Relativistic Physics

lecture notes for a course taught at Prague math-phys

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Franziskus Pfleghart: Genesis

PREFACE

In 1983, I enrolled for the lecture course of Jiří Bičák, then associate professor at the Department of Mathematical Physics, Faculty of Mathematics and Physics, Charles University in Prague. A two-semester course of general relativity for the 4th year of study, about 160 teaching hours, two credits and two exams, three questions each. 220 stairs to the 11th floor of the "Trója" building; we were thirteen, ten theorists and three astronomers. We liked it – the course was interesting, and we had a romantic relation to geometry, to heaven, as well as to professor's daughters.[1] After a few years, I returned to the course, first assisting with exercises and later sharing the lecturing as well.

Department has changed to an Institute – of *Theoretical* Physics – but, hopefully, geometry, heaven and students still have good times there. Our GR lecturer became a full professor long time ago, and it may happen that he starts later then "in our times", including more about string quartets, new antiquarian catch, Bandipur tigers or about an inspiring person who appeared in TV. Several months ago, I joined our 4th-class student at a lunch table. I usually take care of the fall semester, while at that time lilacs were already in blossom, so I asked where they are. He replied: "No, we have not started cosmology yet. ... But we *don't* only concern with physics at the lectures. It even seems like if professor wished to make us cultured people..."

Unfortunately, the cultural thread is hard to capture. We may once get through to cosmology, but the rich branching of digressions to literature and philosophy, historical intermezzi, story from a corridor or from the Learned Society, from Cambridge gallery or from Australian bush, brief (yet complete) assessments of various "discoverers of truth" (including American *Bush*),[2] melancholy as well as oddities in paintings (and the latter's frames, aligned according to the rules of parallel – well, rather Fermi – transport! –But this I have just cut in on myself, as usual...), questions we were embarrassed by (because we did not know the answer), references to R. Penrose, Ya. Zeldovich, Kip (Thorne), B. Russel, A. Whitehead, K. Popper, trees in Chocerady, Březina and Čapek brothers, quivers of idealism in the political misery, dramatic pizzicato in Bartók, relaxed fun as well as urgent message about an inspiring priest – all that must live in remembrances and on textbook edges, it must sound in original.

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[1] I mean, to their study world-lines $x^\mu_{\rm J}(t)$ and $x^\mu_{\rm A}(t)$, about which we also learned much during the course; t denotes cosmic time.

[2] Please note this preface was written in 2002. Had Jiří been able to spot *today's* reality ...

After 56 orbits of the Moon, Jirka Bičák will finish 65 orbits around the Sun (as taken – how else – with respect to distant stars) and a draft of lecture notes to his course might be a good option for a gift. His course brought the applications of general relativity – the relativistic astrophysics – to our country in the period when the black-hole theory was almost completed, but not spoken about in a decent astronomical society. How different the situation is today! Almost every discussion on the structure and evolution of galaxies, on the fate of massive stars or on X-ray sources *revolves* about black holes. The values of cosmological parameters, not long ago somewhat co~~s~~mic due to their error bars, are being finely bounded by numerous independent studies, and the satellite navigation employs formulas that were first presented, by Einstein, from Prague in 1911.

People say that in order to learn something, one should write a textbook about it. Before I manage, gravitational waves may join the electromagnetic ones and begin to bring information about the Universe. It may be necessary to rewrite some parts then. In any case, I hope there's something to that saying.

OS, end of October 2002

Yes, gravitational waves made it to us. 5 years ago, actually. A new class of massive black holes has been discovered thanks to them. Also, our Institute dropped from the 11th to the 10th floor. It was not due to the waves; it was due to a coordinate transformation only, our proper distance from the Earth surface has not changed. More serious (though also not tied to the waves) is that the LaTeX installation has changed several times... Anyway, Jirka is lecturing as ever, and these lecture notes are half-finished as ever. Within the coronavirus home-office, I made an unlucky decision to start once more and in English (half-finished was the Czech version).

OS, end of June 2020

So sadly, on 26th January 2024, Jirka Bičák left us. The whole era closed for us, his students. Jiří designed this course in the 1970s, starting from 1972 when he returned from a half-year stay in Kip Thorne's group at Caltech. He taught the course for 50 years. Originally one-semester, from 1980s it was two-semester, and in 2004 the third, advanced semester was added which Jiří specifically took care of then. We were planning its 2024 run two days before Jirka's demise. Now it is on me, and it is a challenge.

OS, end of February 2024

Exactly as Riemann guessed

"Thus, according to this theory, space is — exactly as Riemann guessed — no longer absolute; its structure depends on physical influences."
[A. Einstein, in *Ideas and Opinions* (Crown Publishers, New York 1954)]

Einstein speaks here about his **general theory of relativity** and about its most fundamental and new feature – the **dynamical role of geometry**. However, he surely means space-*time*, as he himself joined them in the "special" theory already. He had been forced to do so by the requirement of equality of all the inertial systems, inspired by Mach's critics of the absolute system of Newton, and by the assumption/observation that light propagates with respect to all the inertial systems with the same, "absolute" speed. Relativity of time, its interconnection with space as well as geometrical interpretation were already forecasted by Poincaré, but everything only emerged from æther haze after the "Machian" revision of the concept of time, at which Einstein arrived during one early-May night of 1905. Two years later, Minkowski presented the new standpoint in an elegant, geometric form.

While special relativity lives in a "flat" space-time of Minkowski, the *general* theory of relativity is tied to curved space-times. Curved *spaces* were discovered at the beginning of the 19th century, as an alternative to the 2000 years old geometry of Euclid. Riemann generalised it, extended it to a general dimension, and pointed out its possible factual significance. Non-Euclidean concepts then became popular, but they only entered the physical world really after Einstein gave the new meaning to time and Minkowski embedded this new time into the world geometry.

However, no route would have lead the new theory to the physical world, were not gravitation a universal interaction. This aged observation of Galilei and others was generalised by Einstein, and – as the *principle of equivalence* – helped to extend his relativity principle (more accurately, of "covariance") to non-inertial reference systems. In the field of inertial forces, physical processes proceed differently than in the Minkowski geometry, and they proceed in a similar way as in the corresponding gravitational field. Allowing, in addition, for the field's non-homogeneity, it turns out that the gravitational effects can be described in terms of deviation of the space-time geometry from Minkowski. It remains to be specified which properties of matter curve space-time (or, generate gravitation) and how they curve it. Only at this moment (1912) the lines of geometry definitively intersect with the physical world: Einstein learns the results of Riemann and his successors and starts to tackle the gravitational challenge in their language. Soon he almost has it, but a seeming disagreement between the requirements of general covariance, correct Newtonian limit and the Machian, "relativistic" interpretation of inertia postpones the completion of his effort by 3 years.

Einstein was always stating that his theory rests on the equivalence principle and that it describes the relation between matter and space-time geometry. Such a view will also accompany us in these lecture notes. You are hardly going to find in them something which could not be found elsewhere – there are so many books on the relativity theory, on relativistic astrophysics and cosmology; some of them are mentioned in references. Yet not all of them are accessible to everyone and, in addition, more of them would be necessary to cover the topics of the course. It is why we have written this text. We have tried to include all usual parts, and to explicitly compute everything important, but, at the same time, we focused to the basic ideas rather than adding every kind of detail, whether on the geometrical or the astrophysical side.

Current view on the logical starting points and on the content of the relativity theory may differ from the original Einstein's. If only for this reason, it is worth comparing different presentations. Among the GR textbooks, *General Relativity* by R. Wald [50] is being highly acclaimed in the community. In its preface, the author writes: "One of the most difficult issues which arises during the writing of a book on general relativity is where in the book to present the rather substantial amount of mathematical material that is needed. Much of this material (e.g., tensor calculus and curvature) is required even for the formulation of general relativity. ... If all this material were presented at the beginning of the book, it would comprise a truly formidable obstacle to learning general relativity." Yet this obstacle is already *part* of the subject – as a geometric theory of space-time, general relativity *begins* by mathematics, so the university-level textbooks necessarily start by mathematics as well, whether "only" in the form of tensors and curvature, or by a full summary of the (pseudo-)Riemannian geometry.

The GR books rather differ in position in which the mathematics is introduced. C. Møller writes in the preface of [30]: "Certainly the four-dimensional representation, which is based on the symmetry between the space and time variables revealed by the discovery of the Lorentz transformation, is the most elegant way of expressing the principle of relativity in mathematical language, and it has been of the utmost importance for the rapid development of the general theory of relativity particularly. ... However, in a textbook of today I think it is useful to stress again the fundamental physical difference between space and time, which was somewhat concealed by the purely formal four-dimensional representation. In the first three chapters we have, therefore, avoided any reference to the four-dimensional picture ... But in the following chapters also, where the elegant methods of the four-dimensional tensor calculus are developed and applied, a three-dimensional formulation, which gives a better insight into the physical meaning of the theory, is frequently given." However, one may ask: why the four-dimensional tensor approach should be just elegant and important, why it should not be bona fide *natural* as well?[3] The persuasion that "the world in itself is elegant" has for many physicists been not only a vague starting point, but also an ongoing criterion for "correctness" of reasoning, even including concrete calculations.

Whereas Møller introduces the curvature tensor in p. 343, in [46] it already appears in p. 15. And the author of this latter textbook, J. Synge, also added quite a different preface: "It is to support Minkowski's way of looking at relativity that I find myself pursuing the hard path of the missionary." Meant is the *geometrical* view, employing the space-time diagrams (rather than kinematical figures and dynamical terminology of Newtonian mechanics) as means of understanding. "When, in a relativistic discussion, I try to make things clearer by a space-time diagram, the other participants look at it with polite detachment and, after a pause of embarrassment as if some childish indecency had been exhibited, resume the debate in their own terms. Perhaps they speak of the Principle of Equivalence. If so, it is my turn to have a blank mind, for I have never been able to understand this Principle."

[3] If not used in the sense "existing in or derived from nature", the word *natural* feels rather vague ("in accordance with the nature of, or circumstances surrounding someone or something", or "instinctively plausible"), but those who have studied Einstein know that he was using it as a very strong argument, in (natural) science as well as generally in the society.

Synge concedes that the "Principle of Equivalence performed the essential office of midwife at the birth of general relativity", but he suggests "that the midwife be now buried with appropriate honours and the facts of absolute space-time faced". Just on the contrary, S. Weinberg gives as the main reason to write the book [51] that he "became dissatisfied with what seemed to be the usual [i.e., geometric] approach to the subject" – and *exactly oppositely* he also recognises Einstein's position: "Of course, this *was* Einstein's point of view, and his preeminent genius necessarily shapes our understanding of the theory he created. However, I believe that the geometrical approach has driven a wedge between general relativity and the theory of elementary particles. ... the passage of time has taught us not to expect that the strong, weak and electromagnetic interactions can be understood in geometrical terms, and too great emphasis on geometry can only obscure the deep connections between gravitation and the rest of physics. ... I have tried here to put off the introduction of geometric concepts until they are needed [Riemann tensor is put off to p. 133], so that Riemannian geometry appears only as a mathematical tool for the exploitation of the Principle of Equivalence, and not as a fundamental basis for the theory of gravitation." In Weinberg's book, it is even possible to read: "Einstein and his successors have regarded the effects of a gravitational field as producing a change in the geometry of space and time. At one time it was even hoped that the rest of physics could be brought into a geometric formulation, but this hope has met with disappointment, and the geometric interpretation of the theory of gravitation has dwindled to a mere analogy, which lingers in our language in terms like 'metric', 'affine connection', and 'curvature', but is not otherwise very useful. The important thing is to be able to make predictions about images on the astronomers' photographic plates, frequencies of spectral lines, and so on, and it simply doesn't matter whether we ascribe these predictions to the physical effect of gravitational fields on the motion of planets and photons or to a curvature of space and time." Weinberg himself adds in a parenthesis: "The reader should be warned that these views are heterodox and would meet with objections from many general relativists." In the 1970s, such a warning was twice appropriate, as the geometrical approach had brought, in the preceding decade ("golden years" of relativity), a number of crucial results and also – almost concurrently with Weinberg's book – two major textbooks [18] and [29].

One of the authors of [29], K. Thorne, mentions, in "its popular version" [48], another interpretation aspect, namely two possible perspectives on curvature: (i) the space-time is curved while rulers (and clocks) measure true (proper) distances; (ii) the space-time stays flat, but measures are deformed, so the measurement results do not come out "flat". Read on: "What is the real, genuine truth? Is spacetime really flat, or is it really curved? To a physicist like me this is an uninteresting question because it has no physical consequences. Both viewpoints, curved spacetime and flat, give precisely the same predictions for measurements performed with perfect rulers and clocks, and also (it turns out) with any kind of physical apparatus whatsoever. ... Which viewpoint tells the 'real truth' is irrelevant for experiments; it is a matter for philosophers to debate, not physicists." Certainly the perspective (i) has become standard since it is in better tune with the fundamental faith in **background independence**. But yes, physicists *are* used to present their images, humbly, as mere "formalisms", and they are amazed when "it works", namely when these yield predictions which agree with the experiment. Actually, it is very non-trivial that it works quite well, as Einstein himself expressed in his famous quote "The most incomprehensible thing about the world is that it is comprehensible." (cf. the article *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* by E. Wigner, 1960). But we beg to claim that Einstein would deem such a *conventionalism* standpoint a pragmatic deviation, resigning to access to the deepest background of phenomena. W. Heisenberg, for example, was remembering how Einstein did not want to discuss "observables" with him: "You are suddenly speaking of what we know about nature, and no longer about what nature really does. In science we ought to be concerned solely with what nature does."[4]

It belongs to the local (European, Prague, but – we hope – also our-Institute) tradition that the questions about "real truth" are also being included in mathematics and physics. The question whether better insight follows from the four-dimensional "geometry" or the three-dimensional "physics" is, however, the question of personal intuition, and the answer also depends on specific situation. The difference between the two standpoints advocated in the cited classical textbooks, actually, just consists in whether the geometrical formulation is only offered as a handy option *after* analysing the spatial and temporal measurements in a traditional, Newtonian field language, or whether it is firmly stated, at the very beginning, that our physical universe *is* a four-dimensional pseudo-Riemannian manifold, and then the observations are already interpreted in its geometrical terms. R. Wald characterises his textbook position as a "more modern, geometrical viewpoint than Einstein had" (Riemann tensor he first has in p. 37), but the equivalence principle and the general covariance principle he discusses thoroughly, with Mach's principle mentioned as well.

Instead of our own point of view, let us add a quote from the preface of K. Kuchař to his Czech textbook [24] (Riemann tensor in p. 105): "The reader should learn to easily translate the geometrized space-time equations to the language of traditional physics which splits time from space and describes their relation in dynamical terms, so that he can use geometric as well as physical intuition, the interplay of which is one of the most beautiful features of the general theory of relativity."

In order to enjoy and pursue such an interplay, we assume the students/readers know basics of special relativity in real four-dimensional formalism. Who already knows the mathematical foundations of general relativity (tensor analysis on manifolds) will have advantage, but, in contrast to Platón, we also invite those who have not mastered geometry yet. We proceed rather inductively and seldom in the definition-lemma-theorem style. We don't start from the analysis of observations and experiments, yet emphasising fundamental principles – as generalised empirical experience, but also as what for Einstein himself was the starting point. The tensor formulation is thus arrived at very soon, but rather than as an a priori narrative, it is picked as a natural and most economical way of presenting the gravitational problem; the necessary geometric tools we gather in an opportune and "pragmatic" way. It is by no means to stress the non-geometrical story, but (i) in order not to substitute neither repeat the course on geometrical methods (which, at our Faculty, the students of theoretical physics rather go through later than through the first GR course), and (ii) in order that the lessons be

[4] As Heisenberg noticed clearly at this occasion, Einstein had substantially changed his epistemology since the period of struggle with relativity and gravitation. And Einstein himself admitted that elsewhere: "I began with a skeptical empiricism more or less like that of Mach. But the problem of gravitation converted me into a believing rationalist, that is, into someone who searches for the only reliable source of Truth in mathematical simplicity."

also accessible to those whose geometrical background has not yet been completed.

To conclude, let us mention some more recent textbooks. Excellent – friendly yet modern – is the introductory course [39] by L. Ryder; it uses the same conventions as we. More advanced chapters are added in [14] by Ø. Grøn & S. Hervik (same conventions as well), who write in Preface: “We will try to convey the concepts of gravity to the reader as Albert Einstein saw it. ... He saw upon gravity as curved spaces, four-dimensional manifolds and geodesics.” And they also warn the reader that – despite the weakness of gravity – it is dangerous to climb a ladder.

Victoria Ivanova: The envy

Notation

We use the standard index formalism, Einstein summation rule and notation common in the relativity theory. Mostly the indices are abstract – they just indicate type of the quantity, rather than necessarily representing components in any basis. Generally, we adhere to the conventions of the “canonical” textbook MTW [29]. Let us remind them, plus several others:

- with the exception of introductory parts and of some astrophysically/cosmologically oriented chapters, and if not stated otherwise, the quantities are given in geometrized units in which the speed of light in a vacuum c and the gravitational constant G are equal to unity
- metric tensor $g_{\mu\nu}$ has signature $(-{+}{+}{+})$
- Greek indices assume the values 0–3, Latin indices (i, j, ...) run 1–3 (spatial range); tetrad components are denoted by indices with a hat, $\hat{\alpha}$ etc.
- where the indices are *not* just abstract, the symbol X^μ represents all components, or any component, of the quantity X, i.e. $X^\mu := (X^0, X^1, X^2, X^3)$ (similarly for quantities of arbitrary tensor type and order); particular values of the indices are specified by numbers 0–3, or directly by letters representing the respective coordinates (e.g., $u^0 \equiv u^t$, etc.)
- derivatives:
  - partial derivative is denoted by ∂ or by a comma in an index position – e.g.,
    $$\frac{\partial X^\alpha}{\partial x^\lambda} \equiv \partial_\lambda X^\alpha \equiv X^\alpha{}_{,\lambda}$$
  - covariant derivative is denoted by ∇ or by a semicolon in an index position – e.g.,
    $$\nabla_\lambda X^\alpha \equiv X^\alpha{}_{;\lambda}$$
  - just *one* comma or semicolon is given, with *all* the indices which appear after them indicating derivatives
  - absolute derivative (covariant derivative in a specific direction) is denoted by D or – in the case of time direction, in particular – by a dot over the quantity
  - higher mixed derivatives are ordered according to
    $$\frac{\partial}{\partial x^\kappa}\left(\frac{\partial X^\alpha}{\partial x^\lambda}\right) \equiv \frac{\partial^2 X^\alpha}{\partial x^\kappa \partial x^\lambda} \equiv \partial_\kappa \partial_\lambda X^\alpha \equiv X^\alpha{}_{,\lambda\kappa} \qquad \nabla_\kappa \nabla_\lambda X^\alpha \equiv X^\alpha{}_{;\lambda\kappa}$$
- parentheses/brackets in an index position mean symmetrisation/antisymmetrisation in all the enclosed indices:
  $$T_{(\mu_1 \dots \mu_n)} := \frac{1}{n!}\sum_p T_{\mu_{p(1)} \dots \mu_{p(n)}}\,, \qquad T_{[\mu_1 \dots \mu_n]} := \frac{1}{n!}\sum_p \delta_p T_{\mu_{p(1)} \dots \mu_{p(n)}}\,,$$
  where summation goes over all index permutations p, with δ_p being $+1$ for even and -1 for odd permutations (possible other indices of T have not been indicated – those remain untouched)

- braces in an index position mean sum of the terms obtained by cyclic permutation in all the enclosed indices:

$$T_{\{\mu_1 \dots \mu_n\}} := T_{\mu_1 \dots \mu_n} + T_{\mu_n \mu_1 \dots \mu_{n-1}} + T_{\mu_{n-1} \mu_n \mu_1 \dots \mu_{n-2}} + \dots + T_{\mu_2 \mu_3 \dots \mu_n \mu_1} \, ;$$

  in the three-index case, $T_{\alpha\beta\gamma}$, it is useful to know that

$$\begin{aligned} T_{\{\alpha\beta\gamma\}} &= 3\, T_{[\alpha\beta\gamma]} \qquad \text{for a tensor antisymmetric in at least two indices}\, , \\ T_{\{\alpha\beta\gamma\}} &= 3\, T_{(\alpha\beta\gamma)} \qquad \text{for a tensor symmetric in at least two indices} \end{aligned}$$

- generally, the order of indices is important and has to be given distinctly; only when the tensor or matrix is *symmetric* in some indices, one may – in the mixed version – write the respective indices above each other; i.e., if $T^{\mu\nu}$ is symmetric, one may write its mixed version as T^μ_ν (but both $T^\mu{}_\nu$ and $T_\nu{}^\mu$ would be OK as well)

- the Riemann tensor $R^\mu{}_{\nu\kappa\lambda}$ is defined according to the Ricci identity

$$2V_{\nu;[\kappa\lambda]} := V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} = R^\mu{}_{\nu\kappa\lambda} V_\mu \, ,$$

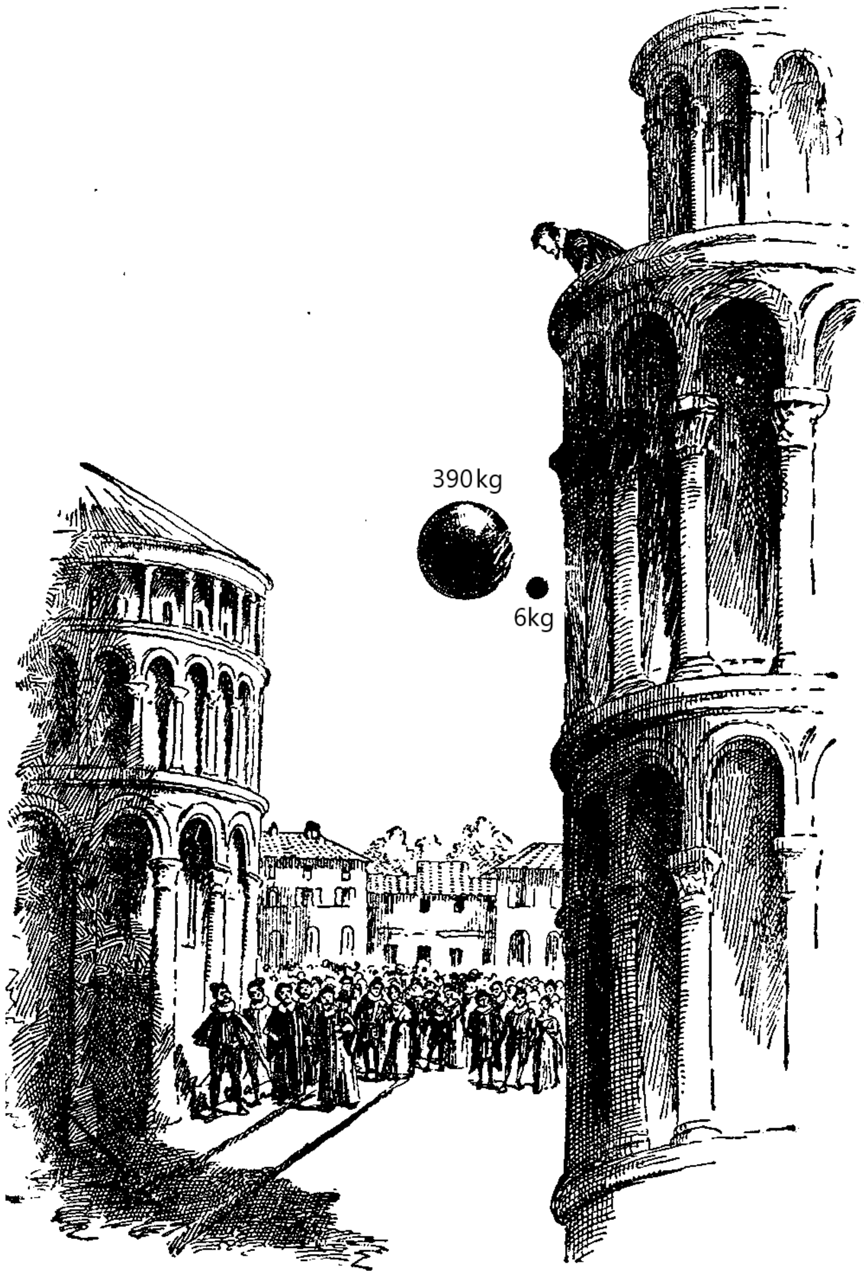
  where V_μ is an arbitrary covector

- the Ricci tensor, the Einstein tensor and the field equations read

$$R_{\nu\lambda} := R^\kappa{}_{\nu\kappa\lambda} \, , \qquad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \, , \qquad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \, .$$

Acknowledgement

We are grateful to our Institute colleagues and to our families for a very kind and inspirational environment. Those who completed the course are thanked for a critical attention under which our account of the theory has been tuned over the years.



Galileo Galilei demonstrating universality of free fall in Pisa.
[Thanks to pngegg.com]

CHAPTER 1

Fundamental principles of general relativity

Old Indians, old Greeks already? No, let's start by falling from a roof.

If all free bodies fall, in a given gravitational field, with the *same* acceleration (as Galilei noticed), their *relative* acceleration with respect to each other has to vanish. Einstein realised this in 1907 when looking at roofers working on a building next to his Bern's office: if a roofer dropped his hammer while falling, the tool would have no acceleration *with respect to him*. Seems trivial, but Einstein was later mentioning this as the luckiest thought of his life. He inferred from it the **principle of equivalence** which enabled him to extend his relativity principle from inertial to accelerated reference frames, and, subsequently, from flat to any space-time, and thus to formulate the **general theory of relativity**.

A short story first. If freely falling bodies have zero acceleration relative to the freely falling roofer, it means that his frame – if spanned by non-rotating Cartesian axes – represents an **inertial frame**. And in an inertial frame, we know (from Einstein, 1905) how physics works: it should obey special relativity. Special relativity only holds in the absence of gravitation, but this simple thought shows that even in a non-zero gravitational field, there exists a reasonable analogue of inertial frames – freely falling, non-rotating Cartesian frames. Does it mean that in *any* space-time (any gravitational field) it is enough to fall freely with non-rotating Cartesian axes, and just keep using special relativity? Not really. If the gravitational field were **homogeneous**, same at every location (and time), then yes, it would be possible, and sufficient, to go over to any of the freely falling and non-rotating Cartesian frames. All bodies freely falling at *any* place (and time) would be, with respect to any such frame, in a uniform linear motion, so the frames would by definition represent **global inertial frames**, and one could resort to special relativity when working in them.

However, real gravitational fields are **non-homogeneous**, so the state of free fall is different at different points (and possibly times), in other words, the family of "inertial frames" is different at different points (and times) – it's impossible to find any single frame that would be "inertial" everywhere, i.e. with respect to which free bodies moving *anywhere* in the space-time would have zero acceleration. Note that this is nothing new, nothing "non-Newtonian"; in extended bodies, the non-homogeneity of gravitational acceleration is well known to generate **tidal forces**. Anyway, for practical reasons, we need *global* coordinate systems, since it is not practical to treat physical problems in the whole *set* of "inertial frames", each only reasonably valid in a limited neighbourhood of some point. Hence, it is necessary to admit more general coordinate frames. This will be addressed by the **principle of general covariance**.

1.1 Principle of equivalence

Galilei found that in a given gravitational field, all free bodies (those only affected by that gravitational field) move with the same acceleration. Admitting that not just free bodies but *"all physics feels the same acceleration"*, one obtains the most elegant formulation of the equivalence principle:

$$\boxed{\text{gravitation is a } \textbf{universal interaction}}\,.$$

Note that the other three fundamental interactions are, in contrast, **differential**, since they *differentiate* between different physical systems. A given electric field, for example, accelerates plus and minus charges in opposite directions, and does not affect neutral particles. In this language, the above principle thus includes the observation that *there are no bodies with negative mass*, because negative masses would be accelerated in the opposite direction than the positive ones.

What does Newton's physics say to the Galilei's finding? Acceleration $\vec{a}$ of a body of mass m in a gravitational field of a spherical body of mass M is given by the equation of motion

$$m\vec{a} = -\frac{GMm}{r^2}\,\vec{r}_0\,, \tag{1.1}$$

with G the gravitational constant, r the distance between centres of the bodies, and $\vec{r}_0$ the unit directional vector pointing from the large-body centre towards the small body. Dividing the equation by m, we obtain for $\vec{a}$ an expression which is really independent of any characteristic of the studied body, in particular, it does not depend on its mass. But the quantity m which we divided by represents totally *different* properties on the two sides of the equation: on the left-hand side, m quantifies the resistance of the body against acceleration (call it the **inertial mass**, $m_{\rm i}$), while on the right-hand side, it quantifies the response of the body to the gravitational field $-GM/r^2$ of the other body (the **gravitational mass**, $m_{\rm g}$). The equality of $m_{\rm i}$ and $m_{\rm g}$ does not follow from anywhere and it is not at all self-evident. Reasoning in the opposite, effective way: since the Newton equation works very well, the two masses must have the same value. The Galilei's observation can thus be expressed by claiming

$$\boxed{m_{\rm i} = m_{\rm g}}\,,$$

more precisely, by claiming that the ratio $m_{\rm g}/m_{\rm i}$ is the same for all free bodies (with its value "tuned" to unity by choosing the value of the constant G appropriately).

Figure 1.1: This man has different experience than Galilei.

A footnote: notice that the mass M has, in the expression $\frac{GMm}{r^2}$ for the gravitational force, a somewhat different role than the mass m – while m *reacts* to the field, M *generates* it, hence m could be called the *passive* gravitational mass, while M represents the *active* gravitational mass. However, this distinction only has a clear meaning in the "instinctive" limit $M \gg m$, with m considered a test mass not affecting the field of M at all. In any case, whatever are the values of the masses, the bodies exert *the same* force on each other,

$$\frac{G\, M_1^{\mathrm{active}} M_2^{\mathrm{passive}}}{r^2} = |F_{1\to 2}| = |F_{2\to 1}| = \frac{G\, M_2^{\mathrm{active}} M_1^{\mathrm{passive}}}{r^2}\ ,$$

so, due to the action-and-reaction law, the active and passive gravitational masses have to be proportional to each other.

1.1.1 Testing the equivalence principle

As soon as in the first paragraph, Newton himself mentions in his *Principia* that the equality $m_{\mathrm{i}} = m_{\mathrm{g}}$ is not obvious and should be tested experimentally (see Figure 1.2). Naturally, considered should be such phenomena where inertia and gravity compete. The simplest is the motion of a mathematical pendulum. If one does *not* assume that $m_{\mathrm{i}} = m_{\mathrm{g}}$ for the swinging body, the period of its small oscillations comes out as $T = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{m_{\mathrm{i}}}{m_{\mathrm{g}}}}$, where l is

Beginning of Newton's Principia :

Definition I

The quantity of matter is the measure of the same arising from its density and bulk conjointly.

Thus air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter.

Comment by S. Chandrasekhar :

It will be noticed that while Newton is careful in defining the notion of *mass* (as a quantitative measure of *quantity of matter*), he leaves the notion of *weight* unspecified except to say that 'by experiments on pendulums, very accurately made', he has shown that mass is proportional to weight. The reason for this partial explanation is that the precise distinction between 'mass' and 'weight' cannot be made without reference to the Second Law of Motion (yet to be formulated in terms of concepts yet to be introduced). This fact is made clear in Newton's account of 'his experiments made with the greatest accuracy' in Proposition XXIV of Book II (with explicit reference to the Second Law of Motion). We shall presently consider (out of context! in §10) this proposition to emphasize that the Definitions and Laws must be read in their totality and not singly.*

* To avoid ambiguity it may be noted explicitly that the distinction that is made here is between *inertial mass* (m_i) and gravitational mass (m_g). By *mass* Newton means the inertial mass m_i and by *weight* he means $g \times m_g$ where m_g denotes the gravitational mass and g the value of *gravity* at the location of the mass. For example, at the surface of the Earth, we should write $g = G \times$ (mass of Earth)/(radius of Earth)^2, where G denotes the constant of gravitation, that is, Newton's constant!

Figure 1.2 *First paragraph of Newton's Principia, in English translation (from original Latin) and commented by S. Chandrasekhar. From S. Chandrasekhar: Newton's Principia for the Common Reader (Clarendon Press, Oxford 1995).*

the length of the massless cord and g is the gravitational-acceleration magnitude. Newton achieved accuracy of about 10^{-3} in measuring the mass ratio. More *accurately*, the accuracy is being expressed in terms of deviation from unity of the parameter (often called the Eötvös parameter)

$$2\,\frac{[m_g/m_i]_1 - [m_g/m_i]_2}{[m_g/m_i]_1 + [m_g/m_i]_2}\;,$$

where indices 1 and 2 denote two different bodies (two different weights compared in the pendulum case). Using the same method, Bessel achieved roughly 100 times better accuracy than Newton. The accuracy rose significantly when *torsion balance* began to be employed by L. von Eötvös (10^{-8} in 1889, 10^{-9} in 1908). Later the team lead by R. H. Dicke reached 10^{-11} in 1963 and the team of V. B. Braginsky reached 10^{-12} in 1971. Only at the turn of the century, the precision was increased to 10^{-13}, still following the Eötvös' method. Current most accurate bound, 10^{-15}, was obtained by direct comparison, on a satellite, of the motion of two free bodies of different compositions in an identical (Earth's) gravity field (the project was called MICROSCOPE, operated 2016-18).

The idea of the torsion-balance experiments is simple but non-trivial. First, the studied process has to be governed by inertia and gravity, with the possible inequality of $m_{\rm i}$ and $m_{\rm g}$ inducing a time-variable effect. (In the opposite, time-independent case, no effect can be revealed, because then the balance can be put in a stationary equilibrium, independently of whether $m_{\rm i}$ and $m_{\rm g}$ are identical for the bodies involved or not.) Therefore, the terrestrial weight field alone (the sum of Earth's gravity and of centrifugal force due to Earth's rotation) is not enough for a test, because, irrespectively of whether the equivalence principle holds or not for any of the masses, equilibrium once tuned stays satisfied for ever (imagine performing it with normal scales for simplicity). Now, however, abstract from this stationary part of the problem, and also notice the second important motion the apparatus undergoes – orbiting the Sun together with the Earth. This is a *free*, Keplerian orbiting, governed by equilibrium between gravitational attraction by the Sun and orbital centrifugal force. If the Sun, the Earth and the balance stayed on one line all the time (as would be the case were the Earth's rotation synchronised with its orbital motion – so called captured rotation or gravitational locking), there again would not be any effect. But the reality is different – the Earth's rotation period is much shorter than the orbital period, having the balance assume an *opposite* position with respect to the Sun every 12 hours (see Figure 1.3). Hence, if the equality of $m_{\rm i}$ and $m_{\rm g}$ were broken for one of the bodies (say), i.e. if the ratio of the Sun's attraction and orbital inertia were different for the two bodies, it would *not* be possible to restore the equilibrium for ever: the balance would suffer a net torque altering its orientation in a sinus-like 12-hour pace, so it would oscillate with the 24-hour period.

High accuracy is ever an experimental ideal, yet here it has a deeper importance, because it enables one to check whether the equivalence principle holds for various subtle (yet fundamental) ingredients contributing to the macroscopic weights. Practically, the bodies are made of nucleons and electrons. Most of the chemical elements contain similar number of protons, neutrons and electrons, hence the estimate of how much these species contribute to the mass of ordinary bodies. Electrons, for example, contribute by some 0.027%, i.e. $2.7\cdot 10^{-4}$. Therefore, one needs the precision of at least 10^{-5} (like Bessel's) in order to be able to check whether the equivalence principle holds for them. Further, the microscopic constituents of the weights are not free, they are subject to several interactions within the matter. The strong nuclear force keeps the nucleons within atomic nuclei, otherwise these would break up due to electric repulsion between protons; electromagnetic force holds together atoms and molecules; a tiny weak force acts there as well; and one might also speak of non-fundamental, mechanical-like interactions, such as particles' low-energy collisions, macroscopically manifested as pressure. Every interaction contributes to the energy of a

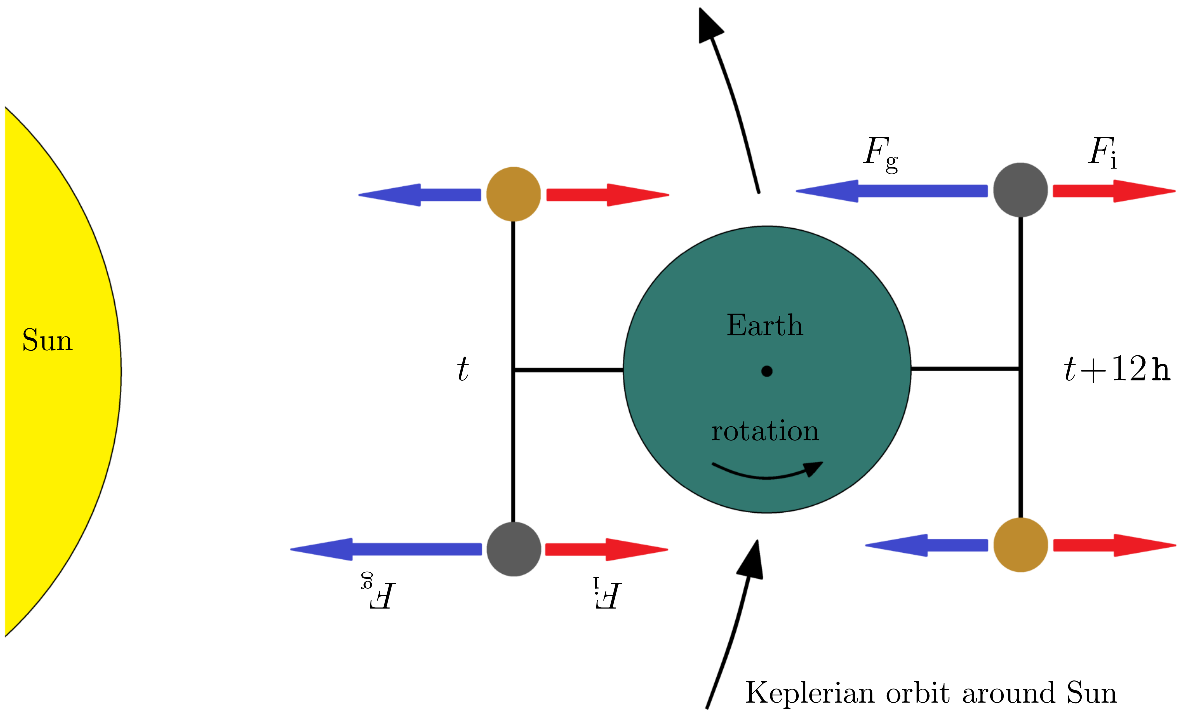


Figure 1.3 *A scheme of the classical torsion-balance tests of the equivalence principle (here depicted on normal balance and somewhat out of scale). Due to the Earth rotation, the balance arms get into an opposite position with respect to the Sun every 12 hours, so if the equivalence principle did not hold for one of the weights (the grey one in the plot), the balance would oscillate with 24-hour period. Please excuse that the Earth's rotation axis is indicated as perpendicular to the ecliptic, it is to keep the scheme simple.*

given particle, so, according to the Einstein formula $E = mc^2$, it changes its mass correspondingly: attractive interactions bind the particles in potential wells, thus decreasing their energy and mass, whereas repulsive interactions lift the particles on potential heights, thus acting oppositely.

The mass-energy contributions of interactions are hard to quantify, because they depend on which ingredients one takes as "elementary". For instance, the strong force acting between nucleons contributes slightly less than 1% to the masses of all compound atomic nuclei. On the other hand, it constitutes as much as 99% of the nucleon masses ("pure quarks" only contribute by about 1%)! And one could continue: the strong force is carried by gluons, but these have zero rest mass... Now a particular piece of knowledge from special relativity must have crossed your mind: even zero-rest-mass particles do have (or may have) energy, so they do have *mass*, according to the $m = E/c^2$, hence $m = m_0\gamma$ formula. The latter holds for *any* particle or body, and many of the microscopic particles existing within bodies move very rapidly. Hence another query: does their m_i as well as m_g rise with velocity according to that same formula? In other words, does the equivalence principle hold for the kinetic contribution to mass-energy?

In order to reduce the probability of accidental cancellations, the tests have been performed with weights of various chemical compositions. The "classical" experiments in the 1960s and 1970s were performed with one weight of aluminium ${}^{27}_{13}\mathtt{Al}$ and one weight of gold ${}^{197}_{79}\mathtt{Au}$. Besides their difference in proton-to-neutron ratio and much bigger contribution (about 0.5%) to the gold-atom mass of the electrostatic repulsion between protons, gold is also known as "the most relativistic" of all stable atoms, because its inner electrons orbit its heavy nucleus with relativistic speeds. Actually, it is $0.53c$ for the innermost $1s$ electrons, making their mass $\gamma \doteq 1.18$ times the rest mass. A simple estimate reveals that about $2.2\cdot 10^{-4}$ of the gold-atom mass is in electrons. Since not only the $1s$, but also the other electrons spend significant time close to the nucleus, the overall Lorentz-factor effect can be of the order of $10^{-6} \div 10^{-5}$ of the gold-atom rest mass. In 2010s, $1\,\mathtt{kg}$ of gold was being sold for about 40 thousand EUR, of which some 10 cents are due to special relativity.

We have not yet mentioned the *contribution of the gravitational interaction itself to the mass-energy*. For ordinary bodies, this contribution, $-\frac{Gm_1m_2}{c^2r}$, is negligible. For example, the mass of two nuclear protons is diminished, due to their mutual gravitational attraction, by $\sim 10^{-39}$ of their rest mass; for two kilogram-mass bodies 1 meter apart, the "gravitational mass defect" amounts to $\sim 10^{-27}$ of their rest mass. Such an accuracy is not likely to be ever reached by experiments. However, the share of the gravitational potential energy is more significant for very massive, astronomical bodies. The Moon, for example, "loses" about 10^{-11} of its mass due to the mutual attraction with the Earth.[1] Were this contribution different for inertial and gravitational masses of the Moon, it would affect the Moon's orbit. Precise tracking of the Moon's distance (referred to as the Lunar laser ranging), together with subtraction of all known effects (such as those due to tides), has supported the validity of the **strong equivalence principle** (the statement which also involves the gravitational potential energy) with the precision of about 10^{-14}.

Recently, further advance in accuracy is mostly being mentioned in connection with the ongoing effort to find the quantum theory of gravitation. Actually, some of the approaches predict violations of the gravitation-inertia equivalence with magnitudes just about to be reached by current technologies. A natural aim, not only restricted to the testing of the equivalence principle, are measurements from regions involving *strong* gravitational fields (rather than such weak ones as in our Solar system). Of course, GR is a non-linear theory (as we shall see), so significant deviations from Newtonian gravity can be expected where the field is strong. From the point of view of the equivalence principle, it is certainly much more interesting if the contribution of the gravitational binding energy is not just of the Newtonian $-\frac{Gm_1m_2}{c^2r}$ type, but if it arises under strong-field conditions. The best stronger-field limit has yet been obtained from long-term observation of the triple stellar system involving a 366-Hz pulsar (PSR J0337+1715) in close orbit with a white dwarf (1.6-day orbital period) and another white dwarf orbiting the binary at larger distance (yet still below 1 astronomical unit – with period of 327 days). Current [2020] accuracy of this strong-equivalence test (in which, specifically, the effect of the outer dwarf on the inner-binary motion is studied) is 10^{-6}.

[1] Worth to realise that the exact value depends on the reference system. In the Earth-Moon rest system, we are done, whereas, for example, in the system in which the Moon orbits with the Keplerian speed given by $v^2 = -\Phi = GM_\oplus/r_☾$, the potential-energy effect $M_☾\Phi$ is partly compensated by the positive kinetic-energy contribution $\frac{1}{2}M_☾v^2 = -\frac{1}{2}M_☾\Phi$.

1.1.2 Local inertial frame

Imagine you are inside a lift with no window. Therefore, you can only perform *quasi-local* measurements, not *global* ones (such as observing distant stars or a possible Earth residing below). If the equivalence principle holds, you cannot recognise whether i) the lift is hanging still in some gravitational field $\vec{g}$ (e.g. that of the Earth), or ii) it is in zero gravitational field (there is no "Earth") but pulled with the acceleration $-\vec{g}$. In both cases, all free bodies will fall, with respect to the lift, with the acceleration $\vec{g}$.

Imagine now that the lift's rope snaps, so that the lift is in a free fall (and non-rotating). As already learnt by our (in fact Einstein's) roofer, all free bodies move without acceleration with respect to such an (*Einstein's*) lift, that is, in the same manner as with respect to unaccelerated, inertial frames in an empty space (without gravitation).[2]

Generalising these observations to *any* physical process, not just the free fall, one arrives at the following reading of the equivalence principle:

- All physical processes run, with respect to a frame kept stationary in a given gravitational field, in the same way as they do with respect to a frame (suitably) accelerated in an empty space without gravitation.

- With respect to freely falling and non-rotating frames ("Einstein's lifts"), the physical processes run in the same way as they do with respect to inertial frames in an empty space.

This is the

$\boxed{\text{equivalence between the gravitational field and the field of inertial forces}}$.

The equivalence principle not only states an important feature of the Universe. It also provides a key for transferring the physical laws from special relativity to a situation where gravitation is not negligible. Actually, if, even in a generic situation, the freely falling and non-rotating frames are equivalent to inertial frames of flat space-time, it means that in such frames one can use special relativity, whatever is the gravitational field.

Well, there still remains one significant flaw to remedy, already touched above: the gravitational field is *non-homogeneous*, that is, the gravitational acceleration $\vec{g}$ is *different* at different places (and possibly times), and thus – among others – it cannot be "eliminated" in an arbitrarily large region by going over to a *single*, large freely falling lift. Therefore,

$$\boxed{\begin{array}{l}\text{the equivalence principle can only hold LOCALLY,}\\ \text{in a neighbourhood of each particular point.}\end{array}}$$

To summarize, the principle of equivalence can be phrased as an analogue of the 1st Newton's law (the definition of inertial frame plus the postulation of its existence), now however only having local validity, i.e. with the inertial frame replaced with the **local inertial frame**:

[2] Since 1979, this course has been taught on the 10th floor. Students are kindly asked to employ, for possible measurements required at practical courses, *other than Faculty* lifts – best of all if Einstein's.

At every (non-singular) point of arbitrary space-time, there exists a Local Inertial FramE (LIFE), i.e. such a locally Cartesian coordinate system with respect to which, in a sufficiently small neighbourhood of the given point, all natural laws hold in the same form as in special relativity.

"Sufficiently small" neighbourhood in principle means *infinitesimally* small; practically, it depends on how much non-homogeneous the field is and on how precise one wants to be. Let us add that – similarly as in the case of the 1st Newton's law – the existence of one LIFE implies the existence of infinitely many such frames at each point; all these (passing through that point, at a given time, in different directions and with different speeds) are related by Lorentz transformations.

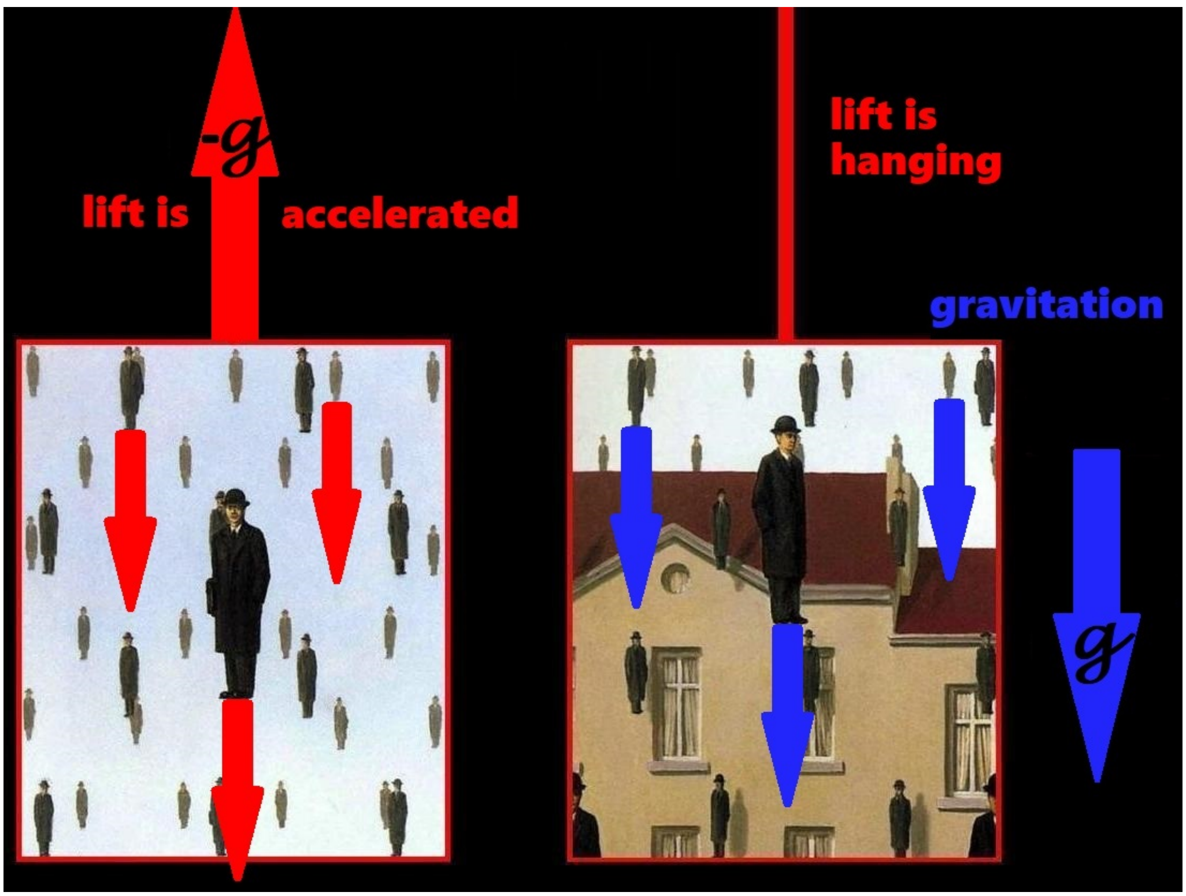


Figure 1.4 *It is not possible to distinguish, by experiments limited to the interior of a lift, whether [right] the lift is hanging still in a homogeneous gravitational field (exerting an acceleration $\vec{g}$), or whether [left] it is accelerated (with acceleration $-\vec{g}$) by some other force in a space without gravitation. In both cases, everything having no acceleration with respect to the lift experiences weight with intensity $\vec{g}$.*

Figure 1.5 *It is not possible to distinguish, by experiments limited to the interior of a lift, whether [right] the lift is freely falling in a given homogeneous gravitational field, or whether [left] it is just standing (or "hanging") still in a space without gravitation. In both cases, everything having no acceleration with respect to the lift feels no weight.*

1.1.3 Terminology: weak, Einstein's and strong principle

Often different names are used for the equivalence principle, depending on how general its statement is understood. The **weak equivalence principle** claims the equivalence for freely falling bodies, i.e., it actually summarizes Galilei's observations. **Einstein's equivalence principle** extends the claim to all matter and fields governed by non-gravitational interactions. The **strong equivalence principle** also includes the gravitational interaction itself. In order to stress that the gravitational interaction energy in fact "non-trivially" (in a non-Newtonian way) contributes in the strong-field regime only, this last term is often reserved to the strong-field situations, while its weak-field limit (most notably tested by Lunar laser ranging) is being called the **gravitational weak equivalence principle**.

1.1.4 *"... I have never been able to understand this Principle."*

Einstein was always claiming that his general theory rests entirely on the equivalence principle, but his followers – most notably those with rather geometrical taste – were having reservations. Illustrative is the preface of the textbook [46]. Its author writes, among others: *"I have never been able to understand this Principle. Does it mean that the signature of the space-time metric is +2 (or −2 if you prefer the other convention)? If so, it is important, but hardly a Principle. Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field, or there is none, according as the Riemann tensor does or does not vanish. This is an absolute property; it has nothing to do with any observer's world line."*[3]

Two responses are immediately at place:

- Synge's reasoning tacitly uses the equivalence principle! Were the inertial and gravitational masses different, the intensity could not be ascribed to the observer's acceleration, and the gravitational field could not be reduced to the Riemann tensor. Sure, the reference-frame acceleration can only "mimic" the effects of intensity, not those of the field's non-homogeneity (i.e., of the space-time curvature). Yet even zero intensity *may* mean a certain gravitational field, even in case when the Riemann tensor vanishes.[4] Einstein himself really understood the field in such a "relativistic" way; also with later hindsight – in an appendix from 1952 to the English reedition of the book *Relativity: The Special and the General Theory* (originally published in 1917 under the title *Über die spezielle und die allgemeine Relativitätstheorie*) – he writes: "Judged from the general-relativity standpoint, $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2$ does not represent a space without field, but a special case of the $g_{\mu\nu}$ field for which – in a certain coordinate system, which in itself has no objective significance – the functions $g_{\mu\nu}$ have values that do not depend on the coordinates. There is no such thing as an empty space, i.e. a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field."

- The equivalence principle is quite powerful actually and it is by no means automatic that the equations of a geometric theory of gravitation satisfy it. The point is that if the gravitational field is non-homogeneous, its derivatives do *not* vanish, irrespectively of how small a region one limits to. Hence, should the principle hold, the equations governing physical processes have to only contain the non-homogeneity in such a way that its effect vanishes locally. This means that the quantity representing the non-homogeneity either must not appear in the equations at all, or it has to be multiplied there by some other quantity that does vanish at a point. (For an example, see the geodesic-deviation equation, Section 6.4.)

  Needless to say, the equivalence principle is a *principle*, so it is only possible to *assume* that it holds (and test it experimentally), not to prove it. We will mention some issues

[3] The Riemann tensor will be the matter of Section 6. Here it is sufficient to say that it describes the space-time curvature (in geometrical language), i.e. the gravitational-field non-homogeneity (in physical language).

[4] Actually, the vanishing of invariants provided by the Riemann tensor and its derivatives (possibly even of *all* such invariants) does *not* necessarily imply that the space-time is "trivial" (flat); it may in fact even contain a singularity.

which are connected with it in Section 9.

Hardly on anything Einstein relied as much as on the equivalence principle when seeking the relativistic theory of gravitation. Yet he himself very soon realised its restrictions. In his first "gravitational" work from Prague (factually his first work as such fully devoted to gravitation), *On the influence of gravitation on the propagation of light* (1911), he writes: "For through a theoretical analysis of processes taking place relative to a uniformly accelerating reference system, we obtain information about the course of processes taking place in a homogeneous gravitational field.", and further specifies in a footnote that "Of course, one cannot replace an arbitrary gravitational field by a state of motion of the system without a gravitational field, just as one cannot transform to rest all the points of an arbitrarily moving medium by means of a relativistic transformation." In one of the following Prague papers – where he attempted, unsuccessfully, to fit together gravitation and special relativity by considering light speed as a function of position playing the role of gravitational potential – Einstein explicitly mentions the *local* validity of the equivalence principle (and he entrusts the measurement of quantities affected by gravitation to small, "pocket" devices only). Note that the early – yet "non-geometric" – history of general relativity is depicted in [2] or in the contribution by J. Bičák in [7] (other contributions therein are devoted to A. Einstein as well).

One more vague caution: we were saying that the LIFE is small, freely falling and non-rotating. Yet what does it mean to *non-rotate*? In Newton's view, it meant to non-rotate relative to the absolute space (relative to inertial frames); Mach considered (non-)rotation relative to a certain "weighted average of momentum" of all the matter in the Universe; in general relativity, what *matters* is not only all the matter (in fact mass-energy), but also the behaviour of space-time at its boundaries (usually at infinities). We'll return to the query of "non-rotation" in Sections 18 and 16.3.3. Anyway, we did not claim that *every* freely falling frame is LIFE, did we? Actually, the above query can well be answered by referring to the equivalence principle itself: "at every point ..., there exists such a locally Cartesian coordinate system with respect to which ... all natural laws hold in the same form as in special relativity." *Those* of the freely falling Cartesian frames are "non-rotating" for which this is true.

1.2 Principle of general covariance

OK, so when trying to describe any physical process, happening at an arbitrary point of arbitrary space-time, one can cut one's rope and … *locally* rely on special relativity. However, in a non-homogeneous field, every LIFE only represents "correctly" an infinitesimal neighbourhood of that point (actually of the whole world-line of the LIFE's origin). Anticipating that the field non-homogeneity (= physical property) is equivalent to the space-time curvature (= geometric property), one can imagine the situation clearly (best on a two-dimensional surface): LIFE's axes, similarly as axes of any Cartesian frame, span the tangent space of the manifold (tangent plane of the surface in the 2D case) at the given point. Therefore, they are *tangent* to the manifold (surface), but do not in general "lie" in it. Now, when studying some process not only happening in a small region, it would be very impractical to constantly change the LIFE when proceeding from one point to the "following" one ... we need a *global coordinate system*! But how to choose global coordinates? This is answered by the general-covariance principle: *any coordinate system (smooth local map) is fine*. Hence the shortest wording of the principle of general covariance:

Physical laws are **generally covariant** – they have the same form (same "content") in all coordinate systems.

Mathematically, it means the laws have to be expressed in terms of equations which preserve their form when transformed, by any diffeomorphism, from one coordinate system to another. Apparently, the form is preserved if both sides of the equation transform in the same way. We already know from special relativity that one of suitable – and standard – options is to write the equations so that their both sides have *tensorial nature*. Actually, tensors are defined "abstractly", without a reference to any particular basis (coordinate system), so they have an invariant meaning and also transform in the same way: a tensor of the (r,s) type, i.e. r-times contravariant and s-times covariant, is represented, in any particular coordinate basis, by 4^{r+s} components $T^{\alpha_1\ldots\alpha_r}{}_{\mu_1\ldots\mu_s}$ which transform, under the basis change $\{x^\kappa\} \to \{x'^\lambda(x^\kappa)\}$, according to

$$T'^{\alpha_1\ldots\alpha_r}{}_{\mu_1\ldots\mu_s}(x') = \frac{\partial x'^{\alpha_1}}{\partial x^{\beta_1}}\cdots\frac{\partial x'^{\alpha_r}}{\partial x^{\beta_r}}\,\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_s}}{\partial x'^{\mu_s}}\,T^{\beta_1\ldots\beta_r}{}_{\nu_1\ldots\nu_s}(x)\,, \tag{1.2}$$

i.e., in such a way that "each upper index transforms via the Jacobi matrix of the transformation" $\frac{\partial x'^{\cdot}}{\partial x^{\cdot}}$, while "each lower index transforms via the corresponding inverse matrix" $\frac{\partial x^{\cdot}}{\partial x'^{\cdot}}$.

Immediate notes:

- Tensorial should be *equations as whole*, not necessarily their individual terms. Naturally, one also uses quantities which are *not* of tensorial nature, as e.g. the mass-energy.
- The principle of general covariance does not claim that equations whose sides are not tensors cannot have a good sense. It only claims that the physical "game rules" are the same with respect to all reference frames. For instance, the equation $E = mc^2$ is completely fine, although its sides are not invariants, i.e. $(0,0)$-type tensors. (This equation even keeps its form, because both its sides transform in the same way.)
- How does the statement "tensors have an invariant meaning" go together with the fact that their components do change under transformation of the reference system? Perfectly: the transformation changes the components in just the proper way to leave invariant the result (a number) the tensor should yield when applied to its variables (vectors and covectors). One particular implication: a tensor can never be made vanish by transformation; if it is non-trivial in some system, it also cannot completely vanish in any other one. For example, the electromagnetic-field tensor $F_{\mu\nu}$ yields two independent invariants, $F_{\mu\nu}F^{\mu\nu}$ and ${}^*F_{\mu\nu}F^{\mu\nu}$; from these it is seen, among others, that for a general EM field, one even cannot transform out the electric (or magnetic) field.

Hence the recipe for transferring the laws of physics from special relativity to general space-time and general coordinates:

- Let a law hold in the absence of gravitation, i.e. in special relativity.
- According to the equivalence principle, it thus holds – in the same form – in any local inertial frame.
- The general-covariance principle now says: if the law is written in a generally covariant form, it holds – in that form – in any space-time and in any coordinates.

In this formulation, the covariance principle is extended (to general frames) thanks to the equivalence principle. It was exactly in such a union how Einstein brought the two principles to the stage at the end of 1907, in the last section *V. The relativity principle and gravitation* of his paper *On the relativity principle and the conclusions drawn from it*, published (in German) in Jahrbuch der Radioaktivität und Elektronik 4, pp. 411-462. It is worth to note, once more, that already in this first formulation, Einstein was well aware of the limitations of the idea of equivalence and spoke explicitly of *homogeneous* field: "At our present state of experience we have thus no reason to assume that the systems Σ_1 and Σ_2 [the system 1 is accelerated in an empty space and the system 2 is at rest in a corresponding homogeneous gravitational field] differ from each other in any respect, and in the discussion that follows, we shall therefore assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system. This assumption extends the principle of relativity to the uniformly accelerated translational motion of the reference system. The heuristic value of this assumption rests on the fact that it permits the replacement of a homogeneous gravitational field by a uniformly accelerated reference system, the latter case being to some extent accessible to theoretical treatment." (Note that right in that paper Einstein employed the principle to show that, in a gravitational field, time should be dilated, frequency should be shifted, and rays should be bending.)

The principle of general covariance extends the principle of special relativity. It is the last step in the history of refutation of the privileged status of certain reference systems (e.g. of those given by the Earth surface and the corresponding up/down directions, geocentric and heliocentric systems, Newton's absolute space and time, or inertial systems of special relativity). Needless to say, the principle only concerns the "game rules" – it does not deny that certain reference systems are more *practical* than others for representing specific situations; for instance, a central, spherically symmetric field is certainly best represented in a spherical system. Nota bene, the configuration of matter and fields in the real world *does prefer* certain systems: passing through different scales, one identifies the rest system of some particular laboratory object, rest system of the Earth, of the Solar-system barycentre, of the Galaxy, etc. Cosmic microwave background radiation is isotropic in a certain system, and the latter appears to be also well followed by the large-scale distribution of matter. Still, the physical *rules* do not distinguish between systems. Having this in mind, one realises how *non-trivial*, *strong* and in fact *unexpected* the principle of general covariance is.

1.2.1 This is not the full story, however

Actually, all the above has been just a facade – the brighter side of life. In fact *every* physical law can be rewritten in a general covariant form (E. Kretschmann, 1917). Loosely speaking, one checks how the equation transforms, and if there is a non-tensorial part, one adds suitable extra (non-tensorial) terms whose non-tensorial part just cancels out, in transformation, with the originally present non-tensorial terms. If the extra added terms contain non-homogeneity of the field (curvature) suitably, they vanish in flat space and the law thus satisfies the special-relativity (or other "flat-space") limit. Clearly, in order that the covariance principle possess factual meaning, one must restrict what is permitted when trying to rewrite a given equation in a covariant manner. Regarding that the equation might have already originally contained terms dependent on non-homogeneity of the field, it may not be easy to distinguish them from those which someone "added by hand in an unguarded moment". Therefore,

> the covariance principle has to be supplemented by a certain restriction on *what kind of quantities may appear in equations*.

In order that the principle have physical meaning, this restriction must be expressed in terms of *physical*, measurable quantities, rather than just in terms of mathematical ones (tensor components). Quite a reliable criterion is that the theory should not contain any **absolute elements** (sometimes called **background fields**), namely such that affect other elements of the theory, but, conversely, are themselves *not* affected by the others; in a sense, such elements violate the action-reaction principle, or relational reciprocity. This kind of requirement has a long history, most notably involving the criticism by E. Mach of the absolute reference system (and also of the "absolute" property of inertia – see below) of the Newtonian mechanics. Anyway, these deep layers of "covariance", "diffeomorphism invariance" and "background independence" (these notions not being synonyms!) are perhaps the most difficult areas of the theory (not only of general relativity),[5] and we do not attempt to treat them properly. At least, it will be more effective to recall them only after we "build" the theory.

Besides that, the above fundamental requirements have to be supplemented, as elsewhere in physics, by a certain practical restriction referred to as "the principle of simplicity": of all conceivable forms a given law or equation may assume in agreement with the above principles, the simplest should be chosen preferably. We will also return to this point later.

1.2.2 Geometrical objects

In GR, one often wants to stress whether a certain property depends on coordinates or not. The invariant, "absolute" properties are also called "geometrical" or "physical". At the same time, in geometry the authors call "geometrical" a wider class of quantities (wider than just tensors). In order to avoid confusion, let us add what these represent. In short, the wider class of objects is useful because they bear *some* features of the tensors, so, correspondingly, *some* operations can be performed with them in a similar manner (as with tensors), while others may have less sense.

Following chiefly [40] (Chapter III, § 3) and [53] (Chapter II, § 1), we say that a quantity Ω is a **geometrical object** (on a given manifold) if it has the following properties:

[5] As an evidence, we quote the title of a paper by T. Teitel which was published in 2019 in Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics: *Background independence: Lessons for further decades of dispute*.

- In each coordinate system x^μ, it is represented by a certain set of components Ω^A which, under a coordinate transformation $x'^\mu = x'^\mu(x^\alpha)$, change to new components which can be expressed as functions of the old components, of the old and the new coordinates, and of the transformation matrix $x'^\mu{}_{,\kappa}$ plus its partial derivatives, i.e.,

$$\Omega'^B = F^B(\Omega^A, x^\alpha, x'^\mu, x'^\mu{}_{,\kappa}, x'^\mu{}_{,\kappa\lambda}, \ldots)\,.$$

- The functions F^B have group properties, namely, they satisfy (without all indices:)
(a) $F^A(\Omega', x', x'') = F^A(\Omega, x, x'')$; (b) $F^A(\Omega', x', x) = \Omega^A$; $\Rightarrow$ (c) $F^A(\Omega, x, x) = \Omega^A$.

