In view of the Jordan form J of ${Y_a}^b$ (equal to its normal form N) and the corresponding Cotton scalars Ψ_{A}, given in tables 5 and 6, we can relate the *complex eigenvalues* of Class I$'$, written as

$$\lambda_1 \equiv \lambda_r + \mathrm{i}\,\lambda_c \qquad \text{and} \qquad \lambda_2 \equiv \lambda_r - \mathrm{i}\,\lambda_c\,, \tag{102}$$

(so that $\lambda_3 = -\lambda_1 - \lambda_2 = -2\lambda_r$) to the canonical values of the Cotton scalars as $\Psi_1 = 0 = \Psi_3$,

$$\Psi_2 = \lambda_r\,, \qquad \Psi_0 = \mathrm{i}\,\lambda_c\,, \qquad \Psi_4 = -\mathrm{i}\,\lambda_c\,. \tag{103}$$

It means that the Cotton scalars Ψ_0 and Ψ_4 *are purely imaginary* (and complex conjugated).

However, in this case it is more appropriate to employ the equivalent form of ${Y_a}^b$ given in table in section 20.5.2 in [25], namely

$${Y_a}^b = \begin{pmatrix} \lambda_r & \lambda_c & 0 \\ -\lambda_c & \lambda_r & 0 \\ 0 & 0 & -2\lambda_r \end{pmatrix}. \tag{104}$$

In view of (79), the corresponding *real* Cotton scalars take the values $\Psi'_1 = 0 = \Psi'_3$ and

$$\Psi_2 = \lambda_r\,, \qquad \Psi_0 = \lambda_c = \Psi'_4\,. \tag{105}$$

In this case the key equation (48) for determining the CANDs becomes

$$\lambda_c K^4 - 6\lambda_r K^2 - \lambda_c = 0\,, \tag{106}$$

and the four solutions to this bi-quadratic equation are

$$K_{\pm}^2 = 3\frac{\lambda_r}{\lambda_c} \pm \sqrt{9\frac{\lambda_r^2}{\lambda_c^2} + 1}\,, \tag{107}$$

so that $K_+^2 > 0$ and $K_-^2 < 0$. There are thus *two complex CANDs corresponding to* $\pm\mathrm{i}\,|K_-|$. This shows that the Class I$'$ defined in [18, 25] is equivalent to the case $4I^3 > -27J^2$ of subtype $\mathrm{I_c}$, introduced here. If (and only if) $\lambda_c = 0$ then the only non-trivial (real) Cotton scalar (103) or (105) is $\Psi_2 = \lambda_r$. The eigenvalues are $\lambda_1 = \lambda_2 = \Psi_2$ and $\lambda_3 = -2\Psi_2$. They are real, and the spacetime is of type D.

Moreover, it can be seen that our subtypes $\mathrm{I_r}$ and $\mathrm{I_c}$ are directly related to the Petrov–Segre types $\mathrm{I}_{\mathbb{R}}$ and $\mathrm{I}_{\mathbb{C}}$ in TMG, as introduced in [22, 23], with real and complex eigenvalues of the Cotton–York/traceless Ricci tensors, respectively.

On the other hand, it follows from table 6 that type II and type D spacetimes have only real eigenvalues and real Cotton scalars Ψ_{A}, so *it is not necessary* to introduce analogous Class II$'$ and Class D$'$. However, the CANDs given by the complex roots of (48) *can be complex*. It thus seems that it is useful to define the subtypes $\mathrm{I_c}$, $\mathrm{II_c}$, $\mathrm{D_c}$, respectively, to denote these subcases.

14. Explicit examples of the new classification method

Finally, to demonstrate the usefulness of our simple classification scheme based on the Cotton scalars $\Psi_{\rm A}$, as summarized in table 1 (see also table 3) and in the flow diagram figure 1, we will apply it to several explicit classes of 2+1 geometries.

14.1. Robinson–Trautman spacetimes with Λ and electromagnetic field

Let us consider a large class of the Robinson–Trautman spacetimes with a cosmological constant Λ and an *aligned* electromagnetic field. Recently in [33] we derived that in the geometrically adapted canonical coordinates r, u, x the *most general form* of such 2+1 solutions to Einstein–Maxwell equations (with a coupling constant $\kappa_0 > 0$) can be written as

$$\mathrm{d}s^2 = \frac{r^2}{P^2}\left(\mathrm{d}x + e\,P^2\mathrm{d}u\right)^2 - 2\,\mathrm{d}u\,\mathrm{d}r + \left(\mu\,Q^2 - \kappa_0\,Q^2\ln\left|\frac{Q}{r}\right| + 2\,(\ln Q)_{,u}\,r + \Lambda\,r^2\right)\mathrm{d}u^2\,, \tag{108}$$

with the Maxwell field potential

$$\mathbf{A} = Q\,\ln\frac{r}{r_0}\,\mathrm{d}u\,, \tag{109}$$

so that $\mathbf{F} = (Q/r)\,\mathrm{d}r\wedge\mathrm{d}u$, see equations (180) and (182) of [33]. Here μ is a constant, $Q(u)$ is *any* function of u, and the metric functions $P(u,x)$, $e(u,x)$ satisfy the field equation

$$\left(\frac{Q}{P}\right)_{,u} = Q\,(e\,P)_{,x}\,. \tag{110}$$

Now, using the general components of the Ricci tensor R_{ab} (see equations (A24)–(A29) of [33]) the Cotton tensor C_{abc} corresponding to the solution (108) can be calculated from the definition (3). Its non-vanishing coordinate components are

$$\begin{aligned}
C_{urr} &= \frac{\kappa_0 Q^2}{2\,r^3}\,,\\
C_{uru} &= L_r\,\frac{\kappa_0 Q^4}{2\,r^3} + A_{uru}\,\frac{1}{r^2} + \left(e^2P^2 - \Lambda\right)\frac{\kappa_0 Q^2}{2\,r}\,,\\
C_{xru} &= e\,\frac{\kappa_0 Q^2}{2\,r}\,,\\
C_{xuu} &= A_{xuu} - \left(P\,(eP)_{,x} + \frac{P_{,u}}{P}\right)_{,x}\left(\frac{\kappa_0}{2} + L_r\right)\frac{Q^2}{r}\,,\\
C_{urx} &= e\,\frac{\kappa_0 Q^2}{2\,r}\,,\\
C_{xrx} &= \frac{\kappa_0 Q^2}{2\,r\,P^2}\,,\\
C_{xux} &= \left((eP)_{,x} + \frac{P_{,u}}{P^2}\right)\left(\frac{3}{2}\,\kappa_0 + L_r\right)\frac{Q^2}{P} - (2\,\kappa_0 + L_r)\,\frac{Q\,Q_{,u}}{P^2}\,,
\end{aligned} \tag{111}$$

