Quantum field theoretical methods in the theory of turbulence

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May 31st, Ondřejov
Overview

1. Kolmogorov theory
   - Fully developed turbulence

2. Quantum field theory

3. QFT and stochastic dynamics

4. Without conclusion
Transition from laminar to turbulent flow

Flow past the cylinder

- Small Reynolds numbers \((R = 0.16)\)
  - Up-down symmetry
  - Left-right symmetry (not exact)
  - \(t\)-invariance (stationarity)
  - \(z\)-invariance (2D character of the flow)
Transition from laminar to turbulent flow

Flow past the cylinder

- Higher Reynolds numbers ($R \approx 40$)
  - Left-right symmetry completely broken
  - Stationarity replaced by periodicity (*Andronov-Hopf bifurcation*)
  - Continuous $t$–invariance replaced by discrete one
Transition from laminar to turbulent flow

Flow past the cylinder

- Even higher Reynolds numbers \((R \approx 140)\)
  - Breakdown of \(z\)–symmetry somewhere between \(R = 40\) and \(R = 70\)
  - Formation of Kármán streets
Transition from laminar to turbulent flow

- Transition to turbulent flow is connected with gradual loss of symmetries of the initial flow.
- However, as the flow becomes more and more chaotic, even higher symmetries are recovered in the limit $R \to \infty$.

\[ R = 240 \quad \text{and} \quad R = 1800 \]
Fully developed turbulence

Fully developed turbulence (FDT) for $R \rightarrow \infty$

- Translational and rotational invariance
- Stationarity, homogeneity and isotropy in statistical sense
Kolmogorov theory

Phenomenological picture of FDT

- **Three scales**
  - Dissipation scale $r_d$
    - Dissipation of energy by viscosity
    - Molecular scale
  - External scale $r_l$
    - Scale of energy injection
    - Creation of large eddies
  - Inertial range $r_d \ll r \ll r_l$
    - Transfer of energy from large scales to the smaller ones
    - Governed by non-linearity of the NS equation
Kolmogorov theory

- Only statistical description is of interest
- Correlation tensor
  \[ S_{ij} = \langle v_i(x_1) v_j(x_2) \rangle \]
- Structure functions
  \[ S_N = \langle [v_r(x_1) - v_r(x_2)]^N \rangle \]
- In appropriate frame
  \[ \langle v_i(x) \rangle = 0 \]
- Homogeneity, isotropy and stationarity means
  \[ S_N = S_N (|x_1 - x_2|) \]
Kolmogorov theory

Characteristic parameters of fully developed turbulence

- Dissipation rate or energy injection rate
  \[ \varepsilon = \frac{\text{energy}}{\text{time} \times \text{mass}}, \quad [\varepsilon] = L^2 T^{-3} \]

- The “mass” (lower bound for wave number \( k \))
  \[ m = \frac{1}{r_l}, \quad [m] = L^{-1} \]

- Inverse dissipation scale (upper bound for \( k \), not a cosmological constant!)
  \[ \Lambda = \frac{1}{r_d}, \quad [\Lambda] = L^{-1} \]
Kolmogorov theory

Kolmogorov hypotheses

First hypothesis

H1. The properties of FDT in the inertial range do not depend on the details of energy injection, only on the total energy injection rate $\varepsilon$.

In other words, $S_N$ does not depend on $r_l, m$

Second hypothesis

H2. The properties of FDT in the inertial range do not depend on the details of the dissipation.

Hence, $S_N$ does not depend on $r_d, \nu$
Kolmogorov theory

- Dimension of the structure function
  \[ [S_N] = L^N T^{-N} \]

- Inside the inertial range \( r_d \ll r \ll r_l \)

- Scale-invariant law
  \[ S_N \propto (\varepsilon r)^{N/3} \]

- Kinetic energy per unit mass
  \[
  E = \frac{1}{2} \langle v_i v_i \rangle = \int_0^\infty E(k) \, dk, \quad [E(k)] = L^3 T^{-2}
  \]

- Kolmogorov “5/3” law
  \[ E(k) = K_0 \varepsilon^{2/3} k^{-5/3} \]
Drawbacks of Kolmogorov theory