The geometrical objects can further be classified into several subtypes:

- When the functions F^B only depend on Ω^A and on the transformation matrix plus its derivatives, not on x and x' themselves, the object is called **differential**.

- When the functions F^B depend linearly on Ω^A, the object is called **linear**. When, in addition, the transformation law is homogeneous (i.e. linear and without the "absolute" term independent of Ω^A), the object is called **linear homogeneous**.
For example, tensors have all the three properties, so they are differential linear homogeneous objects. Indeed, tensors and tensor densities are *the only* such objects. In particular, affine connection is only a differential linear object.

1.3 Mach's principle

Besides criticising Newton's absolute space and time, Ernst Mach also argued that **inertia** should be understood differently – not as an *intrinsic* property of a body (moreover, only revealing itself if the body is accelerated relative to the "absolute space"), but as a consequence of a certain interaction between the body and all other bodies in the Universe. In his view, acceleration of a body – similarly to its velocity – only has a good sense if it is defined and measured *with respect to other bodies*. Hence the idea of an inertial frame as the frame which is not accelerated with respect to a certain (probably distance-weighted) average of momenta of all masses in the Universe.

Mach's view is best illustrated on his reinterpretation of the Newton's rotating-bucket experiment. Instead of attributing the bending of the water surface to acceleration of the water with respect to an absolute space, he emphasises acceleration with respect to "distant cosmic masses". Not only that: he also says i) that the surface would bend similarly if, on the contrary, the bucket stayed at rest while the cosmic masses were orbiting around, and ii) that the water surface would *not* bend significantly if the walls of the bucket were made "several leagues thick". Namely, in that case, it would mainly be the large mass of the bucket itself what would determine, by its state of motion, what it means to be "non-accelerated". Well, we prefer to imagine, rather than the bucket, a dancer under a starry sky: knowing absolutely nothing about any reference frame, she still recognises in two ways whether she is making a pirouette or not – stars are whirling around (global experiment) if and only if her hands tend to rise due to the centrifugal force (quasi-local experiment).

Einstein was very much influenced by Mach's view on inertia (it was him who called it the Mach's principle) when working on his theory of gravitation. And again, it was in Prague where he first tested it explicitly: in the paper *Is there a gravitational effect which is analogous to electrodynamic induction?*, published in 1912 in the medical journal *Vierteljahrsschrift für gerichtliche Medizin und öffentliches Sanitätswesen* 44, pp. 37-40 (it was dedicated to his medical-friend birthday), he considered a test particle inside a massive spherical shell, and asked what the particle "feels" if the shell starts to be linearly accelerated. The answer was – contrary to Newtonian conclusion – that the particle also starts to be accelerated to a certain extent, with the following interpretation: if there is no other mass, it is the sphere's state of motion what determines which motions are "inertial"; if the particle wanted to stay at rest relative to the original rest system, it would have to be accelerated *relative to the sphere* (in the opposite direction), so it would feel a corresponding inertial force acting on it; but the particle is inertial, "free", so it should be dragged, to some extent, along with the sphere. In a letter sent from Zurich on 25 June 1913, Einstein informed Mach about the above conclusion, together with that he also considered a *rotating* shell and found that a Coriolis field arises inside (which should carry along the Foucault-pendulum plane, for example). He writes: "... it follows of necessity that *inertia* has its origin in some kind of *interaction* of the bodies, exactly in accordance with your argument about Newton's bucket experiment."

Soon after it had been finished in 1915, general relativity was shown to really predict "Machian" (dragging) effects. However, already before that, Einstein was much surprised that his theory is *not* strictly "Machian". Yet let's postpone this story to later stages.

1.4 Gravitation is (a non-trivial) geometry

Due to its universality, the effect of gravitation may be viewed as a property of the space-time itself. What property it might be can be inferred with the help of the equivalence principle. The latter claims that the gravitational field (at least the homogeneous one) can be mimicked by means of inertial forces arising in accelerated systems. What happens if one transforms, in flat space-time of special relativity, to an accelerated system? The metric tensor ceases to have the Minkowski form valid in the inertial frames, even in case when the coordinate axes are kept Cartesian: in the accelerated frame, the space-time appears to be *curved*, having a different geometry than the "flat" space-time of Minkowski. If causing the same effect as the reference-frame acceleration, gravitation should thus change the space-time geometry. In addition, since the actual fields are non-homogeneous, "switching them on" should correspond to changing over to a system whose acceleration is different at different places (and/or times). In such a situation, the originally Cartesian coordinates would have to actually become curvilinear, which would still more divert the metric from the Minkowski form. Below, let us illustrate the transformation to an accelerated system on two simple cases.

1.4.1 Linearly accelerated frame

The simplest type of accelerated motion is that due to a constant force; it is a default illustration on the special relativistic equation of motion. If the three-force is constant – with respect to a given inertial time t – in magnitude (f) as well as in direction (chosen to be along the x axis), a particle with rest mass m_0 is known to be accelerated along a hyperbolic world-line,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}=f \quad\Longrightarrow\quad v=c\frac{\frac{ft}{m_0c}}{\sqrt{1+\left(\frac{ft}{m_0c}\right)^2}} \quad\Longrightarrow\quad x=\frac{m_0c^2}{f}\sqrt{1+\left(\frac{ft}{m_0c}\right)^2}\,.$$

The proper time τ along the accelerated world-line is found to read

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\sqrt{-\frac{1}{c^2}\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}=\sqrt{1-\frac{v^2}{c^2}}=\frac{1}{\sqrt{1+\left(\frac{ft}{m_0c}\right)^2}} \quad\Longrightarrow\quad \tau=\frac{m_0c}{f}\,\mathrm{arcsinh}\frac{ft}{m_0c}\,,$$

where $v=v(t)$ from above was used before performing the integration. (Integration constants are chosen so that $\tau=0$, $t=0$ and $x=\frac{m_0c^2}{f}$ at the beginning of acceleration.) Vice versa, expressing from there

$$t=\frac{m_0c}{f}\,\sinh\frac{f\tau}{m_0c} \qquad\Longrightarrow\qquad x=\frac{m_0c^2}{f}\,\cosh\frac{f\tau}{m_0c}\,, \tag{1.3}$$

we can substitute for $\frac{ft}{m_0c}=\sinh\frac{f\tau}{m_0c}$ back to the velocity formula,

$$\frac{v}{c}=\frac{\frac{ft}{m_0c}}{\sqrt{1+\left(\frac{ft}{m_0c}\right)^2}}=\frac{\sinh\frac{f\tau}{m_0c}}{\cosh\frac{f\tau}{m_0c}}\,, \qquad\qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\cosh\frac{f\tau}{m_0c}\,. \tag{1.4}$$

Cartesian frame tied to the accelerating particle/observer (with axes T, X, Y, Z) is related, at every specific moment, to the original inertial system (t, x, y, z) by special Lorentz transformation given by *instantaneous* value of the relative velocity (1.4). Hence,[6]

$$X+\frac{m_0c^2}{f}=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}=\left(\cosh\frac{f\tau}{m_0c}\right)x-\left(\sinh\frac{f\tau}{m_0c}\right)ct\,.$$

Let the accelerated-frame time T be given, naturally, by the proper time τ, i.e. by the time of the clock kept at the accelerate-frame origin. Then the second transformation relation follows directly from (1.3),

$$\tanh\frac{fT}{m_0c}=\tanh\frac{f\tau}{m_0c}=\frac{ct}{x}\,.$$

Inverting these two relations, we obtain the dependence of (t,x) on (T,X):

$$ct=\left(X+\frac{m_0c^2}{f}\right)\sinh\frac{fT}{m_0c}\,, \qquad\qquad x=\left(X+\frac{m_0c^2}{f}\right)\cosh\frac{fT}{m_0c}\,. \tag{1.5}$$

[6] From now on, t and x no longer stand for particular values on the accelerating world-line, but they represent a general location in the accelerated frame.

Finally, the space-time interval in the accelerating system follows by substituting for the differentials

$$\begin{aligned} c\mathrm{d}t &= \mathrm{d}X\ \sinh\frac{fT}{m_0c} + \left(\frac{fX}{m_0c^2}+1\right)c\mathrm{d}T\ \cosh\frac{fT}{m_0c}\,, \\ \mathrm{d}x &= \mathrm{d}X\ \cosh\frac{fT}{m_0c} + \left(\frac{fX}{m_0c^2}+1\right)c\mathrm{d}T\ \sinh\frac{fT}{m_0c} \end{aligned}$$

into the "inertial" form valid in the (ct,x,y,z) system,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 = -\left(1+\frac{fX}{m_0c^2}\right)^2 c^2\mathrm{d}T^2+\mathrm{d}X^2+\mathrm{d}Y^2+\mathrm{d}Z^2. \quad (1.6)$$

In the spatial part, the metric thus remains Euclidean, but the element of proper time of a clock which stays, in the accelerated system, at rest at a position $(X,Y,Z=\mathrm{const})$, reads $\left(1+\frac{fX}{m_0c^2}\right)\mathrm{d}T$, so it only reduces to $\mathrm{d}T\equiv\mathrm{d}\tau$ at the system's origin ($X=0$) while it is $\left(1+\frac{fX}{m_0c^2}\right)$-times greater at a general position X. In other words, clocks at bigger X run faster than those at smaller X. According to the equivalence principle, our frame accelerated in the x-direction is equivalent to an unaccelerated frame in which, instead, gravitation acts in the $(-X)$ direction. Hence, the gravitational field affects the passage of time; specifically, the clocks placed on lower potential ("deeper in the potential well") run more slowly than those placed "higher". In Section 4, we will see this is the case for generic (non-homogeneous) fields as well.

1.4.2 Rotating frame

The second archetype of acceleration is the purely transversal, rotational one. Let us first write the flat metric, valid in the Minkowski form in a certain inertial frame (ct,x,y,z), in terms of polar (cylindrical) coordinates (ct,ρ,ϕ,z) given by $x=\rho\cos\phi$, $y=\rho\sin\phi$:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 = -c^2\mathrm{d}t^2+\mathrm{d}\rho^2+\rho^2\mathrm{d}\phi^2+\mathrm{d}z^2\,.$$

Now, let us transform to a second cylindrical system (cT,ρ,ψ,z), rotating with constant (uniform in time and everywhere same) angular velocity ω with respect to the first one,

$$t=T, \qquad x=\rho\cos(\psi+\omega T), \qquad y=\rho\sin(\psi+\omega T)\,.$$

The metric appears there as

$$\begin{aligned} \mathrm{d}s^2 &= -(c^2-\omega^2\rho^2)\,\mathrm{d}T^2+2\rho^2\omega\,\mathrm{d}T\mathrm{d}\psi+\rho^2\mathrm{d}\psi^2+\mathrm{d}\rho^2+\mathrm{d}z^2 = \\ &= -c^2\mathrm{d}T^2+\rho^2(\mathrm{d}\psi+\omega\mathrm{d}T)^2+\mathrm{d}\rho^2+\mathrm{d}z^2. \end{aligned} \quad (1.7)$$

This is *not* a flat space-time in usual cylindrical coordinates (this would only be the case with $\omega=0$). The deformation is more complicated than in the linear-acceleration case, in particular, it also involves non-diagonal component $g_{T\psi}=\rho^2\omega$.

Suppose to have a disc $\{z=0, \rho \leqslant R\}$ which rotates, rigidly, with the above angular velocity ω with respect to the inertial frame. Imagine observers co-rotating on it (i.e. having constant ρ, z and ψ) and inspecting its geometry. We can expect their result to be "non-flat", because observers orbiting on different radii have *different* acceleration, and we expect – without "technical details" yet – such a non-homogeneous inertial field (equivalent to a non-homogenous gravitational field) to correspond to *curved* geometry. Let us check the situation from the inertial system where we know how things work: i) the measuring rods the co-rotating observers place along the disc's radius move, at every moment, purely transversally, so their length remains the same in both systems; ii) the measuring rods the co-rotating observers place along the disc's circumference move, at every moment, purely longitudinally, so their length is γ times contracted with respect to the inertial frame, with $\gamma = \frac{1}{\sqrt{1-\frac{R^2\omega^2}{c^2}}}$. Hence, the co-rotating observers have to find the radius of the disc to be R, while the disc's circumference to be $2\pi R\gamma$ (since γ-times more co-rotating rods fit along the circumference in comparison with those at rest).

The above is known, for historical reasons, as Ehrenfest's paradox (or rotating-disc paradox), but it is not in fact paradoxical – it just indicates that for co-rotating observers the disc is not flat. Remember that these observers are not inertial, so they are *not* equivalent to the inertial observers – the two views are not symmetric as one is used to from special relativity. Actually, in order to treat the situation from the point of view of co-rotating observers, one needs the formalism of *general* relativity. Yet to check the basic facts is very simple, it just requires to realise that every observer performs all spatial measurements in her *constant* proper time, i.e. within her local 3D space. Such a space is described by the metric

$$h_{\mu\nu} = g_{\mu\nu} + \frac{1}{c^2}\, u_\mu u_\nu\,,$$

where $g_{\mu\nu}$ represents a general metric tensor, which, however, reduces to $\mathrm{diag}(-1, 1, \rho^2, 1)$ in our case (flat space-time in inertial cylindrical coordinates). Actually, $h_{\mu\nu}$ satisfies what a metric should satisfy: it is a symmetric bilinear form, it satisfies correct relation between totally contravariant and totally covariant representation,

$$h_{\mu\nu}h^{\nu\alpha} = \left(g_{\mu\nu} + \frac{1}{c^2}\, u_\mu u_\nu\right)\left(g^{\nu\alpha} + \frac{1}{c^2}\, u^\nu u^\alpha\right) = \delta^\alpha_\mu + \frac{1}{c^2}\, u_\mu u^\alpha \equiv h^\alpha_\mu\,,$$

and its trace equals the dimension,

$$h^{\mu\nu}h_{\mu\nu} \equiv h^\mu_\mu = \delta^\mu_\mu - \frac{c^2}{c^2} = 4 - 1 = 3\,.$$

Besides that, it is by construction orthogonal to u^μ,

$$h_{\mu\nu}u^\nu = u_\mu - u_\mu = 0\,.$$

We in fact know most of the above from special relativity.

Now back to our rotating-disc problem: the co-rotating observer has, in the cylindrical coordinates (ct, ρ, ϕ, z), four-velocity

$$u^\mu = u^t(c, 0, \omega, 0), \qquad u_\alpha = g_{\alpha\mu}u^\mu = u^t(-c, 0, \rho^2\omega, 0), \qquad u^t := \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{u^0}{c}\,,$$

where, from the normalisation $g_{\mu\nu}u^\mu u^\nu = -c^2$,

$$(u^t)^{-2} = -g_{tt} - g_{\phi\phi}\frac{\omega^2}{c^2} = 1 - \frac{\rho^2\omega^2}{c^2}\,.$$

The proper circumference she founds of the disc is given purely by the azimuthal component of her metric,

$$h_{\phi\phi} = g_{\phi\phi} + \frac{(u_\phi)^2}{c^2} = \rho^2 + \frac{(u^t)^2\rho^4\omega^2}{c^2} = \rho^2 + \frac{\frac{\rho^4\omega^2}{c^2}}{1-\frac{\rho^2\omega^2}{c^2}} = \frac{\rho^2}{1-\frac{\rho^2\omega^2}{c^2}}\,,$$

namely

$$o = \int_0^{2\pi} \sqrt{h_{\phi\phi}(\rho\!=\!R)}\,\mathrm{d}\phi = \frac{2\pi R}{\sqrt{1-\frac{R^2\omega^2}{c^2}}} \equiv 2\pi R\gamma\,. \tag{1.8}$$

Neither is symmetric the relation between proper time of the inertial observer (t) and that of the orbiting one (τ).

- With respect to the inertial one, the orbiting clocks has to be ticking γ-times slower because of Lorentz time dilation. Hence, during one complete rotation, the inertial clock measures $\Delta t = 2\pi/\omega$, whereas the orbiting clock measures only $\Delta\tau = \Delta t/\gamma$.

- With respect to the orbiting observer, we employ the general relation $c^2\Delta\tau^2 = -\Delta s^2$, where, in the *rotating* coordinates (cT, ρ, ψ, z), the co-rotating clock does not move, hence we are left with $c^2\Delta\tau^2 = -g_{TT}c^2\Delta T^2$ only. In other words, the orbiting-observer four-velocity has, in the rotating coordinates, only the time component; normalisation yields for it

$$u^T \equiv \frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{1}{\sqrt{-g_{TT}}} = \frac{1}{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}} \equiv \gamma\,.$$

In passing, the above also implies that with respect to the orbiting clock, the inertial clock orbits with angular velocity

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{\mathrm{d}\psi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \omega\gamma\,,$$

rather than ω. Let us add that this last finding is nothing strange; it is known from the issue of speed measurement in special relativity: a correct measurement must use rulers and clocks which are at rest with respect to the reference system (for example, a police lurking in a bush is supposed to measure correctly your-car speed with respect to the local Earth surface; this speed cannot exceed the speed of light); a "hybrid" measurement, using – at least partially – devices at rest with respect to the measured object, does not provide sensible results (for example, using a clock you have in a car + highway milestones may yield however high speed, because the miles are Lorentz contracted with respect to your-car system).

CHAPTER 2

Parallel transport

Intuitively, we treat quantities as if they "lived" directly in the physical space. In the case of a flat, Euclidean or Minkowski space, this does not bring any practical mistake, but actually it is not so. The quantities are defined as mappings which have the physical space (or certain its region) as their domain, but whose "values" (range) are *not* elements of that space. It is already clear from units: electric field, for example, does not have the dimension of length, so it is in fact not correct to depict it as an arrow in the physical space. Such an image, however, well illustrates the main problem. Consider now a *curved* space; in particular – due to the usual 3D limitation of our imagination – consider a curved surface. A vector (the arrow) pointing from a certain point of the surface directs, *locally*, along the surface by definition, but, if the surface is curved and the vector is not infinitesimal, the arrow points *out of* the surface (its endpoint does not lie on the surface). All vectors (of some given dimension) defined at a certain point of the surface thus form a **tangent plane** of the surface at that point. In the case of a generic space (manifold), such vectors form a **tangent space** (to the given underlying manifold) at that point.

Each such tangent space is a linear vector space, so its elements can be added and multiplied by a constant. On the other hand, it is not possible to combine vectors defined at *different* points, because they belong to *different* linear spaces. However, one is seldom interested in just one vector (defined at one particular point), nota bene if considering a *field* theory: typically, one wants to know how the field behaves along its host manifold (thus to know its gradient), or how a certain vector transports along the manifold (for instance, how velocity or angular momentum of a body transport along its world-line). But the derivative (gradient) involves difference between the quantity "at $x+h$" and the quantity "at x", which exactly can*not* be easily performed on a curved manifold – the quantity defined at "$x+h$" has first to be *transported* somehow to "x", and only then one can start subtracting. In the flat space, all reasonable transports reduce to the "self-evident" parallel shift, but in the curved space there are more options, with possibly rather complicated properties. This chapter is devoted to the most fundamental of such transports, the **parallel transport**. Its properties within the pseudo-Riemannian geometry were notably derived by T. Levi-Civita in his 1917' paper, with E. B. Christoffel factually providing the "Levi-Civita connection" (namely *Christoffel* symbols) in 1869 already.

Parallel transport is a solution to the very basic problem of geometry, the transport of a "given" direction from one point to another. In other words, such a transport of a vector in which the latter keeps pointing "in the same direction". In the flat space, the solution is simple, because tangent spaces at all points are mutually isomorphic (they "coincide") – the notion of parallelism is global. In two dimensions, we performed it in an elementary school (*let us draw, through a given point, a straight line which is parallel to the straight line over there*) using a triangle and a ruler. But what if the exercise book was bent? The direction vector would stick out of the page, so the third dimension would come into the play. The most natural approach, not using any additional constructions, would then be to solve the problem in the 3D Euclidean space (in which the exercise book is embedded) while continuously ensuring – by an orthogonal tilt to the local tangent plane of the page – that the vector remains tangent to the page after every infinitesimal step along the transport path. Let us stress we said *tilt*, not *projection* to the tangent plane, because any reasonable transport is required to keep the vector's length.

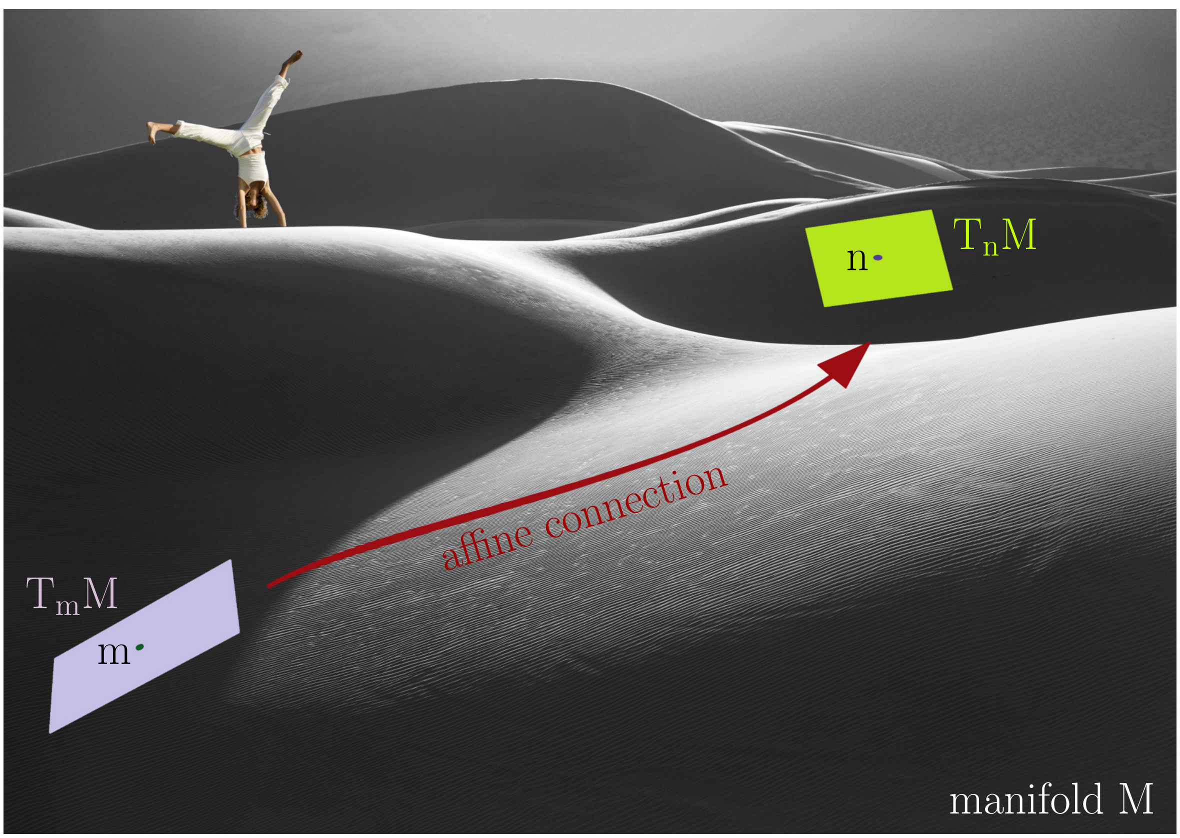


Figure 2.1 *Affine connection connects tangent spaces (e.g. T_mM and T_nM) to a given manifold at different points (here m and n). [The manifold are Eureka Dunes at Death Valley – one of amazing photos by Ian Parker from the Dept. of Neurobiology and Behavior, University of California.]*

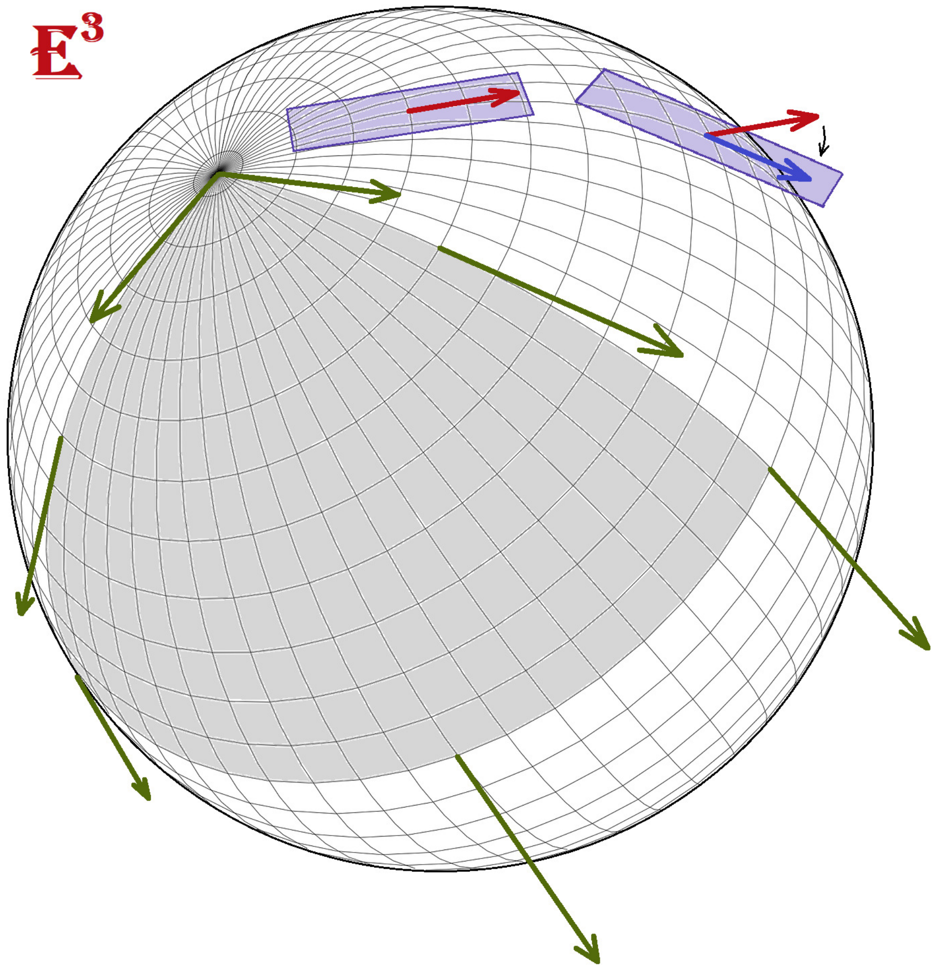


Figure 2.2 *Intuitive concept of the parallel transport, illustrated on a sphere embedded in the 3D Euclidean space. Take a vector (red, closer to the pole) which is tangent to the surface at some point. Shift it to a neighbouring point while keeping its direction in $\mathbb{E}^3$ (red arrow more to the right); after such a shift, it is no longer tangent to the surface, so it has to be tilted (orthogonally) to the local tangent plane (in order to "live in" the chosen manifold, i.e. the sphere). The transport introduced in such a way clearly depends on path along which it is carried out. Consequently, it does* not *in general yield "identity" along a closed path – see the closed-path transport performed with the green vector (starting from the pole). Yet another property is also evident: tangent vectors to curves do not in general transport parallelly; they only do so along meridians and along the equator – actually along all main circles. These are* geodesics *of the spherical surface – see the next chapter.*

2.1 Parallel transport of a vector

From the above image, the parallel transport of a vector is e.g. derived in [24]. Here we follow a simpler way directly using the fundamental principles of the theory. Consider an arbitrary space-time and in it an arbitrary differentiable curve (with uniquely given tangent vector at its every point). Let some vector V^μ be defined at a certain point of that curve. The task is to transport the vector along the curve in such a way that it keeps pointing in the same direction (in some natural sense). According to the equivalence principle, there exists a LIFE (infinitely many LIFEs, actually) at every point. Let us consider one such LIFE at the point where V^μ is defined, and denote its Cartesian axes by ξ^α ($\alpha = 0, 1, 2, 3$). Within an infinitesimal neighbourhood about the given point (LIFE's origin), the parallel transport should work like in inertial frames of special relativity, i.e., it should reduce to its "trivial" Euclidean/Minkowski form (when described with respect to the LIFE). There, however, to keep pointing in the same direction means to keep fixed the (Cartesian) components,

$$\frac{\mathrm{d}V^{\hat{\alpha}}}{\mathrm{d}p} = 0\,, \tag{2.1}$$

where the hat denotes the components with respect to the LIFE and p is the parameter of the curve.[1]

The equivalence principle claims that in the infinitesimal neighbourhood of the given point the above equation holds *exactly*. This is equally true for *any* point on the prescribed curve, but always with respect to some *local* LIFE only. In order to be able to solve the problem globally, we need some coordinate system which would cover the whole region we are interested in; *any* such system must be equally viable. Let us denote its axes by x^μ ($\mu = 0, 1, 2, 3$), and let us transform there the above LIFE-expressed equation (2.1). On the left-hand side we obtain, by standard transformation of a vector (from the LIFE to the global system) and by a composite-function differentiation,

$$\frac{\mathrm{d}V^{\hat{\alpha}}}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\partial \xi^\alpha}{\partial x^\lambda}\,V^\lambda\right) = \frac{\partial \xi^\alpha}{\partial x^\lambda}\frac{\mathrm{d}V^\lambda}{\mathrm{d}p} + \frac{\partial^2 \xi^\alpha}{\partial x^\kappa \partial x^\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda,$$

which, after multiplication by $\frac{\partial x^\mu}{\partial \xi^\alpha}$ (the inverse-transformation matrix), leads to

$$\boxed{\frac{\mathrm{d}V^\mu}{\mathrm{d}p} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda = 0}\,. \tag{2.2}$$

The functions

$$\Gamma^\mu{}_{\kappa\lambda} := \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\kappa \partial x^\lambda} \tag{2.3}$$

represent the **components of the affine connection** in the x^μ coordinates; it is clear from the definition that they are symmetric in the two lower indices. Note that the affine connection need *not* in general have this property (we will return to this point later), but for a theory satisfying the equivalence principle it is the case.

The formula (2.2) represents four 1st-order ordinary differential equations. Recall what must be known in order to be able to solve it:

[1] Note that it is important that the LIFE is Cartesian, similarly as inertial frames of special relativity: it is trivial to see that with respect to non-Cartesian axes, a given direction in general has *different* components at different locations.

- the space-time and the (arbitrary) global coordinates, hence the affine-connection components $\Gamma^{\mu}{}_{\kappa\lambda}$
- the curve; in the LIFE it is written as $\xi^{\alpha} = \xi^{\alpha}(p)$, while in the global coordinate system it writes $x^{\mu} = x^{\mu}(p)$; in the equation, the curve appears through its tangent vector $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}$
- the vector V^{μ} at some initial point $x^{\mu}_{\mathrm{in}} := x^{\mu}(p=p_{\mathrm{in}})$ of the curve, $V^{\mu}(p=p_{\mathrm{in}})$

By solution of the parallel-transport equation (2.2), one obtains a vector function $V^{\mu} = V^{\mu}(p)$ defined along the prescribed curve; this function is called the **parallel vector function** since it arose by parallel transport of a vector (V^{μ}) along that curve.

Let us add that **connection** in general is, loosely speaking, a certain map between spaces of quantities living at different points of a manifold. In particular, the one which connects tangent vector spaces at different points is called the **affine connection**. There typically exist many (in fact infinitely many) affine connections on a given manifold, but we will shortly see that general relativity naturally uses a particular one which exists on pseudo-Riemannian manifolds, i.e. those endowed with the metric. It is tied to the metric in a special way and is called the Levi-Civita connection. Let us embark on this issue.

2.2 Metric tensor and Christoffel symbols

We mentioned in passing that any reasonable transport should leave constant the vector's length (here rather the invariant given by its space-time inner product with itself). Referring again to the equivalence principle, we can compute such a quantity, at the LIFE's origin, in the same way as in special relativity, i.e. using the Minkowski metric tensor $\eta_{\alpha\beta}$, $|V|^2 = \eta_{\alpha\beta}V^{\hat{\alpha}}V^{\hat{\beta}}$, which can again be expressed in terms of the global-coordinate components V^{μ} as

$$|V|^2 = \eta_{\alpha\beta}V^{\hat{\alpha}}V^{\hat{\beta}} = \eta_{\alpha\beta}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}\,V^{\mu}V^{\nu} \equiv g_{\mu\nu}V^{\mu}V^{\nu}, \tag{2.4}$$

where we have denoted

$$g_{\mu\nu} := \eta_{\alpha\beta}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}} \tag{2.5}$$

the object which obviously represents the (covariant) **metric tensor** (of the given – but arbitrary – space-time) in the global coordinates (also arbitrary). Clearly it inherits symmetry from $\eta_{\alpha\beta}$.

In Minkowski space-time, the same invariant can also be expressed in terms of the covector $V_{\hat{\gamma}} = \eta_{\gamma\alpha}V^{\hat{\alpha}}$ dual to the vector $V^{\hat{\alpha}}$,

$$|V|^2 = \eta^{\gamma\delta}V_{\hat{\gamma}}V_{\hat{\delta}} = \eta^{\gamma\delta}\frac{\partial x^{\rho}}{\partial\xi^{\gamma}}\frac{\partial x^{\sigma}}{\partial\xi^{\delta}}\,V_{\rho}V_{\sigma} \equiv g^{\rho\sigma}V_{\rho}V_{\sigma}, \tag{2.6}$$

where we have denoted

$$g^{\rho\sigma} := \eta^{\gamma\delta}\frac{\partial x^{\rho}}{\partial\xi^{\gamma}}\frac{\partial x^{\sigma}}{\partial\xi^{\delta}} \tag{2.7}$$

the object which should represent global-coordinate components of the contravariant metric tensor. The definitions (2.5) and (2.7) are really compatible, as proved by

$$g_{\mu\nu}g^{\nu\sigma} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\,\eta^{\gamma\delta}\frac{\partial x^\nu}{\partial\xi^\gamma}\frac{\partial x^\sigma}{\partial\xi^\delta} = \eta_{\alpha\beta}\eta^{\gamma\delta}\,\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial\xi^\gamma}\frac{\partial x^\sigma}{\partial\xi^\delta} = \eta_{\alpha\beta}\eta^{\gamma\delta}\,\frac{\partial\xi^\alpha}{\partial x^\mu}\,\delta^\beta_\gamma\,\frac{\partial x^\sigma}{\partial\xi^\delta} =$$
$$= \eta_{\alpha\beta}\eta^{\beta\delta}\,\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\sigma}{\partial\xi^\delta} = \delta^\delta_\alpha\,\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\sigma}{\partial\xi^\delta} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\sigma}{\partial\xi^\alpha} = \frac{\partial x^\sigma}{\partial x^\mu} = \delta^\sigma_\mu\,.$$

This has probably been the most spoon-feeding derivation ever made of this automatic result.[2]

Also easy is to check that the metric really lowers and rises global indices, similarly as $\eta_{\alpha\beta}$ does for the LIFE ones,

$$g_{\mu\nu}V^\nu = \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\,V^\nu = \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial x^\mu}\,V^{\hat\beta} = \frac{\partial\xi^\alpha}{\partial x^\mu}\,V_{\hat\alpha} = V_\mu\,,$$
$$g^{\rho\sigma}V_\sigma = \eta^{\gamma\delta}\,\frac{\partial x^\rho}{\partial\xi^\gamma}\frac{\partial x^\sigma}{\partial\xi^\delta}\,V_\sigma = \eta^{\gamma\delta}\,\frac{\partial x^\rho}{\partial\xi^\gamma}\,V_{\hat\delta} = \frac{\partial x^\rho}{\partial\xi^\gamma}\,V^{\hat\gamma} = V^\rho\,.$$

A remark, at last: why we have *not* been hatting the indices of ξ^α and $\eta_{\alpha\beta}$? Because, in these two cases, we even have special letters (ξ and η) for the LIFE coordinates and for the Minkowski tensor, so it is not necessary to hat their indices in addition. On the other hand, it would not have been any mistake to do it.

A running summary: the metric tensor $g_{\mu\nu}$ is a symmetric bilinear form which defines the inner (scalar) product. In particular, when applied to two infinitesimal coordinate-shift vectors $\mathrm{d}x^\mu$, it yields the invariant **space-time interval** $\mathrm{d}s^2$ between the given two events (as separated by $\mathrm{d}x^\mu$),

$$\mathrm{d}s^2 := g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{2.8}$$

In general relativity, however, this quantity is mostly called just **the metric**. In such a manner, the metric completely fixes the local geometry of space-time. (Mathematically, it fixes it *up to a diffeomorphism*, because one must always have a possibility to transform to a different coordinate system, without changing the "true geometry" / "true physics".)

Now, if the metric tells everything about the local space-time geometry, then every characteristic of space-time should somehow be determined by the metric tensor. In particular, this should hold for the affine-connection components. Actually, with the definition of both these quantities in mind, let us write, with the three possible different permutations of free indices, a relation obtained by differentiation of $g_{\kappa\lambda}$,

$$g_{\kappa\lambda,\sigma} = \eta_{\alpha\beta}\,\frac{\partial^2\xi^\alpha}{\partial x^\sigma\partial x^\kappa}\frac{\partial\xi^\beta}{\partial x^\lambda} + \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial x^\kappa}\frac{\partial^2\xi^\beta}{\partial x^\sigma\partial x^\lambda} =$$

[2] The Czech equivalent of "spoon-feeding" is "explaining something *polopatě*" or – originally – *po lopatě*, which literarily means "delivering something *shovel after shovel*" (as opposed to delivering the entire cartload at one stroke). The Czech Wikipedia adds that *such a style of explaining things is instructive, but also degrading for the listeners*. Sorry! –As everyone sees right away, $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$, so (2.5) and (2.7) are really inverse with respect to each other.

$$
\begin{aligned}
&= \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial x^\iota}\Gamma^\iota{}_{\sigma\kappa}\frac{\partial\xi^\beta}{\partial x^\lambda} + \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial x^\kappa}\frac{\partial\xi^\beta}{\partial x^\iota}\Gamma^\iota{}_{\sigma\lambda} = \\
&= g_{\iota\lambda}\Gamma^\iota{}_{\sigma\kappa} + g_{\kappa\iota}\Gamma^\iota{}_{\sigma\lambda} := \Gamma_{\lambda\sigma\kappa} + \Gamma_{\kappa\sigma\lambda}\,, \qquad (2.9)\\
g_{\sigma\kappa,\lambda} &= g_{\iota\kappa}\Gamma^\iota{}_{\lambda\sigma} + g_{\sigma\iota}\Gamma^\iota{}_{\lambda\kappa} = \Gamma_{\kappa\lambda\sigma} + \Gamma_{\sigma\lambda\kappa}\,,\\
g_{\lambda\sigma,\kappa} &= g_{\iota\sigma}\Gamma^\iota{}_{\kappa\lambda} + g_{\lambda\iota}\Gamma^\iota{}_{\kappa\sigma} = \Gamma_{\sigma\kappa\lambda} + \Gamma_{\lambda\kappa\sigma}\,,
\end{aligned}
$$

where we have lowered the first indices of the affine connections.[3] Now, for instance, add the last two equations and subtract the first one: thanks to the symmetry of $\Gamma_{\mu\kappa\lambda}$ in the last two indices and thanks to the symmetry of $g_{\mu\nu}$, two couples of terms cancel out, while the last couple adds up, so we arrive at

$$
\Gamma_{\sigma\kappa\lambda} + \Gamma_{\sigma\lambda\kappa} = 2\Gamma_{\sigma\underline{\kappa\lambda}} = (g_{\sigma\kappa,\lambda} + g_{\lambda\sigma,\kappa} - g_{\kappa\lambda,\sigma}) \qquad (2.10)
$$

and, after "rising the 1st index" by $g^{\mu\sigma}$,

$$
\Gamma^\mu{}_{\kappa\lambda} = \frac{1}{2}\,g^{\mu\sigma}(g_{\sigma\kappa,\lambda} + g_{\lambda\sigma,\kappa} - g_{\kappa\lambda,\sigma})\,. \qquad (2.11)
$$

These expressions given by the first derivatives of the metric are being called the **Christoffel symbols** of the first and of the second kind, respectively.

Christoffel symbols represent the already mentioned special way in which the affine connection is related to the metric in general relativity. The affine connection thus related is called the **Levi-Civita connection**. In the following, we will return to this relation many times, and will also express it in several different (and more elegant) ways.

Let us return to the question of whether the parallel transport keeps constant the invariant "norm" of a transported vector. We will actually address a more general question: let *two* arbitrary vectors, V^μ and W^μ, be parallel transported along some given curve; how does their scalar product behave? By a simple differentiation, using the parallel-transport equation (2.2), some summation-index renaming, and finally by employing the relation (2.9), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\,(g_{\mu\nu}V^\mu W^\nu) &= g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}p}\,V^\mu W^\nu + g_{\mu\nu}\frac{\mathrm{d}V^\mu}{\mathrm{d}p}\,W^\nu + g_{\mu\nu}V^\mu\frac{\mathrm{d}W^\nu}{\mathrm{d}p} = \\
&= g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}p}\,V^\mu W^\nu - g_{\mu\nu}\Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda W^\nu - g_{\mu\nu}V^\mu\Gamma^\nu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,W^\lambda = \\
&= (g_{\mu\nu,\rho} - g_{\iota\nu}\Gamma^\iota{}_{\rho\mu} - g_{\mu\iota}\Gamma^\iota{}_{\rho\nu})\,V^\mu W^\nu\frac{\mathrm{d}x^\rho}{\mathrm{d}p} = 0\,. \qquad (2.12)
\end{aligned}
$$

Since the metric tells how to perform a scalar product while the affine connection tells how to parallel transport the two vectors, the above result might be presented as showing, geometrically, the special relation we found between those two operations. However, a similar result

[3] "Lowering the indices" is only a slang here: we will soon see that Gammas actually do *not* themselves represent tensors, so the action of the metric on them does *not* have its proper geometric sense (mapping between the corresponding tangent and cotangent spaces/bundles).

also holds for other reasonable transports, it is not specific to the parallel transport. (We will e.g. see that the Fermi-Walker transport has this property as well.)

Still it is a very important feature. It implies, in particular (for $W^\mu = V^\mu$), that the space-time invariant connected with a given vector, $g_{\mu\nu}V^\mu V^\nu$, does not change in parallel transport of that vector along *any* curve. Since the invariant says whether the vector is time-like ($g_{\mu\nu}V^\mu V^\nu < 0$), space-like ($\cdots > 0$) or light-like ($\cdots = 0$), it means that

in parallel transport of a vector, the latter's space-time character does not change.

2.3 Parallel transport of a covector and of a generic tensor

So far, we have been transporting vectors. That is the most instructive case, but the parallel transport applies to any tensor. Let us start with a covector. Since the transport keeps constant the scalar product of vectors, (2.12), one just lowers the index of one of the vectors there – say, of W^ν – and that's it,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}p}(g_{\mu\nu}V^\mu W^\nu) = \frac{\mathrm{d}}{\mathrm{d}p}(V^\mu W_\mu) = \frac{\mathrm{d}V^\mu}{\mathrm{d}p}\,W_\mu + V^\mu \frac{\mathrm{d}W_\mu}{\mathrm{d}p} = \\ &= V^\mu \frac{\mathrm{d}W_\mu}{\mathrm{d}p} - \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda W_\mu = \left(\frac{\mathrm{d}W_\mu}{\mathrm{d}p} - \Gamma^\lambda{}_{\kappa\mu}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,W_\lambda\right)V^\mu \end{aligned} \tag{2.13}$$

(changing the name of the summation indices $\lambda \leftrightarrow \mu$ in the second term). As this holds for *any* V^μ, we have (for any W_μ) the covector parallel-transport equation

$$\boxed{\frac{\mathrm{d}W_\mu}{\mathrm{d}p} - \Gamma^\lambda{}_{\kappa\mu}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,W_\lambda = 0}\,. \tag{2.14}$$

Clearly the sign of the affine-connection term is opposite than in the vector case.

Knowing thus "how both the upper and lower indices behave" in parallel transport, one infers the formula valid for a general tensor. Well, rather than writing a fully general, cumbersome formula, let us indicate the logic on an example of a (1,2)-type tensor:

$$\frac{\mathrm{d}T^\mu{}_{\alpha\beta}}{\mathrm{d}p} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\lambda{}_{\alpha\beta} - \Gamma^\iota{}_{\kappa\alpha}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\mu{}_{\iota\beta} - \Gamma^\iota{}_{\kappa\beta}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\mu{}_{\alpha\iota} = 0\,. \tag{2.15}$$

2.4 Parallel transport and the principle of general covariance

It was in fact slightly dogmatic to be presenting the equation (2.2) as the right general relativistic formula, because we have not yet checked whether it is generally covariant. It definitely has a correct special relativistic limit – in any inertial system, $\Gamma^\mu{}_{\kappa\lambda}$ just vanish and one ends with the trivial constancy of components (2.1). More involved is to check, explicitly, whether the equation (2.2) is a tensor equation. In this particular case of the *vector* transport, it should be a *vector* equation.

Let us write the equation in some primed system $\{x'^{\mu}\}$,

$$\frac{\mathrm{d}V'^{\mu}}{\mathrm{d}p} + \Gamma'^{\mu}{}_{\kappa\lambda}\frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}p}\,V'^{\lambda} = 0\,, \tag{2.16}$$

and try to learn, by transforming all the quantities present, how the left-hand side relates to its unprimed components.

- The simplest are the terms V'^{λ} and $\frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}p}$. The former is a vector by assumption, so

  $$V'^{\lambda} = \frac{\partial x'^{\lambda}}{\partial x^{\delta}}\,V^{\delta},$$

  and the latter is the tangent vector to the curve,

  $$\frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}p} = \frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\,\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}p}\,.$$

- Using the transformation of V^{μ}, we obtain for the first term of the equation

  $$\frac{\mathrm{d}V'^{\mu}}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\,V^{\alpha}\right) = \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\partial x^{\beta}}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}p}\,V^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\mathrm{d}V^{\alpha}}{\mathrm{d}p}\,.$$

  This is an important observation: total derivative by parameter of a vector is *not* a vector. It only behaves as a vector if the first term vanishes, which is the case for linear transformations (in particular, for the Lorentz transformations of special relativity).

- Finally, from (2.3), we find that the affine-connection components transform as

  $$\begin{aligned}\Gamma'^{\mu}{}_{\kappa\lambda} \equiv \frac{\partial x'^{\mu}}{\partial \xi^{\iota}}\frac{\partial^2 \xi^{\iota}}{\partial x'^{\kappa}\partial x'^{\lambda}} &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial \xi^{\iota}}\,\frac{\partial}{\partial x'^{\kappa}}\left(\frac{\partial \xi^{\iota}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\lambda}}\right) = \\ &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial \xi^{\iota}}\,\left(\frac{\partial^2 \xi^{\iota}}{\partial x^{\rho}\partial x^{\sigma}}\frac{\partial x^{\rho}}{\partial x'^{\kappa}}\frac{\partial x^{\sigma}}{\partial x'^{\lambda}} + \frac{\partial \xi^{\iota}}{\partial x^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x'^{\kappa}\partial x'^{\lambda}}\right) = \\ &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\rho}}{\partial x'^{\kappa}}\frac{\partial x^{\sigma}}{\partial x'^{\lambda}}\,\Gamma^{\alpha}{}_{\rho\sigma} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\kappa}\partial x'^{\lambda}}\,.\end{aligned} \tag{2.17}$$

  Therefore, these (also) do *not* represent a tensor quantity. Again, affine-connection components only transform as components of a (1,2)-tensor if the second term vanishes, which is the case for linear transformations. However, as indicated by the latter, they still represent a *geometric object* (in the sense of Section 1.2.2).
  Important remark: the non-tensorial character of $\Gamma^{\alpha}{}_{\beta\gamma}$ has already been clear before, from its/their vanishing in the LIFE. Actually, by a coordinate transformation, it is not possible to make completely vanish a non-trivial tensor. It is because a tensor involves a certain invariant information, not dependent on the reference frame; for the electromagnetic-field tensor $F^{\mu\nu}$, for example, this information is embedded in the two invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}{}^{*}F^{\mu\nu}$.

Putting together tensor-like and non-tensor-like terms, we have

$$\begin{aligned}
\frac{\mathrm{d}V'^{\mu}}{\mathrm{d}p} + \Gamma'^{\mu}{}_{\kappa\lambda}\frac{\mathrm{d}x'^{\kappa}}{\mathrm{d}p}\,V'^{\lambda} = {} & \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\,\frac{\mathrm{d}V^{\alpha}}{\mathrm{d}p} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\rho}}{\partial x'^{\kappa}}\frac{\partial x^{\sigma}}{\partial x'^{\lambda}}\,\Gamma^{\alpha}{}_{\rho\sigma}\,\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}p}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\,V^{\delta} + \\
& + \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\partial x^{\beta}}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}p}\,V^{\alpha} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\,\frac{\partial^2 x^{\alpha}}{\partial x'^{\kappa}\partial x'^{\lambda}}\,\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}p}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\,V^{\delta} = \\
= {} & \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\left(\frac{\mathrm{d}V^{\alpha}}{\mathrm{d}p} + \Gamma^{\alpha}{}_{\gamma\delta}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}p}\,V^{\delta}\right) \\
& + \left(\frac{\partial^2 x'^{\mu}}{\partial x^{\gamma}\partial x^{\delta}} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\kappa}\partial x'^{\lambda}}\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\right)\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}p}\,V^{\delta}\,. \qquad (2.18)
\end{aligned}$$

The non-tensorial part (the second row) is identically zero thanks to the identity $\frac{\partial x'^{\mu}}{\partial x^{\iota}}\frac{\partial x^{\iota}}{\partial x'^{\lambda}} = \delta^{\mu}_{\lambda}$. Actually, differentiation of the latter by x^{ρ} yields

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\iota}}\frac{\partial x^{\iota}}{\partial x'^{\lambda}} + \frac{\partial x'^{\mu}}{\partial x^{\iota}}\frac{\partial^2 x^{\iota}}{\partial x'^{\kappa}\partial x'^{\lambda}}\frac{\partial x'^{\kappa}}{\partial x^{\rho}} = 0, \qquad (2.19)$$

which, after multiplication by $\frac{\partial x'^{\lambda}}{\partial x^{\sigma}}$, confirms vanishing of the parenthesis in the second row of (2.18). Therefore, we can conclude that the equation (2.2) is generally covariant (specifically, it is a vector equation), hence it is the correct equation for parallel transport of a vector.

2.5 A few remarks

This chapter is coming to an end, but we will return to parallel transport several times. In particular, it is crucial for the next chapter about geodesics and for the definition of covariant derivative, and we will also very much need it in order to understand another central chapter – on curvature.

Still, parallel transport is definitely not the only reasonable geometrical transport. In particular, bear in mind that it "keeps the direction" *in space-time*, which need not correspond to one's intuition, because we only understand intuitively keeping direction *in space*. This is also the reason why parallel transport is not suitable for a transport of vectors along accelerated world-lines. We will further study this problem in section on Fermi-Walker transport.

CHAPTER 3

Geodesics

Every field theory should provide two major statements: a "passive" law which tells how a given test charge behaves in a given field (so called **equation of motion**), and an "active" law which determines what field is generated by a given distribution of sources (so called **field equations**). Usually easier if the equation of motion. For instance, in Newton's theory, in a gravitational field described by the potential $\Phi(t, \vec{x})$, a test mass moves according to the equation $\ddot{\vec{x}} = -\vec{\nabla}\Phi$. In electrodynamics, in an electromagnetic field described by the vectors $\vec{E}$ and $\vec{B}$, a test charge moves according to the Lorentz equation $\dot{\vec{p}} = q(\vec{E} + \vec{v} \times \vec{B})$.

In general relativity, the mass which is only being affected by the gravitational field is regarded as *free* – it is *freely falling*. According to the principle of equivalence, its world-line should thus be a counter-part of a *straight line* – a world-line of unaccelerated motions in the Minkowski space-time. It is rather problematic to *define* a straight line geometrically (independently of coordinates), whether in Minkowski or Euclidean space, and often it is just taken as a primitive object. Anyway, let's try to identify several basic properties the straight lines should have, and transfer them from the flat space(-time) to a general one. The resulting world-lines will be called **geodesics**.

3.1 Geodesics as straight lines

The most basic property of a straight line is that it is straight :-). In other words, its tangent vector points constantly in the same direction. Since we already know that "to point in the same direction" means, in a general space, "to transport parallelly", we are done:

$\boxed{\text{geodesic is such a world-line whose tangent vector transports along it parallelly}}$.

Hence, it is sufficient to employ the parallel-transport equation (2.2) to the tangent vector of the curve, $V^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}p}$. This yields the **equation of geodesic**

$$\boxed{\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}p^2} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\frac{\mathrm{d}x^\lambda}{\mathrm{d}p} = 0}\,. \tag{3.1}$$

Despite this very simple conclusion, one should realise that the above equation represents quite a different problem than the parallel-transport equation: in parallel transport, the curve is prescribed and the vector is to be found at every its point, whereas here it is exactly the opposite – it is the curve what is to be found, while the parallel vector function (the tangent one) is kind of prescribed.

The geodesic formula represents four 2nd-order ordinary differential equations. As initial conditions, the initial position $x^\mu(p=p_{\rm in})$ and initial tangent vector $\frac{{\rm d}x^\mu}{{\rm d}p}(p=p_{\rm in})$ have to be specified. One may also recall, as the side constraint, that the tangent vector has to keep its "normalisation" $g_{\mu\nu}\frac{{\rm d}x^\mu}{{\rm d}p}\frac{{\rm d}x^\nu}{{\rm d}p}$. Actually, since the parallel transport conserves this quantity (for *any* vector), it specifically implies for geodesics that their space-time character is fixed once for ever (it is e.g. not possible for a geodesic to change from time-like to space-like).

3.2 Geodesics as world-lines of free test particles

In a flat space(-time), zero acceleration means uniform rectilinear motion. Therefore, free test particles should move on geodesics. On *time-like* geodesics, to be precise. Although it should be clear from the above, let us derive this conclusion once more "from scratch", namely from the equivalence principle. In any LIFE, every free test particle moves – as known from special relativity – with zero four-acceleration,

$$a^{\hat\alpha}=\frac{{\rm d}^2\xi^\alpha}{{\rm d}\tau^2}=0\,, \tag{3.2}$$

τ being the particle's proper time. Transforming the left-hand side to some global coordinate system $\{x^\mu\}$, we have

$$\frac{\rm d}{{\rm d}\tau}\left(\frac{{\rm d}\xi^\alpha}{{\rm d}\tau}\right)=\frac{\rm d}{{\rm d}\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\lambda}\frac{{\rm d}x^\lambda}{{\rm d}\tau}\right)=\frac{\partial^2\xi^\alpha}{\partial x^\kappa\partial x^\lambda}\frac{{\rm d}x^\kappa}{{\rm d}\tau}\frac{{\rm d}x^\lambda}{{\rm d}\tau}+\frac{\partial\xi^\alpha}{\partial x^\lambda}\frac{{\rm d}^2x^\lambda}{{\rm d}\tau^2}\,,$$

which, after multiplication by $\frac{\partial x^\mu}{\partial\xi^\alpha}$, yields

$$\frac{{\rm d}^2x^\mu}{{\rm d}\tau^2}+\Gamma^\mu{}_{\kappa\lambda}\frac{{\rm d}x^\kappa}{{\rm d}\tau}\frac{{\rm d}x^\lambda}{{\rm d}\tau}=0\,.$$

Since it is natural, in the case of time-like world-lines, to normalise their tangent vector as four-velocity,

$$u^\mu:=\frac{{\rm d}x^\mu}{{\rm d}\tau}\,,\qquad g_{\mu\nu}u^\mu u^\nu=-c^2\ (\equiv 1\text{ in geometrised units})$$

– and we know that tangent-vector normalisation stays valid along the whole geodesic –, the equation of time-like geodesics can be "shortened" as

$$\boxed{\frac{{\rm d}u^\mu}{{\rm d}\tau}+\Gamma^\mu{}_{\kappa\lambda}u^\mu u^\nu=0}\,. \tag{3.3}$$

3.2.1 Remark on tidal forces

It's probably clear what "free" and "test" mean, but let us mainly emphasise that the above only holds for a *point-like* particle (characterised solely by rest mass). This is nothing new – in the Newtonian treatment, an *extended* body moving in a *non-homogeneous field* is affected not only by the field intensity, but also by the latter's gradient (and possibly also higher derivatives), i.e. by tidal forces. Then it does *not* move according to the equation $\ddot{\vec{x}} = -\vec{\nabla}\Phi$ which takes but the field intensity into account.

Similarly in general relativity: if a free test body is *not* point-like, it is also affected by higher-than-first derivatives of the metric, so it does *not* move on a geodesic. If the body is not too large and too deformed, it can be described by a series of multipoles which however couple with the respective derivatives of the field, making the equation of motion much more complicated. For example, even if restricting to a "pole-dipole" particle, described by just mass and spin (meaning *classical* spin, i.e. rotational angular momentum), the equations contain curvature (the Riemann tensor, see Section 6) and have to also incorporate the evolution law for the spin tensor. For a 2^n-polar body, the problem involves up to $n+1$ derivatives of the metric and the evolution equations for tensors describing all the n multipoles. Ugh!

3.3 Geodesics as extremal connecting lines

A straight line may in fact be determined in two different ways: i) by an initial point and a direction (geometers speak of a *local problem* in this case), or ii) by two points (a *global problem*). The same is also true for geodesics. However, in general space, it seems much harder to solve the global problem, because there is no global, trivial parallelism, so one does not know "in which direction to start" (and in which to approach the second point). Fortunately, the straight line has a third major property, and this also remains true in a general space: among all curves connecting the given two points, straight line is extremal (it is the shortest). In space-time, the generalisation of length is the invariant interval ("metric") $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, integrated along a curve. If the curve is time-like, the interval can be expressed in terms of proper time, $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$ ($\equiv -\mathrm{d}\tau^2$ in geometrised units); if the curve is space-like, the interval can be expressed in terms of proper distance, $\mathrm{d}s^2 = \mathrm{d}l^2$.

Without the loss of generality, let us solve the variational problem for time-like curves. In that case, one is looking for a stationary value of the integral for proper time spent between some given two events (A and B)

$$\Delta\tau \equiv \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\tau \equiv \int_{\mathrm{A}}^{\mathrm{B}} \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} \ .$$

Since time-like curves are most naturally parametrised by proper time, one varies as

$$\delta\Delta\tau = \int_{\mathrm{A}}^{\mathrm{B}} \delta\mathrm{d}\tau = \int_{\mathrm{A}}^{\mathrm{B}} \delta\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} = -\int_{\mathrm{A}}^{\mathrm{B}} \frac{g_{\mu\nu,\rho}\delta x^\rho \mathrm{d}x^\mu \mathrm{d}x^\nu + 2g_{\mu\nu}\delta \mathrm{d}x^\mu \mathrm{d}x^\nu}{2\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}} =$$

$$= -\int\limits_{\mathrm{A}}^{\mathrm{B}} \frac{g_{\mu\nu,\rho}\delta x^\rho \mathrm{d}x^\mu \mathrm{d}x^\nu + 2g_{\mu\nu}\delta \mathrm{d}x^\mu \mathrm{d}x^\nu}{2\,\mathrm{d}\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}\tau} =$$

$$= -\int\limits_{\mathrm{A}}^{\mathrm{B}} \left(\frac{1}{2}\,g_{\mu\nu,\rho}u^\mu u^\nu \delta x^\rho + g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\,u^\nu\right)\mathrm{d}\tau.$$

To get the basic variation δx^ρ out of both terms, we integrate the second term by parts,

$$-\int\limits_{\mathrm{A}}^{\mathrm{B}} g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\,u^\nu \mathrm{d}\tau = -\left[g_{\mu\nu}\delta x^\mu u^\nu\right]_{\mathrm{A}}^{\mathrm{B}} + \int\limits_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}u^\nu)\delta x^\mu \mathrm{d}\tau,$$

where, however, the first term is zero, because $\delta x^\mu = 0$ at both the endpoints A and B (usual step in variational problems "with fixed endpoints"), so, after renaming the summation index at δx^μ, we continue as

$$\delta\Delta\tau = \int\limits_{\mathrm{A}}^{\mathrm{B}} \left[\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\rho\nu}u^\nu) - \frac{1}{2}\,g_{\mu\nu,\rho}u^\mu u^\nu\right]\delta x^\rho \mathrm{d}\tau =$$

$$= \int\limits_{\mathrm{A}}^{\mathrm{B}} \left[g_{\rho\nu}\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} + \left(g_{\rho\nu,\mu} - \frac{1}{2}g_{\mu\nu,\rho}\right)u^\mu u^\nu\right]\delta x^\rho \mathrm{d}\tau\ .$$

Since $u^\mu u^\nu$ is symmetric in μ a ν, we can replace, in the parenthesis in front of it, the term $g_{\rho\nu,\mu}$ by its symmetrisation $\frac{1}{2}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu})$ (while the second term $g_{\mu\nu,\rho}$ we leave without change, since it is symmetric in itself). In such a manner, the parenthesis becomes

$$\frac{1}{2}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \equiv \Gamma_{\rho\mu\nu} \equiv g_{\rho\alpha}\Gamma^\alpha{}_{\mu\nu} \qquad \ldots \quad \text{c.f. } (2.10)\ .$$

Now it's simple already,

$$\delta\Delta\tau = \int\limits_{\mathrm{A}}^{\mathrm{B}} g_{\rho\alpha}\left(\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\mu\nu}u^\mu u^\nu\right)\delta x^\rho \mathrm{d}\tau. \tag{3.4}$$

The variation δx^ρ being arbitrary, we find that $\delta\Delta\tau = 0$ if and only if the geodesic equation (3.3) holds. Hence, the proper time $\Delta\tau$ spent between two events is extremal along a geodesic.

3.3.1 The time spent is maximal, not minimal

In Euclidean space, a straight line is the *shortest* connection, whereas in space-time, geodesic is the *longest* connection. It is due to the indefinite space-time metric. Best to grasp it in a 2D Minkowski space-time (axes t, x): consider a time-like geodesic $x = \mathrm{const}$ and vary it by δx to the side in its centre (thus obtaining a broken line); in such a variation, the value of $|\Delta s^2| = |\Delta\tau^2|$ decreases from $|-\Delta t^2| = |\Delta t^2|$ to $|-\Delta t^2 + 4\delta x^2| = |\Delta t^2 - 4\delta x^2|$. Similarly, for a space-like geodesic $t = \mathrm{const}$, the perpendicular variation by δt in its centre decreases its proper length $|\Delta s^2| = |\Delta l^2|$ from $|\Delta x^2|$ to $|\Delta x^2 - 4\delta t^2|$.

3.4 Affine and non-affine parametrisation of geodesics

Time-like world-lines are usually parametrised by proper time, while space-like world-lines by proper length. Light-like (null) world-lines have to be parametrised differently, because $\mathrm{d}s^2$ vanishes along them. The parametrisation issue can be well illustrated on geodesics. Changing, in a general case, the parameter p to $q = q(p)$ by a sufficiently smooth transformation, we derive

$$\begin{aligned}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}p^2} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\frac{\mathrm{d}x^\lambda}{\mathrm{d}p} &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}p}\right) + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}p}\frac{\mathrm{d}x^\lambda}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}p} = \\ &= \frac{\mathrm{d}^2x^\mu}{\mathrm{d}q^2}\left(\frac{\mathrm{d}q}{\mathrm{d}p}\right)^2 + \frac{\mathrm{d}x^\mu}{\mathrm{d}q}\frac{\mathrm{d}^2q}{\mathrm{d}p^2} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}q}\frac{\mathrm{d}x^\lambda}{\mathrm{d}q}\left(\frac{\mathrm{d}q}{\mathrm{d}p}\right)^2.\end{aligned}$$

Dividing the equation by $\left(\frac{\mathrm{d}q}{\mathrm{d}p}\right)^2$, we obtain

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}q^2} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}q}\frac{\mathrm{d}x^\lambda}{\mathrm{d}q} = -\frac{\mathrm{d}x^\mu}{\mathrm{d}q}\,\frac{\mathrm{d}^2q}{\mathrm{d}p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}q}\right)^2. \tag{3.5}$$

On the left hand side, we see the analogy of the original "acceleration" expression, while the right-hand side is no longer zero but proportional to the tangent $\frac{\mathrm{d}x^\mu}{\mathrm{d}q}$. Therefore, a geodesic may in general be characterised as a world-line whose "acceleration" (quotation marks!) has no perpendicular component.

Such parameters for which the right-hand side vanishes are called the **affine parameters**. In geometry, the term geodesic is sometimes only reserved for the **affinely parametrised** case, i.e. such that is "run with a proper speed". Since the general right-hand side vanishes when $\frac{\mathrm{d}^2q}{\mathrm{d}p^2} = 0$, we see that the affine parametrisation is unique up to a linear transformation.

3.5 Beware of geodesics? (No, rather of LARGE acceleration)

Good to also carry off, from the university, some practical, scalar lesson. Let us open once more the praise for geometry in the Preface of Synge's book [46]:
"I know now that if I break my neck by falling off a cliff, my death is not to be blamed on the force of gravity (what does not exist is necessarily guiltless), but on the fact that I did not maintain the first curvature of my world-line, exchanging its security for a dangerous geodesic. To the ironical mind there is little distinction between the mundane and the exalted, and that is no doubt why Socrates had to drink the hemlock cup."

Our remarks:

- To defend that charming curve with zero curvature ($g_{\mu\nu}a^\mu a^\nu = 0$): one's stomach may not fully enjoy it, but a really serious problem only arises down there, below the cliff, where the geodesic is exchanged back for a world-line with the original value of curvature. That is, $g_{\mu\nu}a^\mu a^\nu \gg 0$ is definitely a bigger issue than the free fall.

- Synge's apparent fear of a free fall (and possibly also of a hemlock cup), as well as his sense of the unity of mundane and exalted, might be related to his dedication of the book "To my friends J. P. and J. J.", read "John Power and John Jameson".

- Irrespectively of all the mundane and exalted influences, including those by J.P. and J.J., or perhaps thanks to them, J. L. Synge achieved many profound results and lived to the age of 98.