where the function $L_r(u,r)$ is

$$L_r \equiv \kappa_0\,\ln\left|\frac{Q}{r}\right| - \mu\,, \tag{112}$$

and the more involved functions A_{uru} and A_{uux} are

$$
\begin{aligned}
A_{uru} &\equiv (\kappa_0 + L_r)\, Q\, Q_{,u} - \left(P\,(eP)_{,x} + \frac{P_{,u}}{P}\right)\left(\frac{3}{2}\,\kappa_0 + L_r\right) Q^2 \,, \\
A_{xuu} &\equiv \left(P\,(eP)_{,x} + \frac{P_{,u}}{P}\right)\left(\frac{3}{2}\,\kappa_0 + L_r\right) e\, Q^2 - (2\kappa_0 + L_r)\, e\, Q\, Q_{,u} \\
&\quad + \left[P\,(eP)_{,x} + \frac{P_{,u}}{P}\right]_{,x} \frac{Q_{,u}}{Q} + P_{,x}\,(P\,e_{,u})_{,x} + P\,(P_{,x}\,e_{,u})_{,x} + P\,(P\,e_{,ux})_{,x} \\
&\quad + \left(\frac{P_{,uu}}{P}\right)_{,x} - 4\,\frac{P_{,u}}{P}\left(\frac{P_{,u}}{P}\right)_{,x} - \left[2e^2P^2P_{,x}^2 + P^4\left(e_{,x}^2 + e\,e_{,xx}\right) + e\,P^3\,(5e_{,x}P_{,x} + e\,P_{,xx})\right]_{,x} .
\end{aligned}
\tag{113}
$$

These coordinate expressions of the Cotton tensor components are very complicated. However, using the *natural null triad* satisfying (7),

$$
\boldsymbol{k} = \partial_r \,, \qquad \boldsymbol{l} = \frac{1}{2}\, g_{uu}\,\partial_r + \partial_u \,, \qquad \boldsymbol{m} = \frac{1}{\sqrt{g_{xx}}}\,(g_{ux}\,\partial_r + \partial_x) \,, \tag{114}
$$

(see equation (6) of [33]), the definition (9) and the field equation (110), we obtain *simple Cotton scalars* Ψ_{A}, namely

$$
\begin{aligned}
\Psi_0 &= 0 \,, \\
\Psi_1 &= -\frac{\kappa_0 Q^2}{2r^3} \,, \\
\Psi_2 &= -eP\,\frac{\kappa_0 Q^2}{2r^2} \,, \\
\Psi_3 &= \left(\kappa_0 \ln\left|\frac{Q}{r}\right| - \mu\right)\frac{\kappa_0 Q^4}{4r^3} + \left(e^2P^2 - \Lambda\right)\frac{\kappa_0 Q^2}{4r} \,, \\
\Psi_4 &= eP\left(\kappa_0 \ln\left|\frac{Q}{r}\right| - \mu\right)\frac{\kappa_0 Q^4}{2r^2} + eP\left(e^2P^2 - \Lambda\right)\frac{\kappa_0 Q^2}{2} \,.
\end{aligned}
\tag{115}
$$

Because $\Psi_0 = 0$, it is obvious from table 1 that *all* such Robinson–Trautman spacetimes with a cosmological constant Λ and an aligned electromagnetic field are (at least) of *algebraic type I*. Moreover, it follows from sections 8 and 9 that the null vector $\boldsymbol{k} = \partial_r$ is CAND. In other words, this CAND $\boldsymbol{k} = \partial_r$ coincides with the privileged null-aligned direction of the electromagnetic field.

In fact, the general Cotton scalars (115) *can considerably be further simplified* just by performing a suitable Lorentz transformation of the triad. In particular, the null rotation (41) with $\boldsymbol{k}$ fixed, changing $\boldsymbol{l}$ and $\boldsymbol{m}$ as

$$
\boldsymbol{k}' = \boldsymbol{k} \,, \qquad \boldsymbol{l}' = \boldsymbol{l} + \sqrt{2}\,L\boldsymbol{m} + L^2\boldsymbol{k} \,, \qquad \boldsymbol{m}' = \boldsymbol{m} + \sqrt{2}\,L\boldsymbol{k} \,, \tag{116}
$$

transforms the Cotton scalars according to the rule (44). Choosing the specific real parameter L,

$$
\sqrt{2}\,L = -eP\,r \,, \tag{117}
$$

and relabeling the constant μ to the function $m(u) \equiv \mu Q^2(u)$, we get a very nice result

$$
\begin{aligned}
\Psi_0' &= 0\,,\\
\Psi_1' &= -\frac{\kappa_0 Q^2}{2r^3}\,,\\
\Psi_2' &= 0\,,\\
\Psi_3' &= -\left(m - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + \Lambda r^2\right)\frac{\kappa_0 Q^2}{4r^3}\,,\\
\Psi_4' &= 0\,.
\end{aligned}
\tag{118}
$$

Interestingly,

$$
2\,\Psi_3' = \left(m - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + \Lambda r^2\right)\Psi_1'\,.
\tag{119}
$$

These are the Cotton scalars expressed with respect to the unique null triad

$$
\begin{aligned}
\boldsymbol{k}' &= \partial_r\,,\\
\boldsymbol{l}' &= \partial_u + \frac{1}{2}\left(m - \kappa_0\, Q^2 \ln\left|\frac{Q}{r}\right| + 2\,(\ln Q)_{,u}\, r + \Lambda r^2\right)\partial_r - eP^2\,\partial_x\,,\\
\boldsymbol{m}' &= \frac{P}{r}\,\partial_x\,.
\end{aligned}
\tag{120}
$$

Clearly, all these scalars vanish when $Q = 0$, which corresponds to *vacuum* solutions with Λ, and thus necessarily are spacetimes of constant curvature (locally Minkowski, de Sitter or anti-de Sitter), which are *conformally flat*.