- Experimental deviations from “5/3” law
  add experimental figure
- Experiments show dependence on $r_l$

$$S_N = (\varepsilon r)^{N/3} \left(\frac{r}{r_l}\right)^{q_N}$$

- Anomalous scaling

$$q_N < 0 \quad \rightarrow \quad \lim_{r/r_l \to 0} S_N = \infty$$
Path integrals: a brief review

- Quantum mechanics in 1D
- Propagator is the amplitude of probability of
  \[ |q_0\rangle \text{ at time } 0 \quad \rightarrow \quad |q_1\rangle \text{ at time } t \]
- Definition
  \[ K(q_1, t; q_0, 0) = \langle q_1 | e^{-it\hat{H}} | q_0 \rangle \]
- Path integral expression
  \[ K(q_1, t; q_0, 0) = \int \mathcal{D}q(t) \ e^{iS[q]} \]
  where \( S[q] \) is classical action
  \[ S[q] = \int \left[ \frac{1}{2} m \dot{q}^2 - V(q) \right] \, dt \]
Quantum field theory

- Action for free scalar field

\[ S_0[\phi] = \int \left[ \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right] d^4x \]

- Green’s correlation functions

\[ G^{(n)}(x_1, \ldots x_n) = \langle 0 | \hat{T}\phi(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle \]

- Path integral expression

\[ G^{(n)}(x_1, \ldots x_n) = \frac{1}{Z} \int \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]} D\phi \]
Quantum field theory

Generating functional for Green functions

\[ G_0[J] = \frac{\int e^{iS_0[\phi] + iJ\phi} \mathcal{D}\phi}{\int e^{iS_0[\phi]} \mathcal{D}\phi}, \quad J\phi \equiv \int \phi(x) J(x) \, d^4x \]

Generating functional

\[ G_0[J] = e^{iS_0[\phi_c] + iJ\phi_c} = e^{-\frac{1}{2} J \Delta_F J} \]

where \( \phi_c \) is classical solution of \( (\Box + m^2)\phi_c = J \):

\[ \phi_c(x) = i \int \Delta_F(x, x') J(x') \, d^4x' \]
Quantum field theory

- Green functions can be derived from $G_0$

$$G^{(n)}(x_1, \ldots x_n) = \frac{1}{i^n} \left. \frac{\delta^n G_0[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0}$$

- Taylor expansion

$$G_0[J] = e^{-\frac{1}{2} J \Delta_F J} = 1 - \frac{1}{2} J \Delta_F J + \frac{1}{8} (J \Delta_F J)^2 + \ldots$$

- One-point Green function

$$G^{(1)}(x) = \left\langle 0 \mid \hat{\phi}(x) \mid 0 \right\rangle = \left. \frac{\delta G_0}{\delta J(x)} \right|_{J=0} = 0$$
Two-point Green function

\[ G^{(2)}(x_1, x_2) = \left\langle 0 \mid \hat{\phi}(x_1) \hat{\phi}(x_2) \mid 0 \right\rangle = \Delta_F(x_1, x_2) \]

or diagramatically

\[ G^{(2)}(x_1, x_2) = x_1 \quad \text{--} \quad x_2 \]
Four-point Green function (free propagator)

\[ G^{(4)}(x_1, x_2, x_3, x_4) = \Delta_F(x_1, x_2)\Delta_F(x_3, x_4) + \Delta_F(x_1, x_3)\Delta_F(x_2, x_4) + \Delta_F(x_1, x_4)\Delta_F(x_2, x_3) \]

or diagrammatically

\[ G^{(4)}(x_1, x_2, x_3, x_4) = \]

\[ + \]

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Quantum field theory

- Self-interacting scalar field

\[ S[\phi] = \int \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right] \]

- 2-point Green’s function (to linear order in \( \lambda \))

\[ G^{(2)} = \Delta_F(x_1, x_2) - i \frac{\lambda}{2} \int \Delta_F(x_1, x) \Delta_F(x, x) \Delta_F(x, x_2) \, dx \]

- Diagrammatically

\[ G^{(2)}(x_1, x_2) = x_1 \hspace{1cm} + \hspace{1cm} x_2 \]

\[ x_1 \hspace{1cm} x \hspace{1cm} x_2 \]
Example: charged scalar field

- Two-component scalar field

\[ \Phi = (\phi, \phi^*) \]