- Were Socrates alive, he might organise a discussion on whether "what does not exist is necessarily guiltless". How is the limit made exactly? Does it also hold in the opposite direction? (To be "guiltless" means not to influence anything, which in turn implies that such a thing cannot be detected by any means.) –Interesting point!

3.6 Covariant form of the geodesic equation

The covariant form of the geodesic equation follows immediately from the parallel-transport formula for covectors (2.14). Substituting there $W_\lambda \equiv \frac{\mathrm{d}x_\lambda}{\mathrm{d}p}$, we have

$$\begin{aligned}\frac{\mathrm{d}^2 x_\lambda}{\mathrm{d}p^2} &= \Gamma^{\mu}{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\frac{\mathrm{d}x_\mu}{\mathrm{d}p} = \Gamma_{\mu\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\frac{\mathrm{d}x^\mu}{\mathrm{d}p} = \frac{1}{2}(g_{\mu\kappa,\lambda}+g_{\lambda\mu,\kappa}-g_{\kappa\lambda,\mu})\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\frac{\mathrm{d}x^\mu}{\mathrm{d}p} = \\ &= \frac{1}{2}\,g_{\mu\kappa,\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,. \end{aligned} \tag{3.6}$$

Antisymmetric part of the parenthesis dropped out after being multiplied by the symmetric term $\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}$. Note that if the metric does not depend on x^λ, then $\frac{\mathrm{d}^2x_\lambda}{\mathrm{d}p^2}=0$, that is, $\frac{\mathrm{d}x_\lambda}{\mathrm{d}p}$ remains constant along any geodesic.

3.7 Newtonian limit of the geodesic equation

Whenever a theory is being sought which should encompass a wider range of phenomena than the old theory, it is natural to demand that it yields the same predictions in situations where the older theory was working well. Theory of relativity should be more appropriate than the Newtonian theory in situations involving very high speeds and/or very strong (non-homogeneous and/or time-varying) gravitational fields; if none of that happens, we want the new theory to agree with the Newtonian results. In this section, we apply such a requirement to the geodesic equation, considering the motion of a *slow particle in a weak field*.

- The field is "weak" if the space-time does not differ much from Minkowski. More accurately, if there exist coordinates in which the metric assumes an almost-Minkowski form,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{where } \; h_{\mu\nu} \; \text{ are very small, including derivatives}\,. \tag{3.7}$$

  "Very small" means that $h_{\mu\nu}$ and their arbitrary derivatives will be left in the equations up to linear order, $O(h)$. The contravariant metric must have the form

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}\,,$$

as easily verified by requiring it to be the inverse of $g_{\mu\nu}$,

$$\delta^\alpha_\nu \stackrel{!}{=} g^{\alpha\mu}g_{\mu\nu} = (\eta^{\alpha\mu} - h^{\alpha\mu})(\eta_{\mu\nu} + h_{\mu\nu}) = \delta^\alpha_\nu + \eta^{\alpha\mu}h_{\mu\nu} - h^{\alpha\mu}\eta_{\mu\nu} + O(h^2)\,.$$

We see the requirement reads

$$h^{\alpha\mu}\eta_{\mu\nu} = \eta^{\alpha\mu}h_{\mu\nu} \quad \text{... multiply by } \eta^{\beta\nu} \quad \Longrightarrow \quad h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}\,.$$

So $h_{\mu\nu}$ behaves like a tensor field *living in the Minkowski space-time*, namely its indices are – in the $O(h)$ accuracy – shifted by the Minkowski metric $\eta_{\mu\nu}$. (Of course: one can shift them by the total metric $g_{\mu\nu}$, but the "$h\cdot h$" terms are $O(h^2)$ and thus drop out.)
–And good to realise right now that Christoffel symbols will be $O(h)$, because

$$\Gamma^\mu{}_{\kappa\lambda} = \frac{1}{2}\, g^{\mu\sigma}(g_{\sigma\kappa,\lambda} + g_{\lambda\sigma,\kappa} - g_{\kappa\lambda,\sigma}) \doteq \frac{1}{2}\,\eta^{\mu\sigma}(h_{\sigma\kappa,\lambda} + h_{\lambda\sigma,\kappa} - h_{\kappa\lambda,\sigma})\,. \tag{3.8}$$

- The motion is slow if the coordinate 3-speed is much smaller than the speed of light, $\left|\frac{\mathrm{d}x^i}{\mathrm{d}t}\right| \ll c\ (\equiv 1)$. Multiplying this by $\left|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|$, we have

$$\left|\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right| \ll \left|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|, \qquad \text{i.e.} \quad \left|u^i\right| \ll \left|u^0\right|. \tag{3.9}$$

  Let us remark that u^0 is almost never zero or infinity (such a circumstance would indicate that the time coordinate t behaves very badly in the given situation).

- The field is also supposed to be stationary in the chosen coordinates, $g_{\mu\nu,0} = 0$ (hence $h_{\mu\nu,0} = 0$). Actually, there is no time derivative in the Newton field equation $\Delta\Phi = 4\pi\rho$, so we expect general relativity to differ from it if the field is not about stationary.

Now, writing out the geodesic equation (3.3)

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{00}(u^0)^2 + 2\Gamma^\mu{}_{0j}u^0u^j + \Gamma^\mu{}_{ij}u^iu^j = 0$$

while keeping in mind that Γs are linearly small, it is reasonable to keep just

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{00}(u^0)^2 \doteq 0\,. \tag{3.10}$$

Since the metric is stationary, (3.8) gives

$$\Gamma^\mu{}_{00} = \frac{1}{2}\,\eta^{\mu\sigma}(\cancel{h_{\sigma 0,0}} + \cancel{h_{0\sigma,0}} - h_{00,\sigma}) = -\frac{1}{2}\,\eta^{\mu\sigma}h_{00,\sigma} = -\frac{1}{2}\,h_{00}{}^{,\mu}\,.$$

Substituting this into (3.10), we have the time component

$$\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = 0\,, \qquad \text{i.e.} \quad \frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = 0\,, \tag{3.11}$$

and the spatial components

$$\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = \frac{1}{2}\, h_{00}{}^{,i}(u^0)^2\,, \qquad \text{i.e.} \quad \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{1}{2}\, h_{00}{}^{,i}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 . \tag{3.12}$$

Consider now the Newtonian equation for motion of a particle in a gravitational field,

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\Phi^{,i}\,. \tag{3.13}$$

In order to compare it to the relativistic equation, we express the latter in terms of the coordinate time t as well, using

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2}\,.$$

The second term vanishes due to (3.11), so (3.12) assumes the form

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \frac{1}{2}\, h_{00}{}^{,i}\,. \tag{3.14}$$

Comparing the latter with the desired limiting equation (3.13), we reach the requirement

$$h_{00}{}^{,i} \overset{!}{=} -2\Phi^{,i} \qquad \Longrightarrow \qquad h_{00} = -2\Phi + \text{const}\,.$$

It is natural to assume the constant to be zero, in order that the gravitational perturbation $h_{\mu\nu}$ be normalised in the same way as it is usual for the Newtonian potential Φ, namely that both these quantities vanish at large distances ("at spatial infinity").

We thus conclude that in the Newtonian situation it has to hold

$$\boxed{g_{00} = -1 - 2\Phi, \qquad \text{or, in standard units,} \quad g_{00} = -1 - \frac{2\Phi}{c^2}}\,. \tag{3.15}$$

This simple relation brings several layers of knowledge:

- In suitable coordinates (in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small and time-independent), the problems not involving large velocities can be treated using $g_{00} = -1 - 2\Phi$, which mediates the Newtonian intuition. We will, for example, use this limit relation in studying the time dilation and frequency shift (where g_{00} is important) in the following section.

- There is a correspondence in physical meaning between the Newtonian potential and the relativistic metric. This confirms the estimation that the components of affine connection $\Gamma^\mu{}_{\kappa\lambda}$ represent, through Christoffel symbols (i.e., derivatives of the metric), the gravitational-field intensity. (Recall again that the latter depends on the reference frame, in particular, it vanishes in freely falling frames such as LIFE.) Let us add in advance that the quantities given by derivatives of affine connection (second derivatives of metric) will thus correspond to non-homogeneity of the field intensity. A preferred representative of such quantities will be the Riemann tensor.

- The "weak-field" condition is now seen to be ensured by smallness of the potential Φ (more accurately, of the dimensionless potential Φ/c^2) with respect to unity. In Table 3.1, we give the order-of-magnitude values of $|\Phi|/c^2$ on the surface of several objects. They indicate that the gravitational field may almost everywhere be taken for weak, except in the vicinity of extremely compact objects (which should form in the final stages of stellar evolution – see Section 21). One can thus estimate the magnitude of relativistic effects and, consequently, to estimate where Newton's theory remains sufficient and where general relativity will have to be applied.

- A caution at the end: the assumption that the derivatives of $h_{\mu\nu}$ (hence of Φ) are negligible as well is non-trivial. Actually, the derivatives may even be large in the vicinity of low-mass bodies (whose potential well is only shallow), if those bodies are sufficiently dense. Also, vice versa, even very low-density bodies, with an almost homogeneous field (as given by the Newton equation $\Delta\Phi = 4\pi\rho$), may be so extended that their total mass is very large and their potential well very deep. Hence, in order to judge how much relativistic the situation is, one has to assess, besides the value of Φ/c^2, how important are its derivatives. In particular, one should constantly bear in mind that *curvature*, i.e. the field's non-homogeneity, is the major "general relativistic" feature.

source	$\frac{\vert\Phi\vert}{c^2} = \frac{GM}{c^2 R}$ on its surface
proton	10^{-39}
Earth	10^{-9}
Sun	10^{-6}
white dwarf	10^{-4}
neutron star	10^{-2}
black hole	$10^{-1} \div 10^{0}$

Table 3.1 *Order-of-magnitude values of the dimensionless Newton gravitational potential on the surface of several types of objects. In the black-hole case, we take its horizon as the "surface", although it is just a mathematical surface, not a solid one; in the spherically symmetric case, we will see the horizon is on $r = \frac{2GM}{c^2}$, so the potential assumes the value $\frac{|\Phi|}{c^2} = \frac{1}{2}$ there. The values indicate that the relativistic effects are significant in the vicinity of neutron stars and black holes.*

Chapter 4

Time dilation and frequency shift in a gravitational field

From the definition of proper time τ it follows... Right, let us *define* proper time properly – by referring to the fundamental principles and starting from LIFE again:

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}\xi^\alpha\mathrm{d}\xi^\beta = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \equiv g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,.$$

Hence, the relation of τ to any coordinate time t reads

$$\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2} = \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,\mathrm{d}t\,. \tag{4.1}$$

The **dilation of time** consists in the fact that the lapses $\mathrm{d}\tau$ and $\mathrm{d}t$ are not the same, the factor between them depending on position and on velocity $\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ of the "carrier" of τ with respect to the given coordinates x^μ. In special relativity where $g_{\mu\nu} = \eta_{\mu\nu}$, the relation reduces to the well known form

$$\mathrm{d}\tau = \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,\mathrm{d}t = \sqrt{1-\delta_{ij}v^iv^j}\,\mathrm{d}t =: \sqrt{1-v^2}\,\mathrm{d}t\,, \qquad \text{where} \quad v^i := \frac{\mathrm{d}x^i}{\mathrm{d}t}\,.$$

This implies, in particular, that it is only possible to realise a global time coordinate using clocks which do not move relative to each other, because those which do move cannot be synchronised. For a quick look at what general relativity brings as a new, gravitational ingredient, it is best to suppress the special-relativity effect by imagining that the above coordinate velocity v^i is zero: the formula still remains non-trivial,

$$\mathrm{d}\tau = \sqrt{-g_{00}(x^\mu)}\,\mathrm{d}t\,.$$

Not only that it is non-trivial, it even yields different relation at different locations (and times) – this is why we emphasised the dependence $g_{00}(x^\mu)$ in it. Hence,

Which is your proper time?
Top: Swarnendu Ghosh: Time keeper. *Bottom:* Sebastian Kisworo: Wasted.

in a general space-time, even such clocks which are mutually at rest do not, in general, tick at the same pace.

This observation is an important appendix to our emphasis on LOCAL in the reasoning about LIFEs. We saw there it is not possible to make the Cartesian system large, because "straight" axes do not represent well the curved space. Now we add that it is also not possible to realise *physical time* in a large area, because clocks which are placed on a different potential tick differently and, consequently, cannot be synchronised.

How to compare the instantaneous pace of clocks placed at different locations (and possibly also moving with respect to each other)? It requires a "global" experiment, because the information must somehow be transferred between the clocks. There's little doubt that the best carrier is the "absolute" one – light.

4.1 Gravitational shift of frequency

Let's have, in some (yet generic) space-time, two observers, A and B, each having her ideal clock showing proper time $\tau_{\rm A}$ and $\tau_{\rm B}$, respectively. Consider the observer A is sending to observer B a monochromatic EM radiation, denoting by $\mathrm{d}\tau_{\rm A}$ and $\mathrm{d}\tau_{\rm B}$ its proper periods respectively measured by them. Assume that these periods are much shorter than any other time characterising the space-time; in particular, assume that the space-time, if non-stationary, only changes negligibly slowly with respect to the above periods, and that the radiation's wavelength is negligibly short with respect to the radius of space-time curvature (loosely speaking, with respect to a scale on which the gravitational field changes significantly). Then the ratio of the corresponding frequencies, measured by A and B, reads

$$\frac{\nu_{\rm B}}{\nu_{\rm A}}=\frac{\frac{1}{\mathrm{d}\tau_{\rm B}}}{\frac{1}{\mathrm{d}\tau_{\rm A}}}=\frac{\sqrt{\left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)_{\rm A}}}{\sqrt{\left(-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\right)_{\rm B}}}\,\frac{\mathrm{d}t_{\rm A}}{\mathrm{d}t_{\rm B}}\,. \tag{4.2}$$

Good to realise two things at this moment:

- Although we consider the "high-frequency limit" (see above), the setting of the problem is still in general non-stationary, because the space-time may be changing and the observers may be changing their state of motion. Hence, it is necessary to specify *one particular event of emission* and *the corresponding single event of reception*. However, we will not complicate the notation beyond "A" and "B", with A automatically meaning the emission event and B meaning the reception event in the following.

- There is a fundamental difference between the two proper times and the coordinate time t: $\tau_{\rm A}$ and $\tau_{\rm B}$ are *physical*, they are really measured by the observers, but they only have clear sense along their respective world-lines; the coordinate time t, on the contrary, need not correspond to anybody's proper time (it's just a smooth function labelling space-time events), yet it is supposed to be defined in the whole region we are interested in (perhaps even on the whole space-time manifold). This means, $\mathrm{d}\tau_{\rm A}$ and $\mathrm{d}\tau_{\rm B}$ are intervals of *different* two times, whereas the corresponding coordinate-time periods $\mathrm{d}t_{\mathrm{A}}$ and $\mathrm{d}t_{\mathrm{B}}$ are intervals of *the same* time t, with $\mathrm{d}t_{\mathrm{A}}$ corresponding to $\mathrm{d}\tau_{\mathrm{A}}$ *in the event of emission of the signal* and $\mathrm{d}t_{\mathrm{B}}$ corresponding to $\mathrm{d}\tau_{\mathrm{B}}$ *in the event of its detection.*

The relation (4.2) is useless if one does not know anything about $\mathrm{d}t_{\mathrm{A}}/\mathrm{d}t_{\mathrm{B}}$. The simplest case is solved by the following extremely simple consideration which is the crucial point of this section: if both the period-defining maxima (or other same-phase points on the wave) spend the same amount of coordinate time t to travel from A to B, then the coordinate-time period does not change, $\mathrm{d}t_{\mathrm{A}} = \mathrm{d}t_{\mathrm{B}}$. It is often difficult to dig this elementary fact from students, but it is nothing but a tautology. Namely, the coordinate time is – nominally (i.e. not necessarily on anyone's clock) – "ticking" in the same pace in the whole region of the experiment. If the first maximum is emitted at $t_{\mathrm{A}}^{(1)}$ and the second maximum at $t_{\mathrm{A}}^{(2)}$, and if both spend the same time Δt on the way, then one has, for the times when the two maxima are received, $t_{\mathrm{B}}^{(1)}$ and $t_{\mathrm{B}}^{(2)}$,

$$t_{\mathrm{B}}^{(1)} = t_{\mathrm{A}}^{(1)} + \Delta t, \quad t_{\mathrm{B}}^{(2)} = t_{\mathrm{A}}^{(2)} + \Delta t \qquad \Longrightarrow \qquad \mathrm{d}t_{\mathrm{B}} := t_{\mathrm{B}}^{(2)} - t_{\mathrm{B}}^{(1)} = t_{\mathrm{A}}^{(2)} - t_{\mathrm{A}}^{(1)} =: \mathrm{d}t_{\mathrm{A}} \; .$$

Under the above circumstance, the ratio (4.2) reduces to

$$\frac{\nu_{\mathrm{B}}}{\nu_{\mathrm{A}}} = \frac{\sqrt{\left(-g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right)_{\mathrm{A}}}}{\sqrt{\left(-g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}\right)_{\mathrm{B}}}} \; . \tag{4.3}$$

We will realise now *when* this is true – when the travel between A and B is time-independent (i.e. the light-travel time remains the same). Along any light-like world-line, we know the interval vanishes,

$$0 = \mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{0j}\mathrm{d}t\mathrm{d}x^j + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \; .$$

Here, $\mathrm{d}t$ stands for an element of t along that world-line, so we may express it from the above quadratic equation and compute the total coordinate time spent on the way from A to B by its integration,

$$\Delta t = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}t = \int_{\mathrm{A}}^{\mathrm{B}} \frac{-g_{0j}\mathrm{d}x^j + \sqrt{(g_{0j}\mathrm{d}x^j)^2 - g_{00}g_{ij}\mathrm{d}x^i\mathrm{d}x^j}}{g_{00}} \; . \tag{4.4}$$

This may be a complicated line integral, but we do not actually need to compute it, we just want to say under which conditions the resulting Δt does not depend on t. Clearly, in general it is true when i) metric is independent of time, i.e. stationary, $g_{\mu\nu,t} = 0$ (at least in the region involved), and ii) if the spatial elements along the trajectory $\mathrm{d}x^i$ do not change as well. This second condition is easy to understand and to satisfy in case that the two observers have zero coordinate velocities v^i, yet such a case is not the only option – for example, the two maxima may follow slightly different trajectories with respect to the coordinates, but these trajectories may still be equivalent thanks to certain symmetries of the problem. More precisely, the condition ii) in fact may not even be automatically fulfilled for the two observers static with respect to the coordinates in which the metric is stationary: as it is the case in gravitational lensing, it may be possible to reach from A to B along two (or more) different null geodesics, so the two successive photons might follow substantially different paths. However, this would require to emit each of the photons in a different direction, and it is clear that we do not consider such a situation here.

4.2 Static case and its Newtonian limit

As already pointed out, the gravitational effect, specific for general relativity, is the most pure if there is no Doppler effect due to the observers' motion. Have the observers zero velocities v^i with respect to the coordinates (those in which the metric is supposed to be stationary), the formula (4.3) reduces to

$$\frac{\nu_{\rm B}}{\nu_{\rm A}} = \sqrt{\frac{(-g_{00})_{\rm A}}{(-g_{00})_{\rm B}}}\,. \tag{4.5}$$

Let us stress that this formula is *exact*, it is only *special* in that it assumes the metric is stationary and the observers are at rest. In the Newtonian limit, we can substitute (3.15) and, since $\Phi \ll 1$, limit ourselves to the linear order,

$$\frac{\nu_{\rm B}}{\nu_{\rm A}} = \sqrt{\frac{1+2\Phi_{\rm A}}{1+2\Phi_{\rm B}}} \doteq \frac{1+\Phi_{\rm A}}{1+\Phi_{\rm B}} \doteq 1+\Phi_{\rm A}-\Phi_{\rm B}\,, \tag{4.6}$$

from where it follows that the relative change of frequency $\Delta\nu \equiv \nu_{\rm B}-\nu_{\rm A}$ is given by potential difference ("gravitational tension") between the place of emission and the place of reception,

$$\frac{\Delta\nu}{\nu_{\rm A}} = \Phi_{\rm A}-\Phi_{\rm B}\,. \tag{4.7}$$

Two examples (given in standard units).
(i) Radiation from a spherical source of radius R is detected at large distance ("at infinity") with the redshift equal to the dimensionless potential on the source's surface,

$$\frac{\Delta\nu}{\nu_{\rm A}} = \frac{\Phi_{\rm A}}{c^2} = -\frac{GM}{c^2 R} \tag{4.8}$$

(one assumes that the potential at the reception point B – e.g. on the Earth – is negligible with respect to $\Phi_{\rm A}$); for Sun it amounts to $-2.12\cdot 10^{-6}$.
(ii) In a homogeneous field (as e.g. in the classical tower experiment by Pound & Rebka, 1960), the potential is normalised to be zero on the surface of a body (rather than at infinity like in the spherically symmetric case), $\Phi = gl$, and the relation for relative shift yields

$$\frac{\Delta\nu}{\nu_{\rm A}} = -\frac{g\Delta l}{c^2}\,, \tag{4.9}$$

where g is the gravitational acceleration on the surface and $\Delta l := l_{\rm B}-l_{\rm A}$ is the height difference between the A and B locations.

Let us check that the last result agrees with common intuition. First, a "lumberjack-like" reasoning. If a photon falls/rises in a gravitational field, it should be gaining/loosing energy,[1] according to $E_{\rm B} = E_{\rm A} + \Delta E_{\rm potential}$. Writing $E = h\nu$ and $\Delta E_{\rm potencial} = -\frac{E_{\rm A}}{c^2}\, g\Delta l$, we obtain

$$h\nu_{\rm B} = h\nu_{\rm A} - \frac{h\nu_{\rm A}}{c^2}\, g\Delta l \qquad \Longrightarrow \qquad \nu_{\rm B} = \nu_{\rm A}\left(1 - \frac{g\Delta l}{c^2}\right)$$

in agreement with (4.9).
A much more reliable derivation follows from the equivalence principle. Let the two observers be in an empty space yet placed in an Einstein lift, Δl from each other. Let the lift be pulled, while a light is emitted by A, in the direction of Δl, with acceleration g. The light has to travel the distance $c\Delta t = \Delta l + \frac{1}{2}g(\Delta t)^2$, where Δt is the travel time. For moderate Δl the time of flight is extremely tiny (the more that g is small – we simulate a weak field), so $\Delta t \doteq \Delta l/c$. During this time interval, the lift acquires the speed $v = g\Delta t = g\Delta l/c$, so the observer B receives the light with a frequency shifted by the Doppler-effect (classical) formula

$$\nu_{\rm B} = \nu_{\rm A}\left(1 - \frac{v}{c}\right) = \nu_{\rm A}\left(1 - \frac{g\Delta l}{c^2}\right) \; .$$

4.3 The case with an orbiting satellite

Consider now a satellite orbiting the (spherical) Earth on a circular trajectory, Δl above the surface. Let the observer A on the satellite send radiation to the observer B on Earth's surface (located exactly below, in the radial direction). In coordinates fixed to the Earth (where B is at rest), the metric is time independent and the paths of any two subsequent wave maxima (defining one period), though *different*, are obviously geometrically equivalent in the leading order of the exercise. Hence, we can use the formula (4.3), with the dilation factor describing the B observer simplified to the static form,

$$\sqrt{\left(-g_{\mu\nu}\frac{{\rm d}x^\mu}{{\rm d}t}\frac{{\rm d}x^\nu}{{\rm d}t}\right)_{\rm B}} = \sqrt{(-g_{00})_{\rm B}} \doteq 1 + \Phi_{\rm B}\,,$$

and the factor describing the A observer written out, in the Newtonian limit (slow satellite in a weak field), as

$$\sqrt{\left(-g_{\mu\nu}\frac{{\rm d}x^\mu}{{\rm d}t}\frac{{\rm d}x^\nu}{{\rm d}t}\right)_{\rm A}} = \sqrt{\left(-g_{00} - 2h_{0j}v^j - g_{ij}v^iv^j\right)_{\rm A}} \doteq$$
$$\doteq \sqrt{\left(1 + 2\Phi - 2h_{0j}v^j - v^2 - h_{ij}v^iv^j\right)_{\rm A}}\;.$$

[1] O.S.: When I was attending the GR course, it was in 1983, the lecturer (J.B.) was saying that with the **redshift** we must have our own experience: *who is climbing is getting red*. Still today students tend to smile at it, but likely because they think the climber gets red due to the toil...

If the satellite orbits *freely* on circular trajectory, its speed is given by

$$\frac{mv_{\rm A}^2}{r_{\rm A}} = \frac{GMm}{r_{\rm A}^2} \qquad \Longrightarrow \qquad v_{\rm A}^2 = -\Phi_{\rm A}\,,$$

so it is of the $O(\Phi^{1/2})$ order. Leaving, in the square root evaluated at A, just terms linear in $\Phi_{\rm A}$,

$$\sqrt{(1+2\Phi-2h_{0j}v^j-v^2-h_{ij}v^iv^j)_{\rm A}} \doteq \sqrt{(1+2\Phi-v^2)_{\rm A}} = \sqrt{1+3\Phi_{\rm A}} \doteq 1+\frac{3}{2}\Phi_{\rm A},$$

we thus have

$$\frac{\Delta\nu}{\nu_{\rm A}} \doteq \frac{1+\frac{3}{2}\Phi_{\rm A}}{1+\Phi_{\rm B}} - 1 \doteq \frac{3}{2}\Phi_{\rm A} - \Phi_{\rm B} = -\frac{3}{2}\frac{M}{R+\Delta l} + \frac{M}{R} = -\frac{M}{2R}\,\frac{R-2\Delta l}{R+\Delta l}\,. \tag{4.10}$$

Restoring standard units and substituting Earth values for M and R, we arrive at the numerical result

$$\frac{\Delta\nu}{\nu_{\rm A}} = -\frac{GM}{2Rc^2}\,\frac{R-2\Delta l}{R+\Delta l} \doteq -3.47\cdot 10^{-10}\,\frac{R-2\Delta l}{R+\Delta l}\,. \tag{4.11}$$

For a very distant satellite ($\Delta l \to \infty$), the orbital speed is negligible, whereas the potential difference is maximal, so one obtains the static-case limit, with the maximal possible blueshift,

$$\frac{\Delta\nu}{\nu_{\rm A}} \doteq 7\cdot 10^{-10}.$$

For a satellite just above the Earth's surface ($\Delta l \to 0$), the orbital speed has to be highest, whereas the potential difference vanishes, so one obtains purely Doppler-caused limit, with maximal possible redshift,

$$\frac{\Delta\nu}{\nu_{\rm A}} \doteq -3.47\cdot 10^{-10}.$$

Between these limiting cases, at the height $\Delta l = R/2$, the gravitational effect just balances the transversal Doppler effect and the frequency is not shifted at all.

Time dilation in satellite navigation

The equation (4.11) says how quickly diverge the times on clock on the Earth surface and on the clock orbiting with the satellite. Returning from frequencies to the proper-time periods,

$$\frac{\Delta\nu}{\nu_{\rm A}} \equiv \frac{\nu_{\rm B}-\nu_{\rm A}}{\nu_{\rm A}} = \frac{\frac{1}{{\rm d}\tau_{\rm B}}-\frac{1}{{\rm d}\tau_{\rm A}}}{\frac{1}{{\rm d}\tau_{\rm A}}} = \frac{{\rm d}\tau_{\rm A}-{\rm d}\tau_{\rm B}}{{\rm d}\tau_{\rm B}}\,,$$

we obtain from there the value by which the two times differ after the ${\rm d}\tau_{\rm B}$ interval of terrestrial proper time,

$${\rm d}\tau_{\rm A} - {\rm d}\tau_{\rm B} \doteq -3.47\cdot 10^{-10}\,\frac{R-2\Delta l}{R+\Delta l}\,{\rm d}\tau_{\rm B}\,. \tag{4.12}$$

For example, after one terrestrial day, i.e. after 86400 seconds, one has

$$|\mathrm{d}\tau_{\mathrm{A}} - \mathrm{d}\tau_{\mathrm{B}}| \doteq (3 \cdot 10^{-5}\,\mathrm{s}) \cdot \left|\frac{R - 2\Delta l}{R + \Delta l}\right| . \tag{4.13}$$

In satellite navigation, the localisation of the receiver is determined from *differences* between arrivals of signals from *several* different satellites orbiting at the same height, so the error caused by time dilation would by far not be as large as given by the above relation. Nevertheless, it is interesting to realise that if just one satellite were employed, and the time dilation were not taken into account, the error in "longitudinal" distance (i.e. in geographical altitude) caused by wrong interpretation of signal arrival times would amount to

$$c\,|\mathrm{d}\tau_{\mathrm{A}} - \mathrm{d}\tau_{\mathrm{B}}| \doteq (9\,\mathrm{km}) \cdot \left|\frac{R - 2\Delta l}{R + \Delta l}\right| \tag{4.14}$$

after a single day. (It simply follows by multiplication of the formula by the speed of light.) The now classical GPS satellites specifically orbit at the height $\Delta l = 20200\,\mathtt{km}$, which leads to the value

$$c\,|\mathrm{d}\tau_{\mathrm{A}} - \mathrm{d}\tau_{\mathrm{B}}| \doteq 11.52\,\mathrm{km} \qquad \text{in one day} .$$

The above conclusion may not be intuitive, regarding that the Earth is a very "classical" body. Actually, when the GPS started to operate and the above issues started to be discussed publicly, even we as relativists were rather sceptical at first sight. The more appreciated should be the design of the positioning systems which already from the very beginning (1960s) correctly took into account the relativistic effect (a calculation similar to the one we have done above was published by F. Winterberg in Astronautica Acta, 1955).

4.4 Derivation using photon-observer projections

We add an alternative derivation of the redshift formula. Consider a completely generic situation – a generic space-time and two observers A and B in arbitrary motion; their four-velocities we denote $(\hat{u}^\mu)_{\mathrm{A}}$, $(\hat{u}^\mu)_{\mathrm{B}}$. Let the photon which A sends to B has four-momentum $(p_\mu)_{\mathrm{A}}$ in emission and $(p_\mu)_{\mathrm{B}}$ in reception; and remember that $p_\mu p^\mu = 0$ for photons of course. (The hats at four-velocities should emphasise that these quantities characterise the observers, whereas the four-momentum belongs to the photon ... $p_\mu = \hbar k_\mu \neq m\hat{u}_\mu$.)

Energy of a photon as measured by an observer at a given event (where their world-lines intersect) is given, as known from special relativity, by minus time component of p_μ as taken with respect to the observer (whose time direction is that of $\hat{u}^\mu$), so it is $\hat{E} = h\hat{\nu} = -p_\mu \hat{u}^\mu$. Hence, the ratio of the emitted to received proper frequencies reads

$$\frac{\hat{E}_{\mathrm{B}}}{\hat{E}_{\mathrm{A}}} = \frac{\hat{\nu}_{\mathrm{B}}}{\hat{\nu}_{\mathrm{A}}} = \frac{(p_\mu \hat{u}^\mu)_{\mathrm{B}}}{(p_\mu \hat{u}^\mu)_{\mathrm{A}}} . \tag{4.15}$$

Restrict now to the special, static situation again, leaving the observers at rest with respect to some chosen coordinates, $\hat{u}^\mu = (\hat{u}^0, 0, 0, 0)$, where, from normalisation

$$g_{\mu\nu}\hat{u}^\mu \hat{u}^\nu = g_{00}(\hat{u}^0)^2 = -1 ,$$

we have specifically

$$(\hat{u}^0)_{\rm A} = \frac{1}{\sqrt{(-g_{00})_{\rm A}}}\,, \qquad (\hat{u}^0)_{\rm B} = \frac{1}{\sqrt{(-g_{00})_{\rm B}}}\,.$$

The redshift formula thus assumes the form

$$\frac{\hat{\nu}_{\rm B}}{\hat{\nu}_{\rm A}} = \frac{(p_0)_{\rm B}}{(p_0)_{\rm A}} \sqrt{\frac{(-g_{00})_{\rm A}}{(-g_{00})_{\rm B}}}\,. \tag{4.16}$$

Finally, let the metric be stationary and let the chosen coordinates be adjusted to this symmetry, i.e. let $g_{\mu\nu,0} = 0$. Then, as it is seen from the covariant form of the geodesic equation, (3.6), $p_0 = {\rm d}x_0/{\rm d}p$ is constant along geodesics.[2] Consequently, $(p_0)_{\rm B} = (p_0)_{\rm A}$ and (4.16) reduces to (4.5),

$$\frac{\hat{\nu}_{\rm B}}{\hat{\nu}_{\rm A}} = \sqrt{\frac{(-g_{00})_{\rm A}}{(-g_{00})_{\rm B}}}\,. \tag{4.17}$$

[2] Photons are by default supposed to travel on (light-like) geodesics, since an “accelerated photon” is a rather special concept – possibly the one sliding along a mirror or that interacting with some environment.

Victoria Ivanova: Stop the time

CHAPTER 5

Covariant derivative

In verifying that the equation (2.2) for parallel transport of a vector,

$$\frac{\mathrm{d}V^\mu}{\mathrm{d}p} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda = 0\,,$$

is general covariant (that it is a vector equation), we encountered two quantities which are *not* of tensor type – the total derivative of a tensor (there, specifically, the vector V^μ) by parameter and the affine-connection components $\Gamma^\mu{}_{\kappa\lambda}$. If having at our disposal some extension of V^μ to a neighbourhood of the curve, we can rewrite the total derivative in terms of partial gradient,

$$\frac{\mathrm{d}V^\mu}{\mathrm{d}p} = \frac{\partial V^\mu}{\partial x^\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} \equiv V^\mu{}_{,\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,,$$

and realise that it is the partial derivative which does not behave in a tensor manner (because the tangent vector really *is* a vector). Sure, one can check that directly by transforming it,

$$\frac{\partial V'^\mu}{\partial x'^\kappa} = \frac{\partial}{\partial x'^\kappa}\left(\frac{\partial x'^\mu}{\partial x^\alpha}\,V^\alpha\right) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\kappa} + \frac{\partial^2 x'^\mu}{\partial x^\gamma \partial x^\alpha}\frac{\partial x^\gamma}{\partial x'^\kappa}\,V^\alpha\,. \tag{5.1}$$

As expected, partial derivative only behaves in a tensor way in linear transformations (like in Lorentz transformations in special relativity). However, it would be great to have a derivative with tensorial behaviour with respect to *any* transformation of coordinates.

5.1 Covariant derivative of a vector

It is very easy to guess a good option from the very equation (2.2). First, we know that the left-hand side as a whole is a vector, so why not to define, as a tensor counter-part of the total derivative $\frac{\mathrm{d}V^\mu}{\mathrm{d}p}$, the **absolute derivative** of V^μ by p (i.e., along a given curve $x^\mu = x^\mu(p)$) by

$$\boxed{\frac{\mathrm{D}V^\mu}{\mathrm{d}p} := \frac{\mathrm{d}V^\mu}{\mathrm{d}p} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,V^\lambda}\,. \tag{5.2}$$

Second, considering some extension of V^μ to a neighbourhood of the curve and rewriting the total derivative in terms of partial derivative, we can rewrite the above as

$$\frac{\mathrm{D}V^\mu}{\mathrm{d}p} = \left(V^\mu{}_{,\kappa} + \Gamma^\mu{}_{\kappa\lambda}V^\lambda\right)\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} \ .$$

We know the whole expression is a vector, and the tangent vector $\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}$ is a vector as well, hence the parenthesis has to be a second-rank tensor. It is thus natural to define, as a tensor counter-part of partial derivative $V^\mu{}_{,\kappa}$, the **covariant derivative** of V^μ by x^κ,

$$\boxed{V^\mu{}_{;\kappa} := V^\mu{}_{,\kappa} + \Gamma^\mu{}_{\kappa\lambda}V^\lambda} \ . \tag{5.3}$$

The correspondence between the derivatives is obvious,

$$\frac{\mathrm{d}V^\mu}{\mathrm{d}p} = V^\mu{}_{,\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} \qquad \longleftrightarrow \qquad \frac{\mathrm{D}V^\mu}{\mathrm{d}p} = V^\mu{}_{;\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} \ .$$

Remark: More general notation:

The covariant derivative is often being denoted by $\nabla_\kappa V^\mu$. This notation enables to write down, economically, the absolute derivative in a general direction (say, W^μ),

$$\nabla_W V \ \longleftrightarrow \ V^\mu{}_{;\kappa}W^\kappa$$

(in the former, "geometrical" form, it is better to write the vector fields without indices).

5.2 Covariant derivative of an invariant, a covector and a general tensor

Since the partial derivative of an invariant $[\Phi'(x') = \Phi(x)]$ is a covector,

$$\frac{\partial\Phi'}{\partial x'^\kappa} = \frac{\partial\Phi}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\kappa} \ ,$$

and the total derivative of an invariant, $\frac{\mathrm{d}\Phi}{\mathrm{d}p} = \frac{\partial\Phi}{\partial x^\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}$, (thus) remains invariant, it follows that for an invariant the covariant/absolute derivative coincides with the partial/total one,

$$\Phi'(x') = \Phi(x) \qquad \Longrightarrow \qquad \Phi_{;\kappa} = \Phi_{,\kappa} \ , \quad \frac{\mathrm{D}\Phi}{\mathrm{d}p} = \frac{\mathrm{d}\Phi}{\mathrm{d}p} \ . \tag{5.4}$$

For covectors, we should define the covariant derivative similarly as we did for vectors, i.e. using the equation for parallel transport. The absolute derivative for covectors is thus given directly by the left-hand side of the equation (2.14),

$$\boxed{\frac{\mathrm{D}W_\lambda}{\mathrm{d}p} := \frac{\mathrm{d}W_\lambda}{\mathrm{d}p} - \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}W_\mu} \ , \tag{5.5}$$

and, correspondingly – after extending W_λ off the transport curve again and rewriting

$$\frac{\mathrm{d}W_\lambda}{\mathrm{d}p} - \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,W_\mu = \left(\frac{\partial W_\lambda}{\partial x^\kappa} - \Gamma^\mu{}_{\kappa\lambda}W_\mu\right)\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,,$$

we define the covariant derivative of W_λ by x^κ as

$$\boxed{W_{\lambda;\kappa} := W_{\lambda,\kappa} - \Gamma^\mu{}_{\kappa\lambda}W_\mu}\,. \tag{5.6}$$

Needless to say, again

$$\frac{\mathrm{d}W_\lambda}{\mathrm{d}p} = W_{\lambda,\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} \qquad \longleftrightarrow \qquad \frac{\mathrm{D}W_\lambda}{\mathrm{d}p} = W_{\lambda;\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,.$$

For a general tensor, the logic is clear now. Rather than to be writing down a complicated general formula, let us look at equation (2.15) and exemplify the logic on a (1,2)-tensor:

$$\frac{\mathrm{D}T^\mu{}_{\alpha\beta}}{\mathrm{d}p} := \frac{\mathrm{d}T^\mu{}_{\alpha\beta}}{\mathrm{d}p} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\lambda{}_{\alpha\beta} - \Gamma^\iota{}_{\kappa\alpha}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\mu{}_{\iota\beta} - \Gamma^\iota{}_{\kappa\beta}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,T^\mu{}_{\alpha\iota}\,, \tag{5.7}$$

which, if one can write $\frac{\mathrm{d}T^\mu{}_{\alpha\beta}}{\mathrm{d}p} = T^\mu{}_{\alpha\beta,\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}$, leads to

$$\frac{\mathrm{D}T^\mu{}_{\alpha\beta}}{\mathrm{d}p} = \left(T^\mu{}_{\alpha\beta,\kappa} + \Gamma^\mu{}_{\kappa\lambda}T^\lambda{}_{\alpha\beta} - \Gamma^\iota{}_{\kappa\alpha}T^\mu{}_{\iota\beta} - \Gamma^\iota{}_{\kappa\beta}T^\mu{}_{\alpha\iota}\right)\frac{\mathrm{d}x^\kappa}{\mathrm{d}p} =: T^\mu{}_{\alpha\beta;\kappa}\frac{\mathrm{d}x^\kappa}{\mathrm{d}p}\,. \tag{5.8}$$

5.3 Basic properties of the covariant derivative

- Firstly, it is very simple to check that the above operation is a derivative: it is linear in the derived quantity (V^μ, say), it is linear in the direction $\frac{\mathrm{d}x^\lambda}{\mathrm{d}p}$, and it follows the Leibniz product rule – e.g., for f some invariant, $\frac{\mathrm{D}}{\mathrm{d}p}(fV^\nu) = \frac{\mathrm{d}f}{\mathrm{d}p}\,V^\mu + f\,\frac{\mathrm{D}V^\mu}{\mathrm{d}p}$, and similarly $\frac{\mathrm{D}}{\mathrm{d}p}(V^\mu W^\nu) = \frac{\mathrm{D}V^\mu}{\mathrm{d}p}\,W^\nu + V^\mu\,\frac{\mathrm{D}W^\nu}{\mathrm{d}p}$.

- Lemma: $\boxed{\text{The metric is constant with respect to the covariant differentiation, } g_{\kappa\lambda;\sigma} = 0.}$

  Proof: From (5.8), we have $g_{\kappa\lambda;\sigma} = g_{\kappa\lambda,\sigma} - \Gamma^\iota{}_{\sigma\kappa}g_{\iota\lambda} - \Gamma^\iota{}_{\sigma\lambda}g_{\kappa\iota}$, which exactly is zero by (2.9).

  This means that $\frac{\mathrm{D}g_{\mu\nu}}{\mathrm{d}p} = 0$ as well, so *the metric is a parallel tensor field (along any curve).* It also implies that lowering and raising of indices commute with the covariant differentiation, i.e. that the indices which are "before the semicolon" can also be standardly shifted by the metric. Please realise securely that this does *not* hold for partial differentiation, namely for indices "before the comma" (very easy to make a mistake in this!), for example,

$$V_{\alpha,\beta} = (g_{\alpha\mu}V^\mu)_{,\beta} = g_{\alpha\mu,\beta}V^\mu + g_{\alpha\mu}V^\mu{}_{,\beta} \neq g_{\alpha\mu}V^\mu{}_{,\beta}\,.$$

- The constancy of metric with respect to covariant differentiation is the shortest (and deepest) expression of the special relation between metric and affine connection which holds in

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GR (and which is in detail represented by the Christoffel symbols). This point is worth a more general geometrical comment:

On a differentiable manifold, there may in general exist neither the affine connection nor the metric. The second possibility is that there is only an affine connection. In such a case, it is possible to parallel transport quantities, to speak about geodesics, and also to describe curvature (tied to that particular connection) – see Chapter 6. On any differentiable manifold (of positive dimension), there exist infinitely many affine connections. When there also exists the metric (so the manifold is called the pseudo-Riemannian one), it may or may not be related to the affine connection. However, there exists then a unique affine connection with the following properties: i) its torsion is zero, and ii) the corresponding parallel transport is an isometry, i.e. it preserves the scalar product. We note that

– In general relativity, the second covariant derivatives commute if applied to a scalar,

$$f_{;\alpha\beta} = (f_{,\alpha})_{;\beta} = f_{,\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha} f_{,\mu} = f_{,\beta\alpha} - \Gamma^{\mu}{}_{\alpha\beta} f_{,\mu} = (f_{,\beta})_{;\alpha} = f_{;\beta\alpha}\,.$$

Clearly, it is thanks to the symmetry of the affine connection.
Were this commutator *non-zero*, the affine connection would be said to have non-zero **torsion**, and it would imply the existence of a certain tensor $T^{\mu}{}_{\alpha\beta}$ such that

$$f_{;\alpha\beta} - f_{;\beta\alpha} = T^{\mu}{}_{\alpha\beta} f_{;\mu}\,.$$

This tensor, called **the torsion tensor**, is a (1,2)-type tensor (standardly viewed as a two-form with vector values) defined, for a given connection, by

$$T(V,W) := \nabla_V W - \nabla_W V - [V,W]\,, \tag{5.9}$$

for any two smooth vector fields V and W, with $[V,W]$ denoting their commutator (Lie bracket). In an arbitrary coordinate basis, this reads

$$\begin{aligned} T^{\mu}{}_{\alpha\beta} V^{\alpha} W^{\beta} &:= W^{\mu}{}_{;\alpha} V^{\alpha} - V^{\mu}{}_{;\alpha} W^{\alpha} - W^{\mu}{}_{,\alpha} V^{\alpha} + V^{\mu}{}_{,\alpha} W^{\alpha} = \\ &= \Gamma^{\mu}{}_{\alpha\beta}(V^{\alpha}W^{\beta} - W^{\alpha}V^{\beta}) = (\Gamma^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha})\, V^{\alpha} W^{\beta}. \end{aligned} \tag{5.10}$$

Hence, vanishing of the torsion tensor is indeed equivalent to the symmetry of the affine-connection coordinate components in the two bottom indices.

– The parallel transport is an isometry if

$$\frac{\mathrm{d}}{\mathrm{d}p}\,(g_{\mu\nu} V^{\mu} W^{\nu}) = \frac{\mathrm{D}}{\mathrm{d}p}\,(g_{\mu\nu} V^{\mu} W^{\nu})$$

vanishes. However, since the parallel transport of V^{μ} and W^{ν} means

$$\frac{\mathrm{D}V^{\mu}}{\mathrm{d}p} = 0, \quad \frac{\mathrm{D}W^{\nu}}{\mathrm{d}p} = 0 \qquad \Longrightarrow \qquad \text{necessarily}\;\; V^{\mu} W^{\nu}\,\frac{\mathrm{D}g_{\mu\nu}}{\mathrm{d}p} = 0\,.$$

Should this be true for any pair of vectors and along any curve, the metric must be covariantly constant, $g_{\mu\nu;\sigma} = 0$, which in turn implies that (2.9) has to hold. And we know the latter is equivalent to the fact that the affine-connection components are represented by Christoffel symbols.

– For those who do not want to study all the above geometrical background, at least a short useful appendix to parallel transport: viewing the last point from the GR perspective already (thus with the metric automatically being covariantly constant), we see that the conservation of scalar product of parallel transported vectors (2.12) is now obvious immediately:

$$\frac{\mathrm{d}}{\mathrm{d}p}\,(g_{\mu\nu} V^{\mu} W^{\nu}) = \frac{\mathrm{D}}{\mathrm{d}p}\,(g_{\mu\nu} V^{\mu} W^{\nu}) = 0\,,$$

because all the terms $g_{\mu\nu}$, V^{μ} and W^{ν} are parallel (V^{μ} and W^{ν} are parallel *along the given curve* by assumption, and $g_{\mu\nu}$ is parallel along *any* curve).

The above affine connection "compatible with the metric" (or "metric connection") is called the **Levi-Civita connection**. General relativity lives on pseudo-Riemannian (specifically, Lorentzian) manifolds where the metric does exist, and it automatically employs this kind of affine connection. A historical remark: the covariant differentiation and Christoffel symbols were already known from E. B. Christoffel (from 1869), but the corresponding affine connection was only formally established by T. Levi-Civita in 1916/17, about a year *after* Einstein had already finished his theory.

- Regarding the short, "geometric" form of the parallel-transport equation, $\frac{\mathrm{D}V^\mu}{\mathrm{d}p}=0$, it is also clear that the geodesic equation (in an affine parametrization) reads $\frac{\mathrm{D}}{\mathrm{d}p}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\right)=0$. In the specific case of time-like world-lines, a natural affine parameter is the proper time and

  $$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}=:u^\mu \quad \text{(four-velocity)}, \qquad \frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}=:a^\mu \quad \text{(four-acceleration)},$$

  so the time-like geodesic equation (describing the motion of free test particles) can be written even shorter, $a^\mu=0$. (Remember again that this is exactly the idea we started from when deriving the geodesic equation, more specifically, we assumed this is the correct equation for free motion in LIFEs.)
  Also clear should be how the equation of motion looks when the test particle is *not* free, i.e. when it is affected by some non-gravitational force – simply that force (F^μ) appears on the right-hand side,

  $$m\,\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}=\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau}=F^\mu$$

  ($p^\mu\equiv mu^\mu$ is the four-momentum; we assume forces which do not change the particle's rest mass m). In the important case of motion in an electromagnetic field, the particle is driven by both the fundamental macroscopic interactions – gravitational and electromagnetic, and it is the Lorentz force which stands on the right-hand side (while gravity is involved in the definition of a^μ, namely in the absolute derivative),

  $$a^\mu=\frac{q}{m}\,F^{\mu\nu}u_\nu\ , \tag{5.11}$$

  so the crucial characteristic of the particle is its specific electric charge q/m.

- It holds $A_{\nu;\mu}-A_{\mu;\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ and, similarly, for an antisymmetric tensor (so called bivector) $F_{\alpha\beta}$ it holds $F_{\{\alpha\beta;\gamma\}}\equiv F_{\alpha\beta;\gamma}+F_{\gamma\alpha;\beta}+F_{\beta\gamma;\alpha}=F_{\{\alpha\beta,\gamma\}}$, because the terms with Gammas cancel out due to the latter's symmetry in the bottom indices.

- The *covariant divergence* can be written in terms of partial divergence, which is mainly advantageous for the Gauss law. We will show it using the well known Jacobi formula for the differentiation of a matrix determinant.

  Lemma: Be $M(x)$ a square matrix depending on λ (it may either be a coordinate, a parameter, or just *some* variable). Then

  $$\frac{(\det M)_{,\lambda}}{\det M}=\mathrm{Tr}\left(M^{-1}\!\cdot M_{,\lambda}\right). \tag{5.12}$$

Proof: Let ${M_\mu}^\alpha$ be a square matrix $n \times n$. Let us differentiate its determinant

$$\det M = {M_\mu}^\alpha {M_\nu}^\beta \dots {M_\sigma}^\eta \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta$$

partially by some variable (we use the notation $X_{,\lambda}$, albeit it need not be a differentiation with respect to a coordinate),

$$\begin{aligned}(\det M)_{,\lambda} &= ({M_\mu}^\alpha)_{,\lambda} {M_\nu}^\beta \dots {M_\sigma}^\eta \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta + \\ &\quad + {M_\mu}^\alpha ({M_\nu}^\beta)_{,\lambda} \dots {M_\sigma}^\eta \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta + \\ &\quad + {M_\mu}^\alpha {M_\nu}^\beta \dots ({M_\sigma}^\eta)_{,\lambda} \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta = \\ &= n \, ({M_\mu}^\alpha)_{,\lambda} {M_\nu}^\beta \dots {M_\sigma}^\eta \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta = \\ &= (\det M) \, {(M^{-1})_\alpha}^\mu ({M_\mu}^\alpha)_{,\lambda} \equiv (\det M) \; \mathrm{Tr} \left(M^{-1} \cdot M_{,\lambda} \right) . \end{aligned} \tag{5.13}$$

We have only employed suitable renaming of the summation indices (thanks to which we have got n manifestly same terms) and, between the last two lines, the fact that the components of the inverse matrix ${(M^{-1})_\alpha}^\mu$ equal co-factors (subdeterminants = minors with the pertinent signs) corresponding to the components ${M_\mu}^\alpha$ divided by $(\det M)$, that is (Cramer's rule)

$${(M^{-1})_\alpha}^\mu = \frac{n}{\det M} \, \delta^{[\mu}_\alpha \delta^\nu_\beta \dots \delta^{\sigma]}_\eta \, {M_\nu}^\beta \dots {M_\sigma}^\eta \, . \tag{5.14}$$

For a particular dimension of the matrix, it is simple to verify the statement explicitly. For example, for a matrix 2×2, one can write, in the MAPLE program:

```
with(linalg):
A:=array([[a(x),b(x)],[c(x),d(x)]]);
dA:=map(diff,A,x);
leftside:=trace(multiply(inverse(A),dA));
rightside:=diff(det(A),x)/det(A);
simplify(leftside-rightside);
```

it is indeed zero.

Corollary: According to the definition of Christoffel symbols, one obtains

$${\Gamma^\mu}_{\mu\lambda} = \frac{1}{2} \, g^{\mu\sigma} (g_{\sigma\mu,\lambda} + g_{\lambda\sigma,\mu} - g_{\mu\lambda,\sigma}) = \frac{1}{2} \, g^{\mu\sigma} g_{\sigma\mu,\lambda}$$

(the last two terms in the parenthesis are antisymmetric in the indices $[\mu,\sigma]$, so they drop out after multiplication by the symmetric $g^{\mu\sigma}$), from where, thanks to the above Lemma, we have (in our case, the square matrix is the metric tensor and $g := \det g_{\mu\nu}$)[1]

$${\Gamma^\mu}_{\mu\lambda} = \frac{1}{2} \frac{g_{,\lambda}}{g} = \frac{1}{\sqrt{-g}} \, \frac{\partial \sqrt{-g}}{\partial x^\lambda} \, . \tag{5.15}$$

[1] In Lorentzian manifolds, the metric has an opposite sign in the time direction than in the three spatial directions – specifically, we use the $(-{+}{+}{+})$ metric signature here –, so the metric determinant $g := \det g_{\mu\nu}$ is negative. It is thus necessary to write it with minus when it appears under the square root.

Owing to this feature, the covariant divergence can be expressed in terms of the partial one:

$$\Rightarrow\ V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,V^{\mu}\right)_{,\mu}\ , \tag{5.16}$$

$$\Rightarrow\ T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\mu\nu}\right)_{,\nu} + \Gamma^{\mu}{}_{\nu\iota}T^{\iota\nu}\ , \tag{5.17}$$

$$\Rightarrow\ \Box\psi \equiv g^{\mu\nu}\psi_{;\mu\nu} = \psi^{;\nu}{}_{\nu} = \left(\psi^{,\nu}\right)_{;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\psi^{,\nu}\right)_{,\nu}\ , \tag{5.18}$$

where apparently the second-rank-tensor result reduces to the partial divergence in the case of an antisymmetric tensor (bivector), because the Γ-term vanishes for such.

5.4 What has been achieved so far

Let us have some drink and summarise shortly what we have done so far.

We started from the **universality of gravitation** – from the observation/assumption that the effect of its intensity is independent of the properties of a system on which is acts. Thanks to this universality, specifically thanks to the universality of free fall, it was possible to claim that the gravitational-intensity effect is locally transformed out ("cancelled") in any local inertial frame (LIFE), i.e. a certain Cartesian frame which is itself freely falling and non-rotating. This assertion – **the equivalence principle** – implied, in particular, that in the LIFE the physics should locally run as in the Minkowski space-time.

In rewriting the laws known from special relativity from the LIFE to general coordinates, we used **the principle of general covariance** which, at its simplest level, requires that the resulting law be represented by a general covariant equation, i.e. such equation whose form is preserved in any diffeomorphic change of coordinates. This requirement we satisfied by writing the equations in a tensorial form. In rewriting the equations, it has gradually turned out that they conceptually did not change from their special-relativistic form, but what did change is the geometry of space-time in which the given physical process takes place. In such a way, gravitation – originally a force interaction – has been ascribed to the geometric properties of the host space-time – it has been *geometrised*.

However, rather than living directly in space-time, tensors form tangent linear spaces which are specific for each space-time point/event. It is thus only simple to operate with tensors *at one particular point/event*, because all these belong to the same linear space. Yet generally, it is necessary to work with tensors living in different linear spaces (in tangent spaces at different points); in particular, in differential equations one needs to specify how to perform the differentiation, namely how to compare tensors at "neighbouring events". The relation between tangent spaces at different events of a manifold is provided by **affine connection**. The connection tells how to transport the tensor quantities, and thus determines a possible "reasonable" (tensorial) concept of differentiation – the **covariant derivative**.

There exists an infinite number of affine connections, but general relativity uses a special one – the one according to which metric transports in a "natural way", namely remains constant with respect to it. Such a connection is most easily obtained by considering the problem of a **parallel transport** of vectors (this is the transport in which the vector "keeps its space-time direction"); it bears the name of Levi-Civita and its special relation to metric is, in any coordinates, represented by Christoffel symbols.

The covariant derivative is the most important derivative on smooth manifolds, and the parallel transport is the most important transport over them. However, tensors need *not* only transport parallelly, and, correspondingly, there also exist other derivatives with tensorial character. The most significant other options are the Lie transport/derivative and the Fermi-Walker transport/derivative. We will come to these later.

Anyway, the crucial task is yet to be tackled. At the present stage, we know (kind of) how to rewrite the physics we know from special relativity to a general space-time (a space-time with general metric $g_{\mu\nu}$ and covered with some arbitrary coordinates). However, we do not know yet i) what deforms the space-time to the shape different from Minkowski, ii) how the space-time is deformed and how that deformation affects its own sources. In short, wanted is the new gravitational law, new "field equations". The following chapter on **curvature** will bring us pretty close to this goal. And parallel transport will again play an important role...

Time for curvature!

CHAPTER 6

Curvature of space-time

6.1 About the Karlov hill and about fools on its slopes

From Vyšehrad, the Prague fortification stretched, across the Botíč valley, to the New-Town walls which still today rise from "Na Slupi" along the western ridge of the Karlov hill. From there, they broke at right angle towards north and – via Svinská gate (the end of Svinská, later Ječná street at today's "Pavlák") – they headed to Poříčí. East of the Karlov rise, above Nusle, the southern Botíč hillside undulates; a small vinery reminds the vineyards that once were founded there by Charles IV and that only went to an end in 1848.[1] They were replaced by a rapid development of Královské Vinohrady (Royal Vineyards) – a town which later (in 1922) joined Prague.

However, we are now mainly interested in the south-eastern corner of the New Town, the region to the north of the conserved section of city walls, today delimited by Sokolská street on the east, by Ječná street on the north, and by Vyšehradská street on the west – the Karlov quarter. Its peeble-paved quiet lanes with gas lamps host respectable buildings under the roofs of which the service to a patient, to god, to science, to nation as well as to emperor meant basically the same. In them, several branches of Czech medicine and natural science experienced their beginnings as well as outstanding moments. The characters who contributed to such a development were at times walking through Karlov in clothes which had never been in fashion, and they were not always keeping a logical direction. Whether heading for their study room, for a seminar, to sketch a picture under the old garden trees, to

[1] It may not be known that the whole surroundings of Prague went through similar evolution. Charles IV, "wishing to breed the Czech Kingdom's honour, good as well as enjoyment", "from Rhinelands, France and Austria had the graceful grapevine conveyed, and, before long, notably the Prague surroundings were a one whole vineyard, as far as 3 miles. In the 16th and 17th centuries, more than 2000 vineries were still around Prague, some of them having up to 30 strychs." (*Strych*, *strich*, *korec* or *měřice* was an old volume unit used in Bohemia and in some parts of Germany; it corresponded to approximately 100 litres.) Since there did not exist computer shooters in those times, a warning had to be posted in 1449 at the entrances to the vineries: "... Did ladies or maidens come walking in the vineyards, they should eschew mischievous young people." [Quotations from F. Ruth: Kronika královské Prahy a obcí sousedních / The Chronicle of Royal Prague and of Neighbouring Municipalities (Körber, Prague 1903)].

reach their patient or a professional dispute, they were contributing to the atmosphere of an area whose charm reverberates till present times.

In Figure 6.1 with aerial photo of the Karlov area, the objects are marked which will be mentioned in the following. The most important of them, the Land maternity hospital, resided, together with a foundling home, in the former chapter of the St. Apollinaire church which looks down at the valley for some 640 years already. In 1875 the hospital moved to new premises built in the neighbourhood by the architect J. Hlávka, in the red-brick pavillon style of English "Gothic Revival" (number 1 in the figure). At its opening, it was the largest maternity hospital in Europe; it still keeps its original purpose, and it is still being admired for aesthetic reasons. From the hospital, it is advisable to enter our Faculty (known as "math-phys") as soon as possible; the Faculty has its dean's office just a bit to the south (number 2). However, the first "graduates" from the hospital could only enrol at our today's dean's office in 1911; in that year, mathematical and zoological departments of the Czech Charles University moved to the then new building. In its northern neighbourhood, a similarly nice workplace had already 4 years before been found by the Physics Institute of the Czech University. Both buildings were raised in a neoclassicist style, with baroque elements, opposite to the (few years older) campus of the Children hospital. They became dominants of the Slupská-garden amphitheatre, where mentally ill worked and which, after 1905, began to give way to the University campus "Albertov".[2] Further to the south, we are approaching the edge of the Nusle valley, close to which stands the Church of the Assumption of the Virgin Mary and St. Charles the Great from 1350-77 (number 3 – namely, there are 3 teeth of that great murderer in its main altar). The nave of this most beautiful Karlov building rises from the ground in regular-octagonal brickwork, roofed by a cupola of 23m diameter, then – allegedly – the boldest in Central Europe, being supported by a subtle and – according to Otto's encyclopædia – "magnificent above every thought" net vault.

However, those who set out, from the maternity hospital (1), in the opposite direction (*than towards our Faculty*), could have ended much worse. Just behind two corners, they would likely resort to the *U kalicha* house (number 4 in the map) where, under the brothel of Mrs. Millerová, there was a smoky pub. Today, after such a visit, one would probably return somewhat to the south, to a urology clinic (number 5), the structure of which neither fits together with the Apollinaire hospital on the west, nor with the Prague business hospital on the south (also neo-gothic, from 1861). Anyway, after the peripeteias undergone, both imaginary characters might have put themselves in the hands of the Land Institute for Mentally Ill which had been located in the neighbouring compound of St. Catherine from 1822. The name of this patron of academy (e.g. of the original Faculty of Liberal Arts of the Charles University) was given to it by the church erected, in 1355-67, at the northern periphery of the park enclosure and which was in 1737-41 supplied with a baroque nave by K. I. Dientzenhofer. From the original, gothic body, a slender tower has been preserved ("Prague minaret"), square at the bottom while higher becoming an octagon (rather typical for Karlov's churches). From 1840 the Institute was based in a "new house" in the south of the garden (number 6), where,

[2] Many voices were heard then regretful about the garden as a reservoir of clean air, and the Club for Old Prague was asking that "ground plan and, in particular, the silhouettes of the structures should meet the requirements of aesthetics and scenic beauty". (A time machine wanted! Already yesterday it was too late...)

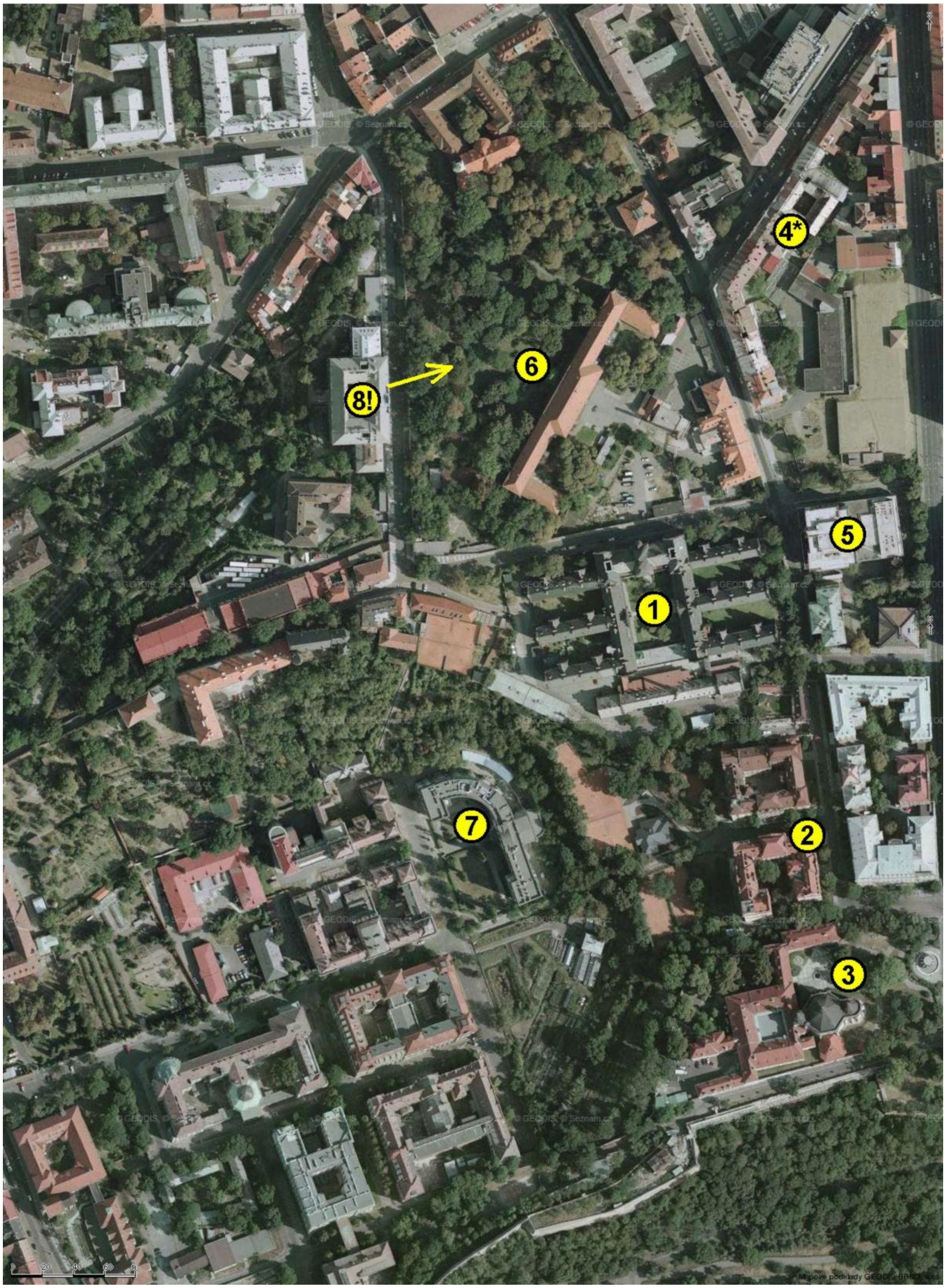

Figure 6.1: Karlov quarter from above. Numbers denote the objects mentioned in the text.

under the direction of J. Riedel, it even reached a reputation of one of the best psychiatric facilities in Europe. After a successful therapy, we may return to the St. Apollinaire church and from there descend the stairs to Albertov, where we finish our excursion in the Pathological Institute (number 7).