In the non-trivial case $Q \neq 0$ with an (aligned) electromagnetic field, the key scalar curvature invariants (94) are

$$
\begin{aligned}
I &= -2\,\Psi_1'\Psi_3' = \left(-m + \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2\right)\frac{\kappa_0^2 Q^4}{4r^6}\,,\\
J &= 0\,.
\end{aligned}
\tag{121}
$$

It is obvious that the fundamental condition (96), that is $4I^3 = -27J^2$, *cannot be satisfied*. Consequently, *all* such spacetimes are of algebraic type I, see also the flow diagram in figure 1. More precisely, here it is necessary to employ table 4 because in this case $\Psi_4' = 0 = \Psi_0'$. Using the fact that $\Psi_1' \neq 0$, $\Psi_2' = 0$, $\Psi_3' \neq 0$, the corresponding row in table 4 determines the type I.

In our work [33] we were able to identify the famous class of (cyclic symmetric) *charged black hole electrostatic solutions* [34] that is the 2+1 analogue to the Reissner–Nordström–(anti-)de Sitter solution, see the metric (192) in [33] and the review given in section 11.2 of [25]. It arises as the special subcase $Q =$ const. and $e = 0$ of the metric (108). In such a situation L given by (117) is trivial ($L = 0$), and (115) is thus identical to (118). In any case, the key invariants (121) remain the same, which implies that these electrostatic black hole spacetimes are of algebraic type I. The same result was obtained already by applying the Petrov classification based on the corresponding Jordan form of the Cotton–York tensor, see section 11.1.5 in [25].

Interestingly, *on the horizons* which are localized by the condition $-m+\kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2 = 0$ the scalar Ψ'_3 vanishes, see (119), so that according to table 4 these horizons are of algebraic type III. Moreover, for (118) the key equation (48) determining the CANDs becomes

$$\left[\left(-m+\kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2\right) K^2 + 2\right] K = 0\,. \tag{122}$$

The square bracket tells us that *above the horizon*, where $-m+\kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2 > 0$, there exist *two complex CANDs*, so that such a region of the spacetime is of algebraic subtype $\mathrm{I_c}$ (which in this case is equivalent to Class I′). Contrarily, below the horizon there are four real CANDs, and therefore the region is of subtype $\mathrm{I_r}$.

14.2. Other examples of 2+1 spacetimes of various algebraic types

In their seminal work [18], García, Hehl, Heinicke and Macías investigated some solutions of Einstein's field equations in 2+1 gravity, as well as solutions of the TMG model of Deser, Jackiw and Templeton, presenting explicit examples for each algebraic class. To further confirm and justify our classification method, we will now apply it to these examples studied in section 7 of [18].

14.2.1. Type I (and type D) spacetime. The line element given by equations (114)–(117) in [18] takes the form

$$\begin{aligned} \mathrm{d}s^2 = &-(a_1+a_2)^2\,\mathrm{d}\psi^2 - 2\,(a_1+a_2)^2 \sinh\theta\,\mathrm{d}\psi\,\mathrm{d}\phi \\ &-\left(a_1^2-a_2^2\right)\sin 2\psi\,\cosh\theta\,\mathrm{d}\theta\,\mathrm{d}\phi + \left(a_1^2\sin^2\psi + a_2^2\cos^2\psi\right)\mathrm{d}\theta^2 \\ &+\left[\left(a_1^2\cos^2\psi + a_2^2\sin^2\psi\right)\cosh^2\theta - (a_1+a_2)^2\sinh^2\theta\right]\mathrm{d}\phi^2\,, \end{aligned} \tag{123}$$

with the natural orthonormal dual basis

$$\begin{aligned} \boldsymbol{\omega}^0 &= -(a_1+a_2)\,(\mathrm{d}\psi + \sinh\theta\,\mathrm{d}\phi)\,, \\ \boldsymbol{\omega}^1 &= a_1\,(-\sin\psi\,\mathrm{d}\theta + \cos\psi\,\cosh\theta\,\mathrm{d}\phi)\,, \\ \boldsymbol{\omega}^2 &= a_2\,(\cos\psi\,\mathrm{d}\theta + \sin\psi\,\cosh\theta\,\mathrm{d}\phi)\,. \end{aligned} \tag{124}$$

In view of (74), the corresponding null triad reads

$$\begin{aligned} \boldsymbol{k} &= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{a_1+a_2} - \frac{\cos\psi\,\tanh\theta}{a_1}\right)\partial_\psi - \frac{1}{a_1}\sin\psi\,\partial_\theta + \frac{1}{a_1}\cos\psi\,\mathrm{sech}\,\theta\,\partial_\phi\right], \\ \boldsymbol{l} &= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{a_1+a_2} + \frac{\cos\psi\,\tanh\theta}{a_1}\right)\partial_\psi + \frac{1}{a_1}\sin\psi\,\partial_\theta - \frac{1}{a_1}\cos\psi\,\mathrm{sech}\,\theta\,\partial_\phi\right], \\ \boldsymbol{m} &= -\frac{1}{a_2}\sin\psi\,\tanh\theta\,\partial_\psi + \frac{1}{a_2}\cos\psi\,\partial_\theta + \frac{1}{a_2}\sin\psi\,\mathrm{sech}\,\theta\,\partial_\phi\,. \end{aligned} \tag{125}$$