- Action of the free field

\[ S_0[\Phi] = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^* \]

- Alternative way of writing the action

\[ S_0[\Phi] = -\frac{1}{2} \Phi \hat{M} \Phi, \]

where

\[ \hat{M} = \begin{pmatrix} 0 & \Box + m^2 \\ \Box + m^2 & 0 \end{pmatrix} \]
Multicomponent fields

- Generating functional

\[ G_0[J] = \frac{1}{Z} \int D\Phi \ e^{-\frac{i}{2} \hat{M} \Phi + iJ\Phi} \]

- Again we introduce classical field \( \Phi_c \) and find

\[ G_0[J] = e^{-\frac{1}{2} \Phi_c \hat{M} \Phi_c + iJ\Phi_c} \]

where classical equations of motion are

\[ \hat{M}_{\alpha\beta} \Phi_\beta = J_\alpha \]

\[ \begin{cases} (\Box + m^2)\phi = J_\phi \\ (\Box + m^2)\phi^* = J_{\phi^*} \end{cases} \]
Multicomponent fields

Solution of classical equation

- Green function – matrix $\Delta_{\alpha\beta}$

$$\hat{M}_{\alpha\beta} \Delta_{\beta\gamma}(x, x') = -i \delta_{\alpha\gamma} \delta(x - x')$$

- Solution

$$\Phi^c_{\alpha} = i \int \Delta_{\alpha\beta}(x, x') J_{\beta}(x') \, dx'$$

- Generating functional

$$G_0[J] = e^{-\frac{1}{2} J \Delta J}$$

where

$$-\frac{1}{2} J \Delta J = -\frac{1}{2} \int d^4 x \int d^4 x' \, J_{\alpha}(x) \Delta_{\alpha\beta}(x, x') J_{\beta}(x')$$
Multicomponent fields

Summary

- Free action is written in the form
  \[ S_0[\Phi] = -\frac{1}{2} \Phi \hat{M} \Phi \]

- The matrix of free propagators is \( \Delta \) defined by
  \[ \hat{M}_{\alpha \beta} \Delta_{\beta \gamma}(x, x') = -i \delta_{\alpha \gamma} \delta(x - x'), \]
  i.e. \( \Delta \) is the “inverse” of \( \hat{M} \)
QFT formulation

Equivalent QFT problem

Stochastic problem given by differential equation

\[ \partial_t \phi = U(x, \phi(x)) + \eta, \]

where \( \eta \) is stochastic noise with

- \( \langle \eta \rangle = 0 \),
- \( \langle \eta(x) \eta(x') \rangle = D(x, x') \),

is equivalent to quantum field theory problem with the action

\[ S[\phi, \phi'] = \frac{1}{2} \phi' D \phi' + \phi' \left[ -\partial_t \phi + U(x, \phi(x)) \right] \]
Stochastic incompressible Navier-Stokes equation

\[ \partial_t v_i + v_j \partial_j v_i = -\partial_i P + \nu \Delta v_i + f_i \]

- \( f \) is stochastic stirring force
- Incompressibility implies

\[ P = -\frac{\partial_i \partial_j}{\Delta} (v_i v_j) \]

so that the transversal part of Navier-Stokes equation is

\[ \partial_t v_i + v_j \partial_j v_i = \nu \Delta v_i + f_i \]
QFT formulation

- QFT formulation of “free” Navier-Stokes equation

\[ S_0[v', v] = \frac{1}{2} v' Dv' + v' \left[ -\partial_t v + \nu \Delta v \right] \]

- \( v' \) is an auxiliary, unphysical field
- \( D \equiv D_{ij}(x, x') \) is the corelatter of stochastic force
- Interaction term

\[ S_I[v', v] = -v' (v \cdot \nabla v) \]
Free action is written in the form
\[ S_0[\Phi] = -\frac{1}{2} \Phi M \Phi, \]
where \( \Phi = (\nu, \nu') \),
\[ M = \mathbb{P} \begin{pmatrix} 0 & -\partial_t - \nu \Delta \\ \partial_t - \nu \Delta & -D \end{pmatrix} \mathbb{P} \]
and \( \mathbb{P} \) is transversal projector
QFT formulation