Yet we are not only mentioning Karlov because of sympathy towards its spiritus loci. Actually, in the St.-Catherine mental asylum, an intriguing intersection of Czech history probably happened, as most notably pointed out by our colleague J. Langer and by the Polish astrophysicist M. Abramowicz; it is connected to the nuances of spherical geometry. In curved spaces (not speaking about space-times), even invariant, "geometrical" quantities may behave in a non-intuitive manner. On a sphere, for instance, the proper circumference of concentric circles first grows with proper radius, but it grows slower and slower compared to how it does in a plane, and, after one reaches the main circle ("equator"), the behaviour reverses – the circumference *shrinks* with growing radius (even to zero finally). The same is true in higher-dimensional spherical geometries: in a closed, spherical universe, when expanding the proper radius from some given point, one first gets larger and larger proper volumes, but at one moment the volumes start to *shrink* back.

Now to the story. As recorded by J. Hašek [17] (a Czech writer of the beginning of 20th century), at the turn of June and July 1914, a Prague servant F. Strašlipka (in the report, he appears as a dog dealer, under the name J. Švejk) was arrested in the dodgy house U kalicha, allegedly for political reasons. He was subject to an investigation in the neighbouring St.-Catherine asylum and, because of somewhat ingenuous behaviour, he spent there a few days. When later describing the asylum life and patients, he stated, among others: "... *One was in a straitjacket all the time so that he shouldn't be able to calculate when world would come to an end. And I also met a certain number of professors there. One of them used to follow me about all the time and expatiate on how the cradle of the gipsy race was in the Krkonoše [Giant Mountains, north-east of Bohemia], and the other explained to me that inside the globe there was another globe much bigger than the outer one.*"

Now, the St.-Catherine clinic is also central to another well known statement, by Prof. A. Einstein, who in 1911-12 was looking down, over a wall, to its park from his office of the director of the Physics Institute of the Prague German University in Viničná street (thus from just an opposite side than where lies the *U kalicha* pub; today the building belongs to biological departments of the Faculty of Natural Sciences – it is Viničná 8). According to his successor and biographer P. Frank, Einstein was once showing the people walking in the garden to one of his guests, pointing from the window (see arrow in the figure) and saying: "These are those of the fools who do *not* deal with the quantum theory." Einstein was troubled by the quantum theory both before and after, but in Prague he was primarily working on gravitation, so it seems plausible that he gave a popular talk at the neighbouring institution[3] that some of the patients remembered and two years later retold to J. Švejk. Actually, tradition has it that it was only later (in July 1912) in Curych that M. Grossmann draw Einstein's attention to the Riemann geometry, but F. Frank claims that such a hint had already been

[3] Actually, contacts between different disciplines were quite lively at those times, in particular, "humanistic" people were attending physics courses and vice versa, public disputes were being organised, etc. No surprise that writers of that period were mentioning new scientific ideas in their works, often with decent understanding; well, sure – if they chiefly wrote them in academic-people bars like Hašek...

provided in Prague, by the mathematician G. Pick. (Pick was a former assistant of E. Mach. He headed the committee which appointed Albert Einstein to his Prague chair. 30 years later, he died in the Terezín camp.)

Besides that, historians should also be alarmed by the first of the above Švejk's sentences – about the end of the world: namely, at that time, a complete general theory of relativity was *not* yet available, let alone the *dynamical* cosmological models. (Friedmann, in particular, found his solutions some 8 years later.) In addition, the end of the world may only happen in closed models, of which the simplest – the homogeneous and isotropic one – has the geometry of a (3D) sphere in its spatial part! (See Chapter 13.)

6.2 The Riemann tensor

When speaking about parallel transport of a vector, we started from a natural image of how to preserve a direction in transporting it along a curved surface. Due to the requirement that the direction stay tangent to the surface, it was necessary, after every step along a chosen transport path (made in an embedding 3D Euclidean space), to tilt the vector down to the surface orthogonally. From the necessity of this tilting, it is clear that the parallel transport in general *depends on the path* if the surface (or host space in general) is curved. Best to illustrate this on a 2D sphere – try, for example, the path going from the north pole to the equator along a meridian, then along the equator and then back to the pole, starting with the vector tangent to the curve. (The final vector is different from the initial one.)

If covariant derivative is the one in which one of the compared vectors is first transported to the other's point by parallel transport, one thus expects, due to the obvious dependence of the parallel transport on path, that the commutator of second covariant derivatives will *not* vanish, and that it should provide information about curvature of the manifold at a given point. In fact it provides information about *both curvature and torsion* (if there was any): torsion manifests itself in that a parallelogram formed by parallel transporting its sides does not close at the opposite vertex (even in an infinitesimal case); and curvature manifests itself in that the parallel transport of any vector along two different paths about such a parallelogram (and thus along any two different paths connecting two points) does not yield the same results. The torsion tensor $T(X,Y)$, or $T^{\mu}{}_{\kappa\lambda}X^{\kappa}Y^{\lambda}$, we already introduced in the previous chapter by (5.9), while the curvature tensor is defined, on an abstract (coordinate-free) level, by

$$R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\,, \tag{6.1}$$

where X, Y and Z are vector fields and $[X,Y]$ is the Lie bracket (commutator) of two of them. Both tensors return a vector field. In an arbitrary coordinate basis, the curvature-tensor definition reads

$$\begin{aligned}
&R^{\sigma}{}_{\nu\kappa\lambda}Z^{\nu}X^{\kappa}Y^{\lambda} := \\
&= (Z^{\sigma}{}_{;\lambda}Y^{\lambda})_{;\kappa}X^{\kappa} - (Z^{\sigma}{}_{;\kappa}X^{\kappa})_{;\lambda}Y^{\lambda} - Z^{\sigma}{}_{;\nu}(Y^{\nu}{}_{,\lambda}X^{\lambda} - X^{\nu}{}_{,\lambda}Y^{\lambda}) = \\
&= (Z^{\sigma}{}_{;\lambda\kappa} - Z^{\sigma}{}_{;\kappa\lambda})X^{\kappa}Y^{\lambda} + Z^{\sigma}{}_{;\nu}(Y^{\nu}{}_{;\lambda}X^{\lambda} - X^{\nu}{}_{;\lambda}Y^{\lambda} - Y^{\nu}{}_{,\lambda}X^{\lambda} + X^{\nu}{}_{,\lambda}Y^{\lambda}) = \\
&= (Z^{\sigma}{}_{;\lambda\kappa} - Z^{\sigma}{}_{;\kappa\lambda})X^{\kappa}Y^{\lambda} + Z^{\sigma}{}_{;\nu}T^{\nu}{}_{\kappa\lambda}X^{\kappa}Y^{\lambda}
\end{aligned}$$

$$\Longrightarrow \quad Z^{\sigma}{}_{;\kappa\lambda} - Z^{\sigma}{}_{;\lambda\kappa} = -R^{\sigma}{}_{\nu\kappa\lambda} Z^{\nu} + T^{\nu}{}_{\kappa\lambda} Z^{\sigma}{}_{;\nu} \,. \tag{6.2}$$

This last relation is called the Ricci identities and we will derive it now, already with zero torsion, i.e. with the symmetric affine connection of the GR theory.

6.2.1 Ricci identities

Let's consider an arbitrary covector and compute the commutator of its second covariant derivatives. Denoting, for lucidity of the first step, $V_{\nu;\kappa} =: W_{\nu\kappa}$, we have from the covariant-derivative definition

$$\begin{aligned} V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} &\equiv W_{\nu\kappa;\lambda} - W_{\nu\lambda;\kappa} = \\ &= W_{\nu\kappa,\lambda} - \Gamma^{\rho}{}_{\lambda\nu} W_{\rho\kappa} - \cancel{\Gamma^{\rho}{}_{\lambda\kappa} W_{\nu\rho}} - W_{\nu\lambda,\kappa} + \Gamma^{\rho}{}_{\kappa\nu} W_{\rho\lambda} + \cancel{\Gamma^{\rho}{}_{\kappa\lambda} W_{\nu\rho}} \,. \end{aligned}$$

The indicated terms are identical and thus cancel out, so it remains

$$\begin{aligned} & V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} = \\ & = (V_{\nu,\kappa} - \Gamma^{\sigma}{}_{\kappa\nu} V_{\sigma})_{,\lambda} - \Gamma^{\rho}{}_{\lambda\nu}(V_{\rho,\kappa} - \Gamma^{\sigma}{}_{\kappa\rho} V_{\sigma}) - (V_{\nu,\lambda} - \Gamma^{\sigma}{}_{\lambda\nu} V_{\sigma})_{,\kappa} + \Gamma^{\rho}{}_{\kappa\nu}(V_{\rho,\lambda} - \Gamma^{\sigma}{}_{\lambda\rho} V_{\sigma}) = \\ & = (\Gamma^{\sigma}{}_{\lambda\nu,\kappa} - \Gamma^{\sigma}{}_{\kappa\nu,\lambda} + \Gamma^{\sigma}{}_{\kappa\rho}\Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\sigma}{}_{\lambda\rho}\Gamma^{\rho}{}_{\kappa\nu})\, V_{\sigma} \,. \end{aligned}$$

The relation can be written as

$$\boxed{V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} = R^{\sigma}{}_{\nu\kappa\lambda} V_{\sigma}}\,, \tag{6.3}$$

where we have introduced[4]

$$\boxed{R^{\sigma}{}_{\nu\kappa\lambda} := \Gamma^{\sigma}{}_{\nu\lambda,\kappa} - \Gamma^{\sigma}{}_{\nu\kappa,\lambda} + \Gamma^{\sigma}{}_{\rho\kappa}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\rho\lambda}\Gamma^{\rho}{}_{\nu\kappa} = 2\left(\Gamma^{\sigma}{}_{\nu[\lambda,\kappa]} + \Gamma^{\sigma}{}_{\rho[\kappa}\Gamma^{\rho}{}_{\lambda]\nu}\right)}\,. \tag{6.4}$$

Though expressed in terms of the affine connection and its partial derivative, this crucial quantity is certainly a (1,3) tensor thanks to the original commutator. The quantity is called **the Riemann curvature tensor**. Note that the relation seems to be slightly different from the torsion-free version of (6.2), but it is OK actually, because the Riemann tensor will before long be found anti-symmetric in the first two indices.

The above relation (6.3) we used as a definition of the Riemann tensor is called the **Ricci identities**. It can actually be generalised to tensors of any rank,

$$T_{\mu\nu\ldots;\kappa\lambda} - T_{\mu\ldots;\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda} T_{\sigma\nu\ldots} + R^{\sigma}{}_{\nu\kappa\lambda} T_{\mu\sigma\ldots} + \ldots \quad \text{(as many terms as indices)}\,. \tag{6.5}$$

In one of the paragraphs below, specifically the version applying to the 2nd-rank tensor will be useful, so let us also derive that (it only requires sense of order):

$$W_{\mu\nu;\kappa\lambda} - W_{\mu\nu;\lambda\kappa} =$$

[4] We are switching the bottom indices at all the Gammas (which makes no difference), simply because it is probably easier to remember the indices in the following arrangement. If torsion were non-zero, use the original order of indices! (The Ricci identity would in such a case contain an additional term $T^{\rho}{}_{\kappa\lambda} V_{\nu;\rho}$ arising from the two terms we cancelled out first; $T^{\rho}{}_{\kappa\lambda}$ being the torsion tensor.)

$$
\begin{aligned}
&= + (W_{\mu\nu;\kappa})_{,\lambda} - \Gamma^{\rho}{}_{\lambda\mu} W_{\rho\nu;\kappa} - \Gamma^{\rho}{}_{\lambda\nu} W_{\mu\rho;\kappa} - \cancel{\Gamma^{\rho}{}_{\lambda\kappa} W_{\mu\nu;\rho}} \\
&\quad - (W_{\mu\nu;\lambda})_{,\kappa} + \Gamma^{\rho}{}_{\kappa\mu} W_{\rho\nu;\lambda} + \Gamma^{\rho}{}_{\kappa\nu} W_{\mu\rho;\lambda} + \cancel{\Gamma^{\rho}{}_{\kappa\lambda} W_{\mu\nu;\rho}} \\
&= + \left(W_{\mu\nu,\kappa} - \Gamma^{\sigma}{}_{\kappa\mu} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\kappa\nu} W_{\mu\sigma}\right)_{,\lambda} \\
&\quad - \Gamma^{\rho}{}_{\lambda\mu} \left(W_{\rho\nu,\kappa} - \Gamma^{\sigma}{}_{\kappa\rho} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\kappa\nu} W_{\rho\sigma}\right) - \Gamma^{\rho}{}_{\lambda\nu} \left(W_{\mu\rho,\kappa} - \Gamma^{\sigma}{}_{\kappa\mu} W_{\sigma\rho} - \Gamma^{\sigma}{}_{\kappa\rho} W_{\mu\sigma}\right) \\
&\quad - \left(W_{\mu\nu,\lambda} - \Gamma^{\sigma}{}_{\lambda\mu} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} W_{\mu\sigma}\right)_{,\kappa} \\
&\quad + \Gamma^{\rho}{}_{\kappa\mu} \left(W_{\rho\nu,\lambda} - \Gamma^{\sigma}{}_{\lambda\rho} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} W_{\rho\sigma}\right) + \Gamma^{\rho}{}_{\kappa\nu} \left(W_{\mu\rho,\lambda} - \Gamma^{\sigma}{}_{\lambda\mu} W_{\sigma\rho} - \Gamma^{\sigma}{}_{\lambda\rho} W_{\mu\sigma}\right) \\
&= - \Gamma^{\sigma}{}_{\kappa\mu,\lambda} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\kappa\nu,\lambda} W_{\mu\sigma} + \Gamma^{\rho}{}_{\lambda\mu} (\Gamma^{\sigma}{}_{\kappa\rho} W_{\sigma\nu} + \cancel{\Gamma^{\sigma}{}_{\kappa\nu} W_{\rho\sigma}}) + \Gamma^{\rho}{}_{\lambda\nu} (\cancel{\Gamma^{\sigma}{}_{\kappa\mu} W_{\sigma\rho}} + \Gamma^{\sigma}{}_{\kappa\rho} W_{\mu\sigma}) \\
&\quad + \Gamma^{\sigma}{}_{\lambda\mu,\kappa} W_{\sigma\nu} + \Gamma^{\sigma}{}_{\lambda\nu,\kappa} W_{\mu\sigma} - \Gamma^{\rho}{}_{\kappa\mu} (\Gamma^{\sigma}{}_{\lambda\rho} W_{\sigma\nu} + \cancel{\Gamma^{\sigma}{}_{\lambda\nu} W_{\rho\sigma}}) - \Gamma^{\rho}{}_{\kappa\nu} (\cancel{\Gamma^{\sigma}{}_{\lambda\mu} W_{\sigma\rho}} + \Gamma^{\sigma}{}_{\lambda\rho} W_{\mu\sigma}) \\
&= - \Gamma^{\sigma}{}_{\kappa\mu,\lambda} W_{\sigma\nu} - \Gamma^{\sigma}{}_{\kappa\nu,\lambda} W_{\mu\sigma} + \Gamma^{\rho}{}_{\lambda\mu} \Gamma^{\sigma}{}_{\kappa\rho} W_{\sigma\nu} + \Gamma^{\rho}{}_{\lambda\nu} \Gamma^{\sigma}{}_{\kappa\rho} W_{\mu\sigma} \\
&\quad + \Gamma^{\sigma}{}_{\lambda\mu,\kappa} W_{\sigma\nu} + \Gamma^{\sigma}{}_{\lambda\nu,\kappa} W_{\mu\sigma} - \Gamma^{\rho}{}_{\kappa\mu} \Gamma^{\sigma}{}_{\lambda\rho} W_{\sigma\nu} - \Gamma^{\rho}{}_{\kappa\nu} \Gamma^{\sigma}{}_{\lambda\rho} W_{\mu\sigma} \\
&= + \left(\Gamma^{\sigma}{}_{\lambda\mu,\kappa} - \Gamma^{\sigma}{}_{\kappa\mu,\lambda} + \Gamma^{\sigma}{}_{\kappa\rho} \Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\sigma}{}_{\lambda\rho} \Gamma^{\rho}{}_{\kappa\mu}\right) W_{\sigma\nu} \\
&\quad + \left(\Gamma^{\sigma}{}_{\lambda\nu,\kappa} - \Gamma^{\sigma}{}_{\kappa\nu,\lambda} + \Gamma^{\sigma}{}_{\kappa\rho} \Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\sigma}{}_{\lambda\rho} \Gamma^{\rho}{}_{\kappa\nu}\right) W_{\mu\sigma}
\end{aligned}
$$

(first the last terms of the two lines cancelled out, then the first covariant derivatives were written out in the parentheses, then all the terms containing derivatives of the tensor cancelled out, and finally the terms containing $W_{\rho\sigma}$ and $W_{\sigma\rho}$ cancelled out as well). We recognize

$$
W_{\mu\nu;\kappa\lambda} - W_{\mu\nu;\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda} W_{\sigma\nu} + R^{\sigma}{}_{\nu\kappa\lambda} W_{\mu\sigma} \,. \tag{6.6}
$$

6.2.2 Fully covariant Riemann tensor

Obviously, the Riemann tensor is purely a property of the affine connection, in particular, one does not need metric for its definition. (In general, the metric need not even exist on the manifold.) However, the metric *is* necessary for the totally covariant form of the tensor, since one has to lower the first index, $R_{\mu\nu\kappa\lambda} = g_{\mu\sigma} R^{\sigma}{}_{\nu\kappa\lambda}$. Substituting from (6.4), we have

$$
R_{\mu\nu\kappa\lambda} = g_{\mu\sigma} \left(\Gamma^{\sigma}{}_{\nu\lambda,\kappa} - \Gamma^{\sigma}{}_{\nu\kappa,\lambda}\right) + g_{\mu\sigma} \left(\Gamma^{\sigma}{}_{\rho\kappa} \Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\rho\lambda} \Gamma^{\rho}{}_{\nu\kappa}\right). \tag{6.7}
$$

Consider the first part first:

$$
\begin{aligned}
g_{\mu\sigma} \left(\Gamma^{\sigma}{}_{\nu\lambda,\kappa} - \Gamma^{\sigma}{}_{\nu\kappa,\lambda}\right) &= (g_{\mu\sigma} \Gamma^{\sigma}{}_{\nu\lambda})_{,\kappa} - g_{\mu\sigma,\kappa} \Gamma^{\sigma}{}_{\nu\lambda} - (g_{\mu\sigma} \Gamma^{\sigma}{}_{\nu\kappa})_{,\lambda} + g_{\mu\sigma,\lambda} \Gamma^{\sigma}{}_{\nu\kappa} = \\
&= \Gamma_{\mu\nu\lambda,\kappa} - \Gamma_{\mu\nu\kappa,\lambda} + g_{\mu\sigma,\lambda} \Gamma^{\sigma}{}_{\nu\kappa} - g_{\mu\sigma,\kappa} \Gamma^{\sigma}{}_{\nu\lambda} \,.
\end{aligned}
$$

Here we further expand the first two terms by plugging in for the Christoffel symbols,

$$
\begin{aligned}
\Gamma_{\mu\nu\lambda,\kappa} - \Gamma_{\mu\nu\kappa,\lambda} &= \frac{1}{2} \left(\cancel{g_{\mu\nu,\lambda\kappa}} + g_{\lambda\mu,\nu\kappa} - g_{\nu\lambda,\mu\kappa} - \cancel{g_{\mu\nu,\kappa\lambda}} - g_{\kappa\mu,\nu\lambda} + g_{\nu\kappa,\mu\lambda}\right) = \\
&= \frac{1}{2} \left(g_{\lambda\mu,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa} - g_{\kappa\mu,\nu\lambda}\right),
\end{aligned}
$$

and the remaining two terms we join up (after renaming the summation index $\sigma \to \rho$) with the second half of (6.7):

$$
g_{\mu\rho,\lambda} \Gamma^{\rho}{}_{\nu\kappa} - g_{\mu\rho,\kappa} \Gamma^{\rho}{}_{\nu\lambda} + g_{\mu\sigma} \left(\Gamma^{\sigma}{}_{\rho\kappa} \Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\rho\lambda} \Gamma^{\rho}{}_{\nu\kappa}\right) =
$$

$$
\begin{aligned}
&= (g_{\mu\rho,\lambda} - \Gamma_{\mu\rho\lambda})\,\Gamma^{\rho}{}_{\nu\kappa} - (g_{\mu\rho,\kappa} - \Gamma_{\mu\rho\kappa})\,\Gamma^{\rho}{}_{\nu\lambda} = \\
&= (\Gamma_{\rho\mu\lambda} + \cancel{\Gamma_{\mu\rho\lambda} - \Gamma_{\mu\rho\lambda}})\,\Gamma^{\rho}{}_{\nu\kappa} - (\Gamma_{\rho\mu\kappa} + \cancel{\Gamma_{\mu\rho\kappa} - \Gamma_{\mu\rho\kappa}})\,\Gamma^{\rho}{}_{\nu\lambda} = \\
&= \Gamma_{\rho\mu\lambda}\Gamma^{\rho}{}_{\nu\kappa} - \Gamma_{\rho\mu\kappa}\Gamma^{\rho}{}_{\nu\lambda} = g_{\pi\rho}(\Gamma^{\pi}{}_{\mu\lambda}\Gamma^{\rho}{}_{\nu\kappa} - \Gamma^{\pi}{}_{\mu\kappa}\Gamma^{\rho}{}_{\nu\lambda})\ ,
\end{aligned}
$$

where its was sufficient to use (in the 2nd line) the $g \leftrightarrow \Gamma$ relation (2.9). The pure covariant form of the Riemann tensor thus reads

$$\boxed{R_{\mu\nu\kappa\lambda} = \frac{1}{2}\,(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) + g_{\pi\rho}(\Gamma^{\pi}{}_{\mu\lambda}\Gamma^{\rho}{}_{\nu\kappa} - \Gamma^{\pi}{}_{\mu\kappa}\Gamma^{\rho}{}_{\nu\lambda})}\,. \tag{6.8}$$

The first part is solely given by the 2nd derivatives of the metric (they appear linearly), while the second part contains the 1st derivatives of the metric (these appear quadratically).

6.2.3 Riemann-tensor symmetries

On any manifold equipped with affine connection, the Ricci identity implies

$$R^{\mu}{}_{\nu\kappa\lambda} = -R^{\mu}{}_{\nu\lambda\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{antisymmetry in } [\kappa,\lambda]\,. \tag{6.9}$$

Also straightforward is to check that

$$R^{\mu}{}_{\{\nu\kappa\lambda\}} \equiv R^{\mu}{}_{\nu\kappa\lambda} + R^{\mu}{}_{\lambda\nu\kappa} + R^{\mu}{}_{\kappa\lambda\nu} = 0 \quad \ldots\ldots\ldots\ldots \quad \text{first Bianchi identities}\,. \tag{6.10}$$

On (pseudo-)Riemannian manifolds (equipped with the metric), one can add that

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{antisymmetry in } [\mu,\nu]\,, \tag{6.11}$$

$$R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu} \quad \ldots\ldots\ldots \quad \text{symmetry in index pairs } ([\mu,\nu],[\kappa,\lambda])\,. \tag{6.12}$$

These properties can also be verified straightforwardly. (By combination of the last symmetry and antisymmetries within the two pairs, one also sees that Riemann stays unchanged if exactly reversing the order of indices, $R_{\mu\nu\kappa\lambda} = R_{\lambda\kappa\nu\mu}$.)

The last symmetry is not independent. Actually, lowering the first index in (6.10) and writing it up for all four possible permutations of indices,

$$
\begin{aligned}
&R_{\mu\nu\kappa\lambda} + R_{\mu\lambda\nu\kappa} + R_{\mu\kappa\lambda\nu} = 0\,, \\
&R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0\,, \\
&R_{\kappa\lambda\mu\nu} + R_{\kappa\nu\lambda\mu} + R_{\kappa\mu\nu\lambda} = 0\,, \\
&R_{\nu\kappa\lambda\mu} + R_{\nu\mu\kappa\lambda} + R_{\nu\lambda\mu\kappa} = 0\,,
\end{aligned}
$$

and adding these equations while using the antisymmetries (6.9) a (6.11) (thanks to which most of the terms cancel out in pairs), we obtain $2(R_{\mu\kappa\lambda\nu} - R_{\lambda\nu\mu\kappa}) = 0$, i.e. the symmetry (6.12).

It may also be noticed that thanks to the last property and to the two anti-symmetries, one may reverse the index order, $R_{\mu\nu\kappa\lambda} = R_{\lambda\kappa\nu\mu}$.

Now independent components of $R_{\mu\nu\kappa\lambda}$ can be summed up. The antisymmetry in the first as well as in the second pair of indices implies that each of these pairs can only assume 6 independent and non-trivial arrangements, $\{01\}$, $\{02\}$, $\{03\}$, $\{12\}$, $\{13\}$ and $\{23\}$, which yields $6\cdot 6 = 36$ possibilities in total. From this should be subtracted the number of independent components of the first Bianchi identities (6.10). Since the expression $R^{\mu}{}_{\{\nu\kappa\lambda\}}$ is antisymmetric in all the three bottom indices $\{\nu, \kappa, \lambda\}$, there are just 4 independent non-trivial possibilities within this group, $\{012\}$, $\{013\}$, $\{023\}$ and $\{123\}$; multiplied by the 4 possible values of the upper index, $\mu = 0, 1, 2, 3$, the Bianchi identities are thus 16. To sum up, Riemann has 36-16=20 independent components. In passing, this number quickly changes with the dimension of the manifold d – in general, it amounts to $\frac{1}{12}d^2(d^2-1)$; hence, it equals 6 for $d\!=\!3$ and just 1 for $d\!=\!2$ (surfaces) – this function is called the Gauss curvature.

6.2.4 Ricci tensor and Ricci scalar

Due to its antisymmetries, the Riemann tensor can only be contracted in one independent way. Actually, contraction over the 1-2 or 3-4 indices trivially vanishes, and contraction over the 1-3 indices yields the same result as contraction over the 2-4 indices, or minus contraction over 1-4 or 2-3. The only non-trivial contraction provides the **Ricci tensor**

$$R^{\kappa}{}_{\nu\kappa\lambda} = g^{\mu\kappa}R_{\mu\nu\kappa\lambda} =: R_{\nu\lambda}. \tag{6.13}$$

This tensor is symmetric thanks to the property (6.12) of Riemann,

$$R_{\lambda\nu} = g^{\mu\kappa}R_{\mu\lambda\kappa\nu} = g^{\mu\kappa}R_{\kappa\nu\mu\lambda} = R_{\nu\lambda}\,.$$

Further contraction of the Ricci tensor is possible and non-trivial,

$$R^{\lambda}{}_{\lambda} \equiv R\,. \tag{6.14}$$

This invariant is called the **Ricci scalar** or scalar curvature / curvature scalar.

6.3 Parallel-transport integrability, and geometric meaning of the Riemann tensor

One of the classical mathematical problems often important in physics is the problem of integrability. Whether literarily or figuratively speaking, the question is whether a certain quantity/feature, known or given at one point and/or instant, can be uniquely determined "everywhere" by *integration* of a known/given rule for its spatial and/or time change. The problem is not whether one knows how to perform the integration (this is assumed to be the case), but whether the integration does not give different results when performed along different paths. Concerning the parallel transport, one can be – even without any formal proof – sure about the following two facts:

- Parallel transport is integrable on flat manifolds (as Euclidean or Minkowskian). Actually, there it is possible to establish global Cartesian systems (e.g. ordinary (x, y, z), or inertial systems (ct, x, y, z) of special relativity), and in the latter "to transport a vector parallelly" (i.e. to keep its direction) means to keep its Cartesian components. Hence, given a certain vector at an arbitrary single point, its parallel "copies" at all other points are uniquely fixed.

- Parallel transport is not in general integrable on curved manifolds. In order to prove this, it is sufficient to give one example – and we already have given that of a sphere –, but, due to the necessity to keep, during the transport, the vector tangent to the manifold, the lack of integrability is intuitively clear for any curved surface (we are saying surface, because geometrical intuition is difficult for higher-dimensional manifolds).

Hence, the integrability/non-integrability of the parallel transport should be equivalent to the flatness/curvature of the manifold. Should the Riemann represent, exhaustively, the curvature, it would have to exclusively decide about the generic integrability of parallel transport on any manifold.

6.3.1 Integrability conditions for a generic differential form

Differential equations typically determine quantities in such a manner that they say how these quantities change in time and/or space. Consider some generic multicomponent quantity (e.g. a tensor) $T^{\cdots}_{\cdots}$ and assume its change to be given by a differential form

$$\delta T^{\cdots}_{\cdots} = f^{\cdots}_{\cdots\alpha}\mathrm{d}x^{\alpha}\,, \tag{6.15}$$

where the "coefficients" $f^{\cdots}_{\cdots\alpha}$ depend on x^{μ}. Specifically, it is a linear differential form (a 1-form) in exact differentials $\mathrm{d}x^{\alpha}$ of independent variables x^{α}. The change of $T^{\cdots}_{\cdots}$ we denoted by δ in order to indicate that it may not be an exact differential. Actually, just this is the clue to an answer: the formula (the differential form) is integrable if and only if it is exact (exact differential of some quantity).[5] Let us derive what this implies for the $f^{\cdots}_{\cdots\alpha}$ coefficients.

Knowing the value of $T^{\cdots}_{\cdots}$ at some point/event (A), one can obtain its value elsewhere by the line integral

$$T^{\cdots}_{\cdots}(\mathrm{B}) = T^{\cdots}_{\cdots}(\mathrm{A}) + \int_{\mathrm{A}}^{\mathrm{B}} f^{\cdots}_{\cdots\alpha}\mathrm{d}x^{\alpha}\,. \tag{6.16}$$

In general, the points A and B are connected by more than one curves (probably even by infinitely many of them). Should the value at B be unique, the integration along any such curve has to yield the same result; in particular, if one integrates – along any curve again – back to the initial point, the initial value $T^{\cdots}_{\cdots}(\mathrm{A})$ would have to be recovered, which means that the integral would have to vanish along any closed curve. The integrability conditions for $f^{\cdots}_{\cdots\alpha}$ are found by using the Stokes theorem and translating the integration along a closed curve (γ) to an integral over the surface (S) enclosed by γ:

$$\begin{aligned}\oint_{\gamma} f^{\cdots}_{\cdots\alpha}\mathrm{d}x^{\alpha} = \int_S f^{\cdots}_{\cdots\alpha,\beta}\mathrm{d}S^{\alpha\beta} &= \int_S f^{\cdots}_{\cdots\alpha,\beta}\left(\mathrm{d}_{(1)}x^{\alpha}\mathrm{d}_{(2)}x^{\beta} - \mathrm{d}_{(2)}x^{\alpha}\mathrm{d}_{(1)}x^{\beta}\right)\\ &= \int_S \left(f^{\cdots}_{\cdots\alpha,\beta} - f^{\cdots}_{\cdots\beta,\alpha}\right)\mathrm{d}_{(1)}x^{\alpha}\mathrm{d}_{(2)}x^{\beta}\,.\end{aligned} \tag{6.17}$$

[5] The word *holonomic* is often used as the synonym of *integrable*, the former being from Greek and the latter from Latin, both meaning "makeable whole", "completeable".

Above, $\mathrm{d}_{(1)}x^\alpha$ and $\mathrm{d}_{(2)}x^\beta$ stand for the representation in independent variables (typically coordinates) of *some* independent elements tangent to S; their "vector product" provides the surface element $\mathrm{d}S^{\alpha\beta}$. The integral has to vanish for *any* closed curve, hence irrespectively of the surface S, which is possible if and only if

$$f^{\cdots}_{\;\;\ldots\alpha,\beta} = f^{\cdots}_{\;\;\ldots\beta,\alpha}\,. \tag{6.18}$$

Linear forms in total differentials $\mathrm{d}x^\mu$ of independent variables x^μ (that need not necessarily be coordinates) are known as Pfaffian forms.[6] The Pfaff form $\delta T^{\cdots}_{\cdots}$ may not be a total differential; yet even if it is not, there may exist such a non-zero function μ that the form $\mu\delta T^{\cdots}_{\cdots}$ already *is* a total differential. The function μ is then called the integrating factor of the form; the Pfaff forms are usually mentioned in connection with the problem of seeking the integrating factor – as e.g. in thermodynamics when seeking the total differential of entropy from the heat element, and, in general, when distinguishing between state and process variables.

Perhaps the most well known case of integrability conditions arises in mechanics, from the requirement of conservativeness of a force field: in order that the work of a force $\vec{f}$ vanish along any closed path, $\oint\vec{f}\cdot\mathrm{d}\vec{r}=0$, it must hold $\mathrm{rot}\vec{f}=\vec{0}$, in components $f_{j,k}=f_{k,j}$. Or, the integrability conditions assume a self-evident form in the case of the differential formula for a coordinate transformation, $\mathrm{d}x'^\mu = f^\mu{}_\nu\mathrm{d}x^\nu$: in that case, $f^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ and the requirement is that any point correspond to unique values of the new coordinates. The conditions (6.18) say that it holds if and only if the second partial derivatives commute, $\frac{\partial^2 x'^\mu}{\partial x^\rho\partial x^\nu} = \frac{\partial^2 x'^\mu}{\partial x^\nu\partial x^\rho}$.

6.3.2 Integrability conditions for parallel transport

Rewriting the parallel-transport equation (2.2) in a "growth" form

$$\mathrm{d}V^\mu = -\Gamma^\mu{}_{\kappa\lambda}V^\lambda\mathrm{d}x^\kappa \tag{6.19}$$

and analogizing this to the general equation (6.15), we see that $f^\mu{}_\kappa = -\Gamma^\mu{}_{\kappa\lambda}V^\lambda$ in this case, so the integrability conditions (6.18) read

$$(\Gamma^\mu{}_{\kappa\lambda}V^\lambda)_{,\nu} = (\Gamma^\mu{}_{\nu\lambda}V^\lambda)_{,\kappa}\,. \tag{6.20}$$

By differentiation and substitution of

$$V^\lambda{}_{,\nu} = -\Gamma^\lambda{}_{\nu\iota}V^\iota, \qquad V^\lambda{}_{,\kappa} = -\Gamma^\lambda{}_{\kappa\iota}V^\iota$$

from the parallel-transport equation,[7] we obtain, after a suitable renaming of the summation index (so that V^ι factor out of all terms properly),

$$R^\mu{}_{\iota\nu\kappa}V^\iota = 0\,. \tag{6.21}$$

Therefore, the parallel transport is integrable if and only if $R^\mu{}_{\iota\nu\lambda}=0$.

---

[6] To the surname of J. F. Pfaff, "f" contributed by 60%. Still it is less than how much "dark energy" contributes to the cosmic energy density. (← Unsolicited advert for cosmology.)

[7] This relation exactly says that $V^\lambda{}_{;\nu}=0$, which is the requirement that V^λ be parallel *along any curve*.

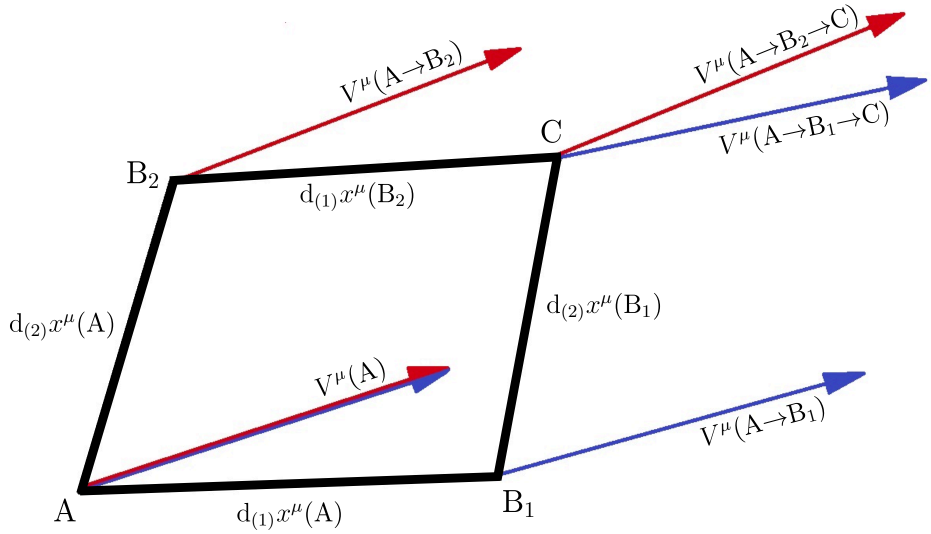

Figure 6.2 *Illustration of the non-integrability of parallel transport on an infinitesimal parallelogram. A vector (V^μ) is transported along two opposite parallelogram branches, and the results obtained at C (blue and red) are compared.*

Let us add a detailed computation of a specific exercise – the parallel transport of some vector V^μ from an arbitrarily chosen point A to some very nearby point C along two opposite branches (A→B$_1$→C, A→B$_2$→C) of a parallelogram composed of infinitesimal shifts $\mathrm{d}_{(1)}x^\mu$ and $\mathrm{d}_{(2)}x^\mu$. One may, for example, use geodesic segments as the shifts, but it is not necessary. In any case, it is assumed that the differences between $\mathrm{d}_{(1)}x^\mu(A)$ and $\mathrm{d}_{(1)}x^\mu(B_2)$, and between $\mathrm{d}_{(2)}x^\mu(A)$ and $\mathrm{d}_{(2)}x^\mu(B_1)$, are of the order $O(d^2)$.

- The transport A→B$_1$, i.e. along $\mathrm{d}_{(1)}x^\mu(\mathrm{A})$:[8]

$$V^\mu(\mathrm{A}\to\mathrm{B}_1) = V^\mu(\mathrm{A}) - (\Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(1)}x^\beta)(\mathrm{A}).$$

- The subsequent transport B$_1$→C, i.e. along $\mathrm{d}_{(2)}x^\mu(\mathrm{B}_1)$:

$$\begin{aligned} V^\mu(\mathrm{A}\to\mathrm{B}_1\to\mathrm{C}) &= V^\mu(\mathrm{A}\to\mathrm{B}_1) - (\Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(2)}x^\beta)(\mathrm{B}_1) = \\ &= V^\mu(\mathrm{A}) - (\Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(1)}x^\beta)(\mathrm{A}) - (\Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(2)}x^\beta)(\mathrm{B}_1)\,. \end{aligned}$$

- The second Gamma-term we express in terms of the values at A, while restricting to quantities of the order $O(d^2)$ at most. Expanding, in a Maclaurin manner from A,

$$\begin{aligned} \Gamma^\mu{}_{\alpha\beta}(\mathrm{B}_1) &= \Gamma^\mu{}_{\alpha\beta}(\mathrm{A}) + (\Gamma^\mu{}_{\alpha\beta,\gamma}\mathrm{d}_{(1)}x^\gamma)(\mathrm{A}) + O(\mathrm{d}^2), \\ (\mathrm{d}_{(2)}x^\mu)(\mathrm{B}_1) &= (\mathrm{d}_{(2)}x^\mu)(\mathrm{A}) + O(\mathrm{d}^2)\,, \end{aligned}$$

[8] We will write the summing indices at Gammas in an opposite order than we did in the parallel-transport equation, but that does not matter of course (we assume the affine connection is symmetric).

and substituting for

$$V^\alpha(\mathrm{B}_1) \equiv V^\alpha(\mathrm{A}\to\mathrm{B}_1) = V^\alpha(\mathrm{A}) - (\Gamma^\alpha{}_{\rho\sigma}V^\rho \mathrm{d}_{(1)}x^\sigma)(\mathrm{A})$$

from above, we have

$$V^\mu(\mathrm{A}\to\mathrm{B}_1\to\mathrm{C}) = V^\mu - \Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(1)}x^\beta - \Gamma^\mu{}_{\alpha\beta}V^\alpha(\mathrm{d}_{(2)}x^\beta)(\mathrm{B}_1) + \\ + \Gamma^\mu{}_{\alpha\beta}\Gamma^\alpha{}_{\rho\sigma}V^\rho \mathrm{d}_{(1)}x^\sigma \mathrm{d}_{(2)}x^\beta - \Gamma^\mu{}_{\alpha\beta,\sigma}V^\alpha \mathrm{d}_{(1)}x^\sigma \mathrm{d}_{(2)}x^\beta\,,$$

where everything is evaluated at A, except the $\mathrm{d}_{(2)}x^\beta(\mathrm{B}_1)$ element in the "linear" term.

- The transport along the other branch A→B$_2$→C proceeds in exactly the same way, just with $\mathrm{d}_{(1)}x^\mu \leftrightarrow \mathrm{d}_{(2)}x^\mu$ and B$_1$ ↔ B$_2$ switched, hence

$$V^\mu(\mathrm{A}\to\mathrm{B}_2\to\mathrm{C}) = V^\mu - \Gamma^\mu{}_{\alpha\beta}V^\alpha \mathrm{d}_{(2)}x^\beta - \Gamma^\mu{}_{\alpha\beta}V^\alpha(\mathrm{d}_{(1)}x^\beta)(\mathrm{B}_2) + \\ + \Gamma^\mu{}_{\alpha\beta}\Gamma^\alpha{}_{\rho\sigma}V^\rho \mathrm{d}_{(2)}x^\sigma \mathrm{d}_{(1)}x^\beta - \Gamma^\mu{}_{\alpha\beta,\sigma}V^\alpha \mathrm{d}_{(2)}x^\sigma \mathrm{d}_{(1)}x^\beta\,.$$

- Renaming the summation indices carefully where necessary, the difference between the vectors obtained by transport along the opposite paths (called the defect of the vector) comes out

$$\delta V^\mu \equiv V^\mu(\mathrm{A}\to\mathrm{B}_2\to\mathrm{C}) - V^\mu(\mathrm{A}\to\mathrm{B}_1\to\mathrm{C}) = \\ = \Gamma^\mu{}_{\alpha\beta}V^\alpha\left[(\mathrm{d}_{(1)}x^\beta)(\mathrm{A}) + (\mathrm{d}_{(2)}x^\beta)(\mathrm{B}_1) - (\mathrm{d}_{(2)}x^\beta)(\mathrm{A}) - (\mathrm{d}_{(1)}x^\beta)(\mathrm{B}_2)\right] + \\ + (\Gamma^\mu{}_{\alpha\beta,\sigma} - \Gamma^\mu{}_{\alpha\sigma,\beta} + \Gamma^\mu{}_{\iota\sigma}\Gamma^\iota{}_{\alpha\beta} - \Gamma^\mu{}_{\iota\beta}\Gamma^\iota{}_{\alpha\sigma})\,V^\alpha \mathrm{d}_{(1)}x^\sigma \mathrm{d}_{(2)}x^\beta\,.$$

However, the paths arrive at the same point (C) by assumption,

$$(\mathrm{d}_{(1)}x^\beta)(\mathrm{A}) + (\mathrm{d}_{(2)}x^\beta)(\mathrm{B}_1) = (\mathrm{d}_{(2)}x^\beta)(\mathrm{A}) + (\mathrm{d}_{(1)}x^\beta)(\mathrm{B}_2)\,,$$

so the parenthesis in the first term *exactly* vanishes, and the result reads

$$\delta V^\mu = R^\mu{}_{\alpha\sigma\beta}V^\alpha \mathrm{d}_{(1)}x^\sigma \mathrm{d}_{(2)}x^\beta\,. \tag{6.22}$$

Hence, the exercise does not depend on the path if and only if the Riemann tensor is zero. (A remark: if transporting along a finite curve of generic shape, one may consider dividing it into infinitesimal parallelograms. An integral form of the result is derived, for example, in [24], Chapter 6,1.)

6.3.3 Flat space-time

Intuitively, some features are "seen" clearly, but it may not be that easy to prove them formally. Since we have already many times referred to "flat space-time", it is worth to make more precise which major properties it has. Sure, flatness means that there exists such a coordinate system in which the metric has everywhere the Minkowski form, $g_{\mu\nu} = \eta_{\mu\nu}$. There are three other, equivalent statements:

- The parallel transport is integrable, i.e. independent of path. In particular, for any two given points and any vector, it does not matter along which connecting curve one transports it, the result is unique. This is also equivalent to saying that any vector does not change in transport along any closed curve.
- There exists, globally, a covariantly constant vector field, i.e. a vector field which is parallel along any curve.
- The Riemann tensor vanishes. (Note that this is an *absolute* statement, not a coordinate-dependent one, because it is a statement about tensor.)

It is obvious that the above three properties hold if $g_{\mu\nu} = \eta_{\mu\nu}$ globally, but more difficult is to prove the converse. We have kind-of shown that the three properties are equivalent, so one can start from any of them. If the parallel transport is integrable, it is possible to uniquely distribute over the space-time, from any point, an orthonormal tetrad $\{e^{\mu}_{\hat{\alpha}}\}^{3}_{\hat{\alpha}=0}$, i.e. such a tetrad of vectors which will – thanks to the fact that parallel transport keeps constant the scalar product – everywhere satisfy the relations

$$g_{\mu\nu} e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} = \eta_{\alpha\beta} \,. \tag{6.23}$$

Independently of the original coordinates used, we can now everywhere change to new ones by the transformation

$$\mathrm{d}x'^{\hat{\alpha}} = e^{\hat{\alpha}}_{\nu} \mathrm{d}x^{\nu} \,.$$

This is a reasonable transformation since – thanks to the parallel property of the vector fields $e^{\mu}_{\hat{\alpha}}$ along *any* curve (will write it down with the index positions switched:)

$$e^{\hat{\alpha}}_{\mu;\rho} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}p} = 0, \qquad e^{\hat{\alpha}}_{\rho;\mu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}p} = 0,$$

from where it follows, by combination,

$$e^{\hat{\alpha}}_{\mu,\rho} = \Gamma^{\iota}{}_{\rho\mu} e^{\hat{\alpha}}_{\iota} = \Gamma^{\iota}{}_{\mu\rho} e^{\hat{\alpha}}_{\iota} = e^{\hat{\alpha}}_{\rho,\mu} \,,$$

i.e. the integrability condition

$$\frac{\partial^2 x'^{\hat{\alpha}}}{\partial x^{\rho} \partial x^{\mu}} = \frac{\partial^2 x'^{\hat{\alpha}}}{\partial x^{\mu} \partial x^{\rho}} \,.$$

However, according to (6.23), the metric tensor is Minkowskian in these new coordinates.

6.4 Deviation of geodesics, and physical meaning of the Riemann tensor

In Newtonian limit of the geodesic equation, we found the correspondence between the metric and the Newtonian gravitational potential. From that it follows that the Christoffel symbols, given by first partial derivatives of the metric, play the role of "gravitational intensity" (as they do not represent a tensor, no surprise that they can be locally made vanish in freely falling systems). The Riemann tensor (6.8) is given by the metric second derivatives (linearly) and the 1st derivatives (quadratically), with the latter actually being redundant since they locally vanish in the LIFE. Consequently, speaking in the Newtonian language, the Riemann tensor should represent non-homogeneity of the gravitational field (gradient of its intensity). In order to verify whether it represents the latter exclusively (completely), let us analyse a thought experiment with two close freely falling particles, i.e. acted upon solely by the given gravitational field. Were the field homogeneous, both particles would feel the same acceleration, so they would not get accelerated *relative to each other*; were the field different along the world-lines of the particles, it would induce some relative acceleration between them.

The exercise is standardly being presented on the whole one-parameter class of (time-like) geodesics $x^\mu(l) = x^\mu(l;\tau)$, where l is a real parameter which, symbolically, "labels" (and thus identifies) the geodesics, while τ is the time parameter having the meaning of proper time along one of the geodesics (this will soon be selected as the reference one). An important assumption is that the mapping $(l,\tau) \to x^\mu(l;\tau)$ be a diffeomorphism, so it should be one-to-one and smooth together with its inverse; in such a case, each point of the pondered region is passed through by exactly one geodesic, and the generated bunch of curves is called the **congruence** of geodesics. (In order to cross all points of some 3D region, it would actually be necessary to use *three* parameters instead of just l. We in fact consider a one-parameter sub-congruence of the whole "bulk" congruence, within which the world-lines are labelled by l.)

Select now the geodesic along which τ is the proper time as the reference one; with respect to it some other nearby geodesic will be tracked. Physically, it means sitting on one of the freely falling particles and tracking the free fall of some other nearby one. Denote by $u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ the four-velocity along the reference geodesic, and by $\delta x^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}l}$ the connecting vector between the geodesics (relative position of the tracked particle with respect to the reference one). Two basic properties follow immediately from the assumption of diffeomorphism:

- Total derivatives of the mapping by τ and by l commute,

  $$\frac{\mathrm{d}u^\mu}{\mathrm{d}l} \equiv \frac{\mathrm{d}}{\mathrm{d}l}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}l} \equiv \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau}\,.$$

  Since both u^μ and δx^μ are actually given as a field, one may remark that the statement can also be expressed more geometrically as vanishing of the Lie derivative of one of the vectors with respect to the other – in other words, as vanishing of their commutator (Lie bracket),

  $$0 = (\pounds_u \delta x)^\mu \equiv [u, \delta x]^\mu \equiv \delta x^\mu{}_{,\nu} u^\nu - u^\mu{}_{,\nu}\delta x^\nu = \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau} - \frac{\mathrm{d}u^\mu}{\mathrm{d}l}$$

  (see Section 11 for the Lie-derivative account). Anyway, due to the symmetry of the affine connection, the commutation property is even inherited by absolute derivatives,

  $$\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\kappa\lambda}u^\kappa\delta x^\lambda = \frac{\mathrm{d}u^\mu}{\mathrm{d}l} + \Gamma^\mu{}_{\kappa\lambda}\delta x^\kappa u^\lambda = \frac{\mathrm{D}u^\mu}{\mathrm{d}l}\,.$$

- When performing an actual measurement, the position of an object is by definition registered *at given* τ (relative position at given τ means that the values at both ends of the measuring ruler are recorded at the same instant of τ), so δx^μ is by construction orthogonal to u^μ. Such a statement really has a good sense, because it turns out that the orthogonality automatically stays valid during the motion,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}u^\mu\delta x^\nu) = \frac{\mathrm{D}}{\mathrm{d}\tau}(g_{\mu\nu}u^\mu\delta x^\nu) = g_{\mu\nu}u^\mu\,\frac{\mathrm{D}\delta x^\nu}{\mathrm{d}\tau} = g_{\mu\nu}u^\mu\,\frac{\mathrm{D}u^\nu}{\mathrm{d}l} = \\ = \frac{1}{2}\frac{\mathrm{D}}{\mathrm{d}l}(g_{\mu\nu}u^\mu u^\nu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}l}(-1) = 0\,. \tag{6.24}$$

  Only the covariant constancy of the metric, geodesic property $\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}=0$ of the congruence, the above equality $\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau}=\frac{\mathrm{D}u^\mu}{\mathrm{d}l}$ and the four-velocity normalisation have been employed.

Now, finally, we will compute the relative acceleration between two neighbouring particles, i.e. the behaviour of the quantity $\frac{\mathrm{D}^2\delta x^\mu}{\mathrm{d}\tau^2}$ along the congruence. Will proceed similarly as above, mainly using the equality $\frac{\mathrm{D}\delta x^\nu}{\mathrm{d}\tau}=\frac{\mathrm{D}u^\nu}{\mathrm{d}l}$, the formula (6.3) for expressing the covariant-derivative commutator in terms of the Riemann tensor – specifically in the form

$$u^\mu{}_{;\alpha\beta} = u^\mu{}_{;\beta\alpha} + R^{\sigma\mu}{}_{\alpha\beta}u_\sigma = u^\mu{}_{;\beta\alpha} - R^\mu{}_{\sigma\alpha\beta}u^\sigma\,,$$

and, finally, the fact that the congruence is geodesic, so its four-acceleration $u^\mu{}_{;\beta}u^\beta$ vanishes:

$$\frac{\mathrm{D}^2\delta x^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau}\right) = \frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{D}u^\mu}{\mathrm{d}l}\right) = \frac{\mathrm{D}}{\mathrm{d}\tau}(u^\mu{}_{;\alpha}\delta x^\alpha) = u^\mu{}_{;\alpha\beta}u^\beta\delta x^\alpha + u^\mu{}_{;\alpha}\frac{\mathrm{D}\delta x^\alpha}{\mathrm{d}\tau} = \\ = u^\mu{}_{;\beta\alpha}u^\beta\delta x^\alpha - R^\mu{}_{\sigma\alpha\beta}u^\sigma\delta x^\alpha u^\beta + u^\mu{}_{;\alpha}\frac{\mathrm{D}u^\alpha}{\mathrm{d}l} = \frac{\mathrm{D}}{\mathrm{d}l}(\cancel{u^\mu{}_{;\beta}u^\beta}) - R^\mu{}_{\sigma\alpha\beta}u^\sigma\delta x^\alpha u^\beta\,,$$

therefore,

$$\boxed{\frac{\mathrm{D}^2\delta x^\mu}{\mathrm{d}\tau^2} = -R^\mu{}_{\sigma\alpha\beta}u^\sigma\delta x^\alpha u^\beta}\,. \tag{6.25}$$

This equation is called **the equation of geodesic deviation**. It confirms that the Riemann tensor really characterises the non-homogeneity of the gravitational field, and thus “tidal forces” which act within the field due to its non-homogeneity (tidal forces we write in quotation marks, because one should bear in mind that here they also include “tides” in the time direction). Note that the Newtonian equation for the same problem reads

$$\frac{\mathrm{d}^2\delta x^i}{\mathrm{d}t^2} = -\Phi^{,i}{}_j\delta x^j\,, \tag{6.26}$$

so the Riemann tensor is seen to be the counterpart of the Newtonian tidal tensor $\Phi^{,i}{}_j$.

The geodesic-deviation equation is in fact one of the most important equations on the way to the new, *geometric* theory of gravitation: according to it, the relative acceleration of particles not affected by any differential influences is *not* due to gravitational forces, but due to the curvature of space-time in which the particles move.

6.4.1 Geodesic-deviation equation in local inertial frame

In order to come closer to "practice", let us express the geodesic-deviation equation in terms of directly measurable components. Actually, the measurement would in reality be naturally made with respect to the LIFE tied to the reference particle (remember that the latter is freely falling). In the LIFE (components indicated by a hat), $g_{\hat{\mu}\hat{\nu}} = \eta_{\mu\nu}$ and the reference particle stays at rest with respect to it, $u^{\hat{\mu}} = (1, 0, 0, 0)$, whereas the second-particle position is purely spatial, $\delta x^{\hat{\mu}} = (0, \delta x^{\hat{\imath}})$. Finally, the tetrad components of tensors (here projections of tensors to the LIFE) are invariant with respect to the change of global coordinates, so the absolute derivative acts on them as the total one. Therefore, the equation reads

$$\frac{\mathrm{d}^2 \delta x^{\hat{\imath}}}{\mathrm{d}\tau^2} = -R^{\hat{\imath}}{}_{\hat{0}\hat{\jmath}\hat{0}} \delta x^{\hat{\jmath}} \,. \tag{6.27}$$

Several observations:

- The correspondence with the Newtonian tidal equation (6.26), hence the one between $R^{\hat{\imath}}{}_{\hat{0}\hat{\jmath}\hat{0}}$ and $\Phi_{,}{}^{i}{}_{j}$, is very clear now.

- This equation can be used to deduce, from actual measurement of the behaviour of $\delta x^{\hat{\imath}}$, (some of) the Riemann-tensor components.

- The equation can be viewed as a precise answer to the query naturally arising in discussion of the equivalence principle, namely how big an error one makes, due to the non-homogeneity of the field (due to the *gradient* of intensity which is even non-zero at a single point), if not restricting to an infinitesimal LIFE: the deviation solely depends on curvature, and grows linearly with $\delta x^{\hat{\jmath}}$.

The linear dependence on $\delta x^{\hat{\jmath}}$ implies that at one single point the effect of non-homogeneity still vanishes. This is also true more generally, provided that the motion is purely translational. On the other hand, rotational degrees of freedom usually couple to non-homogeneity and give rise to terms which – with the non-zero Riemann tensor – need not even vanish in the point limit. A simple example of such a behaviour are the **Mathisson-Papapetrou-Dixon equations** which describe the motion of a free "pole-dipole" particle, i.e. of a point-like body which is endowed, besides the mass, by a proper rotational angular momentum (spin). The equations read

$$\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} \,, \tag{6.28}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = p^\mu u^\nu - p^\nu u^\mu \,, \tag{6.29}$$

where $p^\mu = m u^\mu - u_\sigma \frac{\mathrm{D}S^{\mu\sigma}}{\mathrm{d}\tau}$ is the total momentum of the particle, u^μ is the tangent vector to a certain world-line representing the particle, $m = -p_\sigma u^\sigma$ is the particle's mass in the system with four-velocity u^μ and $S^{\mu\nu}$ is the bivector of spin. The "spin-curvature coupling" term on the right-hand side of (6.28) does not even vanish at a single point. Well, one should add that this feature is a consequence of the approximation: physically, *one point* cannot rotate to generate non-zero spin – a body with spin must have a certain minimal size of the order of $spin/mass$ (otherwise rotation with superluminal speeds would be necessary), so it is understandable that the tidal forces do affect it.

6.5 Bianchi identities

As a consequence of the Ricci (and the first Bianchi) identities, the Riemann tensor satisfies important relations called **second Bianchi identities**. Before long, we will need them in searching for the Einstein field equations.

Lemma The Riemann tensor satisfies the second Bianci identities

$$\boxed{R^{\mu}{}_{\nu\{\kappa\lambda;\rho\}} = 0}\,. \tag{6.30}$$

Proof:

- Let us apply the Ricci identities for a second-rank tensor (6.6), i.e.

$$W_{\mu\nu;\kappa\lambda} - W_{\mu\nu;\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda}W_{\sigma\nu} + R^{\sigma}{}_{\nu\kappa\lambda}W_{\mu\sigma}\,,$$

  to the tensor $W_{\mu\nu} \equiv V_{\nu;\mu}$, where V_{ν} is some covector:

$$V_{\nu;\mu\kappa\lambda} - V_{\nu;\mu\lambda\kappa} = R^{\sigma}{}_{\mu\kappa\lambda}V_{\nu;\sigma} + R^{\sigma}{}_{\nu\kappa\lambda}V_{\sigma;\mu}\,. \tag{6.31}$$

- On the other hand, take the Ricci identities for a covector V_{ν}, i.e.

$$V_{\nu;\kappa\lambda} - V_{\nu;\lambda\kappa} = R^{\sigma}{}_{\nu\kappa\lambda}V_{\sigma}\,,$$

  and differentiate them covariantly by x^{μ},

$$V_{\nu;\kappa\lambda\mu} - V_{\nu;\lambda\kappa\mu} = R^{\sigma}{}_{\nu\kappa\lambda;\mu}V_{\sigma} + R^{\sigma}{}_{\nu\kappa\lambda}V_{\sigma;\mu}\,. \tag{6.32}$$

Subtracting the two obtained equations, the last terms cancel out right away. Make a cyclic permutation in the indices (μ,κ,λ) of what remains: on the left-hand side, the terms sitting "above each other" cancel out easily, and on the right-hand side, the Riemann-tensor term from the upper equation drops out thanks to the first Bianchi identities $R^{\sigma}{}_{\{\mu\kappa\lambda\}} = 0$. Therefore, one arrives at

$$0 = R^{\sigma}{}_{\nu\{\kappa\lambda;\mu\}}V_{\sigma}\,.$$

This, however, are the second Bianchi identities, because the covector V_{σ} is arbitrary. □

How many independent relations the second Bianchi identities represent? First, due to the antisymmetry of Riemann in the second pair of indices, the cyclic permutation in $[\kappa\lambda;\mu]$ results in an expression totally anti-symmetric in all the three indices (similarly as $F_{\{\kappa\lambda;\mu\}}$ in the second set of Maxwell equations). Hence 4 independent configurations of these indices. In the first two indices of Riemann, there are 6 possibilities, due to antisymmetry again. This means $4{\cdot}6 = 24$. However, the remaining independent symmetry of Riemann (not used yet), the first Bianchi identities $R^{\sigma}{}_{\{\nu\kappa\lambda\}} = 0$, make some of the second Bianchi identities dependent. In order to see this, consider once more that the cyclic permutation in both the

Bianchi identities is equivalent to total anti-symmetrization in the enclosed indices (thanks to the anti-symmetry of Riemann in the second pair of indices),

$$0 = R^{\sigma}{}_{\nu\{\kappa\lambda;\mu\}} \equiv 3\, R^{\sigma}{}_{\nu[\kappa\lambda;\mu]}\ , \qquad 0 = R^{\sigma}{}_{\{\nu\kappa\lambda\}} \equiv 3\, R^{\sigma}{}_{[\nu\kappa\lambda]}\,.$$

Because of this, the second Bianchi identities satisfy, automatically, the relation

$$R^{\sigma}{}_{[\nu\kappa\lambda;\mu]} \equiv R^{\sigma}{}_{[\nu[\kappa\lambda;\mu]]} = 0\,,$$

which, however, already follows from the first Bianchi identities,

$$R^{\sigma}{}_{[\nu\kappa\lambda;\mu]} \equiv R^{\sigma}{}_{[[\nu\kappa\lambda];\mu]} = 0\,.$$

Hence, the number of these relations has to be subtracted (from 24). Since the expression $R^{\sigma}{}_{[\nu\kappa\lambda;\mu]}$ is totally anti-symmetric in the bottom four indices, it is only non-trivial if all are different, and since, in addition, the expression yields zero, there is only one independent way how to arrange the indices, e.g. [012;3]. Multiplied by 4 options for the upper index σ, we arrive at 4. Hence, the second Bianchi identities represent $24-4=20$ independent relations.

Bianchi identities, in combination with the symmetries of Riemann, have a number of consequences. Let us show one at least:

$\boxed{\text{Observation}}$ $\quad R_{\nu\lambda} = \Lambda g_{\nu\lambda} \quad (\Lambda = \text{constant}) \quad \Longrightarrow \quad R^{\sigma}{}_{\nu\lambda\mu;\sigma} = 0.$

<u>Proof</u>: Let's write the second Bianchi identities (6.30) in the form $R^{\sigma\nu}{}_{\{\kappa\lambda;\mu\}} = 0$, that is,

$$R^{\sigma\nu}{}_{\kappa\lambda;\mu} + R^{\sigma\nu}{}_{\mu\kappa;\lambda} + R^{\sigma\nu}{}_{\lambda\mu;\kappa} = 0\,.$$

By contraction in ${}^{\sigma}_{\kappa}$ one obtains

$$R^{\nu}{}_{\lambda;\mu} - R^{\nu}{}_{\mu;\lambda} + R^{\sigma\nu}{}_{\lambda\mu;\sigma} = 0\,, \qquad \text{i.e.} \qquad R^{\sigma\nu}{}_{\lambda\mu;\sigma} = 2R^{\nu}{}_{[\mu;\lambda]}\,. \tag{6.33}$$

Hence the assertion of the Observation. □

<u>Consequence</u>: contracting the above result once more – in ${}^{\nu}_{\lambda}$ – we get

$$R_{;\mu} - R^{\nu}{}_{\mu;\nu} - R^{\sigma}{}_{\mu;\sigma} = 0\,, \qquad \text{i.e.} \qquad R_{;\mu} = 2R^{\nu}{}_{\mu;\nu}\,. \tag{6.34}$$

This will be crucial in the field equations.

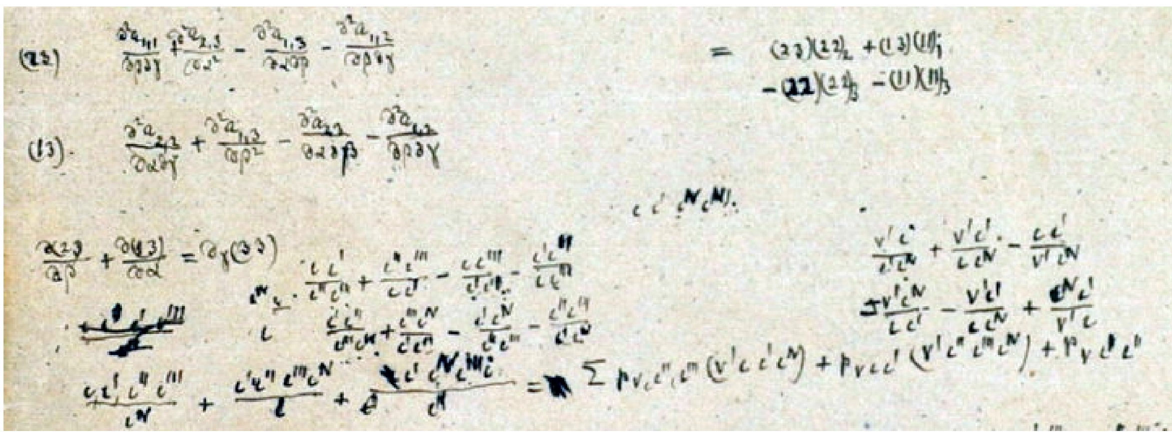

Figure 6.3 *By a careful analysis of Riemann's memoir from 1861 (on propagation of heat in solid bodies [sic!]), historians deciphered that he very probably knew of "the second Bianchi identities". (They were rediscovered by Ricci in 1889 and finally – by Bianchi – in 1902.) See O. Darrigol: The mystery of Riemann's curvature, Historia Mathematica 42 (2015) 47.*

CHAPTER 7

Energy-momentum tensor

And now for something completely different: a tensor describing energy (which nobody knows what it is) as a source of gravitation (which nobody knows what it is).