The non-vanishing Cotton tensor components of the metric (123) are

$$
\begin{aligned}
C_{\psi\theta\theta} &= 2\,\frac{a_1^3-a_2^3}{a_1a_2\,(a_1+a_2)}\,\sin 2\psi\,,\\
C_{\psi\phi\theta} &= 2\,\frac{\left(a_1^2+a_1a_2+a_2^2\right)\left[a_1+a_2-(a_1-a_2)\cos 2\psi\,\right]}{a_1a_2\,(a_1+a_2)}\cosh\theta\,,\\
C_{\psi\theta\phi} &= -2\,\frac{\left(a_1^2+a_1a_2+a_2^2\right)\left[a_1+a_2+(a_1-a_2)\cos 2\psi\,\right]}{a_1a_2\,(a_1+a_2)}\cosh\theta\,,\\
C_{\psi\phi\phi} &= -2\,\frac{a_1^3-a_2^3}{a_1a_2\,(a_1+a_2)}\sin 2\psi\,\cosh^2\theta\,,\\
C_{\theta\phi\psi} &= 4\,\frac{a_1^2+a_1a_2+a_2^2}{a_1a_2}\cosh\theta\,,\\
C_{\theta\phi\theta} &= -2\,\frac{a_1^3-a_2^3}{a_1a_2\,(a_1+a_2)}\sin 2\psi\,\sinh\theta\,,\\
C_{\theta\phi\phi} &= \frac{\left(a_1^2+a_1a_2+a_2^2\right)\left[3\,(a_1+a_2)+(a_1-a_2)\cos 2\psi\,\right]}{a_1\,a_2\,(a_1+a_2)}\sinh 2\theta\,.
\end{aligned}
\tag{126}
$$

The Cotton scalars (9) evaluated with respect to this basis are simply the constants

$$
\begin{aligned}
\Psi_0 &= -2\,\frac{a_1^3+(a_1+a_2)^3}{a_1^2a_2^2\,(a_1+a_2)^2}\,,\\
\Psi_1 &= 0\,,\\
\Psi_2 &= -2\,\frac{a_1^2+a_1a_2+a_2^2}{a_1^2a_2\,(a_1+a_2)^2}\,,\\
\Psi_3 &= 0\,,\\
\Psi_4 &= -\Psi_0\,,
\end{aligned}
\tag{127}
$$

and the scalar invariants (94) reduce to

$$
I=-16\,\frac{\left(a_1^2+a_1a_2+a_2^2\right)^3}{a_1^4a_2^4\,(a_1+a_2)^4}\,,\qquad J=64\,\frac{\left(a_1^2+a_1a_2+a_2^2\right)^3}{a_1^5a_2^5\,(a_1+a_2)^5}\,.
\tag{128}
$$

It is straightforward to check that the key relation $4I^3=-27J^2$ cannot be satisfied in general, so according to the flow diagram in figure 1 the spacetime (123) is of *algebraic type I*.

We can obtain the four distinct CANDs by performing the null rotation (42) to achieve $\Psi_0'=0$. Because $\Psi_1=0=\Psi_3$ and $\Psi_4=-\Psi_0$, the parameter K has to satisfy the bi-quadratic equation

$$
\Psi_0+6\Psi_2K^2+\Psi_0K^4=0\,,
\tag{129}
$$

see (45). Its four distinct roots are $\pm K$ such that the two distinct values of K^2 are

$$
K^2=\frac{1}{B}\left(-\frac{3}{a_1}-\frac{3}{a_2}-\frac{3a_2}{a_1^2}\pm 2\,\frac{a_1^2+a_1a_2+a_2^2}{a_1^2a_2^2}\,\sqrt{2a_2^2-a_1a_2-a_1^2}\right)\,,
\tag{130}
$$

where

$$B = \frac{3}{a_1} + \frac{3}{a_2} + \frac{2a_1}{a_2^2} + \frac{a_2}{a_1^2}\,. \tag{131}$$

Actually, the metric (123) is an example of a subtype $\mathrm{I_c}$ spacetime that is *not* equivalent to Class I′. It is straightforward to check that the relation (98) does not hold. The conditions (99) are satisfied if and only if

$$a_1 < 0 \qquad \text{and} \qquad -\frac{a_1}{2} < a_2 < -2a_1\,, \tag{132}$$

or

$$a_1 > 0 \qquad \text{and} \qquad -2a_1 < a_2 < -\frac{a_1}{2}\,, \tag{133}$$

and the spacetime is of subtype $\mathrm{I_r}$. Otherwise, it is of subtype $\mathrm{I_c}$ but not of Class I′.

Interestingly, in the *special case* $a_1 = a_2$ the invariants (128) reduce to

$$I = -\frac{27}{a_1^6}\,, \qquad J = \frac{54}{a_1^9}\,, \tag{134}$$

so that $4I^3 = -27J^2$. According to the flow diagram in figure 1, the spacetime (123) becomes *algebraically special*. In fact, because $I, J \neq 0$ and $G = N = 0$, it degenerates to *type D*. This is in full agreement with the results of [18]. Actually, the roots (130) are $K = \pm\mathrm{i}$ so that the two double CANDs are *complex*, and the spacetime is of a subtype $\mathrm{D_c}$ (see section 13).

From (130) we also conclude that the spacetime (123) is of *algebraic type D if and only if* $2a_2^2 - a_1a_2 - a_1^2 = 0$. This has only two solutions, namely $a_1 = a_2$ (discussed above) and $a_1 = -2a_2$. In the latter case we get

$$I = -\frac{27}{a_2^6}\,, \qquad J = \frac{54}{a_2^9}\,, \tag{135}$$

again implying $4I^3 = -27J^2$.

14.2.2. Type I′ spacetime. A generic example of the Class I′ metric is a spherically symmetric spacetime

$$\mathrm{d}s^2 = -\psi(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\psi(r)} + r^2\,\mathrm{d}\varphi^2\,, \tag{136}$$

see equation (111) in [18], which suggests a natural orthonormal dual basis (beware of the opposite signature used in [18])

$$\boldsymbol{\omega}^0 = \sqrt{\psi}\,\mathrm{d}t\,, \qquad \boldsymbol{\omega}^1 = \frac{\mathrm{d}r}{\sqrt{\psi}}\,, \qquad \boldsymbol{\omega}^2 = r\,\mathrm{d}\varphi\,. \tag{137}$$

The corresponding null triad reads

$$\boldsymbol{k} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\psi}}\,\partial_t - \sqrt{\psi}\,\partial_r\right), \qquad \boldsymbol{l} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\psi}}\,\partial_t + \sqrt{\psi}\,\partial_r\right), \qquad \boldsymbol{m} = \frac{1}{r}\,\partial_\varphi\,. \tag{138}$$