- In the Fourier space

\[ M = \begin{pmatrix} 0 & (-i\omega + \nu k^2)P_{\alpha\beta} \\ (i\omega + \nu k^2)P_{\alpha\beta} & -P_{\alpha\mu}D_{\mu\nu}P_{\nu\beta} \end{pmatrix} \]

where \( P_{\alpha\beta} = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \)

- Matrix of free propagators is defined by

\[ M_{\alpha\beta} \Delta_{\beta\gamma} = P_{\alpha\gamma} \]
QFT formulation

Free propagators for Navier-Stokes equation

\[ \langle v_\alpha v_\beta \rangle_0 = \Delta_{\alpha\beta}^{vv} = \frac{P_{\alpha\mu}D_{\mu\nu}P_{\nu\beta}}{\omega^2 + \nu k^4} \]

\[ \langle v_\alpha v'_\beta \rangle_0 = \Delta_{\alpha\beta}^{vv'} = \frac{P_{\alpha\beta}}{-i\omega + \nu k^2} \]

\[ \langle v'_\alpha v_\beta \rangle_0 = \Delta_{\alpha\beta}^{v'v} = \frac{P_{\alpha\beta}}{i\omega + \nu k^2} \]

\[ \langle v'_\alpha v'_\beta \rangle_0 = \Delta_{\alpha\beta}^{v'v'} = 0 \]
Interaction term

\[ S_I = -v' (v \cdot \nabla v) \]

In the Fourier space

\[ V_{\alpha\beta\mu} = i k_\beta \delta_{\alpha\mu} \]

Diagramatically

\[ V_{\alpha\beta\mu} = \begin{array}{c}
\nu'_{\alpha} \\
\times \\
\nu_{\beta}
\end{array} = i k_\beta \delta_{\alpha\mu} \]
QFT formulation

- Generating functional

\[ G[J] = \frac{1}{Z} \int d\Phi \ e^{iS_0[\Phi] + iS_I[\Phi] + i J\Phi} \]

- 1-loop approximation

\[ \langle vv \rangle = \quad + \quad + \quad + \frac{1}{2} \]
QFT formulation

Explicit form of the Feynman diagram

\[
\Sigma_{\alpha\beta}^{vv} = \Sigma_{\alpha\beta}(p) = \frac{1}{(2\pi)^4} \int d^4k \, \Delta^{vv'}_{\alpha\gamma}(p) \, V_{\gamma\delta\rho}(p) \, \Delta^{v'v}_{\delta\lambda}(k) \times \\
\times \Delta^{vv}_{\rho\sigma}(p-k) \, V_{\lambda\sigma\mu}(k) \, \Delta^{v'v}_{\mu\beta}(p)
\]

Exercise: compute the integral!
Correlator of stochastic force

- Prescribed statistics for $F$

\[ \langle F_\alpha(x) F_\beta(x') \rangle = D_{\alpha\beta}(x, x') \]

- Isotropy + transversality of velocity field + white noise

\[
D_{\alpha\beta}(x, x') = \frac{\delta(t - t')}{(2\pi)^2} \int dk\ P_{\alpha\beta}(k)\ d_F(|k|)\ e^{i\ k \cdot (x - x')} \]

- Mean energy injection

\[ W \propto \int d_F(k)\ dk \]
Recall: inertial range is $m \ll k \ll \Lambda$

Deviation from Kolmogorov – structure functions depend on $m$

Energy injection should be *infra-red*, i.e. the main contribution to $W$ is from scales $k \sim m$

Physically realistic model

$$d_F(k) = D_0 k^{4-d} (k^2 + m^2)^{-\varepsilon}$$

$\varepsilon$ is the small parameter of perturbative expansion
Without conclusion

What can be investigated?

- Structure functions and scaling laws
- Renormalised quantities
  - Diffusivity
  - Kolmogorov constant (related to energy spectrum)
- Corrections to Kolmogorov laws
- Explanation of anomalous scaling
- Influence of anisotropy, compressibility and helicity
- Simpler models
  - Analysis of Navier-Stokes equation is very difficult
  - Advection of passive admixtures
  - Scalar, vector admixture
- ...

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