In Newton's theory, gravity is generated by mass density, $\Delta\Phi = 4\pi G\rho$. From special relativity we know that relative mass depends on relative speed, and that it is universally linked to energy, $E = mc^2$. The energy is the *total* energy – it includes, besides the kinetic part, also possible contributions from interactions, whether they apply to the body as a whole or to its constituents of any scale (see the story accompanying the equivalence principle). When seeking the description of sources for a new gravitational law, we thus need a quantity which would describe, covariantly and for any system including matter and/or non-gravitational fields, the mass-energy density. This is clearly not that straightforward, because the mass-energy itself is *not* invariant: it is contained in the four-momentum of the system, which already in special relativity transforms in such a way that the mass-energy (time component) mixes with momentum (spatial components). In addition, the above is still not the full story: besides fundamental interactions, the elements of the body are also subject to low-energy, effective, *"mechanical"* interaction, which is being quantified by the pressure/stress variable.

Regarding all the above, the quantity we are looking for has to include, besides the energy density, the momentum density (the energy-density flux) and the momentum-density flux (stresses). For any physical observer, one has to be able to compute from it the energy density locally measured by him/her. Such requirements are satisfied by **the energy-momentum tensor**, $T_{\mu\nu}$. (A thorough discussion of $T_{\mu\nu}$ can e.g. be found in chapter 5 of [29].)

We will first make generally covariant the treatment of the $T_{\mu\nu}$ of a charged dust coupled to (its) electromagnetic field, known from special relativity. Then we will generalise the incoherent-dust $T_{\mu\nu}$ to the case of an ideal fluid which is being employed as the first approximation for interiors of astrophysical bodies and for the "cosmic fluid" in cosmology.

Let us stress there is no abstract "proof" that $T_{\mu\nu}$ is the right description of gravitational sources. We will see later that it can be derived "canonically" from Lagrangian, but this means that the problem of the description of source is just shifted to the problem of the knowledge of Lagrangian. Similarly, in electrodynamics it is also not a priori clear that the current density

J^α is the right description of sources. Like elsewhere in physics, this can only be tested a posteriori, by i) specifying how the letters which appear in the equations should be measured, and by ii) actually measuring them in a significant number of situations.

7.1 Physical interpretation of the energy-momentum tensor

In special relativity, the physical content of $T_{\mu\nu}$ is revealed by its components in some (arbitrary) inertial system. In general relativity, coordinates do *not* have a direct geometrical – and thus physical – meaning, so one should take the components with respect to some really physical bases. For such a purpose serves the so-called tetrad formalism: one considers a suitable family of **physical observers**, i.e. a congruence of time-like world-lines (with a certain tangent four-velocity field $\hat{u}^\mu$), along which it is carried (= transported) a suitable field of orthonormal spatial bases $\{e^\mu_{\hat{\imath}}\}_{\hat{\imath}=1}^{3}$. If denoting $e^\mu_{\hat{0}} := u^\mu$, this is summarised by

$$g_{\mu\nu} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \eta_{\alpha\beta} \,, \qquad \eta^{\alpha\beta} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = g^{\mu\nu} \,. \tag{7.1}$$

The space-time indices are manipulated by $g_{\mu\nu}$, while the indices which number the tetrad vectors = “tetrad indices” (those with the hat) are manipulated by $\eta_{\alpha\beta}$.

If considering some material body or continuum which itself can be characterised by a certain four-velocity or four-velocity field (will be denoted by u^μ), it is reasonable, for interpretation purposes, to also introduce the relative velocity of that source with respect to the local observer, $\hat{v}^\mu$. This is defined by decomposition

$$u^\mu = \hat{\gamma}(\hat{u}^\mu + \hat{v}^\mu) \,, \tag{7.2}$$

where $\hat{v}^\mu$ lies in the instantaneous three-space of the observer ($\hat{u}_\mu \hat{v}^\mu = 0$). Multiplying the decomposition by $\hat{u}_\mu$ and by u_μ , we thus obtain for the relative Lorentz factor, respectively,

$$\hat{u}_\mu u^\mu = -\hat{\gamma} \qquad \text{and} \qquad -1 = \hat{\gamma}\left(-\hat{\gamma} + \hat{\gamma}\hat{v}_\mu \hat{v}^\mu\right) \,,$$

so, in a summary,

$$\hat{\gamma} = -\hat{u}_\mu u^\mu = \frac{1}{\sqrt{1 - \hat{v}_\mu \hat{v}^\mu}} \,. \tag{7.3}$$

Also, multiplying the decomposition by $e^{\hat{\imath}}_\mu$, one has the spatial-triad components of u^μ,

$$u^{\hat{\imath}} \equiv e^{\hat{\imath}}_\mu u^\mu = \hat{\gamma}\hat{v}^{\hat{\imath}} \,,$$

and by projection of the decomposition into the observer’s three-space, one has

$$(g_{\alpha\mu} + \hat{u}_\alpha \hat{u}_\mu)\, u^\mu = u_\alpha - \hat{\gamma}\hat{u}_\alpha = \hat{\gamma}\hat{v}_\alpha \,.$$

Needless to say, if the interpretation congruence $\hat{u}^\mu$ locally follows the motion of the studied body, $u^\mu = \hat{u}^\mu$, then $\hat{v}^\mu = 0$ and $\hat{\gamma} = 1$.

7.2 Charged incoherent dust

Consider an electrically charged continuum without the internal mechanical interaction (without pressure), i.e. such whose "particles" only interact with each other gravitationally and electromagnetically – a charged **incoherent dust**. Each of the particles thus moves according to the equation

$$\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = m\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = qF^\mu{}_\nu u^\nu =: F^\mu_{\mathrm{L}}\,,$$

where m and q are the rest mass and charge of the particle.[1] For a continuous environment, one takes a proper density of the equation (considering of course that the four-acceleration is not an extensive quantity),[2]

$$\rho\,\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = \Phi^\mu_{\mathrm{L}}\,, \qquad \text{where} \qquad \rho := \frac{\mathrm{d}m}{\mathrm{d}V}\,, \quad \Phi^\mu_{\mathrm{L}} := \frac{\mathrm{D}F^\mu_{\mathrm{L}}}{\mathrm{d}V}\,, \quad V = \text{ proper volume}\,. \tag{7.4}$$

On the left-hand side, we can write

$$\rho\,\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = \rho u^\mu{}_{;\nu}u^\nu = (\rho\,u^\mu u^\nu)_{;\nu} - u^\mu(\rho\,u^\nu)_{;\nu} = (\rho\,u^\mu u^\nu)_{;\nu}\,,$$

where the second term has dropped out due to the conservation of rest mass, expressed by the continuity equation

$$(\rho\,u^\nu)_{;\nu} = 0$$

(counterpart of the continuity equation from electrodynamics, there holding for the *electric-current density*). Hence, introducing the *energy-momentum tensor*

$$T^{\mu\nu}_{\mathrm{dust}} := \rho\,u^\mu u^\nu\,, \tag{7.5}$$

we may write the equation of motion in the form

$$(T^{\mu\nu}_{\mathrm{dust}})_{;\nu} = \Phi^\mu_{\mathrm{L}}\,. \tag{7.6}$$

---

[1] Recall that the Lorentz force leaves the rest mass constant, because it is orthogonal to four-velocity,

$$0 = qF^{\mu\nu}u_\nu u_\mu = F^\mu_{\mathrm{L}}u_\mu = \frac{\mathrm{D}p^\mu}{\mathrm{d}\tau}\,u_\mu = \frac{\mathrm{D}(mu^\mu)}{\mathrm{d}\tau}\,u_\mu = \frac{\mathrm{d}m}{\mathrm{d}\tau}\,u^\mu u_\mu + m\,\cancel{\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}\,u_\mu} = -\frac{\mathrm{d}m}{\mathrm{d}\tau}$$

due to the totally general property

$$u^\mu u_\mu = -1 \qquad \Longrightarrow \qquad \frac{\mathrm{D}}{\mathrm{d}\tau}(u^\mu u_\mu) = 2\,\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}\,u_\mu \equiv 2\,a^\mu u_\mu = 0\,.$$

[2] The proper density of force is denoted by Φ, unfortunately - hope there will be no confusion with the Newton's potential.

7.2.1 Physical meaning of the dusty T-mu-nu

In order to reveal the physical content of the $T_{\mu\nu}$ we have just introduced, let us project the tensor onto some observer tetrad we described in Section (7.1). Denoting again by hats the components locally measured by the observer, we arrive at

$$T^{\hat{0}\hat{0}}_{\rm dust} \equiv T^{\mu\nu}_{\rm dust}\hat{u}_\mu\hat{u}_\nu = \rho\, u^\mu u^\nu \hat{u}_\mu\hat{u}_\nu = \rho\,\hat{\gamma}^2\,, \tag{7.7}$$

$$T^{\hat{0}\hat{\jmath}}_{\rm dust} \equiv -T^{\mu\nu}_{\rm dust}\hat{u}_\mu e^{\hat{\jmath}}_\nu = -\rho\, u^\mu u^\nu \hat{u}_\mu e^{\hat{\jmath}}_\nu = \rho\,\hat{\gamma}^2\hat{v}^{\hat{\jmath}}\,, \tag{7.8}$$

$$T^{\hat{\imath}\hat{\jmath}}_{\rm dust} \equiv T^{\mu\nu}_{\rm dust}e^{\hat{\imath}}_\mu e^{\hat{\jmath}}_\nu = \rho\, u^\mu u^\nu e^{\hat{\imath}}_\mu e^{\hat{\jmath}}_\nu = \rho\,\hat{\gamma}^2\hat{v}^{\hat{\imath}}\hat{v}^{\hat{\jmath}}\,, \tag{7.9}$$

which are, respectively,[3]

- $(\hat{0}\hat{0})$ ... mass-energy density (one $\hat{\gamma}$ is for *non*-rest mass, the other is for *non*-proper volume in the measured density)
- $(\hat{0}\hat{\jmath})$ ... energy-density flux, or momentum density
- $(\hat{\imath}\hat{\jmath})$ ... momentum-density flux, specifically, the flux of the $p^{\hat{\imath}}$ component of momentum in the $\mathbf{e}^{\hat{\jmath}}$ direction (in other words, these are the observer-measured components of pressure/stress, because momentum flux = momentum over time = force, and the density of force corresponds to pressure).

Naturally, all the above quantities (mass-energy, momentum, densities and fluxes) are *relative* in the sense that they depend on the observer. In the rest frame of the dust, in particular, $\hat{v}^\mu = 0$ and $\hat{\gamma} = 1$, so the energy-momentum tensor reduces just to $T^{\hat{0}\hat{0}}_{\rm dust} = \rho$.

7.3 Electromagnetic field

On the right-hand side of the equation $(T^{\mu\nu}_{\rm dust})_{;\nu} = \Phi^\mu_{\rm L}$ stands the external force (its density), which indicates that the system under consideration is not closed – it does not only interact with itself (the agent of the external force is not included). Sure, in

$$\Phi^\mu_{\rm L} \equiv \frac{\mathrm{d}F^\mu_{\rm L}}{\mathrm{d}V} = \frac{\mathrm{d}}{\mathrm{d}V}(qF^{\mu\nu}u_\nu) = \frac{\mathrm{d}q}{\mathrm{d}V}\,F^{\mu\nu}u_\nu = F^{\mu\nu}J_\nu$$

(with J^μ the electric-current density) appears the electromagnetic field $F^{\mu\nu}$, about which nothing has been said, though it certainly bears some energy and momentum as well. Physically the point is clear: the charged dust necessarily generates an EM field, but this has not yet been included in the description. In order to obtain a *self-consistent*, closed system, one has to add an EM field – or, more precisely, exactly *the* EM field generated by the dust and, at the same time, producing such a Lorentz force which drives the dust in just the "correct" motion.

[3] In $T^{\hat{0}\hat{\jmath}}_{\rm dust}$, the minus sign is indeed correct, namely, $e^{\hat{0}}_\mu = \eta^{0\beta}\big(e_{\hat{\beta}}\big)_\mu = \eta^{00}\big(e_{\hat{0}}\big)_\mu \equiv \eta^{00}\hat{u}_\mu = -\hat{u}_\mu$. Actually, in order to extract the time component of some quantity with respect to an observer $\hat{u}^\mu$, it is necessary to project it on $(-\hat{u}_\mu)$ – see e.g. the relative energy of some particle $\hat{E} \equiv p^{\hat{0}} = p^\mu e^{\hat{0}}_\mu = -p^\mu\hat{u}_\mu$.

In this part, we will proceed in an opposite way – for the EM field, we will bring from heaven a certain $T^{\mu\nu}$, about which we will show that it has the same physical meaning as that found for the dust, and that it also satisfies similar conservation laws. The tensor reads

$$T^{\mu\nu}_{\rm EM} = \frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\,g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{8\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} + {}^*\!F^{\mu\sigma}\,{}^*\!F^{\nu}{}_{\sigma}\right), \tag{7.10}$$

the second form including the electromagnetic-bivector dual

$${}^*\!F^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,,$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor as introduced in Appendix A.

Notice an important feature of the above tensor: it is traceless,

$$T_{\rm EM} := g_{\mu\nu}T^{\mu\nu}_{\rm EM} = \frac{1}{4\pi}\left(F^{\mu\sigma}F_{\mu\sigma} - \frac{1}{4}\,\delta^{\mu}_{\mu}F^{\rho\sigma}F_{\rho\sigma}\right) = 0\,.$$

7.3.1 Physical meaning of the EM-field T-mu-nu

The physical meaning of the tensors $F^{\mu\nu}$ and $T^{\mu\nu}_{\rm EM}$ again follow by computing their time and spatial components with respect to an orthonormal tetrad tied to some physical observer $\hat{u}^{\mu}$. First, the electric and magnetic fields with respect to that observer are given, covariantly, by

$$\hat{E}_{\mu} := F_{\mu\nu}\hat{u}^{\nu}\,, \qquad \hat{B}_{\mu} := -{}^*\!F_{\mu\nu}\hat{u}^{\nu}\,, \tag{7.11}$$

which corresponds to the reciprocal relations

$$F_{\mu\nu} = \hat{u}_{\mu}\hat{E}_{\nu} - \hat{E}_{\mu}\hat{u}_{\nu} + \epsilon_{\mu\nu\rho\sigma}\hat{u}^{\rho}\hat{B}^{\sigma}\,, \qquad {}^*\!F_{\mu\nu} = \hat{B}_{\mu}\hat{u}_{\nu} - \hat{u}_{\mu}\hat{B}_{\nu} + \epsilon_{\mu\nu\rho\sigma}\hat{u}^{\rho}\hat{E}^{\sigma}\,. \tag{7.12}$$

Let us use the above in the first form of (7.10), remembering that $\hat{E}_{\sigma}\hat{u}^{\sigma}=0$ and $\hat{B}_{\sigma}\hat{u}^{\sigma}=0$ by definition:

$$\begin{aligned}
F^{\mu\sigma}F^{\nu}{}_{\sigma} &= \left(\hat{u}^{\mu}\hat{E}^{\sigma} - \hat{E}^{\mu}\hat{u}^{\sigma} + \epsilon^{\mu\sigma\alpha\beta}\hat{u}_{\alpha}\hat{B}_{\beta}\right)\left(\hat{u}^{\nu}\hat{E}_{\sigma} - \hat{E}^{\nu}\hat{u}_{\sigma} + \epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat{u}^{\kappa}\hat{B}^{\lambda}\right) = \\
&= \hat{u}^{\mu}\hat{u}^{\nu}\hat{E}^{2} - \hat{E}^{\mu}\hat{E}^{\nu} + \hat{u}^{\mu}\epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat{E}^{\sigma}\hat{u}^{\kappa}\hat{B}^{\lambda} + \hat{u}^{\nu}\epsilon^{\mu\sigma\alpha\beta}\hat{E}_{\sigma}\hat{u}_{\alpha}\hat{B}_{\beta} + \epsilon^{\mu\sigma\alpha\beta}\epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat{u}_{\alpha}\hat{B}_{\beta}\hat{u}^{\kappa}\hat{B}^{\lambda} = \\
&= \hat{u}^{\mu}\hat{u}^{\nu}\hat{E}^{2} - \hat{E}^{\mu}\hat{E}^{\nu} + \hat{u}^{\mu}\big(\vec{E}\times\vec{B}\big)^{\nu} + \hat{u}^{\nu}\big(\vec{E}\times\vec{B}\big)^{\mu} + g^{\mu\nu}\hat{B}^{2} + \hat{u}^{\mu}\hat{u}^{\nu}\hat{B}^{2} - \hat{B}^{\mu}\hat{B}^{\nu}\,,
\end{aligned} \tag{7.13}$$

where we used a three-vector notation

$$\big(\vec{E}\times\vec{B}\big)^{\mu} := \epsilon^{\mu\sigma\alpha\beta}\hat{E}_{\sigma}\hat{u}_{\alpha}\hat{B}_{\beta}$$

for the vector product in the observer's three-space, and where we employed the product formula (A.5),

$$\begin{aligned}
&\epsilon^{\mu\sigma\alpha\beta}\epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat{u}_{\alpha}\hat{B}_{\beta}\hat{u}^{\kappa}\hat{B}^{\lambda} = \\
&= \left(-g^{\mu\nu}\delta^{\alpha}_{\kappa}\delta^{\beta}_{\lambda} - g^{\beta\nu}\delta^{\mu}_{\kappa}\delta^{\alpha}_{\lambda} - g^{\alpha\nu}\delta^{\beta}_{\kappa}\delta^{\mu}_{\lambda} + g^{\mu\nu}\delta^{\beta}_{\kappa}\delta^{\alpha}_{\lambda} + g^{\alpha\nu}\delta^{\mu}_{\kappa}\delta^{\beta}_{\lambda} + g^{\beta\nu}\delta^{\alpha}_{\kappa}\delta^{\mu}_{\lambda}\right)\hat{u}_{\alpha}\hat{B}_{\beta}\hat{u}^{\kappa}\hat{B}^{\lambda} =
\end{aligned}$$

$$= g^{\mu\nu}\hat{B}^2 + \hat{u}^\mu\hat{u}^\nu\hat{B}^2 - \hat{B}^\mu\hat{B}^\nu\,.$$

By contraction of (7.13), one easily finds the invariant

$$F^{\rho\sigma}F_{\rho\sigma} = 2\hat{B}^2 - 2\hat{E}^2\,, \tag{7.14}$$

so in total the $T^{\mu\nu}_{\rm EM}$ comes out as

$$T^{\mu\nu}_{\rm EM} = -\frac{1}{4\pi}\big(\hat{E}^\mu\hat{E}^\nu + \hat{B}^\mu\hat{B}^\nu\big) + \hat{u}^\mu\hat{S}^\nu + \hat{S}^\mu\hat{u}^\nu + (g^{\mu\nu} + 2\hat{u}^\mu\hat{u}^\nu)\,\hat{w}\,, \tag{7.15}$$

where

$$\hat{S}^\mu := \frac{1}{4\pi}\big(\vec{E}\times\vec{B}\big)^\mu \qquad \text{is the Poynting vector}\,, \tag{7.16}$$

$$\hat{w} := \frac{1}{8\pi}(\hat{E}^2 + \hat{B}^2) \qquad \text{is the EM energy density}\,. \tag{7.17}$$

Performing now with $T^{\mu\nu}_{\rm EM}$ the same projections as we did with $T^{\mu\nu}_{\rm dust}$ in Section (7.2.1), we have

- $T^{\hat{0}\hat{0}}_{\rm EM}$ is the energy density of the EM field in the observer's system,

  $$T^{\hat{0}\hat{0}}_{\rm EM} = T^{\mu\nu}_{\rm EM}\hat{u}_\mu\hat{u}_\nu = \hat{w}\,. \tag{7.18}$$

  For those more used to the SI units, let us add that in these (μ is the permeability of a vacuum)

  $$T^{\mu\nu}_{\rm EM} = \frac{1}{2\mu}\left(F^{\mu\sigma}F^\nu{}_\sigma + {}^*F^{\mu\sigma}{}^*F^\nu{}_\sigma\right)\,,$$

  hence

  $$T^{\hat{0}\hat{0}}_{\rm EM} = \frac{1}{2\mu}\left(\frac{\hat{E}^2}{c^2} + \hat{B}^2\right) = \frac{1}{2}(\hat{E}^\sigma\hat{D}_\sigma + \hat{H}^\sigma\hat{B}_\sigma) = \hat{w}\,.$$

- In the cross components, only the first of the Poynting terms survives,

  $$T^{\hat{0}\hat{\jmath}}_{\rm EM} = -T^{\mu\nu}_{\rm EM}\hat{u}_\mu e^{\hat{\jmath}}_\nu = \hat{S}^\nu e^{\hat{\jmath}}_\nu = \hat{S}^{\hat{\jmath}}\,. \tag{7.19}$$

  This represents the EM-field energy-density flux in the observer's system.

- The spatial part of $T^{\mu\nu}_{\rm EM}$ is also found easily from (7.15),

  $$T^{\hat{\imath}\hat{\jmath}}_{\rm EM} = T^{\mu\nu}_{\rm EM}e^{\hat{\imath}}_\mu e^{\hat{\jmath}}_\nu = -\frac{1}{4\pi}\left(\hat{E}^{\hat{\imath}}\hat{E}^{\hat{\jmath}} + \hat{B}^{\hat{\imath}}\hat{B}^{\hat{\jmath}}\right) + \delta^{ij}\hat{w}\,. \tag{7.20}$$

  This 3D tensor is known as the Maxwell's stress tensor; it represents the EM momentum-density flux in the observer's frame.

We have thus shown that the locally measured components of $T^{\mu\nu}_{\rm EM}$ are, by physical content, exact counter-parts of the components of $T^{\mu\nu}_{\rm dust}$.

Before continuing, let us also add, to the above $F{\cdot}F$ matrix (7.13) and its trace (the FF invariant), also a similar computation for the ${}^*F\cdot F$ matrix and the ${}^*FF$ invariant. Finding, analogously as for $\epsilon^{\mu\sigma\alpha\beta}\epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat u_\alpha\hat B_\beta\hat u^\kappa\hat B^\lambda$, that

$$\epsilon^{\mu\sigma\alpha\beta}\epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat u_\alpha\hat E_\beta\hat u^\kappa\hat B^\lambda = g^{\mu\nu}\hat E_\beta\hat B^\beta + \hat u^\mu\hat u^\nu\hat E_\beta\hat B^\beta - \hat E^\mu\hat B^\nu\,,$$

we obtain

$$\begin{aligned}{}^*F^{\mu\sigma}F^{\nu}{}_{\sigma} &= \left(\hat B^\mu\hat u^\sigma - \hat u^\mu\hat B^\sigma + \epsilon^{\mu\sigma\alpha\beta}\hat u_\alpha\hat E_\beta\right)\left(\hat u^\nu\hat E_\sigma - \hat E^\nu\hat u_\sigma + \epsilon^{\nu}{}_{\sigma\kappa\lambda}\hat u^\kappa\hat B^\lambda\right) = \\ &= \hat B^\mu\hat E^\nu - \cancel{\hat u^\mu\hat u^\nu\hat B^\sigma\hat E_\sigma} + g^{\mu\nu}\hat E_\beta\hat B^\beta + \cancel{\hat u^\mu\hat u^\nu\hat E_\beta\hat B^\beta} - \hat E^\mu\hat B^\nu\,,\end{aligned} \tag{7.21}$$

hence

$${}^*F^{\rho\sigma}F_{\rho\sigma} = 4\,\vec E\cdot\vec B\,. \tag{7.22}$$

7.3.2 Conservation laws for EM field

Let us calculate the divergence of $T^{\mu\nu}_{\rm EM}$ now,

$$\begin{aligned}4\pi(T^{\mu\nu}_{\rm EM})_{;\nu} &= F^{\mu\sigma}{}_{;\nu}F^{\nu}{}_{\sigma} + F^{\mu\sigma}F^{\nu}{}_{\sigma;\nu} - \frac12\,g^{\mu\nu}F^{\rho\sigma}{}_{;\nu}F_{\rho\sigma} = \\ &= F^{\mu\sigma;\nu}F_{\nu\sigma} + F^{\mu}{}_{\sigma}F^{\nu\sigma}{}_{;\nu} - \frac12 F^{\rho\sigma;\mu}F_{\rho\sigma} = \\ &= F^{\mu[\sigma;\nu]}F_{\nu\sigma} - F^{\mu}{}_{\sigma}F^{\sigma\nu}{}_{;\nu} - \frac12 F^{\nu\sigma;\mu}F_{\nu\sigma} = \\ &= -F^{\mu}{}_{\sigma}\,4\pi J^\sigma + \frac12\left(F^{\mu\sigma;\nu} - F^{\mu\nu;\sigma} - F^{\nu\sigma;\mu}\right)F_{\nu\sigma} = \\ &= -4\pi\Phi^\mu_{\rm L} - 3F^{[\mu\nu;\sigma]}F_{\nu\sigma} = -4\pi\Phi^\mu_{\rm L}\,;\end{aligned} \tag{7.23}$$

in the last-but-one row we used the first set of Maxwell equations, $F^{\sigma\nu}{}_{;\nu} = 4\pi J^\sigma$, and in the last row we used their second set, $F^{[\mu\nu;\sigma]} = 0$. And remember that $F^{\{\mu\nu;\sigma\}} \equiv 3\,F^{[\mu\nu;\sigma]}$.

May be useful to write down the above equations explicitly, in order to see that they really represent **conservation laws** (for energy and momentum of the EM field). Projecting the Lorentz-force density on the right-hand side,

$$\begin{aligned}\Phi^{\hat 0}_{\rm L} &\equiv \Phi^\mu_{\rm L}e^{\hat 0}_\mu = -\Phi^\mu_{\rm L}\hat u_\mu = -F^{\mu}{}_{\nu}J^\nu\hat u_\mu = \hat E_\nu J^\nu \;\left(\equiv \vec E\cdot\vec J\right),\\ \Phi^{\hat\imath}_{\rm L} &= \Phi^\mu_{\rm L}e^{\hat\imath}_\mu = F^{\mu}{}_{\nu}J^\nu e^{\hat\imath}_\mu = -\hat u_\nu\hat E^\mu J^\nu e^{\hat\imath}_\mu + \epsilon^{\mu}{}_{\nu\rho\sigma}\hat u^\rho\hat B^\sigma J^\nu e^{\hat\imath}_\mu \equiv J^{\hat 0}\hat E^{\hat\imath} + \left(\vec J\times\vec B\right)^{\hat\imath},\end{aligned}$$

and substituting for the locally measured components of $T^{\mu\nu}$, we obtain

$$(T^{\hat 0\hat\nu}_{\rm EM})_{;\hat\nu} = -\Phi^{\hat 0}_{\rm L} \qquad\Longleftrightarrow\qquad \frac{\partial\hat w}{\partial\hat\tau} + \vec\nabla\cdot\vec S = -\vec E\cdot\vec J\,, \tag{7.24}$$

$$(T^{\hat\imath\hat\nu}_{\rm EM})_{;\hat\nu} = -\Phi^{\hat\imath}_{\rm L} \qquad\Longleftrightarrow\qquad \frac{\partial\vec S}{\partial\hat\tau} + \vec\nabla\cdot\overleftrightarrow{T}_{\rm EM} = -\vec\Phi_{\rm L}\,, \tag{7.25}$$

where $\hat\tau$ is the reference-observer proper time.

7.3.3 Self-consistent system of charged dust & its EM field

Hitherto, the charged dust and the EM field need not in fact be however coupled to each other, they may have been just two different physical systems. Consider now that they *are* coupled in the tightest possible way (that they are "self-consistent") – that

- the dust is exactly the source of the EM field (as fixed by Maxwell equations), so it generates J_ν which stands in $-\Phi^\mu_{\rm L} = -F^{\mu\nu}J_\nu$ on the right-hand side of (7.23)
- the EM field in turn determines the motion of the dust elements (according to the Lorentz equation of motion), so it provides $F^{\mu\nu}$ for $\Phi^\mu_{\rm L} = F^{\mu\nu}J_\nu$ on the right-hand side of (7.6).

In short, assume the dust generates the field in which it in turn moves. It is clear from the above two equations that such a complete system can be described by the tensor

$$T^{\mu\nu} := T^{\mu\nu}_{\rm dust} + T^{\mu\nu}_{\rm EM}$$

which satisfies simple equations

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0}\,. \tag{7.26}$$

This tensor can be regarded as the energy-momentum tensor of the self-consistent, closed system of charged dust and *its* EM field.

7.4 Ideal fluid

Firstly, an ideal fluid is a *continuum* concept, so it assumes that the mean free path of fluid's particles (molecules) is smaller than the "infinitesimal" length of the macroscopic-approach averaging. Secondly, *ideal* means that the fluid is incompressible and that one can neglect the issue of heat and its transport. This in turn requires that there is no dissipation of kinetic energy into heat within the fluid, which is fulfilled if there is no internal friction between the fluid's elements (forces are purely normal to their surfaces), i.e. if the fluid does not at all resist to shear flow. An effective term for the deviation from this last property is *viscosity*. Therefore, ideal fluid is an **incompressible non-viscous fluid**.

Practically, ideal fluid is a continuum which is characterised, in its rest frame, by two quantities – proper density ρ and proper pressure P (fully represented by a scalar function since it has to be isotropic). In short, it is a dust endowed with isotropic pressure. Specifically, isotropy should hold in the fluid's instantaneous three-space, so the pressure has to enter the $T^{\mu\nu}$ as $Ph^{\mu\nu}$, where $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the metric of that three-space (orthogonal to u^μ), as we already well know. We thus have

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + Ph^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}, \tag{7.27}$$

where u^μ is the four-velocity of the fluid (a tangent field to the world-lines of its macroscopic elements).

7.4.1 Physical meaning of the ideal-fluid T-mu-nu

As in the case of dust and of the EM field, let us check the components of the above $T^{\mu\nu}$ measured by some physical observer (with four-velocity $\hat{u}^\mu$ and an orthonormal spatial basis $e^\mu_{\hat{\imath}}$):

$$T^{\hat{0}\hat{0}} = (\rho+P)u^\mu\hat{u}_\mu u^\nu\hat{u}_\nu + Pg^{\mu\nu}\hat{u}_\mu\hat{u}_\nu = (\rho+P)\hat{\gamma}^2 - P = \hat{\gamma}^2(\rho+\hat{v}^2P)\,, \tag{7.28}$$

$$T^{\hat{0}\hat{\jmath}} = -(\rho+P)u^\mu\hat{u}_\mu u^\nu e^{\hat{\jmath}}_\nu - Pg^{\mu\nu}\hat{u}_\mu e^{\hat{\jmath}}_\nu = (\rho+P)\hat{\gamma}^2\hat{v}^{\hat{\jmath}}\,, \tag{7.29}$$

$$T^{\hat{\imath}\hat{\jmath}} = (\rho+P)u^\mu e^{\hat{\imath}}_\mu u^\nu e^{\hat{\jmath}}_\nu + Pg^{\mu\nu}e^{\hat{\imath}}_\mu e^{\hat{\jmath}}_\nu = (\rho+P)\hat{\gamma}^2\hat{v}^{\hat{\imath}}\hat{v}^{\hat{\jmath}} + P\delta^{\hat{\imath}\hat{\jmath}}\,. \tag{7.30}$$

In the fluid's rest frame (where $\hat{v}^{\hat{\imath}}=0$, $\hat{\gamma}=1$) the tensor naturally reduces to the diagonal matrix $\mathrm{diag}(\rho,P,P,P)$.

7.4.2 Conservation laws and Euler equations of motion for ideal fluid

For an incoherent dust we showed that its equation of motion can be rewritten in terms of the energy-momentum tensor. Here, on the contrary, we show – for the more general case of ideal fluid – how the conservation laws $T^{\mu\nu}{}_{;\nu}=0$ imply the equations of motion. The covariant divergence is easy (remember that $g_{\mu\nu;\alpha}=0$),

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= (\rho+P)_{,\nu}u^\mu u^\nu + (\rho+P)u^\mu{}_{;\nu}u^\nu + (\rho+P)u^\mu u^\nu{}_{;\nu} + P_{,\nu}g^{\mu\nu} = \\ &= \frac{\mathrm{d}\rho}{\mathrm{d}\tau}\,u^\mu + P_{,\nu}h^{\mu\nu} + (\rho+P)a^\mu + (\rho+P)u^\mu u^\nu{}_{;\nu}\,, \end{aligned} \tag{7.31}$$

where $a^\mu \equiv u^\mu{}_{;\nu}u^\nu = \mathrm{D}u^\mu/\mathrm{d}\tau$ is the acceleration of the fluid. Clearly, the first and the last terms are proportional to u^μ, whereas the second and the third terms are normal to u^μ, so we can split the equation into these two independent (orthogonal) components:

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + (\rho+P)u^\nu{}_{;\nu} = 0 \qquad \text{... equation of continuity}}\,, \tag{7.32}$$

$$\boxed{(\rho+P)a^\mu = -P_{,\nu}h^{\mu\nu} \qquad \text{... Euler equations of motion}}\,. \tag{7.33}$$

Classical forms of both equations for comparison:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\mathrm{div}\,\vec{v} = 0\,, \tag{7.34}$$

$$\rho\,\vec{a} = -\rho\,\mathrm{grad}\Phi - \mathrm{grad}P\,. \tag{7.35}$$

The Euler equations fully determine the motion of the fluid subject to a given (in fact its own) gravitational and pressure-gradient fields. Remember that the gravitational force which of course appears "explicitly" in the Newtonian equation of motion is in the relativistic equation "hidden" in the covariant derivative employed in the definition of acceleration.

Note the term $(\rho+P)$ which came from $T^{\mu\nu}$ and which is new in the relativistic equations: since what stands in front of acceleration in the Euler equations represents – as in every equation of motion – the inertial mass (more accurately, the proper density of inertial mass here), it means that *in relativity, pressure contributes to inertial mass*. A ball with positive internal pressure resists acceleration more than a ball without pressure. No surprise, right? Pressure describes mechanical interaction between the fluid particles – and the energy of this interaction necessarily contributes to the fluid's mass. Recalling the discussion about the equivalence principle, this must also mean that *in general relativity, pressure contributes to gravitational mass* – a ball with (positive) pressure should generate stronger gravitational field than the one without pressure. We will confirm this in chapter on stellar equilibria. Let us add that in standard units the inertial term reads $(\rho + P/c^2)$, so the pressure contribution is practically negligible in most situations.

Important remark: When the pressure gradient vanishes, $P_{,\nu} = 0$, the Euler equations imply that the fluid motion is geodesic, $a^\mu = 0$. Well, again no surprise... But it will later be important in understanding Einstein's equations.

Condition for hydrostatic equilibrium

Though an equation of motion primarily determines what motion results from given forces, it may also be used in the opposite way – to find conditions which have to hold in order that a *given* type of motion can happen. In particular, if one requires that the studied system be *at rest* in some sense, one speaks of *equilibrium conditions*. Let us check what conditions have to hold for an ideal fluid to stay at *static equilibrium*. Staticity means that there exists a coordinate system in which a given source, as well as space-time around, are static.

So suppose there are coordinates in which the fluid is not moving, i.e. having four-velocity

$$u^\mu = (u^0, 0, 0, 0), \qquad \text{where} \quad u^0 = \frac{1}{\sqrt{-g_{00}}} \quad (\text{from normalization})\,,$$

and in which the metric is static, i.e. satisfying

$$g_{\mu\nu,0} = 0 \quad \text{and} \quad g_{0i} = 0, \;\; g^{0i} = 0 \quad \left(\Rightarrow \;\; g^{00} = \frac{1}{g_{00}} \quad \text{since} \quad g^{0\iota}g_{\iota 0} = g^{00}g_{00} = \delta^0_0 = 1\right).$$

In such a case, the covariant components of four-velocity are equally simple,

$$u_0 = g_{0\sigma}u^\sigma = g_{00}u^0 = -\sqrt{-g_{00}}\,, \qquad u_i = g_{i\sigma}u^\sigma = g_{i0}u^0 = 0\,.$$

Now we put the above into the Euler equations (7.33). Slightly more comfortable is to evaluate them in a covariant version, $(\rho + P)a_\alpha = -P_{,\nu}h^\nu_\alpha$:

- Left-hand side, without $(\rho + P)$:

$$a_\alpha \equiv u_{\alpha;\kappa}u^\kappa \equiv u_{\alpha,\kappa}u^\kappa - \Gamma^\mu{}_{\kappa\alpha}u^\kappa u_\mu = \cancel{u_{\alpha,0}}\,u^0 - \Gamma_{00\alpha}(u^0)^2 = \frac{1}{2}\frac{(-g_{00,\alpha})}{(-g_{00})} =$$
$$= \frac{1}{2}\left[\ln(-g_{00})\right]_{,\alpha} = \left(\ln\sqrt{-g_{00}}\right)_{,\alpha}\,. \tag{7.36}$$

(Remember that g_{00} is negative, typically, so it is right to keep minus at it.)

- Right-hand side:

$$-P_{,\nu}(\delta^\nu_\alpha + u^\nu u_\alpha) = -P_{,\nu}(\delta^\nu_\alpha - \delta^\nu_0 \delta^0_\alpha)\,.$$

For $\alpha = 0$ both sides are zero, while for $\alpha = i$ the right-hand side reduces to $-P_{,i}$, so the **condition for hydrostatic equilibrium** reads

$$(\rho + P)\,(\ln\sqrt{-g_{00}})_{,i} = -P_{,i}\ . \tag{7.37}$$

Still more illustrative form follows by writing g_{00} as $g_{00} = -e^{2\Phi}$; the latter is often advantageous, namely in highly symmetric problems – apparently because Φ represents a counterpart of the Newtonian potential (recall the Newtonian limit $g_{00} = -1-2\Phi$ which really follows from there for small Φ). With the above parametrization, the equilibrium condition assumes the form

$$(\rho + P)\Phi_{,i} = -P_{,i} \tag{7.38}$$

which clearly reveals that the condition simply requires an equilibrium between the gravitational and the pressure-gradient forces. Note in passing that here $(\rho + P)$ already represents the density of *gravitational* mass.

7.5 Null fluid and null dust

Worth to mention are two limits which are often useful in astrophysics – the "isotropic photon gas", either with pressure or without. The case with pressure (**null fluid**) can be obtained from the $T^{\mu\nu}$ of a fluid. However, photons are the EM field actually, so they should also be described by the EM-field $T^{\mu\nu}$. The latter's main property is the vanishing trace, so let us require this for the fluid: one obtains the condition $\rho = 3P$. The case without pressure (**null dust**) is simply obtained from the incoherent-dust $T^{\mu\nu}$ by considering a null vector (k^μ) instead of the time-like u^μ, i.e. $T^{\mu\nu} = \rho\, k^\mu k^\nu$.

7.6 Notice: essential entanglement of sources and geometry

At the beginning of this chapter, we quoted *"And now for something completely different"* and announced a tensor describing energy as a source of gravitation. Indeed, the charged dust and (its) EM field, as well as the ideal fluid, have been supposed to *affect the geometry of space-time*. So it is *not* a different story, right? It is even more entwined than would seem comfortable. How to specify *where the source is and how it moves*? How to determine its density? How to compute (covariant) derivatives, inner products (how to ensure four-velocity normalization), ...? One needs *metric*! The energy-momentum tensors for EM field and for fluid even contain the metric explicitly. As nicely expressed in [38]: "But here we run into a problem, namely how to treat the sources. For these have no choice but to live in the *curved* spacetime which they at least partly generate. ... we need the sources before we can solve for the spacetime, but we need the spacetime before we can even properly *describe* the sources."

This problem is of course not new – at least in non-linear theories, one may have bad times trying to link the sources and their (own) fields in a self-consistent way. But in GR, "field" means geometry, and geometry practically means "world as such". It means to know how to handle mathematical quantities, and how to interpret observations. In GR, the problem is not only with consistency, but already in *description*: without knowing the "field" (the metric), one is not able to say when, where and how much. Hence, the gravitational law of GR cannot be expected to determine what geometry is generated by *prescribed* sources, and neither vice versa. Instead, we may only expect equations that provide *relations* between the sources and the geometry while leaving a certain freedom on both sides.

7.7 Energy conditions

Besides conservation laws, a physically reasonable source is also usually required to satisfy the **energy conditions**. These are $T_{\mu\nu}$-based restrictions which say, in several different versions, that "any source of gravitation should be attractive and/or should behave in a causal way". They are important assumptions in many classical results of general relativity that require the usage of Einstein equations. They are expressed in terms of what any physical observer can measure, the observer being represented by four-velocity, i.e. by a future-oriented time-like vector field $\hat{u}^\mu$.

- **Weak energy condition**: every physical observer has to detect non-negative energy density,

$$T_{\mu\nu}\hat{u}^\mu\hat{u}^\nu \geqslant 0\,. \tag{7.39}$$

  For an EM field, the condition yields $\hat{w} \geqslant 0$, for an ideal fluid [see (7.28)], $\rho + \hat{v}^2 P \geqslant 0$. The fluid's speed with respect to the observer, $\hat{v}$, may assume values from the interval $\langle 0, 1)$, with the strongest restriction coming from evaluation of the condition at the limit values: $(\rho \geqslant 0)\ \wedge\ (\rho + P \geqslant 0)$.
  The condition should also hold in the limit of *light-like* $\hat{u}^\mu$; often this special case is being presented separately, as the **null energy condition**. For the ideal fluid, this limit ($\hat{v} = 1$) yields $(\rho + P \geqslant 0)$.

- **Strong energy condition** restricts tension (= negative pressure):

$$\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)\hat{u}^\mu\hat{u}^\nu = T_{\mu\nu}\hat{u}^\mu\hat{u}^\nu + \frac{T}{2} \geqslant 0. \tag{7.40}$$

  For an EM field this condition coincides with the weak one (because $T_{\mathrm{EM}} = 0$), for an ideal fluid it implies $(\rho + P)\hat{\gamma}^2 \geqslant \frac{\rho - P}{2}$, or, $\rho + 3P + (\rho - P)\hat{v}^2 \geqslant 0$; combination of the limit cases $\hat{v} = 0$, $\hat{v} \to 1$ thus requires $(\rho + 3P \geqslant 0)\ \wedge\ (\rho + P \geqslant 0)$.

- **Dominant energy condition**: energy-density flux (= density of momentum) measured by any physical observer, i.e. the vector $(-T^\mu{}_\nu\hat{u}^\nu)$, has to be future-directed and time-like or light-like,

$$g_{\alpha\beta}T^\alpha{}_\mu\hat{u}^\mu T^\beta{}_\nu\hat{u}^\nu \leqslant 0\,. \tag{7.41}$$

For an EM field, one writes the $T^{\mu\nu}_{\rm EM}$ as (7.15), to obtain

$$-T^{\alpha}{}_{\mu}\hat{u}^{\mu}=\hat{S}^{\alpha}+\hat{u}^{\alpha}\hat{w}\,, \tag{7.42}$$

hence

$$g_{\alpha\beta}T^{\alpha}{}_{\mu}\hat{u}^{\mu}T^{\beta}{}_{\nu}\hat{u}^{\nu}=g_{\alpha\beta}(\hat{S}^{\alpha}+\hat{u}^{\alpha}\hat{w})(\hat{S}^{\beta}+\hat{u}^{\beta}\hat{w})=\hat{S}^2-\hat{w}^2 \tag{7.43}$$

and the condition assumes the form $\hat{w}^2\geqslant\hat{S}^2$. Should the flux $(\hat{w}\hat{u}^{\alpha}+\hat{S}^{\alpha})$ be future-oriented, $\hat{w}>0$ must hold in addition, so the condition can finally be "square-rooted" to $\hat{w}\geqslant|\hat{S}|$.
For an ideal fluid, $-T^{\alpha}{}_{\mu}\hat{u}^{\mu}=(\rho+P)u^{\alpha}\hat{\gamma}-P\hat{u}^{\alpha}$, so, after an easy manipulation,

$$g_{\alpha\beta}T^{\alpha}{}_{\mu}\hat{u}^{\mu}T^{\beta}{}_{\nu}\hat{u}^{\nu}=\hat{\gamma}^2(\hat{v}^2P^2-\rho^2)\leqslant 0 \qquad\Longleftrightarrow\qquad \rho^2\geqslant\hat{v}^2P^2;$$

the combination of extreme cases $\hat{v}=0$, $\hat{v}\to 1$ gives $(\rho^2\geqslant 0)\wedge(|\rho|\geqslant|P|)$. The requirement that the vector $(-T^{\mu}{}_{\nu}\hat{u}^{\nu})$ be future-oriented, i.e. that $(T^{\mu}{}_{\nu}\hat{u}^{\nu})\hat{u}_{\alpha}=\hat{\gamma}^2(\rho+\hat{v}^2P)>0$, implies, in the extreme cases of $\hat{v}$, that $(\rho>0)$ and $(\rho+P>0)$. Hence, altogether, the dominant energy condition for fluid demands $(\rho>0)\ \wedge\ (|\rho|>|P|)$. As a matter of fact, it says that the energy density should exceed other $T_{\mu\nu}$ components, i.e. pressures/tensions (hence "energy dominance"); especially in the fluid's rest system, $\hat{u}^{\mu}=u^{\mu}$, it reproduces the weak energy condition.

- Relations between the energy conditions: the conditions are independent in general, only the weak condition follows from the energy-dominance condition and, in its null version, it also follows from the strong energy condition.

CHAPTER 8

Einstein field equations

Until now, we have been concerned i) with mathematical properties of the pseudo-Riemannian geometry (specifically, Lorentzian geometry, "$-+++$")[1] which describes curved space-times; ii) with how a *given* geometry enters selected physical problems; and iii) with how gravitational sources could be described in GR (by $T^{\mu\nu}$). Now it's time to address a crucial physical query: how the geometry depends on processes which take place in the space-time? The answer is provided by **the Einstein field equations** – the central point of the relativistic theory of gravitation; they represent the link between the space-time geometry and the behaviour of matter and non-gravitational fields.

A fundamental physical law cannot be expected to be provable as a theorem. Such a "full control" is only possible in working with mathematical structures where statements are derived by logical deduction from definitions (at least within limitations following from Gödel's incompleteness theorems). The meaning of *physical* quantities is only clarified in mutual relations in which they are posed by equations, and by saying whether and how they can be measured. This assignment between "things in themselves" and "letter symbols for us" belongs to the most non-trivial part of physics.

Neither the Einstein equations can be *derived*, strictly speaking. They can only be "found" – on the basis of geometric knowledge, fundamental principles, and Newton's gravitational law. Actually, the fundamental principles themselves would be useless, because there is no gravitational law in special relativity, so there is nothing to refer to in the LIFE. Another thing is that a posteriori, when it is already clear what should come out, the Einstein theory can be arranged in an axiomatic form. We will show later how to "derive" the theory using the Hilbert variational principle or a more general, Palatini-inspired method. Similarly, in electromagnetism, when already knowing Maxwell equations, it is not so difficult to guess which variational problem (Lagrangian) leads to them.

[1] People often write "$- + ++$", can you see the difference? This is obtained by "$-+++$", while our version by "$-$$+$$+$$+$". Also bearable is "$-$+++", though plusses (also) appear slightly different in text mode.

8.1 “Physical derivation”

Our primary goal is to find equations which would determine what gravitational field is generated by a given configuration of sources. In the Newtonian case, the gravitational law is represented by Poisson equation for potential Φ with given mass density ρ,

$$\Delta\Phi = 4\pi\rho. \tag{8.1}$$

This works very well in terrestrial physics and within the Solar system, but it has several features which are clearly not satisfactory:

- It is not *causal*: it does not contain any time derivative, so the field $\Phi(t, \vec{x})$ does not propagate – it is generated *immediately*, in the whole space, according to an *instantaneous* distribution of density $\rho(t, \vec{x})$. Hence, it actually propagates infinitely fast.
- It is not *covariant*: of course, Laplace operator is not invariant, and neither is the mass density. Here, immediate remedy might seem to be to change the Laplace for the d’Alembert operator (which *is* invariant), and to use *proper* density of *rest* mass. However, ...
- ... such a law i) would clearly assume a special form only valid in the rest system of the source, it could not work in other systems (with respect to which the sources are not at rest), because it does not contain any information about how contributes the kinetic mass-energy, or, in other words, how the “rest-mass density” (however effective this notion may be) changes if the source moves. Actually, often it is not possible to adjust coordinates in such a way that *all* the sources stay at rest. Even for a single (extended) source: what about differential rotation? Also, ii) the concept of “proper density of rest mass” is not very clear for a source-free EM or other fields, especially if the field is not static.

In the previous chapter, we saw that a plausible representative of gravity sources could be the energy-momentum tensor – a generalisation of mass-energy density also including kinetic and stress contributions. We will thus employ it on the source side of the field equations.

8.1.1 Uniqueness of the Riemann tensor

The “field” side of the equations, on the contrary, stems almost inevitably from the Newtonian “template” (8.1) and from the properties of the corresponding geometric quantities (and through these, from fundamental principles of GR). In the Newtonian equation, $\Delta\Phi$ contains, linearly, the second derivatives of the potential. From the Newtonian limit of the geodesic equation, we know the potential is paralleled by the metric in GR, so, following the simplicity principle, we wish the left-hand side to be determined by $g_{\mu\nu}$ and by its first and second partial derivatives, with the latter just appearing linearly. In specifying the concrete form of the left-hand side, crucial is the property called the **uniqueness of the Riemann tensor**, meaning that the Riemann tensor is the unique tensor quantity that involves the above metric features.

$\boxed{\text{Theorem}}$ Compose a tensor out of $g_{\mu\nu}$ and its first and second partial derivatives, in such a way that it depends on these three ingredients in the same way in every reference system. Every such tensor can be expressed in terms of just the metric and the Riemann tensor.
Proof:

- First, about the first derivatives we know from (2.9) that

  $$g_{\kappa\lambda,\sigma} = \Gamma_{\lambda\sigma\kappa} + \Gamma_{\kappa\sigma\lambda}\,,$$

  and we also know that the Christoffel symbols vanish in the LIFE. Hence, the sought tensor has to depend trivially on $g_{\kappa\lambda,\sigma}$ in any other system as well. In more general words, the first derivatives of the metric in themselves do not, in general, give rise to any non-trivial tensor. Therefore, they cannot, *independently*, appear in the tensor we are looking for. (This is not to say that they cannot be present in such a tensor at all, only that they can at most appear *within some tensorial quantity*.)

- From (6.8) it follows that in the LIFE (where the affine-connection components vanish) the Riemann tensor is solely given by the second derivatives of the metric,

  $$R_{\mu\nu\kappa\lambda} = \frac{1}{2}\left(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}\right) \qquad \text{in local inertial frame}\,.$$

  Hence, it is clear that the dependence on the second derivatives of $g_{\mu\nu}$ can fully be represented by the Riemann tensor (technically, it is sufficient to invert the above relation).

- Remark: note that in a general system the Riemann tensor also includes the *first* derivatives of the metric (in the quadratic affine-connection terms). So we see the first derivatives *are* really present (quadratically), only that they are solely present within the Riemann tensor. □

Now, should the *right-hand* side of the Einstein equations be proportional to $T_{\mu\nu}$, we have to also form a symmetric second-rank tensor on the left-hand side – out of the metric and the Riemann tensor. And, finally, should the second derivatives of the metric only appear linearly, the Riemann tensor has to appear there linearly. Though it can be proved formally, let us simply refer to the "trial and error" method: it is impossible to invent any other terms which satisfy such constraints than $R_{\mu\nu}$, $Rg_{\mu\nu}$ and $g_{\mu\nu}$. They can appear in a combination whose coefficients can only depend on the metric. However, in the LIFE the metric is constant (it has the Minkowski form), so it should be like that in general – the coefficients have to reduce to constants. Hence the following important consequence of the Riemann-tensor uniqueness:

Consequence The most general symmetric second-rank tensor i) which depends on $g_{\mu\nu}$ and on its first and second partial derivatives in the same way in all reference systems, and ii) in which the second derivatives of $g_{\mu\nu}$ appear linearly, is given by linear combination of the tensors $R_{\mu\nu}$, $Rg_{\mu\nu}$ a $g_{\mu\nu}$. Consequently, we shall be looking for the left-hand side of the Einstein equations in the form

$$R_{\mu\nu} + C_2 R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}\,, \qquad \text{with} \quad C_2, \Lambda, \kappa \quad \text{constants}\,. \qquad \square$$

(Clearly the whole equation can be multiplied by a constant freely, which means one of the coefficients can be chosen; standardly the coefficient at $R_{\mu\nu}$ is being set at unity, $C_1 \equiv 1$.) The C_2 coefficient we will fix from the Bianchi identities and from the requirement that the left-hand side have zero covariant divergence (in order that the equations imply conservation laws for $T_{\mu\nu}$ on the right-hand side). The κ on the right-hand side will follow from the requirement that the equations yield the classical gravitational law (8.1) in the Newtonian limit. Finally, the constant Λ will stay in the equations, mysterious ...

8.1.2 Bianchi-identities contraction, and conservation laws

By double contraction of the second Bianchi identities, we found in (6.34) that

$$R_{;\mu} - 2R^{\nu}{}_{\mu;\nu} = 0\,.$$

A similar relation is also obtained if demanding that the left-hand side of the field equations have zero covariant divergence – in order to comply with the conservation laws which should hold for the right-hand side ($T_{\mu}{}^{\nu}{}_{;\nu} = 0$):

$$(R_{\mu}{}^{\nu} + C_2 R g_{\mu}{}^{\nu} + \Lambda g_{\mu}{}^{\nu})_{;\nu} = 0 \qquad \Longleftrightarrow \qquad R_{\mu}{}^{\nu}{}_{;\nu} + C_2 R_{;\mu} = 0\,.$$

By combination of these two relations, we get

$$\left(\frac{1}{2} + C_2\right) R_{;\mu} = 0 \quad \Longrightarrow \quad C_2 = -\frac{1}{2}\,,$$

so the field equations appear as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}\,. \tag{8.2}$$

The tensor $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =: G_{\mu\nu}$ is called **the Einstein tensor**.

The equations can also be written in a different way (which actually was the first one historically): make their trace (contraction over μ, ν), $R - 2R + 4\Lambda = \kappa T$, then substitute back for $R = 4\Lambda - \kappa T$ from here, and express $R_{\mu\nu}$:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}\,. \tag{8.3}$$

This form is mainly advantageous for a simple $T_{\mu\nu}$ (e.g. the EM field for which $T = 0$), especially for a vacuum ($T_{\mu\nu} = 0$) when only $R_{\mu\nu} = \Lambda g_{\mu\nu}$ remains.

In a general case, still one more thing remains: to fix the coefficient κ between the geometric and the physical sides.

8.1.3 Lovelock's theorem, and the chief attraction of GR

Before doing so, let us mention, without proof, a fundamental result which supplements the above uniqueness property of Riemann: the **Lovelock theorem**. Let us quote the abstract of D. Lovelock's paper (J. Math. Phys., 1972): *All tensors of contravariant valency two, which are divergence free on one index and which are concomitants of the metric tensor, together with its first two derivatives, are constructed in the four-dimensional case. The Einstein and metric tensors are the only possibilities.* After what we have proved or understood before, this seems nothing extra, but consider that this claim does *not* assume linearity in the second metric derivatives. Hence, what we presented as a simplicity assumption, originally motivated by the linearity in Φ of $\Delta\Phi$ in the Newton gravitational equation, actually is *not* necessary – at least not *after* one has already decided that the right-hand side of the field equations should be represented by a divergence-free and symmetric second-rank tensor. Needless to say, this result still more emphasizes that the left-hand side of the Einstein equations is given *uniquely* – given the right-hand side ($T_{\mu\nu}$) with its properties, there is no other option. Einstein expressed that feature – without of course knowing the Lovelock theorem – in the paper *What is the theory of relativity?* which he wrote at the request of The London Times and which appeared there on 28th November 1919: "The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible."

8.1.4 Kappa = 8 pi

The coefficient is fixed from the requirement that the field equations go over to $\Delta\Phi = 4\pi\rho$ in the Newtonian limit. In order to perform the latter, it is natural to select some particularly simple case (remember that the equations have to be valid for *any* physical system and under arbitrary conditions). First, the Newtonian limit involves stationarity, and second, the simplest non-trivial source we know is the incoherent dust, $T^{\mu\nu} = \rho\, u^\mu u^\nu$.

In the Newtonian limit, Christoffel symbols are linearly small, so the two "$\Gamma\Gamma$" terms in the Riemann tensor can be omitted,

$$R^\iota{}_{\mu\kappa\nu} \doteq \Gamma^\iota{}_{\nu\mu,\kappa} - \Gamma^\iota{}_{\kappa\mu,\nu} \,.$$

The R_{00} component of the Ricci tensor thus reduces to

$$R_{00} \equiv R^\iota{}_{0\iota 0} = \Gamma^\iota{}_{00,\iota} - \cancel{\Gamma^\iota{}_{\iota 0,0}} \,,$$

where we have cancelled the second term due to stationarity as the second aspect of the Newtonian limit. Leaving just spatial derivatives in the first term, and recalling that the Newtonian limit (Section 3.7) leads to $\Gamma^i{}_{00} = \Phi^{,i}$, we arrive at

$$R_{00} = \Gamma^i{}_{00,i} = \Phi^{,i}{}_i \equiv \Delta\Phi \,.$$

Now we need to compare this with the Newtonian limit of the right-hand side of the respective field equation

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2} T g_{00}\right) + \Lambda g_{00}$$

(we use the second form (8.3)). Provided that such a limit in no way affects Λ, in the second term one just substitutes $g_{00} = -1 - 2\Phi$. In the first term, one realises that ρ is of the same order of smallness as Φ (as it follows from the equation $\Delta\Phi = 4\pi\rho$), and that the Newtonian limit also means *slow* motion, $|u^i| \ll |u^t|$:

$$T = \rho\, u^\mu u_\mu = -\rho \,,$$

$$T_{00} = \rho(u_0)^2 = \rho\,(g_{00}u^0 + g_{0i}u^i)u_0 \doteq \rho\,g_{00}u^0u_0 = \rho\,g_{00}(-1-u^iu_i) \doteq$$
$$\doteq -\rho\,g_{00} \doteq \rho\,(1+2\Phi) \doteq \rho\,,$$

which makes the field equation (without Λ) look

$$\Delta\Phi = \kappa\left(\rho - \frac{1}{2}\rho\right) = \frac{1}{2}\kappa\rho\,.$$

However, in the Newton's theory, the right-hand side is $4\pi\rho$, so we conclude that $\kappa = 8\pi$, or, in standard units, $\kappa = 8\pi G/c^4$.

That's it. As a culmination of the general theory of relativity, we may write

the Einstein field equations (Einstein's gravitational law):

$$\boxed{R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}\,, \qquad \text{or} \qquad R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}\,.} \tag{8.4}$$

A couple of special cases is worth mentioning when the field equations reduce to a simpler form. If $\Lambda = 0$ and the energy-momentum tensor is traceless (as e.g. that of the EM field), the equations reduce to $R_{\mu\nu} = 8\pi T_{\mu\nu}$ and their trace yields $R = 0$. If $T_{\mu\nu} = 0$, the equations reduce to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and their trace yields $R = 4\Lambda$.

8.1.5 "*At such times one sees to what deplorable breed of brutes we belong.*"[2]

Just as students should learn something about the times when this Prague lecture course was being developed, it is good to realize that Einstein's equations *are*, after all, associated with certain specific coordinates: 25th November 1915, Berlin. The last part of his eight-year struggle their author spent in one of the most tragic periods of history, and, in addition, at the place where he had the roots of most of the horrors in front of his eyes. The waves of pathologic nationalism, stimulated in Germany from the end of the 19th century already, were even clearly resonating within the academic community, including Einstein's close colleagues. Besides devotion to "the starry heavens above", Einstein was keeping "the moral law within", but within Berlin he was apparently next to isolated. Military parades he abhorred since childhood, and to national patriotism he was saying [letter to P. Ehrenfest from 23rd August 1915]: "Isn't this little clutch of busy intellectuals our only 'fatherland' to which such as we have any serious attachment? Should even *these* people have mentalities that are solely a function of residence?" How strong such an adherent of liberty had to be while working, in Berlin, side by side with scientists – not seldom his Jewish compatriots – who were fulminating against "the enemies of white race", who were encouraging students to go to a "fair aggression war", and who were practicing marching on the Academy yard? Einstein was in contact with R. Rolland, published several anti-war articles, he entered a banned peace circle. After the war, he was trying to persuade the student revolutionary boards not to declare the "dictatorship of the proletariat" and to preserve the academic freedoms, and he was also against penalization of Germany included in the Treaty of Versailles.

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[2] From Einstein's letter to P. Ehrenfest, 19th August 1914.

26 years later, from America already – and in spite of usual pacifism –, Einstein was pushing the world to defend democracy against fascism by every means; however, atomic explosions over Japan he deemed an unnecessary massacre, and he was calling for nations' handing over a part of their sovereignty to an international peace administration.

All that *also* concerns Einstein's equations. It illustrates the character of their founder, indispensable in the struggle with gravitation: not only "1% of inspiration and 99% of perspiration", how he (together with Edison) summarized the scientific outcomes, but also the inner independence, sensitiveness to the freedom and justice' getting in jeopardy, and psychic resistance. Actually, you may check the letter Einstein wrote to his Swiss friend, medical doctor H. Zangger, in April 1915: "Why one could not live in it peacefully, similarly as personnel in a madhouse?" A. Fölsing adds, in his brilliant biography [11]: "Deep-rooted inner independence and harsh humour allowed Einstein to endure the most disgusting human turmoil even from closest vicinity with an amused reticence."

However, Germany stayed Nazi-deformed still after the 1st world war, and in 18 years an even stronger wave of violence evicted Einstein from Europe for ever.

8.1.6 Cosmological term: Einstein's biggest blunder?

Originally, the field equations did *not* contain the **cosmological constant** Λ. Einstein added it there, at the end of 1916, in order to "save" the possibility of a static cosmological model. After it became clear that the Universe is expanding, he said the inclusion of the cosmological term was the "biggest blunder" of his life. Let's try to understand it a bit.

First, with the cosmological term, the Newtonian limit of the Einstein equations reads $R_{00} = 4\pi\rho + \Lambda g_{00}$, so

$$\Delta\Phi + 2\Lambda\Phi = 4\pi\rho - \Lambda\,.$$

Experiments (in the Solar system) restrict Λ to some $|\Lambda| < 10^{-50}/\text{cm}^2$. –Yes, Λ has the dimension of [cm^{-2}]: if it is non-zero, it mainly pronounces itself in the largest scales (thus *cosmological*).

The physical role of Λ is revealed from Einstein equations, the most intuitively in case when the source is the ideal fluid: shifting the last term on the left-hand side of

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\left[(\rho + P)u_\mu u_\nu + Pg_{\mu\nu}\right]$$

to the right, we get there $8\pi(\rho + P)u_\mu u_\nu + (8\pi P - \Lambda)g_{\mu\nu}$. Hence, Λ's role is *opposite to that of pressure* – and since pressure acts in the same way as mass density (recall the Euler equations), we see that $\Lambda > 0$ induces mutual *repulsion* between different parts of space, whereas $\Lambda < 0$ induces their attraction. Now we understand that a positive Λ can prevent the static universe from collapsing: if adjusted suitably, it just balances the gravitational attraction of the matter living in the universe.

In order to still better understand the nature of the cosmological term, let us consider the situation with $T_{\mu\nu} = 0$ (vacuum) but with non-zero cosmological term. And imagine we would now understand the cosmological term as a source and would like to interpret it in the "fluid" language. The energy-momentum tensor solely generated by Λ would read

$T_{\mu\nu} = -\frac{\Lambda}{8\pi}\, g_{\mu\nu}$. Comparing this with the general ideal-fluid form $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$, we see that the "$\Lambda$"-fluid would have the equation of state $P = -\rho$ and the energy density $\rho = \frac{\Lambda}{8\pi}$. Therefore, such a source only depends on ρ (which is constant in this case), not on velocity or anything else, and so it only shifts the energy of the system's "basic state". The energy-momentum tensor has trace $T = -4\rho$ and an observer with four-velocity $\hat{u}^\mu$ measures on it the energy density $T_{\mu\nu}\hat{u}^\mu\hat{u}^\nu = \rho$ and the energy flux $-T^\alpha{}_\mu\hat{u}^\mu = \rho\hat{u}^\alpha$. Hence, the source marginally satisfies the null energy condition, and if $\rho \geqslant 0$ ($\Leftrightarrow$ $\Lambda > 0$), it also satisfies the "time-like" weak condition. It satisfies the energy-dominance condition too, but not the strong energy condition – namely, $T_{\mu\nu}\hat{u}^\mu\hat{u}^\nu + \frac{T}{2} = -\rho$.

There is, however, an opposite possibility: to leave the cosmological term on the *left*-hand ("geometric") side of the equations, do *not* understand it as a source – and just state that *gravitation is characterized by two constants*, G and Λ. Sure, no physics aesthete welcomes the appearance of a new free constant in a theory, because every free parameter indicates non-completeness of the theory, or even a complete ignorance of a certain part of reality; also, the more constants in a theory, the more difficult it is to falsify it. Two free constants is actually very few, but still Einstein admitted Λ should be refused after it became evident that the Universe is dynamical.

Yet it has turned out, like many times already, that Einstein was once again right. In the second half of the 20th century, suggestions first began to arrive from particle physics that the "cosmic vacuum" (the basic state of the Universe) might not correspond to zero energy level. The cosmological constant would in such a case describe, as an effective parameter, the density of that vacuum energy (in line with what we mentioned above). Positive Λ would thus correspond to a field state with tension (negative pressure). Anyway, there does not exist any "microscopic theory" for Λ; different particle theories yield different estimates for the value of Λ, typically much larger than the previously mentioned experimental upper bound (really MUCH larger, by some 120 orders of magnitude). The second wave of interest came in the 1980s, together with the hypothesis of **cosmological inflation**. Again, it was/is tied to the occurrence of a field in the negative-pressure state, this time specifically the state of a scalar field (possibly the Higgs field, in particular) just after the Universe had been born. Due to the possibly large positive value of Λ thus induced, the Universe could have been "inflated" extremely rapidly by many orders of magnitude, which would have very much "thinned down" and blurred the cosmological initial conditions. Such an effect could solve some of the problems of standard cosmology.