The only non-vanishing components of the Cotton tensor are

$$C_{rtt} = \tfrac{1}{4}\psi\,\psi'''\,, \qquad C_{r\varphi\varphi} = \tfrac{1}{4}r^2\,\psi'''\,, \tag{139}$$

where prime denotes the derivative with respect to r. The Cotton scalars (9) simply evaluate to

$$
\begin{aligned}
\Psi_0 &= 0\,,\\
\Psi_1 &= -\tfrac{1}{4\sqrt{2}}\sqrt{\psi}\,\psi'''\,,\\
\Psi_2 &= 0\,,\\
\Psi_3 &= -\Psi_1\,,\\
\Psi_4 &= 0\,.
\end{aligned} \tag{140}
$$

According to table 4 the spacetime is of type I. In view of the equation (48) for CANDs which reduces to

$$\left(K^2+1\right)K = 0\,, \tag{141}$$

we conclude that there are two real CANDs, namely $\boldsymbol{k}$ and $\boldsymbol{l}$, and two complex CANDs given by the solutions $K = \pm\mathrm{i}$. Therefore, the spacetime (136) is of algebraic subtype $\mathrm{I_c}$, which in this case is also equivalent to the Class I′.

14.2.3. Type I spacetime with type II hypersurface. The metric

$$\mathrm{d}s^2 = -\mathrm{e}^{-4y}\,\mathrm{d}t^2 - 2\mathrm{e}^{-2y}\,\mathrm{d}t\,\mathrm{d}x + \left(\mathrm{e}^{2y}-1\right)\mathrm{d}x^2 + \mathrm{d}y^2\,, \tag{142}$$

given by the orthonormal dual basis

$$\boldsymbol{\omega}^0 = \mathrm{e}^{-2y}\,\mathrm{d}t + \mathrm{d}x\,, \qquad \boldsymbol{\omega}^1 = \mathrm{e}^{y}\,\mathrm{d}x\,, \qquad \boldsymbol{\omega}^2 = \mathrm{d}y\,, \tag{143}$$

see equation (132) in [18], has the natural null triad

$$\boldsymbol{k} = \tfrac{1}{\sqrt{2}}\left[\mathrm{e}^{y}\left(\mathrm{e}^{y}-1\right)\partial_t + \mathrm{e}^{-y}\,\partial_x\right], \quad \boldsymbol{l} = \tfrac{1}{\sqrt{2}}\left[\mathrm{e}^{y}\left(\mathrm{e}^{y}+1\right)\partial_t - \mathrm{e}^{-y}\,\partial_x\right], \quad \boldsymbol{m} = \partial_y\,. \tag{144}$$

The non-vanishing components of the Cotton tensor are

$$
\begin{aligned}
C_{txy} &= 4\mathrm{e}^{-4y}\left(1-3\mathrm{e}^{2y}\right),\\
C_{tyt} &= 6\mathrm{e}^{-6y}\left(3-\mathrm{e}^{2y}\right),\\
C_{tyx} &= 2\mathrm{e}^{-4y}\left(7-9\mathrm{e}^{2y}\right),\\
C_{xyt} &= 2\mathrm{e}^{-4y}\left(5-3\mathrm{e}^{2y}\right),\\
C_{xyx} &= 6\mathrm{e}^{-2y}\left(1-\mathrm{e}^{4y}\right).
\end{aligned} \tag{145}
$$

The corresponding Coton scalars (9) read

$$
\begin{aligned}
\Psi_0 &= -6\mathrm{e}^{-3y}\left(1-3\mathrm{e}^{y}+\mathrm{e}^{2y}+\mathrm{e}^{3y}\right),\\
\Psi_1 &= 0\,,\\
\Psi_2 &= \mathrm{e}^{-3y}\left(6\mathrm{e}^{2y}-2\right),\\
\Psi_3 &= 0\,,\\
\Psi_4 &= 6\mathrm{e}^{-3y}\left(1+3\mathrm{e}^{y}+\mathrm{e}^{2y}-\mathrm{e}^{3y}\right),
\end{aligned} \tag{146}
$$

and the scalar invariants (94) are

$$I = 12\left(3 - 4e^{-6y} + 27\mathrm{e}^{-4y} - 30\mathrm{e}^{-2y}\right),$$
$$J = 16\mathrm{e}^{-9y}\left(8 - 81\mathrm{e}^{2y} + 225\mathrm{e}^{4y} - 171\mathrm{e}^{6y} + 27\mathrm{e}^{2y}\right). \tag{147}$$

The key expression $4I^3 + 27J^2 = 432^2\,\mathrm{e}^{-14y}(1+\mathrm{e}^{2y})^4(\mathrm{e}^{6y} - 7\mathrm{e}^{4y} + 7\mathrm{e}^{2y} - 1)$ is nonzero, which implies that the spacetime (142) is *generally of algebraic type I*.

However, on the special hypersurface $y = 0$ this expression reduces to $4I^3 + 27J^2 = 0$. In fact, the only non-vanishing Cotton scalars on $y = 0$ are $\Psi_2 = 4$ and $\Psi_4 = 24$, and thus $I = -48$, $J = 128$, $G = 0$, $N = 288^2$. According to the flow diagram in figure 1 the spacetime (142) is of *type II on* $y = 0$. Since the condition (100) is not satisfied, it is of subtype $\mathrm{II_r}$. Actually, the null vector $\boldsymbol{k}$ of the null triad (144) is the Cotton aligned null direction (CAND) on this hypersurface because $\Psi_0 = 0$ there.

14.2.4. Type I spacetime with type III hypersurface. Let us assume a simple metric

$$\mathrm{d}s^2 = -(t-x)^2\,\mathrm{d}t^2 + (t+x)^2\,\mathrm{d}x^2 + \mathrm{d}y^2, \tag{148}$$

given by the orthonormal basis (133) in [18], namely

$$\boldsymbol{\omega}^0 = (x-t)\,\mathrm{d}t, \qquad \boldsymbol{\omega}^1 = (x+t)\,\mathrm{d}x, \qquad \boldsymbol{\omega}^2 = \mathrm{d}y. \tag{149}$$

The corresponding null triad reads

$$\boldsymbol{k} = \frac{1}{\sqrt{2}}\left(\frac{1}{t-x}\,\partial_t - \frac{1}{t+x}\,\partial_x\right), \qquad \boldsymbol{l} = \frac{1}{\sqrt{2}}\left(\frac{1}{t-x}\,\partial_t + \frac{1}{t+x}\,\partial_x\right), \qquad \boldsymbol{m} = \partial_y. \tag{150}$$

The non-vanishing components of the Cotton tensor are

$$C_{txt} = -\frac{4\left(2t^2x + x^3\right)}{(t+x)^4\,(t-x)^2},$$
$$C_{txx} = \frac{4\left(t^3 + 2tx^2\right)}{(t+x)^2\,(t-x)^4},$$
$$C_{tyy} = -\frac{4\left(t^3 + 2tx^2\right)}{(t+x)^4\,(t-x)^4},$$
$$C_{xyy} = \frac{4\left(2t^2x + x^3\right)}{(t+x)^4\,(t-x)^4}, \tag{151}$$

and thus the Cotton scalars (9) take the form

$$\Psi_0 = 0,$$
$$\Psi_1 = -2\sqrt{2}\,\frac{t^4 + 3t^3x + 3tx^3 - x^4}{(t+x)^5\,(t-x)^5},$$
$$\Psi_2 = 0,$$
$$\Psi_3 = -2\sqrt{2}\,\frac{t^4 - t^3x + 4t^2x^2 + tx^3 + x^4}{(t+x)^5\,(t-x)^5},$$
$$\Psi_4 = 0. \tag{152}$$

In this case we have to be careful because $\Psi_0 = 0 = \Psi_4$ and the algorithm presented in figure 1 is thus not applicable. In such an exceptional case we have to use the table 4 instead, which implies that the spacetime (148) is generally of *type I*.

However, applying this table we may notice that there exist specific subcases which are algebraically special. For example, the condition $\Psi_1 = 0$ with $\Psi_3 \neq 0$ leads to the algebraic *type III*. In view of (152), this can be achieved by taking the constraint $x = t(\sqrt{13}+3)/2$, which is exactly the conclusion presented in [18]. Indeed, *on this hypersurface* the Cotton scalars reduce to

$$\Psi_0 = 0\,, \quad \Psi_1 = 0\,, \quad \Psi_2 = 0\,, \quad \Psi_3 = \frac{64\sqrt{2}\,\left(11+3\sqrt{13}\right)}{27\left(3+\sqrt{13}\right)^5\,t^6}\,, \quad \Psi_4 = 0\,. \tag{153}$$

14.2.5. Type D spacetime. Taking the orthonormal basis given by equations (119)–(121) in [18] (with an opposite signature), we obtain the 2+1 Gödel metric

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{9}{\mu^2}\,\mathrm{d}t^2 + \frac{36}{\mu^2}\left(\sqrt{r^2+1}-1\right)\mathrm{d}t\,\mathrm{d}\phi \\ &+\frac{9}{\mu^2}\left(8\sqrt{r^2+1}-3r^2-8\right)\mathrm{d}\phi^2 + \frac{9}{\mu^2}\,\frac{\mathrm{d}r^2}{r^2+1}\,. \end{aligned} \tag{154}$$

The null triad constructed from the natural basis, namely

$$\boldsymbol{\omega}^0 = \frac{3}{\mu}\left[\mathrm{d}t - 2\left(\sqrt{r^2+1}-1\right)\mathrm{d}\phi\right], \qquad \boldsymbol{\omega}^1 = \frac{3}{\mu}\,\frac{\mathrm{d}r}{\sqrt{r^2+1}}\,, \qquad \boldsymbol{\omega}^2 = \frac{3}{\mu}\,r\,\mathrm{d}\phi\,, \tag{155}$$

has the form

$$\begin{aligned} \boldsymbol{k} &= \frac{1}{\sqrt{2}}\,\frac{\mu}{3}\left[\partial_t - \sqrt{r^2+1}\,\partial_r\right], \\ \boldsymbol{l} &= \frac{1}{\sqrt{2}}\,\frac{\mu}{3}\left[\partial_t + \sqrt{r^2+1}\,\partial_r\right], \\ \boldsymbol{m} &= \frac{\mu}{3r}\left[2\left(\sqrt{r^2+1}-1\right)\partial_t + \partial_\phi\right]. \end{aligned} \tag{156}$$

The non-vanishing components of the Cotton tensor are

$$\begin{aligned} C_{t\phi r} &= -\frac{3r}{\sqrt{r^2+1}}\,, \qquad & C_{tr\phi} &= \frac{3r}{\sqrt{r^2+1}}\,, \\ C_{r\phi t} &= -\frac{6r}{\sqrt{r^2+1}}\,, \qquad & C_{r\phi\phi} &= 18r\left(1-\frac{1}{\sqrt{r^2+1}}\right), \end{aligned} \tag{157}$$

so that the Cotton scalars (9) are simply

$$\begin{aligned} \Psi_0 &= \frac{\mu^3}{6}\,, \\ \Psi_1 &= 0\,, \end{aligned}$$

$$\begin{aligned}
\Psi_2 &= \frac{\mu^3}{18}\,,\\
\Psi_3 &= 0\,,\\
\Psi_4 &= -\frac{\mu^3}{6}\,.
\end{aligned}\tag{158}$$

It is easy to evaluate the invariants (94) and (97),

$$I = -\frac{\mu^6}{27}\,,\qquad J = -\frac{2\mu^9}{27^2}\,,\qquad G = 0\,,\qquad N = 0\,.\tag{159}$$

Using the flow diagram in figure 1, we obtain that the Gödel spacetime (154) is of algebraic *type D*, in agreement with the results of [18]. By solving (48) with the Cotton scalars (158) we obtain the complex roots $K = \pm\mathrm{i}$. Therefore, the two double CANDs are *complex*, and the spacetime is actually of a subtype $\mathrm{D_c}$. In fact, using the *complex* null basis

$$\begin{aligned}
\boldsymbol{k} &= \frac{\mu}{3\sqrt{2}\,r}\left[2\mathrm{i}\left(1-\sqrt{r^2+1}\right)\partial_t + r\sqrt{r^2+1}\,\partial_r - \mathrm{i}\,\partial_\phi\right],\\
\boldsymbol{l} &= \frac{\mu}{3\sqrt{2}\,r}\left[2\mathrm{i}\left(1-\sqrt{r^2+1}\right)\partial_t - r\sqrt{r^2+1}\,\partial_r - \mathrm{i}\,\partial_\phi\right],\\
\boldsymbol{m} &= \frac{\mu}{3}\,\mathrm{i}\,\partial_t\,,
\end{aligned}\tag{160}$$

the *real* Cotton scalars take the canonical form

$$\Psi_0 = 0\,,\qquad \Psi_1 = 0\,,\qquad \Psi_2 = -\frac{\mu^3}{9}\,,\qquad \Psi_3 = 0\,,\qquad \Psi_4 = 0\,,\tag{161}$$

with only Ψ_2 non-vanishing. Therefore, *both* $\boldsymbol{k}$ and $\boldsymbol{l}$ given by (160) are *double CANDs*.