Anyway, today what is represented in Einstein equations by the cosmological constant is usually called **the dark energy** and it is mainly being addressed by astronomical techniques. Actually, from about mid 1990s, a growing body of evidence has been collected that if standard (homogeneous and isotropic) cosmological models should reasonably approximate the Universe, they have to include a positive cosmological constant. *Positive* seems to be too strong a word if speaking of the value $\Lambda \doteq 1.1 \cdot 10^{-56}/\mathtt{cm}^2$, but we will see in cosmology that the dark energy forms as much as almost 70% of all the cosmic mass-energy.

8.2 Einstein equations and the mass-geometry relation

The Einstein gravitational law (8.4) represents a system of 10 non-linear second-order partial differential equations for the components of the metric tensor. More specifically, the equations are *quasi-linear*, because they are linear in the highest (second) derivatives of $g_{\mu\nu}$. The number 10 is given by symmetry of their both sides in (μ, ν). For their solution, it is necessary to know/give the energy-momentum tensor $T_{\mu\nu}$, describing the matter and non-gravitational fields present.

It is a novel feature that although the equations are called *field equations*, they do not only determine the field (the metric), but also the behaviour of sources. Actually, we know that thanks to the second Bianchi identities they imply conservation laws $T^{\mu\nu}{}_{;\nu} = 0$ (they have been *composed* so). The conservation laws do not constrain the metric (this appears in them just "passively", in the covariant derivative), they rather constrain the sources. We illustrated this circumstance on ideal fluid – we were in fact able to obtain the Euler equations of motion from the conservation laws, plus the equation of continuity. Now, if 4 of the 10 equations constrain sources, we are left with 6 to determine the metric. It is a well known story that when, at the end of the 19th century, the Czech polymath Jára da Cimrman met A. P. Chekhov and saw Chekhov writing the play *Two Sisters*, he queried: "Isn't it two few, Anton Pavlovich?" Similarly we should query here: aren't 6 equations for 10 unknowns too few? ... [think about it] ... According to the principle of general covariance, it always has to be possible to freely choose the coordinate system. This effectively means that there must always remain a freedom to choose 4 of the metric functions. Therefore, 6 equations for 10 components of the metric is just the right amount. (In other words, had we 10 equations for 10 metric components, the equations would yield an absolutely specific result, i.e. they would even prescribe specific coordinates. But this would be in contradiction with the keystone of the theory.)

This is an important conceptual point. Contrast the above with electrodynamics: there, from Maxwell equations only follows the continuity equation (which says that electric charge has to be conserved), nothing else – in particular not the Lorentz equation of motion. Therefore, the field equations do not dictate how the sources should move; the sources can be prescribed freely (besides satisfying the continuity equation and besides moving sub-luminal). Similarly, in Newton's gravity, the equation of motion $\vec{a} = -\vec{\nabla}\Phi$ does not follow from the field law $\Delta\Phi = 4\pi\rho$. In GR, thanks to the field equations and the Bianchi identities, only such source configurations are allowed which satisfy the conservation laws, and this typically means to evolve precisely according to the equation of motion (which can be derived directly from the conservation laws). Note that the Einstein equations specifically imply, through Euler equations (7.33), that the incoherent dust, i.e. a continuous source made of *free* particles, moves along geodesics. This may seem to be just a "consistency check", but it is not (only) so: here the dust is *not* test, it does generate the field, so the field equations yield a deeper result than just geodesic motion in a *given* background.

So the Einstein equations describe a *coupled system of mass-energy and geometry*. The sources as well as their field are *dynamical* variables and their self-consistent configuration is enforced, by the equations, automatically. In popular words: "The matter tells space-time how to curve, while the space-time, *at the same time*, tells matter how to move." Mathematically, this profound power of the equations is also – besides the link with Bianchi identities – connected with their non-linearity (remember that classical electrodynamics and Newtonian gravity are both linear). Loosely, one may ponder in the following way: in contrast to electrodynamics (for example), where the fields do *not* bear any attributes of the sources (the EM field does not have charge), the *gravitational field acts as a source of gravity*, because it has a certain mass-energy – and every energy contributes to gravitation. However confusing the statement "gravitational field generates gravitation" may sound at first sight, it exactly articulates the non-linearity of the equations: a source acts on itself through a field generated by its [source's] field. Non-linear systems always involve such a feedback – and exactly because of this feedback the space of possible states of non-linear systems is much more restricted than that of linear ones.

Contrary to the initial Einstein's understanding, geometry plays partially an autonomous role in the theory, in the sense that it is *not* fully determined by matter. Actually, similarly as the matter, it has its own degrees of freedom, as best exemplified on the vacuum setting when there is absolutely no source in the space-time, yet the field equations still admit variety of different solutions, not just the flat space-time of special relativity. A natural way how to distinguish between "pure geometry" and "matter-induced geometry" is provided by the Einstein equations $R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}$: matter (and Λ) determines that part of curvature which is given by the Ricci tensor. This has 10 independent components in general, so the remaining 10 components of curvature (the Riemann tensor) should correspond to "pure geometry". If worrying now about how this remaining piece of information in Riemann is determined, recall the second Bianchi identities. These are "pure geometric" (independent of the field equations) and clearly contain more constraints than what we employed in derivation of the Einstein equations. Specifically, the second Bianchi identities are 20, while their twice contracted consequence $R_{;\mu} = 2R^{\nu}{}_{\mu;\nu}$ we used represents 4 relations.

8.2.1 Mass-energy of the gravitational field

Above, we mentioned that gravitational field has a certain mass-energy. Such a remark may easily provoke a lengthy discussion. Namely, there actually does not exist a good notion for the "energy-momentum tensor of a gravitational field", and (thus) for a local density of "gravitational mass-energy" or "gravitational energy flux". Such quantities *can* be defined globally (in the whole space-time), at least if the space-time is asymptotically flat (approaches flat space-time at spatial infinity), but it is impossible to say, uniquely, how much mass-energy is in such and such region. (Hence the frequent statement "gravitational energy is not localizable".)

The simplest hint to expect that is to recall the equivalence principle. It says that in the LIFE, there is locally no gravitational field (affine connection components are locally transformed away). In other words, "gravitational intensity" cannot in general be described by a tensorial quantity. From the side of the conservation laws: these typically follow from symmetries. In special relativity, the symmetry with respect to translation in time implies the conservation of energy while the symmetry with respect to spatial translations implies the conservation of momentum. However, in GR, in contrast to SR, neither of these translations are in general isometries (the space-time is – or may be – different at different times and at different locations).

It is thus worth to repeat once more that the energy-momentum tensor on the right-hand side of Einstein equations really represents just *non-gravitational* contributions.

8.3 Riemann-tensor decomposition, Weyl tensor, and duals

Now that we know the field equations, it's right time for an appendix to curvature (Section 6) – about the degree to which curvature depends on sources (on $T_{\mu\nu}$). The Riemann tensor can be decomposed as

$$\begin{aligned} R_{\mu\nu\kappa\lambda} &= C_{\mu\nu\kappa\lambda} + \frac{1}{2}\left(g_{\mu\kappa}R_{\nu\lambda} + g_{\nu\lambda}R_{\mu\kappa} - g_{\mu\lambda}R_{\nu\kappa} - g_{\nu\kappa}R_{\mu\lambda}\right) - \frac{R}{6}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right) = \\ &= C_{\mu\nu\kappa\lambda} + \underbrace{\frac{1}{2}\left(g_{\mu\kappa}S_{\nu\lambda} + g_{\nu\lambda}S_{\mu\kappa} - g_{\mu\lambda}S_{\nu\kappa} - g_{\nu\kappa}S_{\mu\lambda}\right)}_{=:\,E_{\mu\nu\kappa\lambda}} + \underbrace{\frac{R}{12}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right)}_{=:\,G_{\mu\nu\kappa\lambda}}, \end{aligned} \tag{8.5}$$

where $C_{\mu\nu\kappa\lambda}$ is the **Weyl tensor** and $S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$ is the traceless part of the Ricci tensor. The tensors $C_{\mu\nu\kappa\lambda}$, $E_{\mu\nu\kappa\lambda}$ and $G_{\mu\nu\kappa\lambda}$ have the same symmetries as the Riemann tensor. Besides that, one easily computes their traces,

$$E^{\kappa}{}_{\nu\kappa\lambda} = S_{\nu\lambda}, \quad G^{\kappa}{}_{\nu\kappa\lambda} = \frac{1}{4}Rg_{\nu\lambda} \qquad \Longrightarrow \qquad C^{\kappa}{}_{\nu\kappa\lambda} = 0\,. \tag{8.6}$$

The traceless property of $C^{\mu}{}_{\nu\kappa\lambda}$ means another 10 conditions in addition to the symmetries of $R^{\mu}{}_{\nu\kappa\lambda}$, so the Weyl tensor only has $20-10=10$ independent components. (Hence, vanishing is its contraction over *any* pair of indices.) The Weyl tensor is said to represent a "pure gravitational field" – whereas Riemann also contains an information about sources; this is brought to it by the Ricci tensor, as related to the properties of the sources by the Einstein equations (Section 8). Outside sources, $C^{\mu}{}_{\nu\kappa\lambda} = R^{\mu}{}_{\nu\kappa\lambda}$. In short, Weyl tensor contains that part of Riemann which is *not* in Ricci. However, it is not *completely* free from the sources, due to the second Bianchi identities. Indeed, in (6.33) we found by contracting the Bianchi identities that

$$R^{\sigma\nu}{}_{\lambda\mu;\sigma} = R^{\nu}{}_{\mu;\lambda} - R^{\nu}{}_{\lambda;\mu}\,, \qquad R_{;\mu} = 2R^{\nu}{}_{\mu;\nu}\,.$$

This in turn implies that the divergence of the Weyl tensor has to satisfy

$$\begin{aligned} &C_{\mu\nu\kappa\lambda}{}^{;\mu} \equiv \\ &\equiv R_{\mu\nu\kappa\lambda}{}^{;\mu} - \frac{1}{2}\left(g_{\mu\kappa}R_{\nu\lambda}{}^{;\mu} + g_{\nu\lambda}R_{\mu\kappa}{}^{;\mu} - g_{\mu\lambda}R_{\nu\kappa}{}^{;\mu} - g_{\nu\kappa}R_{\mu\lambda}{}^{;\mu}\right) + \frac{R^{;\mu}}{6}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right) = \\ &= R_{\nu\lambda;\kappa} - R_{\nu\kappa;\lambda} - \frac{1}{2}\left(R_{\nu\lambda;\kappa} - R_{\nu\kappa;\lambda}\right) - \frac{1}{4}\left(g_{\nu\lambda}R_{;\kappa} - g_{\nu\kappa}R_{;\lambda}\right) + \frac{1}{6}\left(g_{\nu\lambda}R_{;\kappa} - g_{\nu\kappa}R_{;\lambda}\right) = \\ &= \frac{1}{2}\left(R_{\nu\lambda;\kappa} - R_{\nu\kappa;\lambda}\right) - \frac{1}{12}\left(g_{\nu\lambda}R_{;\kappa} - g_{\nu\kappa}R_{;\lambda}\right) \equiv R_{\nu[\lambda;\kappa]} - \frac{1}{6}\,g_{\nu[\lambda}R_{;\kappa]}\,. \end{aligned} \tag{8.7}$$

The "vacuum case" should however be made more precise. Actually, we know from Einstein equations that $T_{\mu\nu}=0$ implies $R_{\mu\nu}=\Lambda g_{\mu\nu}$ (thus $R=4\Lambda$), so in such a case we have $S_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}=0$, the decomposition (8.5) reduces to

$$R_{\mu\nu\kappa\lambda}=C_{\mu\nu\kappa\lambda}+\frac{\Lambda}{3}\left(g_{\mu\kappa}g_{\nu\lambda}-g_{\mu\lambda}g_{\nu\kappa}\right),$$

and the divergence of the Weyl tensor (8.7) vanishes. One also notices that in the $T_{\mu\nu}=0$ case all statements involving covariant differentiation of curvature (in particular, the second Bianchi identities) may equivalently be voiced in terms of the Riemann as well as the Weyl tensors.

Weyl tensor $C^{\alpha}{}_{\nu\kappa\lambda}$ is often called the **conformal tensor**, in order to stress that it stays invariant under any conformal transformation of the metric, $g_{\mu\nu}(x)\rightarrow\Omega^2(x)g_{\mu\nu}(x)$. (It is straightforward but lengthy to verify this feature. It is derived thoroughly in Appendix D of [50].)

To tensors antisymmetric in more (than one) pairs of indices, one can define more duals (which are different in general) – see Appendix A.2. For the Riemann tensor, one thus has the **left dual** and the **right dual** (A.28),

$$^*R^{\alpha\beta}{}_{\kappa\lambda}\equiv\frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}R_{\rho\sigma\kappa\lambda}\,,\qquad R^*{}_{\alpha\beta}{}^{\kappa\lambda}\equiv\frac{1}{2}\,R_{\alpha\beta\rho\sigma}\epsilon^{\rho\sigma\kappa\lambda}\,. \tag{8.8}$$

Similarly, one can consider duals to individual terms of its decomposition. In Appendix A, equations (A.36)-(A.39), it is shown that they satisfy

$$^*C_{\alpha\beta\kappa\lambda}=C^*{}_{\alpha\beta\kappa\lambda}\,,\;\; ^*E_{\alpha\beta\kappa\lambda}=-E^*{}_{\alpha\beta\kappa\lambda}\,,\;\; ^*G_{\alpha\beta\kappa\lambda}=G^*{}_{\alpha\beta\kappa\lambda}\,,\;\; ^*R_{\alpha\beta\kappa\lambda}=R^*{}_{\alpha\beta\kappa\lambda}+2\,{}^*E_{\alpha\beta\kappa\lambda}\,.$$

8.3.1 "Electric" and "magnetic" parts of Weyl

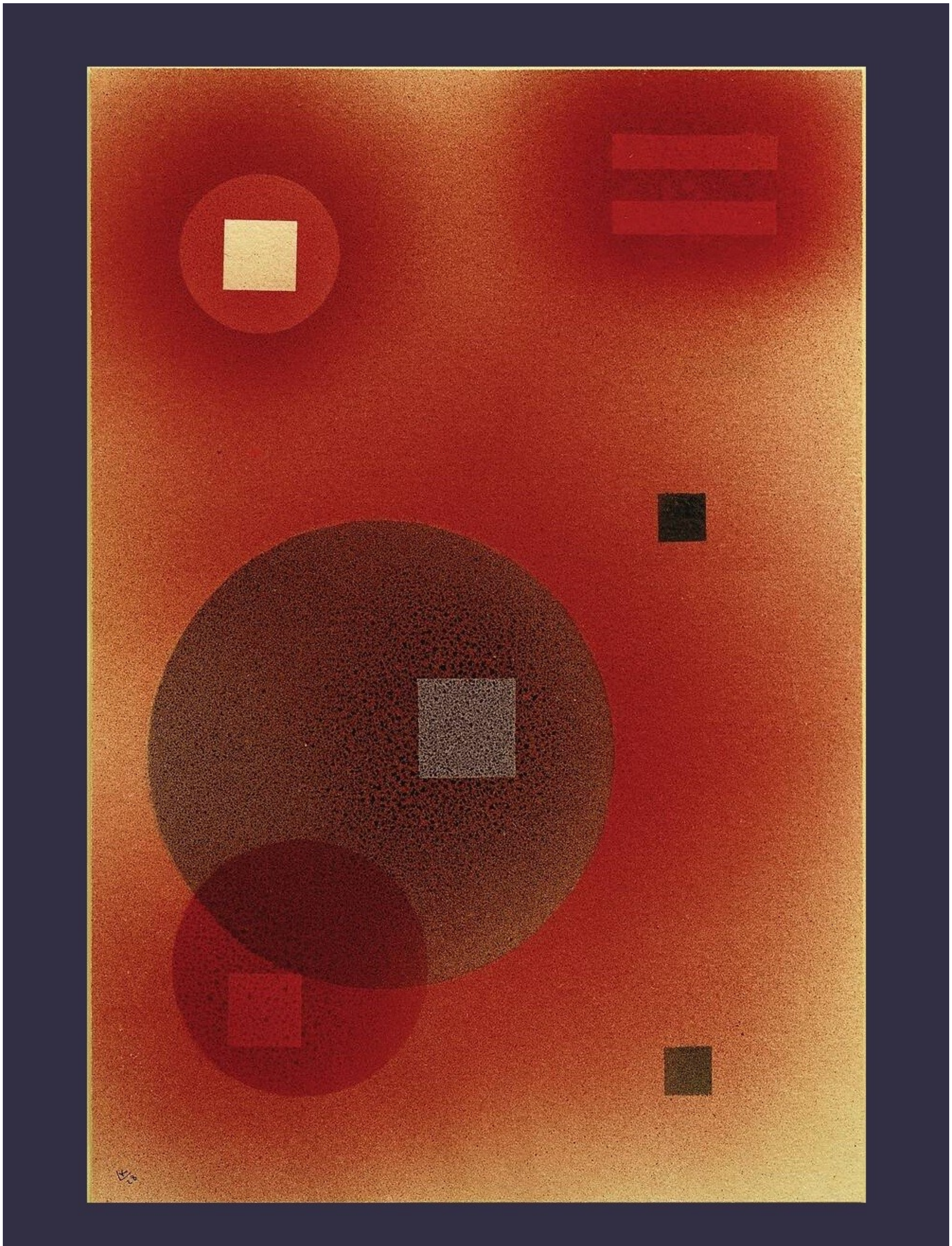
Analogously to how one can obtain, from the EM-field tensor $F_{\mu\nu}$, the electric and the magnetic fields measured by some observer with four-velocity $\hat{u}^\mu$ (see Section 7.3.1),

$$\hat{E}_\mu=F_{\mu\nu}\hat{u}^\nu\,,\quad \hat{B}_\mu=-{}^*F_{\mu\nu}\hat{u}^\nu\,,$$

the tensors

$$\hat{E}_{\mu\kappa}\equiv C_{\mu\nu\kappa\lambda}\hat{u}^\nu\hat{u}^\lambda\,,\qquad \hat{B}_{\mu\kappa}\equiv{}^*C_{\mu\nu\kappa\lambda}\hat{u}^\nu\hat{u}^\lambda \tag{8.9}$$

are being referred to as the respective electric and magnetic parts of the Weyl tensor. These two-tensors are symmetric and traceless, and clearly satisfy $\hat{E}_{\mu\kappa}\hat{u}^\kappa=0$, $\hat{B}_{\mu\kappa}\hat{u}^\kappa=0$; each of them contains $10-4-1=5$ independent components of Weyl.



Wassily Kandinsky : Weighing down

CHAPTER 9

Principle of minimal coupling

Time to also listen to how Einstein himself commented on his field equations: their "geometric" side he appreciated as "marmoreally beautiful", whereas the other side describing matter he assessed as "yet unlovely, wood-made". He thus indicated that the geometry enters the equations in a way that almost inevitably stems from the fundamental principles, whereas the exact role of matter cannot be deduced from anywhere.[1] From Einstein's rating it is obvious that beauty is subjective,[2] but if it is voiced by a theoretician in connection with some thought construction, then they almost always means the same: simplicity of its basic ideas, inevitability of how conclusions follow from them, "rigidity" of the resulting system (including, in particular, minimum of free parameters), and proportions of its formal image – which means, in the physics case, *mathematical elegance*.

General relativity is very much esteemed in these respects (and not only in them); it was actually its aesthetic appeal which maintained the credit of GR during the first decades while the experimental support was yet only very restricted. From the equivalence and covariance principles, the physical laws ensue in a way that is hard to alter without challenging the whole structure of the theory. The resulting construction only contains three constants which do not follow from the theory – the speed of light c, the gravitational constant G and the cosmological constant Λ, the first two of which, however, it is hard to understand otherwise than as "fixed". And the formal face of GR, geometry, stands for ideal since ancient Greece.

9.1 The role of curvature revisited

Yet it's worth to stop here for a while and question the "inevitable ensuing" more carefully. First, as mentioned several times already, it should be clear that statements about Nature cannot emanate just from pure deduction. Leaving this epistemological aspect aside, there

[1] One has to postulate that $T^{\mu\nu}$ is the right representative of gravity sources (of all such sources *besides* the gravitational field itself). After Einstein, various other ways have been suggested how to arrive at the field equations, but none of them is "from scratch" (of course).

[2] What about asbestos, bakelite or polystyrene, for example?

still remains a considerable non-uniqueness in physical theories, which is being solved by combination of the "simplicity principle" and – of course – of an experimental testing. In particular, we already admitted, when discussing "general covariance", that the physical laws might in fact look differently than we suggested, without necessarily violating any of the fundamental principles. The non-uniqueness lies in the role of curvature: since it vanishes in special relativity, it means that any tensor-type term proportional to curvature might be added to equations without violating the special-relativity limit of the theory. Actually, in Section 7.3.2, when processing conservation laws for the EM field, we rather dogmatically assumed that Maxwell equations read, in GR,

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha \,, \qquad F_{\{\mu\nu;\rho\}} = 0 \,,$$

but there might appear in them curvature terms in various arrangements, e.g.

$$F^{\alpha\beta}{}_{;\beta} + F^{\alpha\beta} R_{;\beta} = 4\pi J^\alpha \qquad \text{or} \qquad F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha + R^{\alpha\beta} J_\beta .$$

Two levels of response to such doubts are at place at this moment:

- We actually do not only demand a correct special-relativity limit, we demand much more (Section 1.1.2) – that at every point of *curved* space-time there exists a reference frame (the LIFE) with respect to which the theory assumes, locally, the special relativistic form. Curvature of course vanishes in flat space-time of special relativity, whereas in a curved space-time it does *not* vanish locally, so, in order that the equations for matter and non-gravitational fields satisfy the equivalence principle, their curvature terms have to be multiplied by something that does vanish locally. A good example is the geodesic-deviation equation (6.25) where the local *effect of curvature* vanishes due to the limit vanishing of the relative position δx^μ. On the other hand, the above fictional modifications of the first Maxwell set do *not* satisfy the equivalence principle, because their curvature terms do not disappear in LIFE. This shows the non-trivial demands of the principle of equivalence.

  A footnote: Exactly after discussing the geodesic deviation, we mentioned the Mathisson-Papapetrou-Dixon equations (6.28) describing the behaviour of a small test particle with rotational spin, as an example of a problem which does *not* satisfy the equivalence principle because their curvature term does *not* vanish in a LIFE in that case (the Riemann tensor is multiplied by the spin tensor there, and the latter also does not vanish locally). In that particular case, we understood that the problem actually rests in the approximation employing a *spinning point-like object*, which physically is a nonsense.

- Let us accept we can rest assured that we are able to recognize the acceptable curvature terms (= those whose effect vanishes in the LIFE). However, what if someone came and claimed that a certain equation should contain some more (tensorial) curvature terms of the "acceptable" type? Neither the equivalence nor the covariance principles would be violated, and there are no other arguments for how exactly curvature should enter physical phenomena...

To such queries replies **the principle of minimal coupling**, by saying i) that non-gravitational physics should be coupled to curvature in a minimal way; loosely speaking, this means that curvature should not at all enter *local* physics – that it should only reveal itself over finite spatial or temporal scales; and ii) that no terms should be added to the (originally special relativistic) equations, even not such that would satisfy the i) point, besides those arising by applying the

$$\boxed{\text{“comma-goes-to-semicolon” rule}}\ :$$

when generalising any law from special to general relativity, only change $\eta_{\mu\nu}$ for $g_{\mu\nu}$ and partial derivatives for covariant ones (plus total derivatives for absolute ones, of course). Sure that the above example of point-like image of a spinning particle does *not* satisfy this, because there curvature does couple to purely local physics (to the point-like defined spin tensor); and, actually, the above MPD equations can*not* be derived from special relativity just by “comma goes to semicolon”.

It is worth to think it over a bit. Have some non-gravitational physics (electrodynamics, hydrodynamics, etc.) and study it as an exact problem in GR. “Exact” means that the given physics (the EM field, the fluid, etc.) affects the metric according to the Einstein equations, i.e. it curves the space-time. At the same time, the space-time geometry of course affects the behaviour of the physical system in question (the EM field, the fluid, etc.), and we know this is *also* encoded in the Einstein equations, through the conservation laws (and Bianchi identities). However, the principle of equivalence requires that the physical system is not *locally* affected by curvature, that is, if there appear any curvature-dependent terms in the GR equations governing the system, they have to vanish in the LIFE. And this should be ensured by the minimal-coupling prescription. When approaching a theory or an equation in a Lagrangian way, from a variational principle, the minimal-coupling option translates to a requirement that the Lagrangian does not contain terms where the given non-gravitational field appears *multiplied by curvature*. Actually, such product terms would yield, when varied with respect to the field variables, curvature terms in the corresponding Euler-Lagrange equations.

9.2 The issue of higher covariant derivatives

Still there remains one potentially big issue which can hardly be resolved “canonically”. Partial derivatives commute, whereas covariant derivatives do not – and their commutator brings *curvature*. In the special relativistic equations containing higher partial derivatives, how to order them before changing them to covariant derivatives? Apparently, this is similar to the quantum-theory problem of ordering of the (classically commuting) quantities before changing them for operators.

9.2.1 Electrodynamics in general relativity

As an illustration of the problem with higher derivatives, we give the derivation of the wave equation from Maxwell equations. In special relativity, one substitutes

$$F^{\alpha\beta} = A^{\beta,\alpha} - A^{\alpha,\beta} \qquad \text{into} \qquad F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$$

and obtains

$$A^{\beta,\alpha}{}_{\beta} - \Box A^{\alpha} \equiv A^{\beta}{}_{,\beta}{}^{\alpha} - \Box A^{\alpha} = 4\pi J^{\alpha} \,.$$

Using the comma-goes-to-semicolon rule, however, one obtains two versions which of course are not equivalent:

$$A^{\beta;\alpha}{}_{\beta} - \Box A^{\alpha} = 4\pi J^{\alpha} \qquad , \text{ or } \qquad A^{\beta}{}_{;\beta}{}^{\alpha} - \Box A^{\alpha} = 4\pi J^{\alpha} \qquad \ldots \quad ? \tag{9.1}$$

In general, such uncertainties have to be tackled ad hoc. In this special case, one may rely on the initial equation of the first order, rewrite *that* in the general covariant form, $F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\alpha}$, and only then substitute $F^{\alpha\beta} = A^{\beta;\alpha} - A^{\alpha;\beta}$. In such a way, one uniquely arrives at

$$A^{\beta;\alpha}{}_{\beta} - \Box A^{\alpha} = 4\pi J^{\alpha} \,.$$

Using the Ricci identity (6.3) to commute the covariant derivatives in the first term,

$$A_{\beta;\alpha\gamma} = A_{\beta;\gamma\alpha} + R^{\mu}{}_{\beta\alpha\gamma} A_{\mu} \qquad \Longrightarrow \qquad A^{\beta}{}_{;\alpha\beta} = A^{\beta}{}_{;\beta\alpha} + R^{\mu}{}_{\alpha} A_{\mu} \,,$$

we thus obtain

$$A^{\beta}{}_{;\beta}{}^{\alpha} + R^{\alpha}{}_{\mu} A^{\mu} - \Box A^{\alpha} = 4\pi J^{\alpha} \,,$$

which, after applying the Lorenz condition $A^{\beta}{}_{;\beta} = 0$, assumes the form

$$\Box A^{\alpha} - R^{\alpha}{}_{\mu} A^{\mu} = -4\pi J^{\alpha}, \tag{9.2}$$

where the operator acting as

$$(\Box_{\mathrm{dR}} A)^{\alpha} := \Box A^{\alpha} - R^{\alpha}{}_{\mu} A^{\mu}$$

is referred to as the de Rham (vector) wave operator – the general relativistic generalization of the d'Alembert wave operator $\Box \equiv g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}$.

The above wave equation clearly does not satisfy the equivalence principle (neither Ricci tensor nor the four-potential vanish locally), and it also does not satisfy the minimal-coupling prescription, since it cannot be derived from the wave equation of special relativity just by changing commas for semicolons. Yet it *is* a correct wave equation for EM field in GR. To confirm this, let us show that *this* equation – and not the one lacking the curvature term – yields the charge conservation. Should this be true ($J^{\alpha}{}_{;\alpha} = 0$), the divergence of the left-hand side would have to be zero. Proceed straightforwardly,

$$\left(\Box A^{\alpha} - R^{\alpha}_{\mu} A^{\mu}\right)_{;\alpha} = A^{\alpha;\beta}{}_{\beta\alpha} - R^{\alpha}_{\mu;\alpha} A^{\mu} - R^{\alpha}_{\mu} A^{\mu}{}_{;\alpha} \,, \tag{9.3}$$

where in the first term we first commute the last two indices while using the Ricci identity (6.6),

$$A^{\alpha;\beta}{}_{\beta\alpha} = A^{\alpha;\beta}{}_{\alpha\beta} + R^{\sigma\alpha}{}_{\beta\alpha} A_{\sigma}{}^{;\beta} + R^{\sigma\beta}{}_{\beta\alpha} A^{\alpha}{}_{;\sigma} = A^{\alpha;\beta}{}_{\alpha\beta} + R^{\sigma}_{\beta} A_{\sigma}{}^{;\beta} - R^{\sigma}_{\alpha} A^{\alpha}{}_{;\sigma} = A^{\alpha;\beta}{}_{\alpha\beta} \,,$$

and in this we commute the second and the third indices while using the Ricci identity (6.3),

$$A^{\alpha;\beta}{}_{\alpha\beta} = (A^{\alpha}{}_{;\beta\alpha})^{;\beta} = (A^{\alpha}{}_{;\alpha\beta} + R^{\sigma\alpha}{}_{\beta\alpha}A_{\sigma})^{;\beta} = \Box\,(A^{\alpha}{}_{;\alpha}) + R^{\sigma;\beta}_{\beta}A_{\sigma} + R^{\sigma}_{\beta}A_{\sigma}{}^{;\beta} \;.$$

Employing now the Lorenz condition $A^{\alpha}{}_{;\alpha} = 0$ and plugging the above to (9.3), the 4 terms cancel out in pairs and one really obtains zero. It is clear too that zero would not come out if (9.3) did not contain the two Ricci-tensor terms, i.e. if the wave equation did not involve the curvature term.

In special relativity, the wave equation is also satisfied by the electromagnetic tensor itself, as can easily be obtained by divergence of the second set of Maxwell equations (and using the first set then),

$$F_{\{\mu\nu,\rho\}} = 0 \qquad \Longrightarrow \qquad \Box F_{\mu\nu} = 4\pi(J_{\mu,\nu} - J_{\nu,\mu})\,.$$

One suspects that also here the shift to GR will bring extra curvature terms, not "covered" by the comma-goes-to-semicolon rule. Actually,

$$0 = (F_{\mu\nu;\rho} + F_{\rho\mu;\nu} + F_{\nu\rho;\mu})^{;\rho} = \Box F_{\mu\nu} + F_{\rho\mu;\nu}{}^{\rho} + F_{\nu\rho;\mu}{}^{\rho} = \Box F_{\mu\nu} + F^{\rho}{}_{\mu;\nu\rho} + F_{\nu}{}^{\rho}{}_{;\mu\rho}\,,$$

where in the last two terms we again commute covariant derivatives according to (6.6) and use the first Maxwell equation,

$$F^{\rho}{}_{\mu;\nu\rho} = F^{\rho}{}_{\mu;\rho\nu} + R^{\sigma\rho}{}_{\nu\rho}F_{\sigma\mu} + R^{\sigma}{}_{\mu\nu\rho}F^{\rho}{}_{\sigma} = -4\pi J_{\mu;\nu} + R^{\sigma}_{\nu}F_{\sigma\mu} + R_{\sigma\mu\nu\rho}F^{\rho\sigma}\,,$$
$$F_{\nu}{}^{\rho}{}_{;\mu\rho} = F_{\nu}{}^{\rho}{}_{;\rho\mu} + R^{\sigma}{}_{\nu\mu\rho}F_{\sigma}{}^{\rho} + R^{\sigma\rho}{}_{\mu\rho}F_{\nu\sigma} = 4\pi J_{\nu;\mu} + R_{\sigma\nu\mu\rho}F^{\sigma\rho} + R^{\sigma}_{\mu}F_{\nu\sigma}\,,$$

and then in summing the two Riemann-tensor terms we use the first Bianchi identity to shorten $R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = -R_{\sigma\rho\mu\nu}$,

$$R_{\sigma\mu\nu\rho}F^{\rho\sigma} + R_{\sigma\nu\mu\rho}F^{\sigma\rho} = (R_{\sigma\mu\nu\rho} - R_{\sigma\nu\mu\rho})\,F^{\rho\sigma} = (R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu})\,F^{\rho\sigma} = -R_{\sigma\rho\mu\nu}F^{\rho\sigma}\,.$$

Hence, we arrive at wave equation

$$\Box F_{\mu\nu} = 4\pi(J_{\mu;\nu} - J_{\nu;\mu}) + F_{\mu\sigma}R^{\sigma}_{\nu} - F_{\nu\sigma}R^{\sigma}_{\mu} - F^{\rho\sigma}R_{\rho\sigma\mu\nu}\,. \tag{9.4}$$

The curvature terms in the wave equations bring the interesting effect of *scattering of EM radiation on space-time curvature*.

A warning concerning F-mu-nu with upper indices

We have already been warning that one should be careful when rising/lowering indices "under" partial derivatives (whereas it is no problem with covariant derivatives). A special addendum is the case of $F^{\mu\nu}$. Namely,

$$F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}\,, \quad \text{BUT} \quad F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu} \neq A^{\nu,\mu} - A^{\mu,\nu}\,. \tag{9.5}$$

Let us show it in detail:

$$A^{\nu;\mu} - A^{\mu;\nu} = g^{\mu\alpha}g^{\nu\beta}(A_{\beta;\alpha} - A_{\alpha;\beta}) = g^{\mu\alpha}g^{\nu\beta}(A_{\beta,\alpha} - A_{\alpha,\beta}) =$$

$$
\begin{aligned}
&= g^{\mu\alpha}\left[(g^{\nu\beta}A_\beta)_{,\alpha} - g^{\nu\beta}{}_{,\alpha}A_\beta\right] - g^{\nu\beta}\left[(g^{\mu\alpha}A_\alpha)_{,\beta} - g^{\mu\alpha}{}_{,\beta}A_\alpha\right] = \\
&= g^{\mu\alpha}A^\nu{}_{,\alpha} - g^{\mu\alpha}g^{\nu\beta}{}_{,\alpha}A_\beta - g^{\nu\beta}A^\mu{}_{,\beta} + g^{\nu\beta}g^{\mu\alpha}{}_{,\beta}A_\alpha = \\
&=: A^{\nu,\mu} - g^{\mu\alpha}g^{\nu\beta}{}_{,\alpha}A_\beta - A^{\mu,\nu} + g^{\nu\beta}g^{\mu\alpha}{}_{,\beta}A_\alpha = \\
&= A^{\nu,\mu} - A^{\mu,\nu} - g^{\mu\alpha}g^{\nu\beta}{}_{,\alpha}g_{\beta\iota}A^\iota + g^{\nu\beta}g^{\mu\alpha}{}_{,\beta}g_{\alpha\iota}A^\iota = \\
&= A^{\nu,\mu} - A^{\mu,\nu} + g^{\mu\alpha}g^{\nu\beta}g_{\beta\iota,\alpha}A^\iota - g^{\nu\beta}g^{\mu\alpha}g_{\alpha\iota,\beta}A^\iota = \\
&= A^{\nu,\mu} - A^{\mu,\nu} - 2g^{\mu\alpha}g^{\nu\beta}g_{\iota[\alpha,\beta]}A^\iota\,.
\end{aligned}
$$

Note that it may actually not be clear what the upper index for partial derivative should mean, but it is natural to *define* $A^{\nu,\mu} := g^{\mu\alpha}A^\nu{}_{,\alpha}$, i.e. $\partial^\mu := g^{\mu\alpha}\partial_\alpha$.

9.3 Live with the equations

Hence, after all, the GR equations are *not* always ensuing clearly and uniquely from fundamental principles, even with the principle of minimal coupling added. A plethora of papers exist on the logical structure of GR (not speaking about still more difficult underlying layers concerning the "ontological" nature of the metric), where these issues are addressed, possibly together with suggestions how to supplement the principles in order to make the transfer to curved-stage physics completely axiomatized. On the other hand, having experienced complications of such efforts, many respected authors finally admit that *one has to live with that* (with the non-uniqueness), trying to resolve the remaining queries via physical insight and – ultimately – via experiment. Regarding the almost miraculous strength and richness of general relativity and of the Einstein equations in particular, the classical bible MTW [29] nicely comments on the above issues, in §17.5., by saying, casually:

> *"In the beginning axioms told what equation is acceptable.*
> *By now the equation tells what axioms are acceptable."*

9.4 A possible summary of general relativity

Einstein's field equations are the culmination of the general theory of relativity. Let us try, now when we have reached this turning point, to briefly summarize its message – now already in a kind-of axiomatic wording:

- Space-time is a four-dimensional pseudo-Riemannian manifold with the Lorentzian metric ($\equiv$ Lorentzian manifold), that is, it is a four-dimensional smooth manifold,
  – on which it is defined a smooth symmetric tensor field g of the (0,2)-type which is non-degenerate (i.e., it is invertible, which is equivalent to having non-zero determinant) and has the $+2$ signature (equivalently, it may also have -2 signature, but we adhere to the first option); and
  – whose tangent spaces at different points are connected by Levi-Civita connection induced by the above metric.

- The (Einstein's) equivalence principle holds, that is, there exists at every point a local reference system (called LIFE) with respect to which all matter and non-gravitational fields behave in the same way as in special relativity. In other words, there exist a reference frame in which curvature, *locally*, does not have any effect on non-gravitational physical laws.

- The principle of general covariance and the principle of minimal coupling hold, that is,
  – there are *no "absolute elements"* in the theory (which are not *dynamically* coupled to others); somewhat loosely speaking, this is achieved if there appear no other "space-time properties" than those derivable from the physical metric
  – the physical laws are represented by equations whose form is invariant under coordinate transformations given by general diffeomorphisms
  – of all forms compatible with general covariance, the one is chosen which *minimally* differs from its special-relativity version, i.e., which only differs by the change $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ and by "comma goes to semicolon" (see above for possible remaining queries).

- The metric is interconnected with the mass-energy content of space-time through the Einstein field equations (8.4).

9.5 Mathematical layers of space-time

In order for a set to be able to represent the underlying "stage" (the space-time), it has to be equipped with several layers of structure:

- Topological structure
  - It must be equipped with topology, i.e. it has to be a topological space. This is also supposed to be separable (its topology is supposed to have a countable base).
  - It must be a topological manifold, i.e. the topological space has to be Hausdorff and locally Euclidean.

- Differentiable structure
  In order that calculus can be performed on it, it has to be a differentiable (smooth) manifold, that is, it has to be equipped with a differentiable structure (a complete atlas of smooth coordinate maps).

- Geometrical structure
  - An affine connection has to exist on the manifold (which connects its tangent spaces at different points, thus defining parallel transport and covariant derivative).
  - The manifold has to be a (pseudo-)Riemannian manifold, i.e. a smooth manifold on which a smooth and non-degenerate metric exists of a chosen signature (Lorentzian in the case of space-time). The affine connection need not necessarily be tied to the metric. However, in GR, in follows from the equivalence principle that the connection has to be compatible with the metric and to have zero torsion, i.e. that it has to be the Levi-Civita connection.

Of the above structures, only the geometrical one is subject to “field equations”. The field equations do not constrain topology, in particular. (And GR also does not “comment” on differentiable structure, because it considers all coordinate maps acceptable.) Interestingly, we will see in Section 23.5, in the Lagrangian, variational formulation of the theory, that besides the field equations themselves, one also obtains the requirement that the connection be the Levi-Civita connection.

Finally, within “geometry”, one may further distinguish the **conformal part** which is given by (or which gives) **causal structure** (as represented by light cones), plus a **scaling factor** which practically is represented by the **volume element** (thus the metric determinant, see Appendix A).

CHAPTER 10

Introduction

> "Of course, the correct theory of gravitation is general relativity.
> And if it isn't, it certainly should be."

This opinion is being ascribed to I. Robinson, one of the GR protagonists of the second half of the 20th century. But the precise author is not important. It is for long already that relativity has been enchanting by elegance of its geometrical formulation, by an inner consistence within extremely wide range of physical conditions, and by a minimum of free parameters. It has been admired as an extraordinary triumph almost exclusively achieved by a single person – and in particularly heavy circumstances.

However, today it's already justified to also claim firmly that Einstein's theory is respected for its extraordinary resistance against experimental falsification. To better put it positively, GR has anticipated previously unknown effects observed in distant cosmic sources as well as tiny deviations from Newtonian rules recognized in the weak gravitational field around us; it predicted black holes about which astronomical data have already brought a variety of convincing evidence, and, in addition, "there is an ace of all aces up its sleeve" (by R. Blandford) – the cosmic microwave background radiation as a relict of big bang from which our Universe sprung according to it. The confidence in the last crucial prediction of GR – the existence of gravitational waves – was, already before decades, so high that many experts were asking why to build so expensive giant detectors. Actually, it was so obvious that pulsars lose energy in exact agreement with the relativistic formula. At the same time, however, it was clear that the point is not just the direct detection itself, but rather the whole "gravitational window to the universe", through which astrophysics could hope to acquire new, direct information about dramatic interactions of extremely compact objects, as well as about properties of the very early Universe.

10.1 Gravis

It may seem late for an Introduction, but you will enjoy it better if *already knowing* :-).

Of the **four fundamental physical interactions**, gravitation accompanies us on every step (and penalizes every stumble). No surprise that the word root *gravis*, presumedly of ancient proto-Indo-European origin, has stayed unchanged for millennia. It denotes *heaviness* (*weight*), but also *seriousness* (*significance*), *gravidity* (*pregnancy*) or *difficulty*[1].

However, *gravity is very weak*. Actually, the **four fundamental physical interactions** – strong nuclear, electromagnetic, weak nuclear and gravitational – have different strengths. As inferred from their coupling constants or, more accurately, from dimensionless parameters composed from these constants, the "strengths" (intensities) are in order-of-magnitude ratios

$$\mathrm{strong} : \mathrm{EM} : \mathrm{weak} : \mathrm{G} \quad \sim \quad 1 : 10^{-3} : 10^{-15} : 10^{-42} \, .$$

Several remarks are at place.

- The nuclear interactions are short-range, they only act within "nucleon size" and then fall of exponentially, so they must be evaluated within such a micro-scale in order to be non-zero.
- The EM : G proportion is easy to estimate by dividing the EM and G forces which act between two electrons,

  $$\frac{F_\mathrm{G}}{|F_\mathrm{EM}|} = \frac{\frac{Gm_\mathrm{e}^2}{r^2}}{\frac{e^2}{r^2}} = \frac{Gm_\mathrm{e}^2}{e^2} \sim 10^{-40}$$

  (in SI units, it has to be multiplied by $4\pi\epsilon$, with ϵ permitivity).
- Electromagnetism is thus much stronger than gravitation. And it is long-range like gravitation (falling off as $1/r^2$). But *it is not universal*. The electric force even has an "auto-neutralizing" character – electric charges of a given sign do not like to be close to each other, so the plus and minus charges tend to distribute equally. Due to this, cosmic bodies are electrically close to neutral. With gravitation, it is just the opposite: *all* massive bodies gravitationally attract each other, and the more mass has already accumulated at some region, the more it attracts further mass from the surroundings.

Consequently, the physical picture of the universe is – at least on large scales – almost exclusively determined by the gravitational interaction. The best theory of gravitation still is the Einstein's one. Plenty of alternative theories have been developed, but GR still passes the observational and experimental tests with high accuracy. It is by no means automatic. C. Will, who already for decades keeps updated his "Was Einstein Right?" summary of GR testing, wrote in the "Centenary-Assessment" version prepared for the 2015 anniversary:
"Having spent almost half of the century of general relativity's existence being astonished by its continuing agreement with observation, I might be permitted a personal reflection at this point on the future of the subject: It would not at all surprise me if general relativity turned out to be perfectly valid at all scales, from the cosmological to the astrophysical to the microscopic, failing only at the Planck scale where one naturally expects quantum gravity to take over."

[1] In an extreme case, the latter even leads to the *grave*. We apologize for not having told you earlier.

Leaving unaddressed the Planck scale only means an incredible range of validity. Already for decades, a fierce struggle is going on at the border of the Planck scale. Still the numerous arrays of theorists have yet only achieved partial success in their effort to **quantize gravitation**.

10.2 What chiefly is new in general relativity

Chiefly new is the role of the space-time "stage" which hosts the physical events. In other theories, the properties of this stage are *decided a priori*, not following from the given theory or from anywhere at all, and they do not depend on what is happening on the stage. In Newtonian physics, the stage is decided to be the **"absolute space and time"**; in special relativity, it is the **Minkowski space-time**. In GR, "decided" is only that the stage (the space-time) is supposed be some 4D Lorentz-type manifold, which just means that the stage is mathematically sufficiently cultivated for handling the physical quantities properly, plus that it is equipped with the metric (in order to know how to make scalar product). However, the space-time geometry remains unspecified, *it has to be found* as a part of the problem solution, namely as entangled with the distribution and behaviour of matter which is present in the space-time. The matter-geometry "entanglement" is encoded in the Einstein equations.

We should once again recall E. Mach at this place, because he was criticising any *absolute* elements in physical theories – such elements which influence, in a conceptional sense (but possibly even "physically"), other parts of the theory, but themselves remain unaffected by the rest of the ideological construction. In particular, Mach was criticising absolute reference systems, "put there by hand", stressing that *the world should itself provide structures in terms of which it will be described*. As mentioned in Chapter 1, he also deemed inertia to be such an absolute property, and suggested how to understand it in a relational (perhaps *relativistic*?) way. Today, this line of thought is being rendered by the notion of **background independence** of a theory.

GR is also new in the description (thus understanding) of the interaction. In Newtonian theory as well as in electrodynamics, the mutual influence between sources is described by the **force** – a vector field defined on a given space-time background. In GR, objects do not act on each other by "gravitational force" (such a notion actually does not exist in the theory), but through how they curve the space-time. Thus the term **geometrization of the (gravitational) interaction** – the interaction is being communicated by geometry of the "stage" itself, so there is no need to define any additional structure (the force field). We know already that such a shift has been possible thanks to the **universality of gravitation**. It should be emphasized once more, however, that the geometrical language itself is *not* the crucial point: actually, we realized that almost any theory (the Newtonian one, for example) can be re-formulated in geometrical terms. The crucial point is that in GR the geometry is a **dynamical part of the problem**.

10.2.1 What does it mean *dynamical*?

Several words are often used in GR without proper explanation. We think the most important of them are *local* (which however should be understandable: it simply is the opposite of *global*) – and *dynamical*. The latter is less clear: when saying that the Universe is dynamical, one means that it is not static, that is changes in time. When saying that the geometry (or metric) is dynamical, it probably also means that it changes with time. But when claiming that in GR the geometry is a *dynamical part of a problem*, it means that it is *not absolute* in a sense that it *mutually interacts with the rest of the problem* rather than being "decoupled". When something is *not* a dynamical part of a theory, it need not mean that it is given a priori, and it even need not mean that it stays fixed (there may exist some "evolution equations" for it), but it means that it is not "coupled" (mutually interconnected) with the rest of the theory. In special relativity, for example, the space-time is *not* dynamical, because it keeps its Minkowski character irrespectively of what is happening in it, whereas in GR the space-time *is* dynamical – it is entangled with every mass-energy through the Einstein equations.

This little section is to stress that the word *dynamical* represents the most important feature of GR. And that it is *this* sense in which GR is very "Machian", because its ideal is to be free of "absolute" concepts, in particular of a fixed space-time background ("background independence"). The properties of being "absolute" vs. "dynamical" can be more formalised, but let us rather give a few examples, in order to also distinguish the *background independence* from *diffeomorphism invariance* (*covariance*); we follow the excellent book [45] by N. Straumann (section 3.5):

Have a differentiable manifold and consider three metric (g) theories, *all of them covariant under 4D real diffeomorphisms* (having the same covariance group):

- $R_{\mu\nu\kappa\lambda}(g) = 0$
  ... has no dynamical degrees of freedom, because every solution of this theory is equivalent to the Minkowski metric (it is Minkowski "modulo diffeomorphism") – only flat space-time satisfies this theory (and is thus "absolute" according to it).

- $R(g) = 0$ and $C_{\mu\nu\kappa\lambda}(g) = 0\,,$
  where $C_{\mu\nu\kappa\lambda}$ is the Weyl tensor. Recall that the Weyl tensor has three properties: i) it has the same symmetries as Riemann, plus it is trace-free (in all index pairs), so it has only 10 independent components; ii) it is often called *conformal* tensor, for it behaves very simply under conformal transformations of the metric – for $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ (with the conformal factor Ω^2 a scalar function), one has $\tilde{C}_{\mu\nu\kappa\lambda} = \Omega^2 C_{\mu\nu\kappa\lambda}$; and iii) it vanishes for a conformally flat metric, i.e. for a metric which differs from the Minkowski metric just by a conformal factor (and possibly a diffeomorphism).
  The above theory thus implies that the metric *is* conformally flat, $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, up to a diffeomorphism, so the only degree of freedom rests in the scalar field Ω (equation $R = 0$ yields the flat wave equation for it). Therefore, in such a theory the metric is again not dynamical – it is "conformally absolute".

- $R_{\mu\nu} = 0$
  ... the vacuum (and Λ-free) Einstein equations – involving $g_{\mu\nu}$ as a completely *dynamical* object.

10.3 Non-Euclidean geometries

Similarly as it seems insufficient to easily state, at the beginning of special relativity, that at one May night Einstein realized what it means *time* and *simultaneity* (while before it was clear for ages...), one "obvious" page on curved space-time stage cannot convey anything of more than two millennia during which Euclid's *Elements* "absolutely" ruled the geometry, impressing as kind of a miracle for the admirers of logical and aesthetic perfection.[2] During which their *fifth postulate*[3] was provoking a query whether it could not be proved from the remaining four axioms, while finally it turned out that this is not possible and that Eukleides correctly listed it as independent. This finding came at the beginning of the 19th century, together with the discovery of consistent geometries where the fifth postulate does not hold – the **non-Euclidean geometries**.

As it is clear from correspondence, the non-Euclidean geometry was first – during the first decades of the 19th century – approached by K. F. Gauss. However, he deemed the topic too controversial to officially publish anything on it. In 1820-25, a similar finding was made, independently, by a Hungarian graduate from the Vienna University, later an army worker J. Bolyai; in 1831 he published an essay on that matter as an appendix to the book of his father. In the meantime (around 1825), the same result was achieved independently by the rector of the Kazan University N. Lobachevsky; an official paper about the new geometry he published in the university journal Kazan Messenger. Specifically, Gauss, Bolyai and Lobachevsky discovered the surface of constant negative curvature.

Gauss then found quantities which permitted to describe *generic* surfaces and classify them in terms of their *intrinsic* properties (independently of how they appear from outside, as immersed in a three- or higher-dimensional Euclidean space): the metric and the scalar curvature (called Gauss' curvature today) determined by the metric tensor and by its first and second derivatives.

K. F. Gauss was also present at another, definitive breakthrough: he was a chairman of the committee for habilitation of G. F. B. Riemann. Riemann did not live to 40, but he left his name in every area he touched. In his doctorate which he worked out, as an assistant of W. E. Weber, under Gauss' supervision, he developed the theory of complex manifolds known today as Riemann surfaces. Then he started working at the Göttingen University and devoted his habilitation thesis to the relationship between integrability of a function and the possibility to expand it into a Fourier series (Riemann conditions...). For the habilitation talk, he had to prepare *three different topics* [sic]; two concerned electricity and one concerned geometry. Gauss selected the last one. On Saturday, June 10, 1854, Riemann gave the lecture *"On the hypotheses that lie at the foundation of geometry"* which opened a pathway to curved spaces of arbitrary dimension – the Riemann manifolds. The lecture was successful,

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[2] Einstein was remembering how, when he was 12, a student of medicine Talmud lent him a book on Euclidean geometry: "Here were assertions, as for example the intersection of the three altitudes of a triangle in one point which – though by no means evident – could never-the-less be proved with such certainty that any doubt appeared to be out of the question. This lucidity and certainty made an indescribable impression on me."

[3] Let a straight line intersect two other straight lines, thus forming two interior angles on both sides. Then the two straight lines, if extended arbitrarily, meet on that side of the intersecting line where the interior angles sum to less than two right angles.

though only Gauss was reported to really enjoy it. Actually, according to H. Weyl, the Riemann's habilitation essay "was not grasped by his contemporaries, and his words died away almost unheard (with the exception of a solitary echo in the writings of W. K. Clifford)". Looking back, one would also mention E. B. Christoffel who further developed Riemann's ideas already in 1860s (the curvature tensor if often called the Riemann-Christoffel one).

Riemann discovered that while the 2D-surface curvature can be fully represented by just one function (the Gauss curvature), for a 3D space 6 functions are needed and for a 4D space it is 20 functions; they form the components of the Riemann curvature tensor. In the written version of the lecture, one can also read: "The space of constant curvature is necessarily finite, if the curvature is positive, even if arbitrarily small... It is conceivable that within very small scales the space does not satisfy the axioms of [Euclidean] geometry... Properties which distinguish the space from other 3D entities have to be derived from experience..."

Actually, it was already from Gauss' times that ideas began to appear about whether the real world might have something in common with the new geometry, and that it might be tested by measurement. "Between Riemann and Einstein" it was most notably W. K. Clifford who, e.g. in his Cambridge talk "On space theory of matter"(1870), considered that matter and energy may in fact be symptoms of spatial geometry curved in various ways. Clifford's view was kind-of a foretaste to **unitary theories** of the 20th century (Hilbert, Weyl, Einstein) which attempted at a unified explanation of natural interactions (the gravitational and the EM ones, in particular). Their aspirations are by far not forgotten;[4] today, it has notably been the string theory which tries to accomplish such a programme in a different, "microscopic" way. In the unitary theories, primary is the field, the question being how sources are formed out of that field. GR keeps the view of a classical field theory where sources and the field are distinct entities, but the questions of how space-time is deformed by the sources and of how sources are affected by space-time are *coupled*. The field keeps its own degrees of freedom, not fully determined by behaviour of the sources.

Einstein was born exactly 11 days after the death of Clifford. It was his 1907-1915 effort what definitively interlinked the geometrical properties of space-time with physics which is taking place in it.

10.4 What chiefly is unusual in general relativity

At the beginning of his route to the new theory of gravitation, Einstein had no clue about such a link. It was at the end of 1907 when he first formulated the principle of equivalence and predicted, on its basis, the light bending and the gravitational shift of frequency. In his *Autobiographic notes*, he then explained: "Why were another seven [in fact eight] years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning."

Still today, for a student coming with the knowledge of "classical physics" and special relativity, the main "issue" is the non-trivial geometry of space-time. Not so much because

[4] See e.g. [29]: "What else is there out of which to build an 'elementary particle' except the geometry? And what else is there to give discreteness to such an object except the quantum principle?"

one wouldn't be able to imagine, after reducing the number of dimensions (i.e., typically on a curved 2D surface), the objects and rules of the Riemann's theory, rather it is difficult to link these abstract notions, via their coordinate or basis representations, to measurable physical quantities. Namely, in special relativity the coordinate components of tensors do in general have direct physical sense – typically, if a certain quantity has certain components in some inertial system, then an observer at rest in that system really measures such values. In GR it is only exceptionally so; the numbers which coordinates assign to quantities at every point of space-time need not correspond to the measurements of *any* observer. The physical meaning only have invariants which can be computed from the coordinate components using the metric tensor. However, even these sometimes may not indicate "what is much and what is little", because in curved space even invariants may not correspond well to our intuition.

If missing the simplicity of special relativity, it's good to realize it is not entirely at place. Things were only simple *in Cartesian coordinates* (denote them by ξ^α). Whenever transforming to curvilinear coordinates (x^μ; we will assume they are orthogonal), it is necessary – even without any gravity/curvature – to use the metric tensor

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\,\eta_{\alpha\beta}$$

to compute scalar products and (thus) invariants. This is in fact even necessary in the 3D Euclidean space if one does not use Cartesian coordinates. In classical physics, if the coordinates are orthogonal, this point is usually being solved using the Lamé coefficients given by

$$(h_i)^2 = \delta_{jk}\frac{\partial \xi^j}{\partial x^i}\frac{\partial \xi^k}{\partial x^i}\,.$$

These correspond to the metric tensor through the simple relations $(h_i)^2 = g_{ii}$. For example, in spherical coordinates (r, θ, ϕ) one has $g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$, while in the cylindrical coordinates (ρ, ϕ, z) one has $g_{ij} = \mathrm{diag}(1, \rho^2, 1)$.

As an illustration, let us recall that the covariant divergence and Laplace operators in an ordinary Euclidean space read

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[(h_1h_2h_3A^1)_{,1} + (h_1h_2h_3A^2)_{,2} + (h_1h_2h_3A^3)_{,3}\right],$$
$$\Delta\psi = \frac{1}{h_1h_2h_3}\left[\left(\frac{h_2h_3}{h_1}\,\psi_{,1}\right)_{,1} + \left(\frac{h_3h_1}{h_2}\,\psi_{,2}\right)_{,2} + \left(\frac{h_1h_2}{h_3}\,\psi_{,3}\right)_{,3}\right],$$

while in the language of the metric tensor, we know they appear as

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,A^i\right)_{,i} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,g^{ij}A_j\right)_{,i}\ ,$$
$$\Delta\psi = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,\psi^{,i}\right)_{,i} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,g^{ij}\psi_{,j}\right)_{,i}\ .$$

Translation: realize that for orthogonal coordinates the metric is diagonal, so

$$\sqrt{g} = \sqrt{g_{11}g_{22}g_{33}} = h_1h_2h_3, \qquad \sqrt{g}\,g^{ij} \to \sqrt{g}\,g^{ii} = \frac{\sqrt{g}}{g_{ii}} = \frac{h_1h_2h_3}{(h_i)^2}\ .$$

10.5 New predictions of general relativity

GR brings two main general predictions: that *our physical space-time is curved* and that

$$\boxed{\text{the space-time is not just a passive "stage", it actively participates in physical events}}\,.$$

Actually, since possessing its own degrees of freedom, it may even be entirely responsible for certain happenings (like those induced by gravitational waves). Turning to more specific novelties now, let us mention black holes, dynamical universe and gravitational waves.

In 1783, a polymath J. Michell published, in Philosophical Transactions, a very natural consideration: given that the speed of light is finite (it was known thanks to Rømer 1676 and Bradley 1729) and that, according to the corpuscular theory, gravity should also act on the light corpuscles, some of the celestial bodies might be so compact (massive and dense) that the escape speed on their surface would exceed the speed of light. Hence, some of the cosmic objects – and most likely those of greatest mass! – might be unobservable. Newtonian result is based on comparison of the kinetic and potential energy. So have a spherically symmetric body of mass M and radius R, and let a particle of mass m be launched just radially outwards from some radius r. It is very simple: the escape speed reads

$$\frac{1}{2}mv^2 = \frac{GmM}{r} \qquad \Longrightarrow \qquad v_{\rm esc}^2 = \frac{2GM}{r} \equiv -2\Phi\,,$$

so it is larger than c if

$$\frac{2GM}{r} > c^2 \qquad \Longleftrightarrow \qquad r < \frac{2GM}{c^2}\,.$$

In particular, the "star" is invisible (from infinity at least) if its radius is below that value, $R < \frac{2GM}{c^2}$. To summarize, if a mass M is concentrated in a spherical ball of radius smaller than $\frac{2GM}{c^2}$, photons emitted from its surface do not fly arbitrarily far. The critical value is nowadays called the Schwarzschild radius, because in GR it comes out exactly the same (there, however, the whole semester is needed to derive it). Worth to note, however, that in the Newtonian picture photons *can travel at least somewhat outwards from any radius*, only that from some radii they may not be able to reach arbitrarily far. In GR, on the contrary, the region below the Schwarzschild radius has totally new properties, unknown from the Newtonian physics.

The regions below the Schwarzschild radius are called **black holes**. The gravitational field is so strong there that even the fastest signal, light, cannot travel outwards; not only that it cannot escape to infinity, it even cannot move to larger radius at all. Actually, even the photons emitted "outwards" from anywhere there have to move to *smaller* radii. When inside a black hole, everything thus has to descend towards its centre. Therefore, a **space-time singularity** occurs there – a point-like or ring-like region where mass density diverges, and consequently also curvature (infinite tidal forces). Such a singularity resembles the one occurring, in any classical field theory, in the field quantities evaluated at the very point-like source ("charge"), i.e., in $\Phi = -GM/r$, $E = Q/r^2$, etc. However, there (in Newtonian gravity or electrodynamics in the Minkowski background), the singularity only concerns the *given* field, because that field is *not* considered to affect the properties of the underlying space(-time), so one can safely study any *other* physics at that point. In GR, on the contrary, the space-time singularity really means a pathology in the underlying manifold, so it is not possible to pursue *any* physics there.

Gravitational waves are ripples in the gravitational field (= in the space-time geometry) which can transport energy between distant areas. Again, they may (correctly) be regarded as a counter-part of the EM waves, yet in a standard-electrodynamics picture the latter do not affect the underlying (Minkowski) space-time, so they are not being felt by any non-EM physics possibly happening in a given region. On the contrary, the gravitational waves represent oscillation of the underlying "stage" itself, so *any physics* which may happen there does feel them. Just to be on the safe side, it should be stressed that within an exact, self-consistent GR view, both the EM and gravitational waves carry energy (and thus deform space-time), so they actually induce each other (if there is any EM field present, of course). Gravitational waves were first predicted in 1916 when Einstein tackled his linearised equations describing a weak gravitational field.

Finally, the idea that the **Universe is dynamical**, i.e. that its properties vary in time, has both the observational and theoretical history, starting about when GR was born. V. Slipher, G. Lemaître and E. P. Hubble measured, between 1915 and 1929, that most other galaxies recede from our one (better to say, from each other), evidencing the cosmic expansion. On the theoretical side, the first example of dynamical solution was obtained by W. de Sitter in 1917, but the first thorough account of basic, homogeneous and isotropic cosmological scenarios including matter as well as cosmological constant was given by A. Friedmann in 1922.

Not that the above novelties would have been welcome enthusiastically. At the time when GR was born, the image was preferred of a static Universe infinite in space as well as in time, where "suns shine for ever" and nothing is changing. And, suddenly, there came a dynamical Universe, perturbed by gravitational waves and hosting matter prone to a collapse to black holes, in whose interiors it is being totally destroyed in singularities... In spite of these uncomfortable predictions, the theory flattered the theorists. Even S. Weinberg writes in [52]: "I believe that it was this intrinsic attractiveness that preserved physicists' belief in general relativity during the decades when the evidence from successive eclipse expeditions continued to prove so disappointing." Apparently, first it worked for Einstein himself; on February 8, 1916, he wrote to A. Sommerfeld: "Of the general theory of relativity you will be convinced once you have studied it. Therefore I am not going to defend it with a single word."

However, besides the GR's appeal, Einstein's faith was – already a week or two before delivering the final version of the theory – strongly supported by calculation of the perihelion precession of Mercury. And it is also not fully true that the eclipse expeditions brought but disappointment...

10.6 Classical tests of general relativity

10.6.1 Advance of pericentre

It is known from Newtonian celestial mechanics that the bound orbits in the potential $-GM/r$ are closed ellipses, whereas if the potential is perturbed by some other term, the ellipses rotate in space. If the field is stronger than GM/r^2, the particle is more attracted towards the centre, which results in an advance of the ellipse's pericentre (shift in the sense of orbital motion). This is the case in GR, so one might expect such an effect to occur. However, if speaking of our Solar system, the effect should be very tiny, and it would also be necessary to first subtract purely Newtonian perturbations due to other planets plus, possibly, that due to the oblateness of the Sun. A sufficiently accurate calculation was first performed by U. J. Le Verrier in the middle of the 19th century; it showed that 43 arc seconds per century *[sic]* cannot be explained within the Newtonian theory.

Einstein had been thinking about this effect since his first steps towards general relativity at the end of 1907. From that time, he was testing on it consecutive versions of his theory. However, only on 18th November 1915 he showed to the Prussian Academy definitively that the Newtonian limit to GR explains the tiny 43 seconds with great accuracy.[5] Einstein's correspondence reveals that it was this particular result which probably brought him the deepest professional satisfaction ever. He experienced heart palpitations and on 17th January 1916 he confirmed to P. Ehrenfest: "I was beside myself with joy and excitement for days." We will enjoy this result in Section 17.1.2.

10.6.2 Bending of light, and its frequency shift

Already before Einstein reached his final field equations, there appeared serious efforts to find whether, during a Solar eclipse, the light of distant stars bends – and *how much* it bends. The measurements organized for October 10, 1912 (at Córdoba, Argentina), were – like quite some others which were to come – spoiled by bad weather. [The crew already included A. S. Eddington.] At the following occasion, on 21st August 1914, a team of E. Finlay-Freundlich, also supported by Einstein, prepared for observation at Crimea, but because the First World War broke out, they were for weeks detained in Odessa on the suspicion of espionage, with instruments confiscated. [100 years later, they would likely have ended worse.]

Weather was also an issue during the successive eclipse, on 29th May 1919, yet it still permitted two teams lead by A. S. Eddington and F. W. Dyson, located at Sobral (Brazil) and at Príncipe island (Gulf of Guinea), to make several photographs. These revealed that the light rays from distant stars, when passing by the Sun on their way to us, are being bent by Sun's gravity according to then novel Einstein's theory, rather than according to the Newtonian gravity and corpuscular theory of light (and rather than not being bent at all). For decades, this result was being quoted then as inconclusive or even biased, but recent re-analysis of the measurements confirmed high quality of the original expedition judgements [22]. The result, officially announced on 6th November 1919, was a crucial launch pad for GR. The New York Times was writing "Lights All Askew in the Heavens", "but Nobody Need Worry", and that it had verified "the prediction of Dr. Einstein, Professor of physics in the University of Prague" (which had no longer been true for more than 7 years then). When Einstein came to visit America, he was welcome as a film top star.