14.2.6. Type N spacetime. An example of type N metric was given by equations (128)–(130) in [18], namely the orthonormal basis

$$\begin{aligned}
\boldsymbol{\omega}^0 &= \mathrm{e}^{\mu y/2}\left[\left(1+\tfrac{1}{2}\mathrm{e}^{-\mu y}\right)\mathrm{d}t + \left(1-\tfrac{1}{2}\mathrm{e}^{-\mu y}\right)\mathrm{d}x\right],\\
\boldsymbol{\omega}^1 &= \tfrac{1}{2}\mathrm{e}^{-\mu y/2}\left(\mathrm{d}t - \mathrm{d}x\right),\\
\boldsymbol{\omega}^2 &= \mathrm{d}y\,,
\end{aligned}\tag{162}$$

which yields the line element

$$\mathrm{d}s^2 = -\left(1+\mathrm{e}^{\mu y}\right)\mathrm{d}t^2 - 2\mathrm{e}^{\mu y}\,\mathrm{d}t\,\mathrm{d}x + \left(1-\mathrm{e}^{\mu y}\right)\mathrm{d}x^2 + \mathrm{d}y^2\,.\tag{163}$$

The corresponding null triad reads

$$\begin{aligned}
\boldsymbol{k} &= \tfrac{1}{\sqrt{2}}\,\mathrm{e}^{\mu y/2}\left(\partial_t - \partial_x\right),\\
\boldsymbol{l} &= \tfrac{1}{\sqrt{2}}\,\mathrm{e}^{-\mu y/2}\left[\left(\mathrm{e}^{\mu y}-1\right)\partial_t - \left(\mathrm{e}^{\mu y}+1\right)\partial_x\right],\\
\boldsymbol{m} &= \partial_y\,.
\end{aligned}\tag{164}$$

The only non-vanishing components of the Cotton tensor are

$$C_{tyt} = C_{tyx} = C_{xyt} = C_{xyx} = -\tfrac{1}{2}\mu^3\mathrm{e}^{\mu y}\,,\tag{165}$$

which projected onto the triad (164) give

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,,\qquad \Psi_4 = -\mu^3\,.\tag{166}$$

In view of the definition of invariants (94) and (97) we obtain

$$I = 0 = J\,, \qquad G = 0 = H\,, \tag{167}$$

and using the flow diagram in figure 1 we immediately see that the spacetime (163) is of type N, in agreement with [18]. Moreover, it is clear that the Cotton scalars (166) are already in the canonical form with only the scalar Ψ_4 non-trival, see table 3. The vector $\boldsymbol{k}$ given by (164) is thus a *quadruple CAND*.

Notice that by performing transformations $u = t + x$ and $r = \frac{1}{2}(t - x)$, the metric (163) takes the canonical Brinkmann form of pp-waves [33, 35]

$$\mathrm{d}s^2 = \mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}r + a\,\mathrm{d}u^2\,, \tag{168}$$

with $\boldsymbol{k} \propto \partial_r$ and specific metric function $a = -\mathrm{e}^{\mu y}$ which depends only on the transverse spatial coordinate y. Actually, it is a VSI spacetime with pure radiation (the only non-trivial component of the energy-momentum tensor is $T_{uu} = \frac{\mu^2}{16\pi}\,\mathrm{e}^{\mu y}$), see equations (101) and (102) in [35].

14.2.7. Type O spacetime. Finally, we consider spherically symmetric solution with perfect fluid of a constant density ρ and pressure $p(r)$,

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F^2} + r^2\,\mathrm{d}\phi^2\,. \tag{169}$$

The metric functions are

$$\begin{aligned} N(r) &= \frac{c_1}{\rho + p(r)}\,,\\ F^2(r) &= c_2 - (\ell\rho + \Lambda)\,r^2\,,\\ p(r) &= \frac{c_3\,(\ell\rho + \Lambda)\,F(r) + c_3^2\,\ell\,\Lambda + \rho F^2(r)}{c_3^2\,\ell^2 - F^2(r)}\,, \end{aligned} \tag{170}$$

where c_1, c_2, c_3, ℓ are constants, see equations (134), (139), (140) and (142) in [18]. The Cotton tensor for this metric identically vanishes ($C_{abc} = 0$), so that its projections onto a null triad give

$$\Psi_{\mathrm{A}} = 0 \quad \text{for all A}\,. \tag{171}$$

The spacetime is *conformally flat*, that is of *algebraic type O*.

15. Summary

We introduced a useful approach to algebraic classification of 2+1 geometries, assuming no particular field equations. It is based on projecting the Cotton tensor onto a null triad. The corresponding five real Cotton scalars Ψ_{A} (which are the 2+1 analogue of well-known 3+1 Newman–Penrose curvature scalars constructed from the Weyl tensor) then simply determine the algebraic types I, II, III, N, D and O by their gradual vanishing, starting with those of the highest boost weight, see table 1. Moreover, such a classification is directly related to the specific multiplicity of the CANDs and to the Bel–Debever criteria, see tables 2 and 3, respectively. We also derived a synoptic algorithm of the algebraic classification based on the polynomial curvature invariants (94) and (97), see figure 1 (or table 4 when $\Psi_4 = 0 = \Psi_0$).

Using the bivector decomposition, we showed that our method is equivalent to the previously introduced Petrov-type classification of 2+1 spacetimes based on the eigenvalue problem and respective canonical Jordan form of the Cotton–York tensor, see tables 5 and 6.

In addition, we introduced a refinement of the algebraic types into the subtypes $\mathrm{I_r}$, $\mathrm{II_r}$, $\mathrm{D_r}$ (for which all CANDs are real) and subtypes $\mathrm{I_c}$, $\mathrm{II_c}$, $\mathrm{D_c}$ (for which some of the CANDs are complex). The subtype $\mathrm{I_c}$ is related to the Class I' defined in [18, 25], and the subtypes $\mathrm{I_r}$ and $\mathrm{I_c}$ correspond to the Petrov–Segre types $\mathrm{I}_\mathbb{R}$ and $\mathrm{I}_\mathbb{C}$ in TMG defined in [22].

In final section 14 we demonstrated the practical usefulness of our novel method on several explicit examples of various algebraic types. We hope that it will prove to be helpful for classification and analysis of other interesting spacetimes in 2+1 gravity.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

This work was supported by the Czech Science Foundation Grant No. GAČR 23-05914S.

ORCID iDs

Matúš Papajčík https://orcid.org/0000-0002-5664-1670
Jiří Podolský https://orcid.org/0000-0003-2998-6830

References

[1] Petrov A Z 1954 Classification of spaces defined by gravitational fields *Uch. Zapiski Kazan Gos. Univ.* **144** 55–75
[2] Géhéniau J 1957 Une classification des espaces riemanniens *C. R. Acad. Sci., Paris* **244** 723–4
[3] Pirani F A E 1957 Invariant formulation of gravitational radiation theory *Phys. Rev.* **105** 1089–99
[4] Bel L 1959 La radiation gravitationnelle *Sém. méc. analytique méc. céleste* **2** 1–16
[5] Debever R 1959 Super-energy in general relativity *Bull. Soc. Math. Belg.* **10** 112–47
[6] Debever R 1959 Tenseur de super-énergie, tenseur de Riemann: cas singuliers *C. R. Acad. Sci., Paris* **249** 1744–6
[7] Penrose R 1960 A spinor approach to general relativity *Ann. Phys., NY* **10** 171–201
[8] Newman E and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients *J. Math. Phys.* **3** 566–79 Newman E and Penrose R 1963 *J. Math. Phys.* **4** 998 (erratum)
[9] Penrose R and Rindler W 1984 *Spinors and Space-Time* (Cambridge University Press)
[10] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* (Cambridge University Press)
[11] Coley A, Milson R, Pravda V and Pravdová A 2004 Classification of the Weyl tensor in higher dimensions *Class. Quantum Grav.* **21** L35–L41
[12] Milson R, Coley A, Pravda V and Pravdová A 2005 Alignment and algebraically special tensors in Lorentzian geometry *Int. J. Geom. Methods Mod. Phys.* **2** 41–61
[13] Coley A 2008 Classification of the Weyl tensor in higher dimensions and applications *Class. Quantum Grav.* **25** 033001
[14] Ortaggio M, Pravda V and Pravdová A 2013 Algebraic classification of higher dimensional spacetimes based on null alignment *Class. Quantum Grav.* **30** 013001
[15] Krtouš P and Podolský J 2006 Asymptotic structure of radiation in higher dimensions *Class. Quantum Grav.* **23** 1603–15

[16] Cotton É 1899 Sur les variétés à trois dimensions *Ann. Fac. Sci. Toulouse* II **1** 385–438
[17] Barrow J D, Burd A B and Lancaster D 1986 Three-dimensional classical spacetimes *Class. Quantum Grav.* **3** 551–67
[18] García A A, Hehl F W, Heinicke C and Macías A 2004 The Cotton tensor in Riemannian spacetimes *Class. Quantum Grav.* **21** 1099–118
[19] Hall G S and Capocci M S 1999 Classification and conformal symmetry in three-dimensional space-time *J. Math. Phys.* **40** 1466–78
[20] Torres del Castillo G F and Gómez-Ceballos L F 2003 Algebraic classification of the curvature of three-dimensional manifolds with indefinite metric *J. Math. Phys.* **44** 4374–80
[21] Sousa F C, Fonseca J B and Romero C 2008 Equivalence of three-diemnsional spacetimes *Class. Quantum Grav.* **25** 035007
[22] Chow D D K, Pope C N and Sezgin E 2010 Classification of solutions in topological massive gravity *Class. Quantum Grav.* **27** 105001
[23] Chow D D K, Pope C N and Sezgin E 2010 Kundt spacetimes as solutions of topological massive gravity *Class. Quantum Grav.* **27** 105002
[24] Plebański J F 1964 The algebraic structure of the tensor of matter *Acta Phys. Polon.* **26** 963–1020
[25] García-Díaz A A 2017 *Exact Solutions in Three-Dimensional Gravity* (Cambridge University Press)
[26] Schouten J A 1954 *Ricci-Calculus* (Springer)
[27] Ortaggio M 2009 Bel–Debever criteria for the classification of the Weyl tensor in higher dimensions *Class. Quantum Grav.* **26** 195015
[28] Senovilla J M M 2010 Algebraic classification of the Weyl tensor in higher dimensions based on its 'superenergy' tensor *Class. Quantum Grav.* **27** 222001
[29] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 Metric-Affine gauge theory of gravity: field equations, Noether identities, world spinors and breaking of dilation invariance *Phys. Rep.* **258** 1–71
[30] York Jr J W 1971 Gravitational degrees of freedom and the initial-value problem *Phys. Rev. Lett.* **26** 1656
[31] Arnowitt R, Deser S and Misner C W 1962 The dynamics of general relativity *Gravitation: An Introduction to Current Research* ed L Witten (Wiley)
[32] d'Inverno R A and Russell-Clark R A 1971 Classification of the Harrison metrics *J. Math. Phys.* **12** 1258–63
[33] Podolský J and Papajčík M 2022 All solutions of Einstein–Maxwell equations with a cosmological constant in 2+1 dimensions *Phys. Rev.* D **105** 064004
[34] Peldan P 1993 Unification of gravity and Yang-Mills theory in (2+1)-dimensions *Nucl. Phys.* B **395** 239–62
[35] Podolský J, Švarc R and Maeda H 2019 All solutions of Einstein's equations in 2+1 dimensions: Λ-vacuum, pure radiation, or gyratons *Class. Quantum Grav.* **36** 015009