Among "classical" experiments, we should also mention the measurements of the light's frequency shift. After a series of failures in the early years of GR (often connected with the

[5] At that time, the theory still was not finished: "the last fallacies in this struggle" Einstein corrected by the following Thursday, but these did not affect the Newtonian limit.

name of enthusiastic E. Finlay-Freundlich), R. V. Pound and G. A. Rebka were finally able to detect the effect in the Earth's gravitational field in 1960. The experiment was made in a 22.6-metre high tower, which corresponded to the relative frequency shift $|\Delta\nu/\nu| \doteq 2.46 \cdot 10^{-15}$ (!) – see Chapter 4. Recall that the prediction follows just from the equivalence principle, so no "GR" (the field equations) is actually necessary; the result thus mainly was a technical achievement.

10.6.3 Dynamical Universe

Another triumph had been coming in the meantime. Though Einstein himself preferred – because of his Machian view of inertia, seeming staticity of the Universe as well as for curvature reasons – a closed static cosmological model, his field equations clearly prefer dynamical scenarios. Within the period 1915–1929, the spectroscopy of distant galaxies revealed that the Universe is expanding (V. Slipher, G. Lemaître, E. P. Hubble). In parallel (in 1922 and 1924), A. Friedmann wrote two papers where general relativistic homogeneous and isotropic cosmologies were introduced thoroughly, with Einstein's static universe only appearing as an (unstable) marginal case.

10.6.4 Cosmic microwave background radiation

Expansion immediately evokes further question: what was in the past? If playing the cosmic-expansion film backwards, at a certain moment one arrives at a very dense state. From the 1940s, G. Gamow argued that for thermodynamic reasons the Universe had to also be very hot then, so hot that radiation and matter were coupled in equilibrium. After the temperature fell below a certain value due to the expansion, radiation decoupled from matter and, from those times, it should be "left" in the Universe, just cooling down gradually and not any more interacting with matter effectively. Subsequently, the properties of such a relict radiation were also estimated by others, most notably by R. Alpher and R. Herman.

Already in 1941, A. McKellar effectively discovered this radiation (at 2.3K) in stellar spectroscopic data, but interpreted it as a rotational temperature of interstellar molecules. In the 1950s, several detections were reported by experiments really focused on searching for the relict microwave background, but they apparently were not deemed convincing enough. In 1964, A. Penzias and R. Wilson were looking for something completely different – a suitable frequency range for a radio communication by means of bouncing off the metalized spherical-baloon satellites (project called Echo). After fighting, for some time, with "excess of antenna temperature at 4080 megacycles per second" (meaning an extra radiation of unknown source at 3.5K), Penzias was informed by his friend B. F. Burke about a preprint where J. Peebles was discussing the possibility of detecting the **cosmic microwave background radiation** (CMBR) left over from the hot birth of our Universe (the **"big bang"**). Actually, J. Peebles together with R. H. Dicke and D. Wilkinson were just at the same time – and just 60km away from Penzias & Wilson's antenna – preparing to measure the expected microwave relic of big bang. The two groups were thus put together and the discovery was out. By the time when Penzias and Wilson were awarded Nobel Prize (in 1978), Gamow had already been deceased for 10 years. Yet theorists do not forget about his prediction.

10.6.5 Black holes

Though leading the light-bending 1919 expedition and though being the author of the first English textbook on GR, the picture of gravitational collapse was already too much for Eddington. When, due to the Chandrasekhar's calculation of the maximum mass which can be supported by degenerate fermion gas (1930), such a picture ceased to be just mathematical scenario, Eddington protested: "I think there should be a law of Nature to prevent a star from behaving in this absurd way!" He also added more technical arguments, but these gradually turned out to be wrong.[6] In 1939, J. R. Oppenheimer and H. Snyder calculated, exactly, the general relativistic gravitational collapse of a spherically symmetric ball of dust. Yet of much more concern were *explosions* then, so the study of collapse was postponed.

In the astronomical community, black holes remained an extravagant topic until the 1970s. Despite the "golden years" of black-hole theory (1963-1974, say), eventually it was thanks to the astronomical discoveries that this status came to an end. X-ray sources were discovered in 1962, quasars in 1963 and pulsars in 1967. Before long, the unprecedented luminosity of quasars was connected with accretion onto black holes, pulsars were identified with fast rotating magnetized neutron stars, and many of the X-ray sources turned out to be located in binary systems containing a black hole or a neutron star. The ultracompact objects of GR quickly spread to the astrophysical literature.[7]

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[6] The controversy between Eddington and Chandrasekhar was quite harsh. After some years, they met in Cambridge. At dinner at the high table Chandrasekhar, Eddington, Dirac and Maurice Pryce were seated together. Chandrasekhar reminiscences follow: *Pryce expressed surprise at seeing me and asked me whether I would join them in discussion with Eddington after Hall on the matter of relativistic degeneracy. After Hall, we adjourned to Pryce's room in Neville's Court. The discussion began with Pryce trying to tell Eddington his version of Eddington's arguments against relativistic degeneracy, so that Eddington could be satisfied that he, Pryce, understood Eddington's arguments. After Pryce had completed his narration, Eddington remarked that Pryce's account was entirely fair and accurate, and asked, 'What was the argument about? Pryce turned to Dirac and asked him, 'Did you agree with any of the things I have said?' Dirac said, 'No.' Pryce added, 'I do not either.' Eddington became very angry – in fact, it was the only occasion when I saw him really angry. He got up from his chair, walked back and forth, and said, 'This matter is not for joking!' He went on finding fault with Pryce's argument even though he had agreed with it a moment earlier and, for the next hour of so, it was Eddington's monologue. Next day, after Hall, Eddington came up to me and said that he was very disappointed that Dirac did not seem to understand the implications of his own relativity theory of the electron. I did not assent or dissent with Eddington's remark but asked instead, 'How much of your fundamental theory depends on your ideas on relativistic degeneracy?'; He replied, 'Why, all of it!'. Since I did not react to that remark, he asked me why I had asked the question. My response was, 'I am only sorry' – not a polite remark to have made but by that time I was really enraged with Eddington's supreme confidence in himself and his own ideas.*

[7] That year was not only a milestone due to the discovery of quasars, but also due to the discovery by R. P. Kerr of an exact metric describing a vacuum space-time containing a stationary (rotating) black hole. At the end of 1963, the 1st *Texas Symposium on Relativistic Astrophysics* was thus organized in Dallas in order to link the efforts of relativists and astrophysicists. In his book [48], K. S. Thorne recalls the atmosphere of that "crossover": "The astronomers and astrophysicists had come to Dallas to discuss quasars; they were not at all interested in Kerr's esoteric mathematical topic. So, as Kerr got up to speak, many slipped out of the lecture hall and into the foyer to argue with each other about their favorite theories of quasars. Others, less polite, remained seated in the hall and argued in whispers. Many of the rest catnapped in a fruitless effort to remedy their sleep deficits from late night science. Only a handful of relativists listened, with rapt attention. This was more than Achilles Papapetrou, one of the world's leading relativists, could stand. As Kerr finished, Papapetrou demanded the floor, stood up, and with deep feeling explained the importance of Kerr's feat. He, Papapetrou, had been

The “golden years” of black holes have not in fact ended yet. These objects have been most probably recognized in most of the galactic nuclei (including that of our Galaxy) and in many X-ray sources. In 2015, a new, high-mass type of stellar-size black holes has been discovered thanks to the gravitational waves. In 2019, a first-ever image of a black-hole silhouette was provided by radio-interferometric study of the supermassive black hole in the nucleus of the M87 galaxy. A similar image, or even a video, of our-nucleus black hole (Sgr A* source) is awaited.

10.6.6 Gravitational waves

In 1916, Einstein developed a linearized version of his theory, arriving at the wave equation for a deviation of the metric from the Minkowski form. He also derived the famous *quadrupole formula* for a power generated by a source in the form of gravitational waves. The topic has experienced long evolution vacillating between doubts (also of Einstein himself) and keen attempts. Finally, after a long and extensive effort which started at the beginning of 1980s, the first gravitational signal was detected in 2015 by interferometric detectors of the LIGO-VIRGO collaboration. (The leading personalities of the effort – B. C. Barish, K. S. Thorne and R. Weiss – were awarded the Nobel Prize in 2017.) The detection was interpreted as generated by inspiral and merging of two black holes of about $30M_\odot$. The detected pulse evidenced the most energetic event ever observed, with about $3M_\odot c^2$ turned into the waves in fraction of a second, producing $3.6\cdot10^{49}\,\mathtt{W}$ of peak power in the last milliseconds (which was estimated to exceed 50 times the combined radiative power of all the stars in the observable universe).[8] As of 2020, an event with even $7.6M_\odot c^2$ energy output has been recorded, and also one or two events which were interpreted as produced by the merger of two neutron stars (with not yet clear nature of the resulting object).

10.7 Application range of general relativity

However, in most astrophysical situations the GR deviations from Newtonian predictions are very small, so Newton’s theory remains a sufficient tool. There are three exceptions:

- Regions of very strong and non-homogeneous gravitational field, typically occurring (permanently) in the vicinity of extremely compact objects, such as black holes or neutron stars.

trying for thirty years to find such a solution of Einstein’s equation, and had failed, as had many other relativists. The astronomers and astrophysicists nodded politely, and then, as the next speaker began to hold forth on a theory of quasars, they refocussed their attention and the meeting picked up pace.”

[8] Were you more used to ergs per second, the relation reads $1\,\mathtt{kW}=10^{10}\,\mathtt{ergs/s}$. Cannot resist a reminiscence: *some years ago*, I (O.S.) was reporting about my master thesis on a seminar of our class. The seminar leader (our lecturer J.B.) brought a paper and a pen as usual, but was not writing down anything then. When speaking about power of quasars, I used ergs/s and added the above conversion relation. J.B. suddenly picked a pen and made a note, asking me to repeat the relation once more. We were laughing, but he responded, seriously: “Always good to learn something new.”

- Space-time in a very large scale, i.e. cosmology. Namely, even if deviations from the Euclidean/Minkowskian geometry are negligible in local scales, the large-scale character of the manifold may be totally different from the flat case.

- Situations with significantly time-variable gravitational field, as e.g. in the gravitational collapse or in gravitational waves. This can be expected from the fact that the Newtonian field equation $\Delta\Phi = 4\pi\rho$ contains no time derivatives at all.

Where, on the other hand, lie the limits of applicability of the "classical" (non-quantum) GR? These are given by such a small scale on which the classical notions of space (distance) and time cease to have good sense, because fluctuations of the space-time geometry are no more negligible. An order-of-magnitude estimate of that scale is given by

$$l_{\rm Planck} := \sqrt{G\hbar/c^3} \doteq 1.6 \cdot 10^{-33}\,\texttt{cm} \qquad \ldots \quad \text{Planck length}\,, \tag{10.1}$$

$$t_{\rm Planck} := l_{\rm Planck}/c \doteq 5.4 \cdot 10^{-44}\,\texttt{s} \qquad \ldots \quad \text{Planck time}\,. \tag{10.2}$$

These "quanta of space and time" are usually incomparably smaller than resolution in which it is necessary to address problems. The enormous gap between the nuclear scale ($1\,\texttt{fermi} = 10^{-13}\texttt{cm}$) and the Planck scale incites discussions whether it is reasonable, in looking for the quantum theory of gravitation, to rely on standard tools such as the quantum field theory. On the other hand, the Planck mass is very large. It is such mass whose reduced Compton wavelength equals the Planck length,

$$\frac{\hbar}{Mc} = \sqrt{\frac{G\hbar}{c^3}} \qquad \Longrightarrow \qquad M = \sqrt{\hbar c/G} \doteq 2.2 \cdot 10^{-5}\texttt{g} := M_{\rm Planck}\,. \tag{10.3}$$

Finally, the Planck density naturally follows as

$$\rho_{\rm Planck} := \frac{M_{\rm Planck}}{l^3_{\rm Planck}} = \frac{c^5}{G^2\hbar} \doteq 5.2 \cdot 10^{93}\texttt{g}/\texttt{cm}^3\,. \tag{10.4}$$

This is really an enormous number: the largest density which "usually" appears in physics is the nuclear density $\sim 2 \cdot 10^{14}\texttt{g}/\texttt{cm}^3$ (present in atomic nuclei and, macroscopically, in neutron stars). It can be concluded from here that *general relativity is valid under extremely broad range of physical conditions*; it only ceases to be adequate in the closest vicinity of what it regards as space-time singularities – in particular, in the first 10^{-43} second after the big bang and in the very final stages of gravitational collapse.

CHAPTER 11

Lie derivative and space-time symmetries

In order to properly enjoy Einstein's equations and their solutions, it is worth to insert a chapter on Lie derivative which is the key to space-time isometries ("symmetries"). Besides, before addressing this goal, we will first *enjoy* another type of transport (the transport along vector fields), different from the parallel transport we know from one of the first chapters.

We began the chapter on parallel transport by stressing the need to know how to transport, reasonably, quantities living in (or acting on) tangent spaces of the manifold (i.e. tensors), between different manifold points. A default example when such a transport is necessary is a derivative, where, according to basic definition, a given quantity has to be subtracted after evaluation at two different points, and that is not possible without first transporting the two "values" to the *same* point, because on a curved manifold the tangent spaces at different points are different, so the direct subtraction is not a well defined operation. From the picture of "keeping direction", we derived the parallel transport as *one possible* answer to the transport problem. The corresponding derivative operation is the covariant derivative. Here we derive another possible transport of tensorial quantities, and the corresponding derivative. It is more primitive in the sense that it does not require any special structure on the manifold (whereas the parallel transport requires affine connection).

A simple image first. In chapter on curvature, we physically understood the Riemann tensor thanks to the equation of geodesic deviation. There, we considered a congruence of time-like geodesics, given by diffeomorphism $x^\mu = x^\mu(l;\tau)$, where the parameter l identified the geodesics while the parameter τ specified where on a given geodesic one is. It's worth to realize again that for such a construction one needs affine connection (for geodesics as well as for Riemann); later in the geodesic-deviation section, we also needed the metric in computing scalar products. Imagine now a bare smooth manifold, not necessarily equipped with any other structure (it need not be a pseudo-Riemannian manifold). Still there always exists one simple way how to transport quantities: in every smooth manifold, there exist congruences of curves, and the quantities can be transported along them, including those which live in tangent spaces.

11.1 Flowing along congruences of curves

So let us have some congruence given by diffeomorphism $x^\mu = x^\mu(l;\lambda)$ parametrized by λ; the latter is a generalization of proper time τ which we used in geodesic deviation, while l again "labels the curves", i.e. orbits of the mapping. The mapping is supposed to cover the whole manifold, or a certain region we are interested in. The diffeomorphism maps from some real intervals of l and λ to the manifold (which in coordinates means to $\mathbb{R}^4$). Recall that diffeomorphism is one-to-one and smooth, together with its inverse; in such a case, each point is passed through by exactly one curve (the curves do not intersect and "fill" the whole region under study).

Imagine now to shift, along any of the curves (any l), by some finite interval of λ, call it $\Delta\lambda$. Such a shift represents a different diffeomorphism, now mapping between manifold points; call it ϕ:

$$\phi(\Delta\lambda): \quad x^\mu(l;\lambda) \longrightarrow x^\mu(l;\lambda+\Delta\lambda) \qquad \text{for any real } \ l,\lambda,\Delta\lambda\,.$$

Shifts by various possible $\Delta\lambda$ along a given curve (given l) form a one-parameter group of diffeomorphisms (parameter is the increase $\Delta\lambda$). Actually,
i) by composition of two shifts, one again gets a shift represented by a given type of diffeomorphism: if $\Delta\lambda = \Delta_1\lambda + \Delta_2\lambda$, then, for any l and λ (we do not write l explicitly)

$$x^\mu(\lambda+\Delta\lambda) \equiv \phi(\Delta\lambda)[x^\mu(\lambda)] = \phi(\Delta_2\lambda)\,\{\phi(\Delta_1\lambda)[x^\mu(\lambda)]\} = \phi(\Delta_1\lambda)\,\{\phi(\Delta_2\lambda)[x^\mu(\lambda)]\}\,,$$

ii) the shifting is surely associative (and it is also commutative),
iii) the identity mapping (zero shift) is obtained, for any l and λ, by taking $\Delta\lambda = 0$,
iv) the inverse mapping (reverse shift) is obtained, for any l, λ and $\Delta\lambda$, by taking $-\Delta\lambda$.

In analogy with motion along fluid streamlines, such a diffeomorphism group is being called a **flow**. There is a unique correspondence between the congruence (the "streamlines") and its tangent field

$$\xi^\mu := \frac{\mathrm{d}x^\mu(l;\lambda)}{\mathrm{d}\lambda}\,,$$

so one also speaks of the **flow of a vector field**; the vector field ξ^μ is the **generator of the flow**. (We might actually have started from considering a smooth vector field and only then draw the congruence as given by its integral curves, and thus *arrive* at the flow.)

How to naturally transport quantities along such a flow? As expected, scalar functions will be transported in such a way that, for any curve (l), any λ and any $\Delta\lambda$, their value at $x^\mu(l;\lambda+\Delta\lambda)$ be identical to the "initial" value at $x^\mu(l;\lambda)$. The transport of tensors requires a bit more reasoning, but the picture remains simple.

11.2 Mappings of tangent spaces induced by the flow

Having a flow along some congruence in a manifold, i.e. knowing how to transport *points*, one also finds easily how to naturally transport objects living in tangent spaces. Imagine to

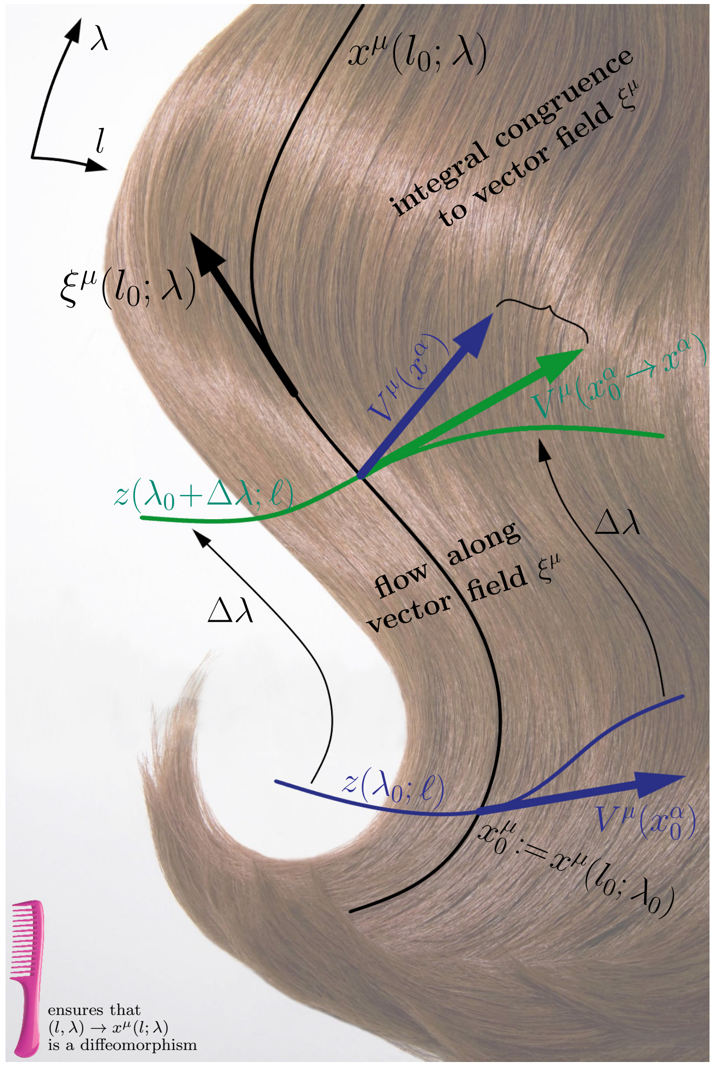

Figure 11.1 *On a smooth manifold, integral curves to every smooth vector field ξ^μ form a congruence $x(l;\lambda)$ along which a one-parameter group of diffeomorphisms (a* **flow** *of the field) exists. It shifts the manifold points, $x^\mu(l_0,\lambda_0) \to x^\mu(l_0,\lambda_0+\Delta\lambda)$, thus also inducing a natural way how to transport geometrical objects. Vectors (V^μ), for example, transport as tangent vectors to smooth curves (here denoted as $z(\lambda;\ell)$). The vector given by the field value at a certain point (blue) differs in general from the vector dragged there from elsewhere by the field flow (green). On this difference rests the Lie-derivative. [Background image by Andreas Kuehn @ Getty Images.]*

have some vector at some particular point. Take *any* smooth curve such that the vector is its tangent vector at that point.[1] Use the selected flow to shift that curve by some chosen $\Delta\lambda$, i.e. shift every its point by applying the diffeomorphism $x^\mu(l;\lambda+\Delta\lambda)$. The transported vector is given by tangent to the transported curve. And, the transport of any covector from that point

[1] This curve does *not* belong to the congruence, it is given – locally – by the vector one wants to transport.

is determined by requiring that the action of that covector on *any* vector at the initial point yields the same number as the action of the transported covector on the transported vector. Let us write it down.

11.2.1 Transport of vectors

Given the flow $\phi(\Delta\lambda)$ and a vector V^μ at some point $x_0^\alpha := x^\alpha(l = l_0; \lambda = \lambda_0)$, denote by $z^\mu(\lambda_0; \ell)$ a certain curve to which V^μ is tangent, with $z^\alpha(\lambda_0; \ell = 0) \equiv x_0^\alpha$. The vector transported to $x^\alpha := x^\alpha(l_0; \lambda_0 + \Delta\lambda)$ is defined by tangent to the transported curve, i.e.

$$V^\mu(x_0^\alpha) = \left.\frac{\mathrm{d}z^\mu(\lambda_0;\ell)}{\mathrm{d}\ell}\right|_{\ell=0} \qquad \Longrightarrow \qquad V^\mu(x^\alpha) := \left.\frac{\mathrm{d}z^\mu(\lambda_0+\Delta\lambda;\ell)}{\mathrm{d}\ell}\right|_{\ell=0}, \tag{11.1}$$

where $z^\mu(\lambda_0 + \Delta\lambda; \ell)$ is the curve obtained by transport of $z^\mu(\lambda_0; \ell)$ along the congruence $x^\mu(l; \lambda)$ by $\Delta\lambda$. The above mapping of vectors is called **push-forward** (along a given flow). Notice that of the curve $z^\mu(\lambda_0; \ell)$ one has to transport at least a small neighbourhood of $\ell = 0$, so, contrary to the transport of scalars for which a single curve $x^\alpha(l = l_0; \lambda)$ suffices, the transport of vectors really needs a congruence.

Note that the parameter of the curve $z^\alpha(\lambda_0; \ell)$ determined locally by the vector V^μ we denoted by ℓ, which should have suggested that l might in principle be used in this role. Sure, l would not be a good parameter if V^μ was parallel to ξ^μ, but that case is anyway trivial.

11.2.2 Transport of covectors

Consider now, at our starting point $x_0^\alpha \equiv x^\alpha(l_0; \lambda_0)$, some covector C_μ. Its action on any vector $V^\mu(x_0^\alpha)$ yields a real number, $C_\mu(x_0^\alpha)V^\mu(x_0^\alpha)$. The covector transported to $x^\alpha \equiv x^\alpha(l_0; \lambda_0 + \Delta\lambda)$ by the flow is defined so that its action on the corresponding transported vector yields the same number as at the starting point, i.e.

$$\forall\ V^\mu(x_0^\alpha): \qquad C_\mu(x^\alpha)V^\mu(x^\alpha) = C_\mu(x_0^\alpha)V^\mu(x_0^\alpha)\,. \tag{11.2}$$

Now we know how to transport any tensor, since tensors are multilinear mappings acting on vectors and covectors and returning, at every point, a number as the result.

11.2.3 Geometrical remark

The mappings between tangent spaces are in fact induced by *any* (smooth) mapping on a manifold (or between different manifolds), it need not be a diffeomorphism. The difference is that a generic mapping need not be invertible, so one then has to be careful in which direction such and such quantity can be transported. It can easily be seen graphically: have some smooth mapping ϕ from M to N (the image is more distinct if considering *two* manifolds, but it may be two regions of the same manifold, or $M \equiv N$, of course), and some scalar functions on each of them (f and g, respectively),

$$\mathbb{R} \xleftarrow{f} M \xrightarrow{\phi} N \xrightarrow{g} \mathbb{R}\,.$$

Apparently, it is possible to compose $g\circ\phi$ and thus make a new function on M, different from f, whereas the opposite suggestion, $f\circ\phi^{-1}$, is not guaranteed since ϕ^{-1} need not exist. In the former, always existing possibility, the domain of the function (g) originally defined on N is extended to M thanks to ϕ, so g is factually transported in the direction *opposite to* ϕ. Such an induced mapping is being called **pull-back** (a mapping cotangent to ϕ; standard notation is ϕ^*). On the other hand, vectors are transported as tangents to curves, so crucial is to transport the respective curve, which means to transport its points, thus *direct* mapping ϕ is involved and the resulting, tangent mapping is being called **push-forward** (denoted ϕ_*). Covectors are transported on the basis of giving numbers on vectors, which leads to the transport of a function, so they are effectively pulled back as well.

However, if the mapping represents a differentiable and invertible flow (diffeomorphism), any tensor can be transported in both directions.

11.3 Lie derivative

Lie derivative assumes one has such a flow and employs the above transport in an infinitesimal version (thus we will use ϵ instead of $\Delta\lambda$ for the parameter shift). Here the idea:

- have some congruence $x^\mu(l;\lambda)$ and its generating smooth vector field $\xi^\mu := \frac{\mathrm{d}x^\mu(l;\lambda)}{\mathrm{d}\lambda}$; denote by $\phi(\epsilon)$ the corresponding diffeomorphism $x^\mu(l;\lambda) \longrightarrow x^\mu(l;\lambda+\epsilon)$

- take a quantity (a general tensor field) you wish to differentiate, call it T (indices omitted), and select, arbitrarily, a certain point where to perform the differentiation; let it correspond to the point $x_0^\alpha := x^\alpha(l_0;\lambda_0)$ of the congruence

- take T at $x_0^\alpha \equiv x^\alpha(l_0;\lambda_0)$ and at $x^\alpha := x^\alpha(l_0;\lambda_0+\epsilon)$, with ϵ small; no transport yet!

- in order to compare the two at the point x_0^α, take the tensor at x^α and transport it, *against* the flow, to x_0^α; let us denote this symbolically as $T(x_0^\alpha \leftarrow x^\alpha)$: in the case of scalars, covectors and covariant tensors, it is natural to interpret it as a pull-back, $T(x_0^\alpha \leftarrow x^\alpha) = \phi^*(\epsilon)[T(x^\alpha)]$, while in the case of vectors and contravariant tensors, it is natural to interpret it as an inverse push-forward, $T(x_0^\alpha \leftarrow x^\alpha) = \phi_*(-\epsilon)[T(x^\alpha)] \equiv \phi_*^{-1}(\epsilon)[T(x^\alpha)]$

- compute the difference between T transported to x_0^α from x^α and the “actual” value of T at x_0^α,

$$\pounds_\xi T := \lim_{\epsilon\to 0} \frac{T(x_0^\alpha \leftarrow x^\alpha) - T(x_0^\alpha)}{\epsilon} \ . \tag{11.3}$$

By the above diffeomorphism-induced transport, one spreads the given quantity over the manifold in a way which is “natural under the given flow”. Hence, the Lie derivative quantifies how such a natural “standard” deviates, in the given direction ξ^μ, from the *actual* course of the quantity (supposed to itself exist as a field).

11.3.1 Index formulas for the Lie derivative

Let us apply the above generic recipe to quantities of particular types. It will be useful to realize that the shift now is infinitesimal, so we may write, from the equation for the tangent vector field ξ^μ,

$$\xi^\mu = \frac{\mathrm{d}x^\mu(l;\lambda)}{\mathrm{d}\lambda} \qquad \Longrightarrow \qquad x^\mu(l_0;\lambda_0+\epsilon) = x^\mu(l_0;\lambda_0) + \left.\frac{\mathrm{d}x^\mu(l_0;\lambda)}{\mathrm{d}\lambda}\right|_{\lambda_0}\epsilon + O(\epsilon^2)$$

which in the abbreviating notation reads

$$x^\mu = x_0^\mu + \epsilon\,\xi^\mu(x_0^\alpha) + O(\epsilon^2)\,.$$

Consider first a scalar function, f. For it, the transport term in (11.3) means to take $f(x^\alpha)$ and pull it back to x_0^α in such a way that it has there the same value, that is,

$$T(x_0^\alpha \leftarrow x^\alpha) \equiv \phi^*(\epsilon)[T(x^\alpha)] \quad \longrightarrow \quad \phi^*(\epsilon)[f(x^\alpha)] \equiv f(x^\alpha)\,.$$

Using then the above expansion of the infinitesimal diffeomorphism $x_0^\mu \to x^\mu$, correspondingly to which any function expands as

$$f(x^\alpha) = f(x_0^\alpha) + f_{,\mu}(x_0^\alpha)(x^\mu - x_0^\mu) + O\big(|x^\mu - x_0^\mu|^2\big) = f(x_0^\alpha) + \epsilon\,(f_{,\mu}\xi^\mu)(x_0^\alpha) + O(\epsilon^2)\,,$$

one finds that the Lie derivative of a scalar function along a vector field ξ^μ is given by its directional derivative along that field,

$$\pounds_\xi f(x_0^\alpha) = \lim_{\epsilon\to 0}\frac{f(x^\alpha) - f(x_0^\alpha)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\epsilon\,(f_{,\mu}\xi^\mu)(x_0^\alpha)}{\epsilon}\,, \qquad \text{i.e.} \quad \boxed{\pounds_\xi f = f_{,\mu}\xi^\mu}\,. \tag{11.4}$$

Recall the definition of push-forward now: it says that vectors are transported as tangents to curves. In the special case of the flow-generating field ξ^μ itself, every such curve is automatically transported so that its tangent remains parallel to the field, so the transport simply yields ξ^μ at the new point. For a vector (V^μ), however, the transport term in (11.3) is interpreted as a push-forward of the vector from x^α to x_0^α by interval $-\epsilon$ (or inverse push-forward by $+\epsilon$),

$$T(x_0^\alpha \leftarrow x^\alpha) \equiv \phi_*(-\epsilon)[T(x^\alpha)] \quad \longrightarrow \quad \phi_*(-\epsilon)[V^\mu(x^\alpha)]\,.$$

For ξ^μ itself, the formula (11.3) thus yields

$$(\pounds_\xi \xi^\mu) = 0 \quad \text{... a tautology actually } (\xi^\mu \text{ behaves along itself as it behaves})\,.$$

Consider now an arbitrary covector C_μ and an arbitrary generating field ξ^μ. Since $\xi^\mu C_\mu$ yields a function, its Lie derivative is

$$\pounds_\xi(\xi^\mu C_\mu) = (\xi^\mu C_\mu)_{,\nu}\xi^\nu = \xi^\mu{}_{,\nu}C_\mu\xi^\nu + \xi^\mu C_{\mu,\nu}\xi^\nu \quad \ldots \overset{\mu\leftrightarrow\nu}{=} \xi^\mu\left(C_{\nu,\mu}\xi^\nu + \xi^\nu{}_{,\mu}C_\nu\right)\,,$$

while, on the other hand, we can express the result using the Leibniz rule,

$$\pounds_\xi(\xi^\mu C_\mu) = (\pounds_\xi\xi^\mu)C_\mu + \xi^\mu(\pounds_\xi C_\mu) = \xi^\mu(\pounds_\xi C_\mu)\,.$$

Comparing the two results and regarding that ξ^μ has been arbitrary, we see that the Lie derivative of covectors should read

$$(\pounds_\xi C_\mu) = C_{\nu,\mu}\xi^\nu + \xi^\nu{}_{,\mu} C_\nu \,. \tag{11.5}$$

Sorry, this equation is WRONG (or it *may be* wrong at least)![2] The "tautological" case of ξ^μ is really special and one has to be careful. Look at the first result for $\pounds_\xi(\xi^\mu C_\mu)$ and notice the term $\xi^\mu C_{\nu,\mu}\xi^\nu$: because of the symmetric term $\xi^\mu\xi^\nu$, one could have put there, without any change, $C_{\mu,\nu}$ instead of $C_{\nu,\mu}$ – but these expressions are *not* equivalent!

However, the above derivation *can* be employed for one special subclass of covectors – gradients of functions ($C_\mu = f_{,\mu}$). Actually, for those, $C_{\nu,\mu} \equiv f_{,\nu\mu}$ is symmetric by itself, so its multiplication by $\xi^\mu\xi^\nu$ does not cancel any information. Substituting above, we thus have

$$\pounds_\xi(f_{,\mu}) = f_{,\nu\mu}\xi^\nu + \xi^\nu{}_{,\mu} f_{,\nu} \,.$$

But this exactly equals gradient of the Lie derivative of f,

$$(\pounds_\xi f)_{,\mu} = (f_{,\nu}\xi^\nu)_{,\mu} = f_{,\nu\mu}\xi^\nu + f_{,\nu}\xi^\nu{}_{,\mu} \,,$$

so we observe that

$$\pounds_\xi(f_{,\mu}) = (\pounds_\xi f)_{,\mu} \,. \tag{11.6}$$

(This in fact is a special consequence of a more general truth that pull-back commutes with gradient.)

The formula for vectors is now easy, because we know how to differentiate functions and their gradients. Hence, for any vector V^μ, any function f and any field ξ^μ, we have

$$\begin{aligned}\pounds_\xi(V^\mu f_{,\mu}) &\stackrel{1}{=} (\pounds_\xi V^\mu) f_{,\mu} + V^\mu \pounds_\xi(f_{,\mu}) = (\pounds_\xi V^\mu) f_{,\mu} + V^\mu\left(f_{,\nu\mu}\xi^\nu + \xi^\nu{}_{,\mu} f_{,\nu}\right)\\ &\stackrel{2}{=} \left(V^\mu f_{,\mu}\right)_{,\nu}\xi^\nu = V^\mu{}_{,\nu} f_{,\mu}\xi^\nu + V^\mu f_{,\mu\nu}\xi^\nu \,.\end{aligned}$$

Comparing the two expressions, the double-derivative terms cancel out and, after factoring out $f_{,\mu}$ (exchange the names of indices in $V^\mu\xi^\nu{}_{,\mu} f_{,\nu}$), we arrive at

$$\boxed{(\pounds_\xi V^\mu) = V^\mu{}_{,\nu}\xi^\nu - \xi^\mu{}_{,\nu} V^\nu = [\xi, V]^\mu = -[V,\xi]^\mu = -(\pounds_V \xi^\mu)} \,. \tag{11.7}$$

This result is worth memorizing: the Lie derivative of a vector field with respect to another vector field is given by Lie bracket (commutator) of those vector fields, in the order how they are naturally written. Note also that the automatic property $(\pounds_\xi\xi^\mu) = 0$ (which we assumed above) really conforms with it.

With this knowledge, similarly easy is the general formula for covectors. For any C_μ, any V^μ and any ξ^μ, we have

$$\pounds_\xi(V^\mu C_\mu) \stackrel{1}{=} (\pounds_\xi V^\mu) C_\mu + V^\mu(\pounds_\xi C_\mu) = \left(V^\mu{}_{,\nu}\xi^\nu - \xi^\mu{}_{,\nu} V^\nu\right) C_\mu + V^\mu(\pounds_\xi C_\mu)$$

[2] Tempting to say that it may be a LIE...

$$\stackrel{2}{=} (V^\mu C_\mu)_{,\nu}\, \xi^\nu = V^\mu{}_{,\nu} \xi^\nu C_\mu + V^\mu C_{\mu,\nu} \xi^\nu \ .$$

Again, comparison yields

$$\boxed{(\pounds_\xi C_\mu) = C_{\mu,\nu}\xi^\nu + \xi^\nu{}_{,\mu} C_\nu} \,. \tag{11.8}$$

BTW, we see that equation (11.5) really was a lie...
One more BTW: having already memorized the commutator formula for the Lie derivative of vectors, one can also infer the covector formula – just by remembering that now there is the *plus* sign between the two terms. Actually, the only mistake one might thus make is to write $\xi_\mu{}^{,\nu} C_\nu$ (or even $\xi_{\mu,\nu} C^\nu$) instead of the second term, but consider that we do not at all know what ξ_μ (and C^ν) is, because we do *not* assume the metric to exist on the manifold – we only know the *vector* ξ^μ and the *covector* C_μ. Hence, there is but one reasonable option of how to arrange the indices in the second term.

The recipe for the Lie derivative of a general tensor is thus obvious – for a (1,2)-tensor, for example,

$$\boxed{(\pounds_\xi T^\mu{}_{\alpha\beta}) = T^\mu{}_{\alpha\beta,\nu}\xi^\nu - \xi^\mu{}_{,\nu} T^\nu{}_{\alpha\beta} + \xi^\nu{}_{,\alpha} T^\mu{}_{\nu\beta} + \xi^\nu{}_{,\beta} T^\mu{}_{\alpha\nu}} \,. \tag{11.9}$$

11.3.2 Basic properties of the Lie derivative

- It is clear from construction that generally the Lie derivative results in a *tensor of the same type as was the original one*.

- Directly from the introductory ideas, as well as from the coordinate expressions, it is clear that the Lie derivative is linear in both its arguments,

  $$\pounds_\xi (S + \beta T) = \pounds_\xi S + \beta \pounds_\xi T, \qquad \pounds_{\eta + b\xi} T = \pounds_\eta T + b \pounds_\xi T,$$

  with S, T arbitrary tensor fields, η, ξ arbitrary vector fields, and β, b constants.

- Also straightforward is to prove that it follows the Leibniz rule. One finds, in particular, for a scalar f and a tensor T,

  $$\pounds_\xi (fT) = (\pounds_\xi f) T + f(\pounds_\xi T) = f_{,\mu}\xi^\mu T + f \pounds_\xi T \ .$$

  In fact we already assumed the Leibniz-rule in derivation, since we wanted a *derivative*.

- The Lie derivative satisfies the Jacobi identity,

  $$\pounds_\xi \left([V, W]^\mu\right) = [\pounds_\xi V, W]^\mu + [V, \pounds_\xi W]^\mu \tag{11.10}$$

  (proof is absolutely straightforward, just a bit tedious – one has to write out both sides carefully). This, together with the linearity and anti-commutativity, means that the space of vector fields over the manifold, equipped with the Lie bracket, forms a Lie algebra.

- The Lie derivative satisfies, for every tensor field T, the identity

$$\pounds_{[\eta,\xi]}T = \pounds_\eta(\pounds_\xi T) - \pounds_\xi(\pounds_\eta T)\,, \tag{11.11}$$

  where η and ξ are arbitrary vector fields. Hence, Lie derivative with respect to a commutator of vector fields is equal to the commutator of Lie derivatives with respect to these fields. (Proof is again straightforward, just from definitions.)

- The Lie derivative commutes with contraction. Let us verify this, without loss of generality, on the above (1,2)-tensor. By contraction of (11.9) in ${}^{\mu}_{\alpha}$ we have

$$T^\mu{}_{\mu\beta,\nu}\xi^\nu - \cancel{\xi^\mu{}_{,\nu}T^\nu{}_{\mu\beta}} + \cancel{\xi^\nu{}_{,\mu}T^\mu{}_{\nu\beta}} + \xi^\nu{}_{,\beta}T^\mu{}_{\mu\nu} = T^\mu{}_{\mu\beta,\nu}\xi^\nu + \xi^\nu{}_{,\beta}T^\mu{}_{\mu\nu}\,,$$

  which really is $\pounds_\xi(T^\mu{}_{\mu\beta})$.

- Lie derivative has been introduced as a tensor operation, but from its coordinate expression this is not obvious, since there appear partial derivatives. However, suppose now the manifold is endowed with the affine connection, so one knows a covariant derivative. Evaluating (11.9) with *covariant* derivatives, one has

$$\begin{aligned}
&T^\mu{}_{\alpha\beta;\nu}\xi^\nu - \xi^\mu{}_{;\nu}T^\nu{}_{\alpha\beta} + \xi^\nu{}_{;\alpha}T^\mu{}_{\nu\beta} + \xi^\nu{}_{;\beta}T^\mu{}_{\alpha\nu} = \\
&= \left(T^\mu{}_{\alpha\beta,\nu} + \cancel{\Gamma^\mu{}_{\nu\kappa}T^\kappa{}_{\alpha\beta}} - \bcancel{\Gamma^\kappa{}_{\nu\alpha}T^\mu{}_{\kappa\beta}} - \xcancel{\Gamma^\kappa{}_{\nu\beta}T^\mu{}_{\alpha\kappa}}\right)\xi^\nu - \\
&\quad - \xi^\mu{}_{,\nu}T^\nu{}_{\alpha\beta} - \cancel{\Gamma^\mu{}_{\nu\kappa}\xi^\kappa T^\nu{}_{\alpha\beta}} + \xi^\nu{}_{,\alpha}T^\mu{}_{\nu\beta} + \bcancel{\Gamma^\nu{}_{\alpha\kappa}\xi^\kappa T^\mu{}_{\nu\beta}} + \xi^\nu{}_{,\beta}T^\mu{}_{\alpha\nu} + \xcancel{\Gamma^\nu{}_{\beta\kappa}\xi^\kappa T^\mu{}_{\alpha\nu}} \\
&= T^\mu{}_{\alpha\beta,\nu}\xi^\nu - \xi^\mu{}_{,\nu}T^\nu{}_{\alpha\beta} + \xi^\nu{}_{,\alpha}T^\mu{}_{\nu\beta} + \xi^\nu{}_{,\beta}T^\mu{}_{\alpha\nu}\,,
\end{aligned}$$

  where all the extra six terms with Gammas cancel out in pairs thanks to the symmetry of Gammas in lower indices. Therefore, all the coordinate expressions for the Lie derivative can equally well be written with covariant derivatives. (In a space-time with torsion, it thus would not work. Sure – the torsion tensor expresses exactly the difference between the covariant and the partial commutator of two vector fields.)

- Smooth manifolds are defined as those covered by (complete atlas of) coordinates. Let us assume we adapt a coordinate system to the given vector field ξ^μ so that the field can be written $\xi^\mu = \frac{\partial x^\mu}{\partial x^K} = \delta^\mu_K$, where x^K is some particular coordinate. In these coordinates, $\xi^\mu{}_{,\nu} = 0$, so, for any tensor T,

$$\pounds_\xi T = T_{,\nu}\xi^\nu = T_{,\nu}\delta^\nu_K = T_{,K}\,. \tag{11.12}$$

  This result can help in understanding the Lie derivative: it is kind-of *covariant expression for partial derivative* (even more basic than the covariant derivative).

11.3.3 More special properties of the Lie derivative

Let us add some more advanced properties of the Lie derivative. They are mutually related, some of them simple (commutator with the covariant derivative), some of them more involved. For some of them, we will need "knowledge from the future" (of these lecture notes only), yet still we place them here rather than shifting them to an appendix. A first-semester (in fact *any*) reader may skip the section safely and go to Killing vectors.

- Lie derivative of the Levi-Civita tensor (A.3):

$$\pounds_\xi \epsilon_{\mu\nu\kappa\lambda} = \cancel{\epsilon_{\mu\nu\kappa\lambda;\iota}}\xi^\iota + \xi^\iota{}_{;\mu}\epsilon_{\iota\nu\kappa\lambda} + \xi^\iota{}_{;\nu}\epsilon_{\mu\iota\kappa\lambda} + \xi^\iota{}_{;\kappa}\epsilon_{\mu\nu\iota\lambda} + \xi^\iota{}_{;\lambda}\epsilon_{\mu\nu\kappa\iota} = \xi^\iota{}_{;\iota}\epsilon_{\mu\nu\kappa\lambda}$$

(it follows just by considering how it works for *any* particular non-trivial set of indices, i.e. for all indices different).

- Lie derivative does *not* commute with the covariant derivative:

$$\begin{aligned}(\pounds_\xi C_\mu)_{;\alpha} - \pounds_\xi(C_{\mu;\alpha}) &= (C_{\mu;\sigma}\xi^\sigma + \xi^\sigma{}_{;\mu}C_\sigma)_{;\alpha} - (C_{\mu;\alpha\sigma}\xi^\sigma + \xi^\sigma{}_{;\mu}C_{\sigma;\alpha} + \xi^\sigma{}_{;\alpha}C_{\mu;\sigma}) = \\ &= (C_{\mu;\sigma\alpha} - C_{\mu;\alpha\sigma})\,\xi^\sigma + \cancel{C_{\mu;\sigma}\xi^\sigma{}_{;\alpha}} + \xi^\sigma{}_{;\mu\alpha}C_\sigma + \cancel{\xi^\sigma{}_{;\mu}C_{\sigma;\alpha}} - \cancel{\xi^\sigma{}_{;\mu}C_{\sigma;\alpha}} - \cancel{\xi^\sigma{}_{;\alpha}C_{\mu;\sigma}} = \\ &= R^\iota{}_{\mu\sigma\alpha}C_\iota\xi^\sigma + \xi^\sigma{}_{;\mu\alpha}C_\sigma = (R^\iota{}_{\mu\sigma\alpha}\xi^\sigma + \xi^\iota{}_{;\mu\alpha})\,C_\iota\,, \end{aligned} \tag{11.13}$$

$$\begin{aligned}(\pounds_\xi V^\mu)_{;\alpha} - \pounds_\xi(V^\mu{}_{;\alpha}) &= (V^\mu{}_{;\sigma}\xi^\sigma - \xi^\mu{}_{;\sigma}V^\sigma)_{;\alpha} - (V^\mu{}_{;\alpha\sigma}\xi^\sigma - \xi^\mu{}_{;\sigma}V^\sigma{}_{;\alpha} + \xi^\sigma{}_{;\alpha}V^\mu{}_{;\sigma}) = \\ &= (V^\mu{}_{;\sigma\alpha} - V^\mu{}_{;\alpha\sigma})\,\xi^\sigma + \cancel{V^\mu{}_{;\sigma}\xi^\sigma{}_{;\alpha}} - \xi^\mu{}_{;\sigma\alpha}V^\sigma - \cancel{\xi^\mu{}_{;\sigma}V^\sigma{}_{;\alpha}} + \cancel{\xi^\mu{}_{;\sigma}V^\sigma{}_{;\alpha}} - \cancel{\xi^\sigma{}_{;\alpha}V^\mu{}_{;\sigma}} = \\ &= -R^\mu{}_{\iota\sigma\alpha}V^\iota\xi^\sigma - \xi^\mu{}_{;\sigma\alpha}V^\sigma = -(R^\mu{}_{\iota\sigma\alpha}\xi^\sigma + \xi^\mu{}_{;\iota\alpha})\,V^\iota\,. \end{aligned} \tag{11.14}$$

For higher-rank tensors, there are more terms (with obvious logic).

- **Lie derivative of an affine connection**

We will find, in equation (28.8), that the change of a tensorial quantity induced by an infinitesimal coordinate shift $x'^\mu = x^\mu - \epsilon\,\xi^\mu(x)$ is related to its Lie derivative by

$$\delta\psi^{\mu\nu\dots}{}_{\kappa\lambda\dots} = \epsilon\,(\pounds_\xi\psi^{\cdots}{}_{\dots} - \psi^{\cdots}{}_{\dots,\alpha}\,\xi^\alpha) = \bar\delta\psi^{\mu\nu\dots}{}_{\kappa\lambda\dots} - \epsilon\,\psi^{\cdots}{}_{\dots,\alpha}\,\xi^\alpha\,,$$

where $\delta\psi := \psi'(x') - \psi(x)$, while $\bar\delta\psi := \psi'(x) - \psi(x)$. Let us *define* the Lie derivative of Gammas by the same relation, where $\delta\Gamma^\mu{}_{\kappa\lambda} \equiv \Gamma'^\mu{}_{\kappa\lambda}(x') - \Gamma^\mu{}_{\kappa\lambda}(x)$ on the l.h. side we fix from the known transformation of Gammas (2.17),

$$\Gamma'^\mu{}_{\kappa\lambda}(x') = \frac{\partial x'^\mu}{\partial x^\iota}\frac{\partial x^\gamma}{\partial x'^\kappa}\frac{\partial x^\delta}{\partial x'^\lambda}\,\Gamma^\iota{}_{\gamma\delta}(x) + \frac{\partial x'^\mu}{\partial x^\iota}\frac{\partial^2 x^\iota}{\partial x'^\kappa\partial x'^\lambda}\,.$$

For the infinitesimal coordinate shift $x'^\mu = x^\mu - \epsilon\,\xi^\mu(x)$, one has, specifically,

$$\begin{aligned}\Gamma'^\mu{}_{\kappa\lambda} &= (\delta^\mu_\iota - \epsilon\,\xi^\mu{}_{,\iota})(\delta^\gamma_\kappa + \epsilon\,\xi^\gamma{}_{,\kappa})(\delta^\delta_\lambda + \epsilon\,\xi^\delta{}_{,\lambda})\,\Gamma^\iota{}_{\gamma\delta} + (\delta^\mu_\iota - \epsilon\,\xi^\mu{}_{,\iota})(\delta^\iota_\lambda + \epsilon\,\xi^\iota{}_{,\lambda})_{,\kappa} = \\ &= \Gamma^\mu{}_{\kappa\lambda} + \epsilon\left(\xi^\delta{}_{,\lambda}\,\Gamma^\mu{}_{\kappa\delta} + \xi^\gamma{}_{,\kappa}\,\Gamma^\mu{}_{\gamma\lambda} - \xi^\mu{}_{,\iota}\,\Gamma^\iota{}_{\kappa\lambda} + \xi^\mu{}_{,\kappa\lambda}\right), \end{aligned} \tag{11.15}$$

using which, therefore,

$$\begin{aligned}\pounds_\xi\Gamma^\mu{}_{\kappa\lambda} &= \frac{\bar\delta\Gamma^\mu{}_{\kappa\lambda}}{\epsilon} \equiv \frac{\Gamma'^\mu{}_{\kappa\lambda}(x) - \Gamma^\mu{}_{\kappa\lambda}(x)}{\epsilon} = \\ &= \frac{\delta\Gamma^\mu{}_{\kappa\lambda}}{\epsilon} + \Gamma^\mu{}_{\kappa\lambda,\alpha}\xi^\alpha \equiv \frac{\Gamma'^\mu{}_{\kappa\lambda}(x') - \Gamma^\mu{}_{\kappa\lambda}(x)}{\epsilon} + \Gamma^\mu{}_{\kappa\lambda,\alpha}\xi^\alpha = \\ &= \xi^\delta{}_{,\lambda}\,\Gamma^\mu{}_{\kappa\delta} + \xi^\gamma{}_{,\kappa}\,\Gamma^\mu{}_{\gamma\lambda} - \xi^\mu{}_{,\iota}\,\Gamma^\iota{}_{\kappa\lambda} + \xi^\mu{}_{,\kappa\lambda} + \Gamma^\mu{}_{\kappa\lambda,\alpha}\xi^\alpha\,. \end{aligned} \tag{11.16}$$

This formula has e.g. been derived by [15], equation (2.77). It is straightforward to arrange it so as to see, explicitly, that this actually *is a tensor*. Let us expand, "by definition",

$$\begin{aligned}\xi^\mu{}_{;\kappa\lambda} &= (\xi^\mu{}_{,\kappa} + \Gamma^\mu{}_{\kappa\delta}\xi^\delta)_{,\lambda} + \Gamma^\mu{}_{\lambda\gamma}(\xi^\gamma{}_{,\kappa} + \Gamma^\gamma{}_{\kappa\delta}\xi^\delta) - \Gamma^\iota{}_{\kappa\lambda}(\xi^\mu{}_{,\iota} + \Gamma^\mu{}_{\iota\delta}\xi^\delta) = \\ &= \xi^\mu{}_{,\kappa\lambda} + \Gamma^\mu{}_{\kappa\delta,\lambda}\xi^\delta + \Gamma^\mu{}_{\kappa\delta}\xi^\delta{}_{,\lambda} + \Gamma^\mu{}_{\lambda\gamma}\xi^\gamma{}_{,\kappa} + \Gamma^\mu{}_{\lambda\gamma}\Gamma^\gamma{}_{\kappa\delta}\xi^\delta - \Gamma^\iota{}_{\kappa\lambda}\xi^\mu{}_{,\iota} - \Gamma^\iota{}_{\kappa\lambda}\Gamma^\mu{}_{\iota\delta}\xi^\delta .\end{aligned}$$

After substituting for $\xi^\mu{}_{,\kappa\lambda}$ to (11.16), all the terms with the first derivatives of ξ^μ cancel in pairs, and one is left with

$$\begin{aligned}\pounds_\xi \Gamma^\mu{}_{\kappa\lambda} &= \xi^\mu{}_{;\kappa\lambda} + \Gamma^\mu{}_{\kappa\lambda,\alpha}\xi^\alpha - \Gamma^\mu{}_{\kappa\delta,\lambda}\xi^\delta + \Gamma^\iota{}_{\kappa\lambda}\Gamma^\mu{}_{\iota\delta}\xi^\delta - \Gamma^\mu{}_{\lambda\gamma}\Gamma^\gamma{}_{\kappa\delta}\xi^\delta = \\ &= \xi^\mu{}_{;\kappa\lambda} + R^\mu{}_{\kappa\delta\lambda}\xi^\delta . \end{aligned} \tag{11.17}$$

This is indeed the result derived, in a similar manner, in the classical books [40] (formula (5.47)) and [53] (Chapter I, § 2); see e.g. also [21], Exercise 8.4, or [26], end of section 4.4. (A more general formula also includes a torsion term.) It is also good to check the last points of Section 11.3.2 to see how the present issue relates to the non-commutation of the Lie derivative with the *covariant* derivative.

The expression $\pounds_\xi \Gamma^\mu{}_{\kappa\lambda} = \xi^\mu{}_{;\kappa\lambda} + R^\mu{}_{\kappa\delta\lambda}\xi^\delta$ found in equation (11.17) clearly is a *tensor*. This is *not* a surprise, because $\pounds_\xi \Gamma^\mu{}_{\kappa\lambda}$ represents the change between two Gammas computed *at the same point* (one brought there from elsewhere and one existing there). In such a case, however, the non-tensorial part of the transformation of Gammas is *the same* for both, so it cancels out.

On the contrary, the tensorial character does *not* apply to

$$\pounds_\xi \Gamma_{\alpha\kappa\lambda} = \pounds_\xi (g_{\alpha\mu}\Gamma^\mu{}_{\kappa\lambda}) = (\xi_{\alpha;\mu} + \xi_{\mu;\alpha})\Gamma^\mu{}_{\kappa\lambda} + \xi_{\alpha;\kappa\lambda} + R_{\alpha\kappa\delta\lambda}\xi^\delta . \tag{11.18}$$

- The above result can be checked from the formula (23.22) we will derive in Chapter (23). It says how the variation of the Christoffel symbol at a given coordinate point is related to the same kind of variation of the metric,

$$\bar\delta\Gamma^\mu{}_{\kappa\lambda} = \frac{1}{2}\, g^{\mu\nu}\left[(\bar\delta g_{\nu\kappa})_{;\lambda} + (\bar\delta g_{\lambda\nu})_{;\kappa} - (\bar\delta g_{\kappa\lambda})_{;\nu}\right].$$

At that time, we will already know that if the variation is induced by the infinitesimal coordinate translation $x'^\mu = x^\mu - \epsilon\,\xi^\mu$, then $\bar\delta g_{\mu\nu;\iota} := (\bar\delta g_{\mu\nu})_{;\iota} = \epsilon \pounds_\xi g_{\mu\nu} = \epsilon(\xi_{\mu;\nu} + \xi_{\nu;\mu})$. Substituting, one obtains

$$\begin{aligned}\bar\delta\Gamma^\mu{}_{\kappa\lambda} &= \frac{\epsilon}{2}\, g^{\mu\nu}\left[(\xi_{\nu;\kappa} + \xi_{\kappa;\nu})_{;\lambda} + (\xi_{\lambda;\nu} + \xi_{\nu;\lambda})_{;\kappa} - (\xi_{\kappa;\lambda} + \xi_{\lambda;\kappa})_{;\nu}\right] = \\ &= \frac{\epsilon}{2}\, g^{\mu\nu}\left[2\xi_{\nu;\kappa\lambda} + (\xi_{\nu;\lambda\kappa} - \xi_{\nu;\kappa\lambda}) + (\xi_{\kappa;\nu\lambda} - \xi_{\kappa;\lambda\nu}) + (\xi_{\lambda;\nu\kappa} - \xi_{\lambda;\kappa\nu})\right] = \\ &= \epsilon\,\xi^\mu{}_{;\kappa\lambda} + \frac{\epsilon}{2}\, g^{\mu\nu}(R^\delta{}_{\nu\lambda\kappa} + R^\delta{}_{\kappa\nu\lambda} + R^\delta{}_{\lambda\nu\kappa})\,\xi_\delta = \\ &= \epsilon\,\xi^\mu{}_{;\kappa\lambda} + \epsilon\, g^{\mu\nu} R_{\delta\lambda\nu\kappa}\xi^\delta = \epsilon\,\xi^\mu{}_{;\kappa\lambda} + \epsilon R^\mu{}_{\kappa\delta\lambda}\xi^\delta .\end{aligned}$$

- A geometrical formula for the Lie derivative of an affine connection is also being offered: for X, Y and Z smooth vector fields, let it be introduced by

$$\begin{aligned}(\pounds_X\nabla)(Y,Z) &:= \pounds_X(\nabla_Y Z) - \nabla_Y(\pounds_X Z) - \nabla_{\pounds_X Y} Z = \\ &= [X,\nabla_Y Z] - \nabla_Y[X,Z] - \nabla_{[X,Y]}Z\,. \end{aligned} \tag{11.19}$$

Yes, it somewhat resembles the definition of the Riemann tensor (6.1), and it also involves the torsion tensor (5.9). Recalling both,

$$\begin{aligned} &R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\,,\\ &T(X,Z) = \nabla_X Z - \nabla_Z X - [X,Z]\,,\end{aligned}$$

we first express, from Riemann, $-\nabla_{[X,Y]}Z = R(X,Y)Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z)$, to obtain, from (11.19),

$$(\pounds_X\nabla)(Y,Z) = [X,\nabla_Y Z] - \nabla_Y[X,Z] + R(X,Y)Z - \nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z)\,.$$

Then we express, from torsion, $[X,\nabla_Y Z] - \nabla_X(\nabla_Y Z) = -T(X,\nabla_Y Z) - \nabla_{\nabla_Y Z}X$ and $-\nabla_Y[X,Z] + \nabla_Y(\nabla_X Z) = \nabla_Y T(X,Z) + \nabla_Y(\nabla_Z X)$, to get

$$(\pounds_X\nabla)(Y,Z) = R(X,Y)Z - T(X,\nabla_Y Z) - \nabla_{\nabla_Y Z}X + \nabla_Y T(X,Z) + \nabla_Y(\nabla_Z X)\,.$$

Back in indices, we thus have

$$\begin{aligned}(\pounds_X\Gamma^\mu{}_{\kappa\lambda})Y^\kappa Z^\lambda &= R^\mu{}_{\lambda\nu\kappa}Z^\lambda X^\nu Y^\kappa - T^\mu{}_{\nu\lambda}X^\nu Z^\lambda{}_{;\kappa}Y^\kappa - X^\mu{}_{;\lambda}Z^\lambda{}_{;\kappa}Y^\kappa + \\ &\quad + (T^\mu{}_{\nu\lambda}X^\nu Z^\lambda)_{;\kappa}Y^\kappa + (X^\mu{}_{;\lambda}Z^\lambda)_{;\kappa}Y^\kappa = \\ &= R^\mu{}_{\lambda\nu\kappa}Z^\lambda X^\nu Y^\kappa + (T^\mu{}_{\nu\lambda}X^\nu)_{;\kappa}Z^\lambda Y^\kappa + X^\mu{}_{;\lambda\kappa}Z^\lambda Y^\kappa\,,\end{aligned}$$

that is, since Z^λ and Y^κ are generic,

$$\pounds_X\Gamma^\mu{}_{\kappa\lambda} = R^\mu{}_{\lambda\nu\kappa}X^\nu + (T^\mu{}_{\nu\lambda}X^\nu)_{;\kappa} + X^\mu{}_{;\lambda\kappa}\,. \tag{11.20}$$

An alternative form follows from Ricci identities (6.2), i.e.

$$X^\mu{}_{;\lambda\kappa} = X^\mu{}_{;\kappa\lambda} + R^\mu{}_{\nu\kappa\lambda}X^\nu - T^\nu{}_{\kappa\lambda}X^\mu{}_{;\nu}\,,$$

and using $R^\mu{}_{\lambda\nu\kappa} + R^\mu{}_{\nu\kappa\lambda} = -R^\mu{}_{\kappa\lambda\nu} = R^\mu{}_{\kappa\nu\lambda}$ (first Bianchi):

$$\pounds_X\Gamma^\mu{}_{\kappa\lambda} = X^\mu{}_{;\kappa\lambda} + R^\mu{}_{\kappa\nu\lambda}X^\nu + (T^\mu{}_{\nu\lambda}X^\nu)_{;\kappa} - X^\mu{}_{;\nu}T^\nu{}_{\kappa\lambda}\,. \tag{11.21}$$

With zero torsion, this formula clearly reduces to (11.17).

- Since $g_{\mu\nu,\alpha} = \Gamma_{\nu\alpha\mu} + \Gamma_{\mu\alpha\nu} = \Gamma_{\nu\mu\alpha} + \Gamma_{\mu\nu\alpha}$, it is easy to learn from above that

$$\begin{aligned}\pounds_\xi g_{\mu\nu,\alpha} &= \pounds_\xi\Gamma_{\nu\mu\alpha} + \pounds_\xi\Gamma_{\mu\nu\alpha} = \pounds_\xi\left(g_{\nu\iota}\Gamma^\iota{}_{\mu\alpha} + g_{\mu\iota}\Gamma^\iota{}_{\nu\alpha}\right) = \\ &= \Gamma^\iota{}_{\mu\alpha}\pounds_\xi g_{\nu\iota} + \Gamma^\iota{}_{\nu\alpha}\pounds_\xi g_{\mu\iota} + g_{\nu\iota}\pounds_\xi\Gamma^\iota{}_{\mu\alpha} + g_{\mu\iota}\pounds_\xi\Gamma^\iota{}_{\nu\alpha} = \\ &= \Gamma^\iota{}_{\mu\alpha}\pounds_\xi g_{\nu\iota} + \Gamma^\iota{}_{\nu\alpha}\pounds_\xi g_{\mu\iota} + g_{\nu\iota}(\xi^\iota{}_{;\mu\alpha} + R^\iota{}_{\mu\delta\alpha}\xi^\delta) + g_{\mu\iota}(\xi^\iota{}_{;\nu\alpha} + R^\iota{}_{\nu\delta\alpha}\xi^\delta) = \end{aligned}$$

$$
\begin{aligned}
&= \Gamma^{\iota}{}_{\mu\alpha}(\xi_{\nu;\iota} + \xi_{\iota;\nu}) + \Gamma^{\iota}{}_{\nu\alpha}(\xi_{\mu;\iota} + \xi_{\iota;\mu}) + (\xi_{\nu;\mu} + \xi_{\mu;\nu})_{;\alpha} + \cancel{(R_{\nu\mu\delta\alpha} + R_{\mu\nu\delta\alpha})}\,\xi^{\delta} = \\
&= (\xi_{\mu;\nu} + \xi_{\nu;\mu})_{,\alpha} \;\equiv\; (\pounds_{\xi} g_{\mu\nu})_{,\alpha} \,.
\end{aligned}
$$

Note that a "naive" computation "according to the indices" would *not* lead to the correct answer:

$$
\begin{aligned}
[\pounds_{\xi} g_{\mu\nu,\alpha}]_{\text{naive}} &= g_{\mu\nu,\alpha\beta}\xi^{\beta} + \xi^{\beta}{}_{,\mu} g_{\beta\nu,\alpha} + \xi^{\beta}{}_{,\nu} g_{\mu\beta,\alpha} + \xi^{\beta}{}_{,\alpha} g_{\mu\nu,\beta} \neq \\
&\neq (g_{\mu\nu,\beta}\xi^{\beta} + \xi^{\beta}{}_{,\mu} g_{\beta\nu} + \xi^{\beta}{}_{,\nu} g_{\mu\beta})_{,\alpha} = \\
&= g_{\mu\nu,\beta\alpha}\xi^{\beta} + g_{\mu\nu,\beta}\xi^{\beta}{}_{,\alpha} + \xi^{\beta}{}_{,\mu} g_{\beta\nu,\alpha} + \xi^{\beta}{}_{,\nu} g_{\mu\beta,\alpha} + \xi^{\beta}{}_{,\mu\alpha} g_{\beta\nu} + \xi^{\beta}{}_{,\nu\alpha} g_{\mu\beta} = \\
&= (\pounds_{\xi} g_{\mu\nu})_{,\alpha} = \pounds_{\xi} g_{\mu\nu,\alpha}
\end{aligned}
$$

(the last two terms $\xi^{\beta}{}_{,\mu\alpha} g_{\beta\nu} + \xi^{\beta}{}_{,\nu\alpha} g_{\mu\beta}$ of the correct result are missing in the "naive" one). So, on the metric, the first partial derivative commutes with the Lie derivative. Assuming that the same also applies to the first partial derivatives of Gammas,[3] one finds that

$$
\pounds_{\xi} g_{\mu\nu,\alpha\beta} = \pounds_{\xi}(\Gamma_{\nu\mu\alpha,\beta} + \Gamma_{\mu\nu\alpha,\beta}) = (\pounds_{\xi}\Gamma_{\nu\mu\alpha} + \pounds_{\xi}\Gamma_{\mu\nu\alpha})_{,\beta} = (\pounds_{\xi} g_{\mu\nu,\alpha})_{,\beta} \,.
$$

- Notice that $\pounds_{\xi}\Gamma^{\iota}{}_{\mu\alpha}$ of (11.17) exactly represents the commutators (11.13), (11.14):

$$
(\pounds_{\xi} C_{\mu})_{;\alpha} - \pounds_{\xi}(C_{\mu;\alpha}) = C_{\iota}\pounds_{\xi}\Gamma^{\iota}{}_{\mu\alpha} \,, \qquad (\pounds_{\xi} V^{\mu})_{;\alpha} - \pounds_{\xi}(V^{\mu}{}_{;\alpha}) = -V^{\iota}\pounds_{\xi}\Gamma^{\mu}{}_{\iota\alpha} \,.
$$

Substituting for the $\pounds_{\xi}\Gamma$ terms, one finds that the Lie and the partial derivatives commute (e.g. on a covector):

$$
\begin{aligned}
&(R^{\iota}{}_{\mu\sigma\alpha}\xi^{\sigma} + \xi^{\iota}{}_{;\mu\alpha})\, C_{\iota} = (\pounds_{\xi} C_{\mu})_{;\alpha} - \pounds_{\xi}(C_{\mu;\alpha}) = \\
&= (\pounds_{\xi} C_{\mu})_{,\alpha} - \pounds_{\xi}(C_{\mu,\alpha}) - \cancel{\Gamma^{\iota}{}_{\alpha\mu}\pounds_{\xi} C_{\iota}} + (\pounds_{\xi}\Gamma^{\iota}{}_{\mu\alpha})C_{\iota} + \cancel{\Gamma^{\iota}{}_{\alpha\mu}\pounds_{\xi} C_{\iota}} \,.
\end{aligned} \tag{11.22}
$$

Note that in the adapted coordinates where $\xi^{\mu} = \delta^{\mu}_{K}$ and $\pounds_{\xi} C_{\mu} = C_{\mu,K}$, $\pounds_{\xi}(C_{\mu,\alpha}) = C_{\mu,\alpha K}$, one has it "naturally" since $(\pounds_{\xi} C_{\mu})_{,\alpha} = C_{\mu,K\alpha} = C_{\mu,\alpha K} = \pounds_{\xi}(C_{\mu,\alpha})$.[4]

11.4 Killing vectors and space-time symmetries

The concept of the vector-field flow and of the Lie derivative permit to decide, invariantly (without reference to coordinates), whether there exist any symmetries in the space-time. By symmetry we mean here an existence of such a vector field ξ^{μ} along which certain quantities do not change. To be more accurate, the question is different actually: the mapping of tangent spaces induced by a given flow (pull-backs and push-forwards) represents the most natural way how to spread the quantity "without change". Hence, given some quantity as a field,

[3] As e.g. shown in [40] (Chapter II, § 10), this applies to all geometric objects (which Gammas are).

[4] However, as e.g. pointed out by [40] (Chapter II, § 10), it is only safe to commute the Lie derivative with the partial one *for geometric objects*, while, in addition, bearing in mind that the partial derivative of a geometric object is mostly *not* a geometric object any more, so its Lie derivative can only be considered as providing a *set of* the latter's *components*. Sure, for an invariant it is "safe", and the commutation works straightforwardly: $(\pounds_{\xi}\psi)_{,\alpha} = (\psi_{,\iota}\xi^{\iota})_{,\alpha} = \psi_{,\iota\alpha}\xi^{\iota} + \psi_{,\iota}\xi^{\iota}{}_{,\alpha} = \psi_{,\alpha\iota}\xi^{\iota} + \xi^{\iota}{}_{,\alpha}\psi_{,\iota} = \pounds_{\xi}(\psi_{,\alpha})$.

one can compare its *actual* space-time behaviour against the above standard of "constancy" – this is exactly the message provided by the Lie derivative. A quantity whose Lie derivative vanishes for some ξ^μ is said to **Lie-transport** along ξ^μ. And if symmetries of the space-time itself are in question, one naturally turns to the **Lie derivative of the metric**.

Hence, we suppose to have a Lorentzian manifold again, equipped with every luxury (metric and the corresponding Levi-Civita connection), and calculate

$$\pounds_\xi g_{\mu\nu} = \cancel{g_{\mu\nu;\kappa}}\,\xi^\kappa + \xi^\kappa{}_{;\mu} g_{\kappa\nu} + \xi^\kappa{}_{;\nu} g_{\mu\kappa} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \equiv 2\xi_{(\mu;\nu)}\,. \tag{11.23}$$

Important note: in the result, the covariant derivatives can no more be changed for partials, because we have already used the fact that $g_{\mu\nu;\kappa} = 0$ – with partial derivatives, we would have had to leave there the term $g_{\mu\nu,\kappa} = \Gamma_{\nu\kappa\mu} + \Gamma_{\mu\kappa\nu}$. So we have a simple result: the metric "does not change" in the direction of ξ^μ if

$$\boxed{\pounds_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0}\,. \tag{11.24}$$

This is called **the Killing equation**.[5] If such an equation is satisfied for some ξ^μ, that vector field is called the **Killing vector field** and the corresponding flow is called an **isometry** – the metric remains the same along ξ^μ, i.e., the space-time properties does not change along ξ^μ, they are independent of its flow ... thus the term **space-time symmetry**.

11.4.1 Some properties of Killing vector fields

- Whether written as $\phi^*(\epsilon)[g_{\mu\nu}(x^\alpha)] = g_{\mu\nu}(x_0^\alpha)$, or $\pounds_\xi g_{\mu\nu} = 0$, or $\xi_{(\mu;\nu)} = 0$, the Killing equation represents 10 equations, since it has two indices and it is symmetric in them. The unknown components of ξ^μ are only 4, so the set is overdetermined and does not necessarily have any solution.

- By contraction of the Killing equation one finds that the Killing vector fields have zero divergence, $\xi^\mu{}_{;\mu} = 0$.

- A linear combination of Killing vector fields (η_μ, ξ_μ) is also a Killing field,

$$(\eta_\mu + b\xi_\mu)_{;\nu} + (\eta_\nu + b\xi_\nu)_{;\mu} = \eta_{\mu;\nu} + \eta_{\nu;\mu} + b\,(\xi_{\mu;\nu} + \xi_{\nu;\mu}) = 0\,.$$

- Commutator of Killing fields is again a Killing field,

$$[\eta,\xi]_{\mu;\nu} + [\eta,\xi]_{\nu;\mu} =$$

[5] Don't be scared... –A. J. Coleman, a Canadian mathematician, documents in his paper *The greatest mathematical paper of all time* [The Mathematical Intelligencer 11 (1989) 29]: "His students loved and admired Killing because he gave himself unsparingly of time and energy to them, never being satisfied for them to become narrow specialists, so he spread his lectures over many topics beyond geometry and groups." Still more interestingly, Coleman shows there how extremely influential were papers "Die Zusammensetzung der stetigen, endlichen Transformationsgruppen I, II, III, IV" which W. Killing published in Mathematische Annalen 31 (1888) 252, 33 (1889) 1, 34 (1989) 57, and 36 (1890) 161, respectively. Coleman mainly stresses the paper II and writes: "... if you can name one paper in the past 200 years of equal significance..." –Well, we would certainly name Riemann's habilitation.

$$
\begin{aligned}
&= (\xi_{\mu;\kappa}\eta^{\kappa} - \eta_{\mu;\kappa}\xi^{\kappa})_{;\nu} + (\xi_{\nu;\kappa}\eta^{\kappa} - \eta_{\nu;\kappa}\xi^{\kappa})_{;\mu} = \\
&= (\xi_{\mu;\kappa\nu} + \xi_{\nu;\kappa\mu})\eta^{\kappa} - (\eta_{\mu;\kappa\nu} + \eta_{\nu;\kappa\mu})\xi^{\kappa} + \cancel{\xi_{\mu;\kappa}\eta^{\kappa}{}_{;\nu}} + \cancel{\xi_{\nu;\kappa}\eta^{\kappa}{}_{;\mu}} - \cancel{\eta_{\mu;\kappa}\xi^{\kappa}{}_{;\nu}} - \cancel{\eta_{\nu;\kappa}\xi^{\kappa}{}_{;\mu}} = \\
&= (\cancel{\xi_{\mu;\nu\kappa} + \xi_{\nu;\mu\kappa}} + R^{\sigma}{}_{\mu\kappa\nu}\xi_{\sigma} + R^{\sigma}{}_{\nu\kappa\mu}\xi_{\sigma})\eta^{\kappa} - (\cancel{\eta_{\mu;\nu\kappa} + \eta_{\nu;\mu\kappa}} + R^{\sigma}{}_{\mu\kappa\nu}\eta_{\sigma} + R^{\sigma}{}_{\nu\kappa\mu}\eta_{\sigma})\xi^{\kappa} = \\
&= (\cancel{R_{\sigma\mu\kappa\nu}} + \cancel{R_{\sigma\nu\kappa\mu}} - \cancel{R_{\kappa\mu\sigma\nu}} - \cancel{R_{\kappa\nu\sigma\mu}})\,\eta^{\kappa}\xi^{\sigma} = 0\,,
\end{aligned}
$$

where the curvature terms have appeared due to Ricci identities (6.3). Note that the above effort actually was superfluous since the statement follows from the general property (11.11), i.e. from

$$\pounds_{[\eta,\xi]}g_{\mu\nu} = \pounds_{\eta}(\cancel{\pounds_{\xi}g_{\mu\nu}}) - \pounds_{\xi}(\cancel{\pounds_{\eta}g_{\mu\nu}}) = 0\,.$$

The linear space of Killing vectors endowed with the commutator (in the role of a multiplication operation) thus itself forms a Lie algebra. It is a subalgebra of the Lie algebra of all vector fields.

- Along the flow of a Killing vector field ξ^{μ}, the scalar $\xi_{\alpha}\xi^{\alpha}$ is constant,

$$(\xi_{\alpha}\xi^{\alpha})_{,\mu}\xi^{\mu} = (\xi_{\alpha}\xi^{\alpha})_{;\mu}\xi^{\mu} = 2\xi_{\alpha;\mu}\xi^{\alpha}\xi^{\mu} = 2\xi_{(\alpha;\mu)}\xi^{\alpha}\xi^{\mu} = 0\,.$$

- If ξ^{μ} is a Killing vector field, then, along any geodesic (with $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}$ its tangent), the projection $\xi_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}$ remains constant.
Proof: Regarding the geodesic equation $\frac{\mathrm{D}}{\mathrm{d}p}\left(\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}\right)=0$ and the Killing equation $\xi_{(\mu;\nu)}=0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\xi_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}\right) = \frac{\mathrm{D}}{\mathrm{d}p}\left(\xi_{\mu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}\right) = \frac{\mathrm{D}\xi_{\mu}}{\mathrm{d}p}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p} = \xi_{\mu;\nu}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}p}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p} = \xi_{(\mu;\nu)}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}p}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}p} = 0\,.$$

The corresponding coordinate version of this statement we already know from Section 3.6.

- Intuitively, if there is some symmetry in space-time and if one chooses coordinates adapted to it, i.e. so that the respective Killing field be tangent to some of the coordinate lines, the metric will not depend on that "Killing" coordinate. The two statements are actually equivalent: the Killing vector field exists if and only if there exists such a coordinate system in which the metric is independent of some of the coordinates.
Proof: Choose the coordinates adapted to the Killing vector field ξ^{μ}, i.e. let $\xi^{\mu} = \frac{\partial x^{\mu}}{\partial x^{K}} = \delta^{\mu}_{K}$, where x^{K} is some particular coordinate (it actually represents parameter of the isometry). We know from (11.12) that in these coordinates

$$\pounds_{\xi}g_{\alpha\beta} = g_{\alpha\beta,K}\,. \tag{11.25}$$

Hence, the field ξ^{μ} is Killing if and only if $g_{\alpha\beta,K}=0$.

- When deriving how to transport quantities (vectors in particular) along the ξ^{μ} flow, we have not at all enquired how their *norm* behaves. Sure: such a question only has sense if one has a *metric* on the manifold. If one does, scalar products evolve along ξ^{μ} according to

$$(g_{\mu\nu}V^{\mu}W^{\mu})_{,\nu}\xi^{\nu} = \pounds_{\xi}(g_{\mu\nu}V^{\mu}W^{\mu})\,.$$

If both the vectors are Lie-transported along ξ^μ, i.e. if $\pounds_\xi V^\mu = 0$ and $\pounds_\xi W^\mu = 0$, one is left with

$$(g_{\mu\nu} V^\mu W^\mu)_{,\nu} \xi^\nu = (\pounds_\xi g_{\mu\nu})\, V^\mu W^\mu = (\xi_{\mu;\nu} + \xi_{\nu;\mu})\, V^\mu W^\mu \,,$$

which in general is only zero if ξ^μ is a Killing field.

- Let us also mention an important relation between Killing vector fields and curvature. In the Ricci identity for ξ_ν,

$$\xi_{\nu;\kappa\lambda} - \xi_{\nu;\lambda\kappa} = R^\sigma{}_{\nu\kappa\lambda}\xi_\sigma,$$

we anti-commute, by the Killing equation, ν and λ in the second term, and then we write the relation together with its cyclic permutations:

$$\begin{aligned} \xi_{\nu;\kappa\lambda} + \xi_{\lambda;\nu\kappa} &= R^\sigma{}_{\nu\kappa\lambda}\xi_\sigma \,, \\ \xi_{\lambda;\nu\kappa} + \xi_{\kappa;\lambda\nu} &= R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma \,, \\ \xi_{\kappa;\lambda\nu} + \xi_{\nu;\kappa\lambda} &= R^\sigma{}_{\kappa\lambda\nu}\xi_\sigma \,. \end{aligned}$$

Now add the first and the last equation, while subtracting the middle one (for example),

$$\begin{aligned} & 2\xi_{\nu;\kappa\lambda} = (R^\sigma{}_{\nu\kappa\lambda} + R^\sigma{}_{\kappa\lambda\nu} - R^\sigma{}_{\lambda\nu\kappa})\xi_\sigma = \left(R^\sigma{}_{\{\nu\kappa\lambda\}} - 2R^\sigma{}_{\lambda\nu\kappa}\right)\xi_\sigma = -2R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma \\ & \Longrightarrow \quad \xi_{\nu;\kappa\lambda} = -R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma = R_{\nu\kappa\lambda\sigma}\xi^\sigma \,. \end{aligned} \tag{11.26}$$

  - Corollary: by contraction of this equation, one has

$$\xi_{\nu;}{}^\kappa{}_\kappa \equiv \Box\xi_\nu = -R^\sigma{}_\nu \xi_\sigma \,. \tag{11.27}$$

  If the Ricci tensor vanishes, this correspond to the wave equation (9.2) for the electromagnetic four-potential (otherwise the signs at the curvature terms are opposite). Regarding also that the Killing fields automatically satisfy the "Lorenz condition" $\xi^\mu{}_{;\mu} = 0$, one infers the following: in space-times with $R^{\mu\nu} = 0$, the knowledge of a Killing vector implies the knowledge of a possible EM four-potential. (Note that the corresponding EM field must be a *test* field, because otherwise the Ricci tensor would be $R^{\mu\nu} = 8\pi T^{\mu\nu}_{\rm EM}$ rather than zero.)

  - Another corollary: projecting (11.26) twice on a tangent vector $u^\mu = \frac{{\rm d}x^\mu}{{\rm d}\tau}$ of any geodesic ($\frac{{\rm D}u^\mu}{{\rm d}\tau} = 0$), we find

$$\xi_{\nu;\kappa\lambda}u^\kappa u^\lambda = \frac{{\rm D}\xi_{\nu;\kappa}}{{\rm d}\tau}\,u^\kappa = \frac{{\rm D}(\xi_{\nu;\kappa}u^\kappa)}{{\rm d}\tau} = \frac{{\rm D}^2\xi_\nu}{{\rm d}\tau^2} = R_{\nu\kappa\lambda\sigma}u^\kappa u^\lambda \xi^\sigma \,. \tag{11.28}$$

  That means, the Killing vectors satisfy the geodesic-deviation equation.

  - Yet another corollary: differentiating equation (11.26), one obtains an equation symbolically looking as $\nabla\nabla\nabla\xi = -\xi\nabla R - R\nabla\xi$; differentiating once more, one has $\nabla\nabla\nabla\nabla\xi = -\xi\nabla\nabla R - 2\nabla R\nabla\xi - R\nabla\nabla\xi$, where $\nabla\nabla\xi$ can be expressed from (11.26); etc etc...: whenever the 2nd derivative arises on the r.h. side, one substitutes from

(11.26), thus gradually expressing all the derivatives ($\geqslant$ 2nd) in terms of ξ^μ and its gradient. In other words, thanks to equation (11.26), the entire Taylor expansion of ξ^μ is fully determined by ξ^μ and its gradient. Hence, the Killing exercise can in principle have as many independent solutions as the number of the "initial conditions" ξ^μ and $\xi_{\mu;\nu}$; and these are 4+6, since $\xi_{\mu;\nu}$ is antisymmetric. So, in a 4D space-time, there may at most exist 10 independent Killing fields. In a general dimension d, it is $d(d+1)/2$.

- The existence of Killing fields also brings an important knowledge about the gravitational sources. The latter need *not* necessarily follow the symmetries of their gravitational field, but the conservation laws ${T^{\mu\nu}}_{;\nu}=0$ imply, in such a case, conserved quantities. Actually, a divergence of the "current" $\xi_\mu T^{\mu\nu}$ vanishes,

$$\left(\xi_\mu T^{\mu\nu}\right)_{;\nu} = \xi_{\mu;\nu} T^{\mu\nu} + \xi_\mu \cancel{{T^{\mu\nu}}_{;\nu}} = \xi_{(\mu;\nu)} T^{\mu\nu} = 0 \tag{11.29}$$

(remember that $T^{\mu\nu}$ is symmetric), so the corresponding "charge" is being conserved.

CHAPTER 12

Schwarzschild solution of Einstein equations

About a month after finishing his theory, Einstein received a letter from K. Schwarzschild. This German theoretical physicist, professor of the Göttingen University and director at the Göttingen Observatory, later director of the Potsdam Observatory and member of the Prussian Academy of Sciences, father of 3 children, voluntarily enrolled to the army when the war started. He headed a meteorological station in Belgium, calculated trajectories for artillery in France, and then worked at the rear in Russia. It was from there that he wrote, on 22th December 1915, to Einstein that he found an exact spherically symmetric solution of the field equations for a point-mass source.[1] Einstein was surprised: "I would not have thought that the strict treatment of the point problem was so simple." Yet he was more surprised for a much deeper reason. At that time he shared Mach's belief that masses have inertia – and thus also generate gravitation – due to their interaction with all the other masses in the Universe. If there is just one point mass in the Universe, it has nothing to interact with, so it should either have no inertia, or the corresponding solution should have no sense.[2] Hence, by its mere existence, the Schwarzschild solution indicated that, for the concept as well as value of inertia, also important are the conditions at the space-time boundaries or infinities. During the years to come, Einstein was to discuss this point notably with W. de Sitter, who later even found a totally vacuum (but with $\Lambda \neq 0$), non-singular, yet still non-trivial (non-flat) solution. But this we will only return to in the cosmology chapter.

[1] At the end of the letter, he writes: "As you see, the war is kindly disposed toward me, allowing me, despite fierce gunfire at a decidedly terrestrial distance, to take this walk into this your land of ideas."

[2] Actually, in the essay called *Cosmological considerations in the General Theory of Relativity* (in German) from 1917, Einstein writes: "In a consistent theory of relativity there can be no inertia *relatively to 'space'*, but only an inertia of masses *relatively to one another*. If, therefore, I have a mass at a sufficient distance from all other masses in the universe, its inertia must fall to zero."

Schwarzschild solution is an exact spherically symmetric solution to the Einstein equations with $T^{\mu\nu}=0$ and $\Lambda=0$.

We will not tackle the field equations right away. It would be a hopeless strategy to try to find some "generic" solution and only then restrict it to the required properties, while possibly using the freedom in the choice of coordinates. Practically, the procedure is just opposite: one first specifies the main features the space-time should have, then chooses the coordinates well adapted to the prescribed properties, constrains the solution (the metric) as much as possible on the basis of the required properties (geometrical arguments), and *only then* tries to solve the field equations for the thus obtained, restricted form of the metric. (When solving the equations with pure mathematical motivation, this story can be shortened to "looking for a solution in a given form", i.e. for some metric *ansatz*.) The derivation of the Schwarzschild solution is a good illustration of such an approach.

12.1 Metric of a spherically symmetric space-time

L. Ryder writes in [39], section 6.6: "We could, however, if we were perverse, re-express the Schwarzschild solution in a very different coordinate system [than in the spherical one] ..." Without any intention to degrade our readers, let us settle for the spherical-type coordinates (usually called the Schwarzschild coordinates in GR). Imagine to have a world-line of the centre of symmetry, parametrized by its proper time τ. Imagine to send out from there, at any value of τ, radial geodesics in every spatial direction, i.e. in every direction orthogonal to the centre's four-velocity. Assign, to all points passed through by each single of these geodesics, unique values of θ and ϕ (and τ of course), with θ covering $\langle 0,\pi\rangle$ and ϕ covering $\langle 0,2\pi)$, as usual for the angles on a sphere. It is clear that the metric expressed in such coordinates may contain neither $\mathrm{d}\theta$ nor $\mathrm{d}\phi$ linearly, because the interval between any two events has to be independent of the signs of these angular shifts. Hence, the metric can be written as

$$\mathrm{d}s^2 = g_{\tau\tau}\mathrm{d}\tau^2 + 2g_{\tau\rho}\mathrm{d}\tau\mathrm{d}\rho + g_{\rho\rho}\mathrm{d}\rho^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2\,.$$

Note that τ only surely represents proper time at the origin, so $g_{\tau\tau}$ has to be left general. On the other hand, $g_{\rho\rho}$ can be set to unity by assuming that the radial geodesics are (*everywhere*) parametrized by proper distance (arc length), call it ρ. Formally, one obtains this from normalization

$$1 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\rho} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\rho}\frac{\partial x^\nu}{\partial\rho} = g_{\mu\nu}\delta^\mu_1\delta^\nu_1 = g_{11} \equiv g_{\rho\rho}\,.$$

Even more simplification follows for $g_{\tau\rho}$. Namely, $g_{\tau\rho}$ is certainly zero along the world-line of the origin, because the radial geodesics have been sent in directions orthogonal to that world-line. More accurately, orthogonal are – *along that world-line** – the respective tangent vectors

$$\frac{\partial x^\mu}{\partial\tau} = \delta^\mu_\tau \quad \text{and} \quad \frac{\partial x^\mu}{\partial\rho} = \delta^\mu_\rho\ : \qquad g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\rho} = g_{\tau\rho} \stackrel{*}{=} 0\,.$$

Now, the radial geodesics are described by equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\rho^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\rho}\frac{\mathrm{d}x^\beta}{\mathrm{d}\rho} = 0 \qquad \Longrightarrow \qquad \Gamma^\mu{}_{\rho\rho} \equiv \Gamma^\mu{}_{11} = 0\,,$$

where the implication follows by regarding that $\frac{\mathrm{d}x^\alpha}{\mathrm{d}\rho} = \frac{\partial x^\alpha}{\partial \rho} = \delta^\alpha_\rho\ (\equiv \delta^\alpha_1)$. Writing the Γs in terms of the Christoffel symbols, we thus have

$$0 = \Gamma^\mu{}_{11} = \frac{1}{2}\,g^{\mu\sigma}(g_{\sigma1,1} + g_{1\sigma,1} - \cancel{g_{11,\sigma}}) = g^{\mu\sigma}g_{\sigma1,1} \quad \Longrightarrow \quad g_{\alpha\mu}g^{\mu\sigma}g_{\sigma1,1} = g_{\alpha1,1} = 0\,. \tag{12.1}$$

This means that the values of $g_{\alpha1}$ do not change along the radial geodesics, so one can use everywhere the values they have along the central world-line. In particular, it applies to $g_{01} \equiv g_{\tau\rho} (=0)$ and to $g_{11} \equiv g_{\rho\rho} (=1)$ (the latter is automatic due to the choice of ρ).

To summarize, the spherically symmetric metric can be written as

$$\mathrm{d}s^2 = g_{\tau\tau}\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2\,.$$

Further, we demand that every set $\{\tau = \mathrm{const}, \rho = \mathrm{const}\}$ have the geometry of a two-dimensional sphere, which means

$$g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 \stackrel{!}{=} r^2(\tau,\rho)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

where the function $r(\tau,\rho)$ plays the role of the Euclidean radius of the sphere, namely, it is given from proper area of the sphere by the Euclidean formula

$$\int_0^{2\pi}\int_0^{\pi} \sqrt{(g_{\theta\theta}g_{\phi\phi})_{\tau,\rho=\mathrm{const}}}\,\mathrm{d}\theta\,\mathrm{d}\phi =: 4\pi r^2(\tau,\rho)\,. \tag{12.2}$$

Being given by proper area (and thus often called the "area radius"), this radius has a clear, invariant meaning. It represents "circumferential radius" at the same time, because the proper circumference of any circle $\{\tau = \mathrm{const}, \rho = \mathrm{const}, \theta = \mathrm{const}\}$ reads

$$\int_0^{2\pi} \sqrt{(g_{\phi\phi})_{\tau,\rho,\theta=\mathrm{const}}}\,\mathrm{d}\phi = 2\pi\sqrt{g_{\phi\phi}(\tau,\rho)} = 2\pi r(\tau,\rho)\sin\theta\,,$$

especially the equatorial circle ($\theta = \pi/2$) yields $2\pi r(\tau,\rho)$.

Finally, the term $g_{\tau\tau}\mathrm{d}\tau^2$ represents proper time of someone standing at rest at constant ρ, θ and ϕ. Under spherical symmetry, such a contribution has to be independent of θ and ϕ, so $g_{\tau\tau} = g_{\tau\tau}(\tau,\rho)$. So the result is

$$\mathrm{d}s^2 = g_{\tau\tau}(\tau,\rho)\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + r^2(\tau,\rho)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{12.3}$$

Wow! The spherically symmetric space-time can be described by mere two functions of time and radius! This has been found solely from the symmetry requirements themselves, *without* *any reference to the field equations*, so it even holds more generally than just for general relativity.

In order to reach the "canonical" form of the spherically symmetric element, we will perform one more transformation $(\tau, \rho) \to (T, R \equiv r)$ whose reason is to use $r(\equiv R)$ as the radial coordinate (instead of the proper distance ρ). In such a step, it may not be obvious that one can keep the cross-term g_{TR} zero, so let us first write

$$\mathrm{d}s^2 = g_{TT}(T,R)\mathrm{d}T^2 + 2g_{TR}(T,R)\mathrm{d}T\mathrm{d}R + g_{RR}(T,R)\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

However, the diagonal form can be restored by adjusting the coordinates to (t, r) by

$$\mathrm{d}t = f(T,R)(g_{TT}\mathrm{d}T + g_{TR}\mathrm{d}R)\,, \qquad \mathrm{d}r = \mathrm{d}R\,,$$

with $f(T, R)$ standing for an integrating factor which ensures that the first expression is really a total differential of some scalar. The transformation yields

$$\begin{aligned} &\frac{\mathrm{d}t^2}{f^2 g_{TT}} - \frac{(g_{TR})^2}{g_{TT}}\,\mathrm{d}r^2 = g_{TT}\mathrm{d}T^2 + 2g_{TR}\mathrm{d}T\mathrm{d}R \\ \Longrightarrow \quad & g_{TT}\mathrm{d}T^2 + 2g_{TR}\mathrm{d}T\mathrm{d}R + g_{RR}\mathrm{d}R^2 = \frac{\mathrm{d}t^2}{f^2 g_{TT}} + \left[g_{RR} - \frac{(g_{TR})^2}{g_{TT}}\right]\mathrm{d}r^2\,. \end{aligned}$$

In such a way, one arrives at

$$\boxed{\mathrm{d}s^2 = g_{tt}(t,r)\mathrm{d}t^2 + g_{rr}(t,r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)}\,, \tag{12.4}$$

where

$$g_{tt} := \frac{1}{f^2 g_{TT}}\,, \qquad g_{rr} := g_{RR} - \frac{(g_{TR})^2}{g_{TT}}\,.$$

We may thus summarize the observations of this section in the following

$\boxed{\text{Theorem}}$: The metric of every spherically symmetric space-time can be written in the form (12.4) containing two undetermined functions, $g_{tt}(t,r)$ and $g_{rr}(t,r)$. (These are to be fixed by particular form of the field equations.) The coordinate r represents the area (and circumferential) radius.

12.2 Schwarzschild solution: Birkhoff theorem

Only now we will require that the spherically symmetric metric satisfy the Einstein equations. Specifically, let it satisfy *vacuum Einstein equations without the cosmological constant*. In such a situation, the equations reduce to $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$ or $R_{\mu\nu} = 0$ (these forms are equivalent since the scalar curvature R necessarily vanishes as well). Computing, from the general metric (12.4), the necessary quantities, i.e. $g_{\mu\nu} \to \Gamma^{\mu}{}_{\kappa\lambda} \to R^{\sigma}{}_{\nu\kappa\lambda} \to R_{\nu\lambda}$ (possibly using programs for tensor computations, such as MAPLE or MATHEMATICA), we find that most of the Einstein tensor $G_{\mu\nu}$ is trivially zero, except for the components

$$G^t{}_t = -\frac{r\frac{\partial g_{rr}}{\partial r} + g_{rr}(g_{rr}-1)}{r^2(g_{rr})^2}\,, \quad G_{tr} = \frac{\frac{\partial g_{rr}}{\partial t}}{r g_{rr}}\,, \quad G^r{}_r = \frac{r\frac{\partial g_{tt}}{\partial r} - g_{tt}(g_{rr}-1)}{r^2 g_{tt} g_{rr}} \tag{12.5}$$

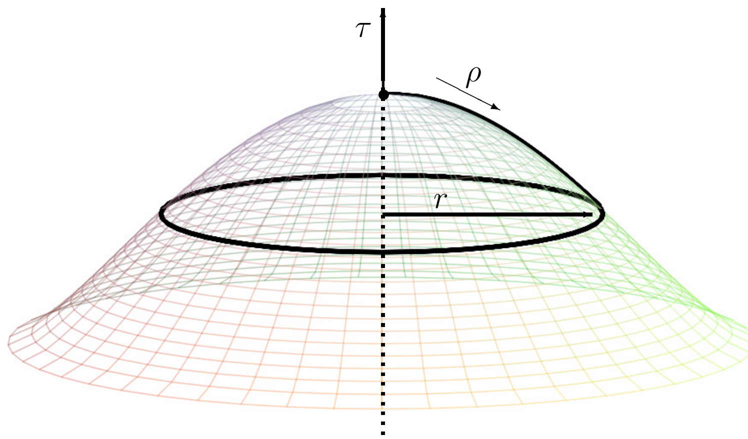

Figure 12.1 *A spherically symmetric hypersurface $\tau = \mathrm{const}$ (here depicted as a 2D surface). The affine parameter of radial geodesics ρ represents length of their arc, so it determines proper distance in the radial direction. The Schwarzschild radius r is defined as the Euclidean radius of the sphere $\{\tau = \mathrm{const}, \rho = \mathrm{const}\}$.*

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and $G^{\theta}{}_{\theta} = G^{\phi}{}_{\phi}$ (for these two components, one obtains the same, longer expressions).

From the equation $G_{tr} = 0$ it follows immediately that g_{rr} does not depend on t. Therefore, in the equation $G^{t}{}_{t} = 0$ we can write total derivative instead of partial,

$$\frac{\mathrm{d}g_{rr}}{\mathrm{d}r} = -\frac{g_{rr}}{r}(g_{rr} - 1) \,. \tag{12.6}$$

By separation of variables, we get

$$\begin{aligned} \frac{\mathrm{d}g_{rr}}{g_{rr}(g_{rr}-1)} = -\frac{\mathrm{d}r}{r} \quad &\Longrightarrow \quad \ln\frac{g_{rr}-1}{g_{rr}} = -\ln r + \ln(\mathrm{const}) = \ln\frac{\mathrm{const}}{r} \quad \Longrightarrow \\ &\Longrightarrow \quad \frac{g_{rr}-1}{g_{rr}} = \frac{\mathrm{const}}{r} \quad \Longrightarrow \quad g_{rr} = \frac{1}{1-\frac{\mathrm{const}}{r}} \,. \end{aligned} \tag{12.7}$$

Finally, divide the equation $G^{r}{}_{r} = 0$, i.e. $\frac{\partial g_{tt}}{\partial r} = \frac{g_{tt}}{r}(g_{rr} - 1)$, by equation (12.6):

$$\begin{aligned} \frac{\frac{\partial g_{tt}}{\partial r}}{\frac{\mathrm{d}g_{rr}}{\mathrm{d}r}} = -\frac{g_{tt}}{g_{rr}} \quad &\Longrightarrow \quad \frac{g_{tt,r}}{g_{tt}} = -\frac{g_{rr,r}}{g_{rr}} \quad \Longrightarrow \quad (\ln|g_{tt}|)_{,r} = -(\ln|g_{rr}|)_{,r} \\ &\Longrightarrow \quad (\ln|g_{tt}g_{rr}|)_{,r} = 0 \quad \Longrightarrow \quad (g_{tt}g_{rr})_{,r} = 0 \quad \Longrightarrow \quad g_{tt}g_{rr} = -f(t). \end{aligned}$$

The integration time function $f(t)$ is arbitrary, it is not restricted by the field equations; it fixes the scaling of the time coordinate. It may be chosen $f(t) = 1$; in other words, in the obtained $g_{tt}\mathrm{d}t^2 = -\left(1-\frac{\mathrm{const}}{r}\right) f(t)\mathrm{d}t^2$, one may rescale the time by $f(t)\mathrm{d}t^2 \to \mathrm{d}t^2$. So the resulting metric reads

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathrm{const}}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{\mathrm{const}}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \,. \tag{12.8}$$

Last step is to interpret the integration constant. Recall Section 3.7 where, by Newtonian limit of the geodesic equation, we found that in a weak stationary field, $g_{tt} \doteq -1-2\Phi$; in the spherically symmetric field, it means, specifically, $g_{tt} \doteq -1 + \frac{2M}{r}$, where M is the mass contained within the given radius r. Comparing with our $g_{tt} = -1 + \frac{\mathrm{const}}{r}$, we see that $\mathrm{const} = 2M$.

$\boxed{\text{Birkhoff theorem}}$ (1923): In general relativity, every vacuum spherically symmetric region of space-time can be described by the Schwarzschild metric

$$\boxed{\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)}\,. \tag{12.9}$$

In standard units, the metric reads

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2GM}{c^2 r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{12.10}$$

What about $G^\theta{}_\theta = 0$ and $G^\phi{}_\phi = 0$?

We have not at all employed the remaining non-trivial Einstein equations $G^\theta{}_\theta = G^\phi{}_\phi = 0$. Do not they further restrict the metric functions? No, because they are not independent. In order to show that, one has to derive them *only using the other Einstein equations* (so also without using the relation $G^\mu{}_\mu = -2R = 0$). The only relation the left-hand side of the field equations satisfies completely generally is $G^\nu{}_{\mu;\nu} = 0$, so let us start from there:

$$0 = G^\nu{}_{\mu;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,G^\nu{}_\mu\right)_{,\nu} - \Gamma^\kappa{}_{\mu\lambda}G^\lambda{}_\kappa\,. \tag{12.11}$$

For $\mu = 1$, the first term drops out since $G^\nu{}_r = 0$, and the second term we write out

$$\Gamma^\kappa{}_{\mu\lambda}G^\lambda{}_\kappa = \Gamma_{\kappa\mu\lambda}G^{\lambda\kappa} = \frac{1}{2}\left(g_{\kappa\mu,\lambda} + g_{\lambda\kappa,\mu} - g_{\mu\lambda,\kappa}\right)G^{\underline{\lambda\kappa}} = \frac{1}{2}\,g_{\lambda\kappa,\mu}G^{\lambda\kappa}\,. \tag{12.12}$$

If all the other components of $G^{\lambda\kappa}$ (other than $G^{\theta\theta}$ a $G^{\phi\phi}$) are zero, one has, for $\mu = 1 \equiv r$,

$$g_{\theta\theta,r}G^{\theta\theta} + g_{\phi\phi,r}G^{\phi\phi} = 2r(G^{\theta\theta} + G^{\phi\phi}\sin^2\theta) = \frac{2}{r}\left(G^\theta{}_\theta + G^\phi{}_\phi\right) = 0\,. \tag{12.13}$$

However, we know that $G^\theta{}_\theta = G^\phi{}_\phi$ directly from computation of the Einstein tensor, so these components have to be zero individually.

12.2.1 Basic features of the Schwarzschild metric

One-parameter family of metrics

The Birkhoff theorem says that the field outside of any spherically symmetric source is characterized by just one parameter representing mass contained within a given radius. Actually, when interpreting the integration constant using the Newtonian limit of g_{tt}, one should make more precise that in the Newtonian case M stands for the mass *found below a given radius*. Hence, the field does not depend on an exact radial dependence of density, only on total mass present in the sphere below the given radius.

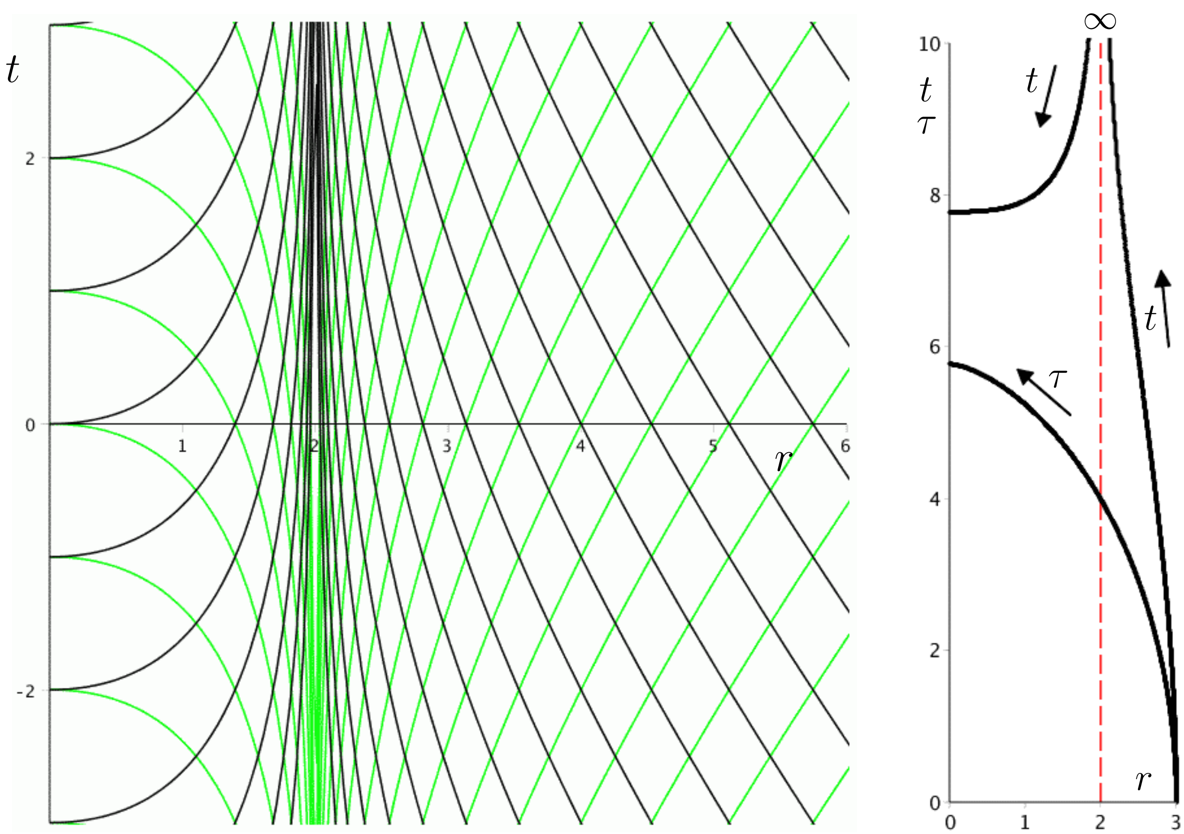

Figure 12.2 *Left:* *Radial light-like world-lines in the Schwarzschild space-time. The values along both axes are given in the units of M. Black are "ingoing" world-lines, given by the slope $\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{1-\frac{2M}{r}}$; they start at $r \to \infty$ and everywhere point in the direction of decreasing r, passing through the horizon $r = 2M$ via $t = +\infty$; below $r = 2M$ they travel against the direction of t [sic]. Light green are "outgoing" world-lines, given by the slope $\frac{\mathrm{d}t}{\mathrm{d}r} = +\frac{1}{1-\frac{2M}{r}}$; they start on the horizon $r = 2M$ at $t = -\infty$ and everywhere point in the direction of increasing time t; below $r = 2M$ they also travel against the direction of r (!). Tangents to the outgoing and ingoing world-lines at a given radius limit the local radial-motion light-cone. It is seen that (i) far from the horizon ($r \gg M$), $\mathrm{d}t/\mathrm{d}r = \pm 1$, so the light-cones are $\pm 45°$ as in special relativity; (ii) towards the horizon they narrow down, so that every ingoing time-like and light-like trajectories intersect the horizon at infinite values of t (the slopes diverge there, $\mathrm{d}t/\mathrm{d}r \to \pm\infty$); (iii) below the horizon the future of the cones points "inwards"; closely below $r = 2M$ the cones are widely open, $\mathrm{d}t/\mathrm{d}r = \mp\infty$, and then towards $r = 0$ they narrow down to $\mathrm{d}t/\mathrm{d}r = 0^{\mp}$, so all the time-like and light-like world-lines enter the singularity horizontally.*

Right: *Example of a radial time-like world-line – a fall of a massive particle to the black hole. The particle is released from rest from $r = 3M$ at zero coordinate time t and zero proper time τ. At the top is the wild behaviour in t, at the bottom is the monotonous behaviour in τ. The plot also illustrates Section 14.1.4 where we will treat the radial free fall in more detail in order to confirm that nothing special happens at the horizon.*

Flat limit

If there is no source ($M = 0$), the metric (12.9) reduces to the metric of flat space-time represented in spherical coordinates, $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$.

Asymptotic flatness

For any M, the metric (12.9) goes over to the flat metric (in spherical coordinates) at large distances, $r \gg M$.

Nature of the coordinates

This also confirms the spherical nature of the Schwarzschild coordinates. In particular, θ and ϕ are angles on any sphere $\{t = \mathrm{const}, r = \mathrm{const}\}$, with r being proportional to the proper area of that sphere (and to the circumference of circles on that sphere). A new information concerns the time t: obviously, it represents the proper time of clocks staying at rest at radial infinity. Actually, being at rest means $\mathrm{d}r = 0$, $\mathrm{d}\theta = 0$, $\mathrm{d}\phi = 0$, so the interval reduces to $\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2$, and this further reduces to $\mathrm{d}s^2 = -\mathrm{d}t^2$ at $r \to \infty$.

Metric is static

The Schwarzschild metric is *static*, namely, it is stationary ($g_{\mu\nu,t} = 0$) plus it is independent of the *direction* of time, i.e. invariant under the change $t \to -t$, which is equivalent to $g_{0i} = 0 = g^{0i}$ (because $\mathrm{d}t$ cannot appear in the interval in linear order). In comparison with the general spherically symmetric metric (12.4), the staticity is the main new feature, only fixed by the vacuum field equations. Note that we did not *assume* staticity of the sources – and actually, they need not be such. For example, a pulsating ball generates static metric outside, provided it keeps the spherical symmetry. (This means, in particular, that it must not rotate.) Hence, in passing, a pulsating ball cannot generate gravitational waves (it is the same with EM waves). This circumstance is a special example of a general fact that the field always has the *same or higher* symmetry than its source.

Killing symmetries

We know from Section 11.4 that the space-time symmetries can also be stated in a coordinate-free language – in terms of the existence of Killing vector fields. The stationarity corresponds to the existence of a time-like Killing field t^μ (at least time-like at radial infinity); this field can be written $t^\mu = \frac{\partial x^\mu}{\partial t}$, so it has components $t^\mu = \delta^\mu_0$ in the Schwarzschild coordinates.[3] The second apparent symmetry – independence of the metric on the azimuth ϕ – corresponds to

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[3] The metric is not only stationary, but even static, which in geometrical language means that the field t^μ is *hypersurface-orthogonal*, namely orthogonal to the hypersurfaces $t = \mathrm{const}$. Actually, it can be written as $t_\mu = g_{tt}t_{,\mu}$. See Section 24.4. Note that "hypersurface" is a name for a submanifold whose dimension is only by 1 less than that of the host manifold (geometers say it has codimension one). Hence, in the 4D manifolds, hypersurfaces are 3D; in a space-like case, they correspond to a "space at certain time", $t = \mathrm{const}$.

the Killing field ϕ^μ which is space-like and its integral curves are closed (circles); it can be written as $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$ and in Schwarzschild coordinates it has components $\phi^\mu = \delta^\mu_3$.

You may point out now that the main assumption of this whole chapter has been spherical symmetry, so there should actually exist *three* space-like Killing symmetries. Yes, they do exist, each corresponding to an invariance with respect to rotation about one of the three orthogonal spatial axes. Choosing one of these rotations to be parameterized by the azimuthal coordinate ϕ, thus having the above Killing field $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$, the two remaining spatial Killing fields read, in the Schwarzschild coordinates,

$$\vartheta^\mu_{(1)} = (0, 0, \sin\phi, \cot\theta\cos\phi)\,, \qquad \vartheta^\mu_{(2)} = (0, 0, -\cos\phi, \cot\theta\sin\phi)\,. \tag{12.14}$$

Metric only holds outside sources

Let us stress once more that the Schwarzschild metric describes the external field of *any* spherically symmetric source – it may be a spherical ball, a spherical shell (or a set of such shells), thick spherically symmetric layer(s), or a combination of these. In such cases, the Schwarzschild metric only covers a certain part of the manifold. We will, in the following, suppose *the whole* space-time to be vacuum (this will shortly be made more precise).

Light-cones

Basic intuition about a given space-time is provided by light-cones. Since we are in a spherically symmetric field, radial motion is clearly privileged, so let us find the light-cones for it. Radial photon world-lines are given by

$$0 = \mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} \quad \Longrightarrow \quad \mathrm{d}t = \pm\frac{\mathrm{d}r}{1 - \frac{2M}{r}} = \pm g_{rr}\mathrm{d}r \equiv \pm\mathrm{d}r^*,$$

where $r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$ is often being called the "tortoise coordinate". The radial light trajectories are drawn in the left part of Figure 12.2; the behaviour of the light-cones is clear from there, or just from the above slope: at large radii they are "45°" as in Minkowski, while when going down to smaller radii, they get more and more narrow (in the direction of t); at $r = 2M$, the cone shrinks to a vertical line and then, below $r = 2M$, opens in the perpendicular (radial) direction. Towards $r = 0$, the cones narrow in the r direction and finally shrink to a horizontal line at the very centre.

As the tangential part of the metric is positive, it is also seen that the light-cones for non-radial motion are narrower than the radial ones (above $r = 2M$, their slope $|\mathrm{d}t/\mathrm{d}r|$ is larger, while below $2M$ it is smaller than for radial motion); this is natural – the radial component of speed has to be smaller then, since there is also some tangential component (and the total speed is fixed to c).

Singularities of the metric

Two singularities apparently exist in the metric (12.9), at $r = 2M$ and $r = 0$. Their precise nature we will inspect later (just a coordinate, or a real, space-time singularity, that is the question), but already now we can see the following. In order to decide between the coordinate and "genuine" singularity, one either turns to some invariants or to some "physical" (really measurable) quantities. In the case of Schwarzschild, due to the latter's vacuum character, the simplest scalars R and $R_{\mu\nu}R^{\mu\nu}$ automatically vanish, so one has to compute the Kretschmann scalar $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$. Although looking scary, it comes out as simple as $48M^2/r^6$. Clearly $r=0$ is a true singularity.[4] Since the proper area of the surface $\{t=\text{const}, r=\text{const}\}$ is $4\pi r^2$, hence it vanishes for $r\to 0$, one can claim the singularity is spatially point-like.

On the other hand, $r=2M$ (called the Schwarzschild radius, r_S) does not seem to make any problem, at least not on the level of curvature. However, we see from the light-cones that it is a one-way causal membrane (causal motions can only cross it in the inward direction). Also, we suspect that it is a light-like hypersurface. In order to check this, consider that the simplest verification of the space-time character of any hypersurface is to analyse the character of its normal (rather than of the tangent vectors); namely, normal to a time-like hypersurface is space-like and vice versa, with an apparent light-like ⇔ light-like limit. The hypersurfaces $r=\text{const}$ have normal $\frac{\partial r}{\partial x^\mu}$ which in the Schwarzschild coordinates reduces to $\frac{\partial r}{\partial x^\mu}=\delta^r_\mu$, so

$$g^{\mu\nu}\frac{\partial r}{\partial x^\mu}\frac{\partial r}{\partial x^\nu}=g^{\mu\nu}\delta^r_\mu\delta^r_\nu=g^{rr}=\frac{1}{g_{rr}}=1-\frac{2M}{r}\ .$$

Hence, the hypersurfaces $r=\text{const}>2M$ are time-like, the hypersurfaces $r=\text{const}<2M$ are space-like, and $r=2M$ is light-like (null).

Killing horizon; static limit; infinite redshift

The null hypersurface $r=2M$ is called a **horizon**. We have seen already it is a boundary of the region where causal connection is in principle possible between any two spatial points. This outer region ($r>2M$) is often called the **domain of outer communications**, while the inner region (from where it is not possible to escape and where it is not even possible to stay in the same place) is called the **black hole**. The horizon has several further properties. First, it is a **Killing horizon**, because the "time" Killing vector field $t^\mu=\frac{\partial x^\mu}{\partial t}$ becomes light-like there (and space-like beneath):

$$g_{\mu\nu}t^\mu t^\nu=g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial t}=g_{\mu\nu}\delta^\mu_t\delta^\nu_t=g_{tt}=-1+\frac{2M}{r}\ .$$

Second, $r=2M$ is a bottom boundary of the region where it is possible to stay at rest with respect to infinity, i.e. to stay at $r=\text{const}$, $\theta=\text{const}$, $\phi=\text{const}$. This is best seen from the light-cones (Figure 12.2), or by realizing that the four-velocity of the static observer is proportional to t^μ; more specifically, it has Schwarzschild components $(g_{tt})^{-1/2}\delta^\mu_t$. The horizon thus represents the **static-limit surface**. Finally, the horizon is also an **infinite-redshift surface**. Actually, as we know from equation (4.5), in stationary fields the frequency shift between two observers at rest reads

$$\frac{\nu(r_\text{B})}{\nu(r_\text{A})}=\sqrt{\frac{-g_{00}(r_\text{A})}{-g_{00}(r_\text{B})}}\qquad\Longrightarrow\qquad\frac{\nu(r_\text{B}>2M)}{\nu(r_\text{A})}\ \xrightarrow{r_\text{A}\to 2M^+}\ 0^+\ . \tag{12.15}$$

[4] The result is no surprise: in treating geodesic deviation, we saw the Riemann tensor is a counterpart of the Newtonian tidal tensor $\Phi_{,ij}$. And in the spherically symmetric case, that yields $\Phi_{,ij}\Phi^{,ij}=(\Phi_{,rr})^2=4M^2/r^6$.

Metric is only static above the horizon!

Look once more at the metric (12.9). We were saying above that it is static, but that is not entirely true. Firstly, it is according to the sign of the g_{tt} and g_{ii} terms that one recognizes whether a given coordinate is time(-like) or space(-like). For Schwarzschild, $g_{tt} < 0$ and $g_{rr} > 0$ at $r > 2M$, so there t and r represent what we have been declaring, BUT inside the black hole it is $g_{tt} > 0$ and $g_{rr} < 0$, so the roles of t and r reverse! Well, these are just coordinates, so who cares, but it indicates something important: the metric depends on r, and this represents *time* below the horizon, so *there* the metric is *dynamical* rather than static. In a geometrical language, there exists no time-like Killing vector field below the horizon.

12.3 Geodesic motion in the Schwarzschild field

The causal structure of space-time (light-cones) limits what motions are possible in principle. Another piece of intuition is provided by time-like and light-like geodesics – these tell how looks the unaccelerated motion, solely driven by the given gravitational field. Central point naturally is the geodesic equation, supplemented by appropriate initial conditions. Integration of this equation is necessary when one wishes to find an exact evolution of specific trajectories. In this section, however, we rather focus on qualitative discussion of generic properties of free motion. Similarly as in the Newtonian discussion of motion in the central field (the Kepler problem), this is best addressed using the effective-potential method. Its starting point is to ask whether there exist any constants of geodesic motion.

12.3.1 Space-time symmetries and constants of geodesic motion

From Section 11.4.1 we know that the projections of the test-particle four-momentum p^μ on Killing vector fields are constant along any geodesic. We also know, from Section 3.6 already, that the equivalent coordinate criterion is the independence of metric on some ("Killing") coordinate. The Schwarzschild metric being independent on t and ϕ, we thus immediately have two constants of geodesic motion,

$$p_t = p_\mu t^\mu, \qquad p_\phi = p_\mu \phi^\mu \,. \tag{12.16}$$

In order to learn their physical meaning, let us make use of the fact that they are constant, so that they can be evaluated at any point of the geodesic. In asymptotically flat space-times, one standardly tries to interpret the quantities at spatial infinity, because there they assume their special-relativistic form. Important will be the relation for time dilation,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}} = \frac{1}{\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}} = \frac{1}{\sqrt{-g_{tt} - g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}}} = \frac{1}{\sqrt{1-\frac{2M}{r} - v^2(r)}}\,,$$

where $v^2(r) \equiv g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}$ is the square of the coordinate three-velocity $v^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}t}$. Now to the constants $p_t \equiv m u_t$, $p_\phi \equiv m u_\phi$:

- The time component

$$-u_t = -g_{t\sigma}u^\sigma = -g_{tt}u^t = -g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1-\frac{2M}{r}}{\sqrt{1-\frac{2M}{r}-v^2(r)}} \tag{12.17}$$

yields, very far from the centre, a clear physical meaning,

$$\lim_{r\to\infty}(-p_t) = \frac{m}{\sqrt{1-v^2(\infty)}} \equiv m\gamma(\infty) =: E\,. \tag{12.18}$$

The expression γ represents the Lorentz factor given by speed with respect to a rest observer at radial infinity. Actually, t is the proper time of that observer, so $v(\infty)$ is really the speed measured by her. Therefore, $E := -p_t = -mu_t$ is the energy of the particle with respect to the rest observer at radial infinity – particle's **energy at infinity** in short. Three cases may happen:

  - $E > m$ ... the particle reaches infinity having still some velocity $v(\infty) > 0$ there (in the Newtonian case, such a particle follows a hyperbolic trajectory)
  - $E = m$ ... the particle reaches infinity and exactly stops there, $v(\infty) = 0$ (in the Newtonian case, such a particle follows a parabolic trajectory)
  - $E < m$ ... the particle does not reach infinity (in the Newtonian case, such a particle is on an elliptic orbit); in this case, the observer at infinity naturally does not measure E, she rather could measure the particle's *binding energy* $m-E$ which is the energy that would have to be *added* to the particle in order that this can reach infinity.

- The azimuthal component

$$u_\phi = g_{\phi\sigma}u^\sigma = g_{\phi\phi}u^\phi = g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = g_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{v^\phi r^2\sin^2\theta}{\sqrt{1-\frac{2M}{r}-v^2(r)}} \tag{12.19}$$

yields, far from the centre,

$$\lim_{r\to\infty}(p_\phi) = E\lim_{r\to\infty}(v^\phi r^2\sin^2\theta) =: L\,, \tag{12.20}$$

which is the angular momentum with respect to the $\theta = 0$ axis, as measured by the rest observer at infinity; standard abbreviation is the **angular momentum at infinity**.

- Note: the above meaning of $-p_t$ and p_ϕ does not solely apply to geodesics and to spherically symmetric fields. However, along a general world-line and/or in a space-time without the respective symmetries, these quantities need not be conserved. On the other hand, they may (of course) even be conserved along accelerated orbits, if those orbits "fit in" the space-time symmetries well, as for example do the stationary circular orbits in stationary and axially symmetric space-times.

- Still another "constant of the motion" follows from spherical symmetry, similarly as in the Newtonian case: **the motion is planar**. Actually, choose the equatorial plane ($\theta = \pi/2$) to be the plane defined by the centre of symmetry and by the particle's momentary (or initial) velocity. In spherical symmetry, such a plane is surely a main plane of the space, so the particle has no reason to leave it. In the following, we will thus assume $\theta = \pi/2$, $g_{\phi\phi} = r^2$, $u^\theta = 0$, $\hat{v}^{\hat{\theta}} = 0$. In passing, the planarity of motion really follows from the geodesic equation (of course), as can best be checked from its covariant θ component, with $\theta = \pi/2$ and $u^\theta = 0$ inserted,

$$\frac{\mathrm{d}u_\theta}{\mathrm{d}\tau} = \frac{1}{2}\, g_{\mu\nu,\theta} u^\mu u^\nu = \frac{1}{2}\, g_{\phi\phi,\theta} (u^\phi)^2 = r^2 \sin\theta \cos\theta\, (u^\phi)^2 = 0\,.$$

One is in fact interested in the behaviour of *contravariant* component u^θ, but that is similar, because

$$u^\theta = g^{\theta\theta} u_\theta \qquad \Longrightarrow \qquad \frac{\mathrm{d}u^\theta}{\mathrm{d}\tau} = g^{\theta\theta}{}_{,\iota} u^\iota \cancel{u_\theta} + g^{\theta\theta} \frac{\mathrm{d}u_\theta}{\mathrm{d}\tau} = \frac{1}{g_{\theta\theta}} \frac{\mathrm{d}u_\theta}{\mathrm{d}\tau} = \frac{1}{r^2} \frac{\mathrm{d}u_\theta}{\mathrm{d}\tau}\,.$$

12.3.2 Four-momentum normalization and the equation for radial motion

Since the motion is planar ($\theta = \mathrm{const} \equiv \pi/2$), it has just 3 degrees of freedom, (t, r, ϕ). We thus need not directly solve the geodesic equation, because we have 3 constants of the motion – E, L, plus the rest mass m, as fixed (for *any* world-line) by the four-momentum normalization $g^{\mu\nu} p_\mu p_\nu = -m^2$. Actually, p_t and p_ϕ are obtained immediately from the constants

$$E \equiv -p_t = -g_{t\sigma} p^\sigma = -g_{tt} p^t\,, \qquad\qquad L \equiv p_\phi = g_{\phi\sigma} p^\sigma = g_{\phi\phi} p^\phi\ :$$

$$p^t = \frac{E}{-g_{tt}} = \frac{E}{1 - \frac{2M}{r}}\,, \qquad\qquad p^\phi = \frac{L}{g_{\phi\phi}} = \frac{L}{r^2}\,. \tag{12.21}$$

Finally, from $g^{\mu\nu} p_\mu p_\nu = -m^2$, we have (using $g^{\mu\nu} = 1/g_{\mu\nu}$ given by diagonality of the metric)

$$g^{tt} (p_t)^2 + g^{rr} (p_r)^2 + g^{\phi\phi} (p_\phi)^2 = \frac{E^2}{g_{tt}} + g_{rr} (p^r)^2 + \frac{L^2}{g_{\phi\phi}} = -m^2\,,$$

hence

$$(p^r)^2 = \frac{E^2}{-g_{tt} g_{rr}} - \frac{1}{g_{rr}} \left(m^2 + \frac{L^2}{g_{\phi\phi}} \right). \tag{12.22}$$

This form is valid for the equatorial motion in any static and axially symmetric space-time. Specifically for the Schwarzschild metric (in which $g_{tt} g_{rr} = -1$), we obtain

$$\boxed{(p^r)^2 = E^2 - \left(1 - \frac{2M}{r}\right) \left(m^2 + \frac{L^2}{r^2}\right)}\,. \tag{12.23}$$

This formula is the key to the discussion of radial behaviour of time-like as well as light-like geodesics in the Schwarzschild field. Similarly as it is common in the Newtonian case, the discussion employs the method of effective potential.

12.3.3 Radial motion of free massive test particles

For particles with $m \neq 0$, following time-like geodesics, it's natural to divide p^μ by m and go over to four-velocity components $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$. In the case of equation (12.23), it means

$$(u^r)^2 = \frac{E^2}{m^2} - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right) =: \tilde{E}^2 - \tilde{V}^2\,, \tag{12.24}$$

where $\tilde{E} := E/m$, $\tilde{L} := L/m$, and we have introduced the **effective potential** (per unit m) by

$$\tilde{V}^2 \equiv -g_{tt}\left(1 + \frac{\tilde{L}^2}{g_{\phi\phi}}\right) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right). \tag{12.25}$$

Meaning of the effective potential

The right-hand side of (12.24) clearly has to be non-negative, so, at a given radius, the particle must have $\tilde{E}^2 \geqslant \tilde{V}^2$. The effective potential thus represents *the minimal value of energy* $\tilde{E}$ with which the given particle (i.e., the particle with a given angular momentum $\tilde{L}$) can exist at the given location.

Let us check whether the effective potential corresponds, in the Newtonian limit, to the classical form known from the Kepler problem. For $r \gg M$, $r^2 \gg \tilde{L}^2$ ($\sim$ small tangential speed),

$$\tilde{V} = \sqrt{1 - \frac{2M}{r}}\sqrt{1 + \frac{\tilde{L}^2}{r^2}} \doteq \left(1 - \frac{M}{r}\right)\left(1 + \frac{\tilde{L}^2}{2r^2}\right) \doteq 1 - \frac{M}{r} + \frac{\tilde{L}^2}{2r^2}\,. \tag{12.26}$$

On the other hand, the classical motion in the central field $\Phi = -M/r$ has constants

$$\tilde{E} = \frac{1}{2}\left[(v^r)^2 + r^2(v^\phi)^2\right] + \Phi\,, \qquad \tilde{L} = r^2 v^\phi\,, \tag{12.27}$$

from where

$$v^\phi = \frac{\tilde{L}}{r^2}\,, \qquad (v^r)^2 = 2(\tilde{E} - \tilde{V}_{\text{eff}})\,; \qquad \tilde{V}_{\text{eff}} := -\frac{M}{r} + \frac{\tilde{L}^2}{2r^2}\,. \tag{12.28}$$

The $\tilde{V}_{\text{eff}}$ exactly coincides with the above limit form of $\tilde{V}$, only it does not include the rest energy $\tilde{E}_{\text{rest}} \equiv 1$ which does not exist in classical physics.

Radial motion of a particular particle

The properties of radial motion we will learn on diagrams where the dimensionless energy quantities $\tilde{E}$ and $\tilde{V}$ will be plotted against the Schwarzschild radius r (given in the units of M). In such a diagram, each single particle has its specific curve of effective potential given by its angular momentum $\tilde{L}$. Radial motion with a given (constant) energy $\tilde{E}$ is represented there as the motion along a horizontal straight line (or a certain part of it), *above* the graph of $\tilde{V}(\tilde{L};r)$. Where the straight line of given energy "hits" the effective potential ($\tilde{E}=\tilde{V}$), the radial velocity vanishes ($u^r=0$), so either it is a turning point of radial motion, or – in the special case when $\partial\tilde{V}/\partial r=0$ at that radius – the radial motion asymptotically (at $t\to\infty$) stops there.

In a general situation,[5] 3 types of trajectories exist about the centre (see the right-hand plot in figure 12.3:

- **0.** Trajectories which have *no turning point*. These correspond to straight lines with $\tilde{E}>\tilde{V}_{\rm max}$ which nowhere hit the $\tilde{V}(\tilde{L};r)$ graph. Along such trajectories, the particle either arrives from infinity and plunges to the centre, or, on the contrary, it starts from some radius ($r>0$) and escapes to infinity.

- **1.** Trajectories which have *one turning point*. These correspond to straight lines with energy in the interval $\tilde{V}_{\rm max}\geqslant\tilde{E}\geqslant 1$, which, at a certain point, hit the $\tilde{V}(\tilde{L};r)$ curve; in addition, they also include trajectories with $\tilde{E}<1$ entirely lying below the radius of $\tilde{V}_{\rm max}$. Along this type of trajectories, the particles either arrive from infinity, turn back at a certain radius and return to infinity, or, on the contrary, they travel out from the centre, stop at a certain radius and fall back.

- **2.** Trajectories with $\tilde{E}<1$ lying above the radius of $\tilde{V}_{\rm max}$. These have *two turning points* – the corresponding horizontal line $\tilde{E}={\rm const}$ only has a certain limited part above the curve of $\tilde{V}(\tilde{L};r)$. The motion is thus restricted to that radial interval only.

As we have also seen analytically, at large radii $\tilde{V}$ is very close to $\tilde{V}_{\rm eff}$ (just larger by 1 due to the rest energy). The main difference from the Newtonian situation (left plot of figure 12.3) is found near the centre: the relativistic potential $\tilde{V}$ has *two* local extremes in general – minimum and (on smaller radius) maximum, whereas the newtonian potential $\tilde{V}_{\rm eff}$ has the minimum only. Towards the centre, $\tilde{V}(r\to 2M)\to 0$ and $\tilde{V}(r\to 0)\to-\infty$, whereas $\tilde{V}_{\rm eff}(r\to 0)\to+\infty$. Let's look at what this means for the above types of trajectories:

- **0.** These trajectories *do not exist* in the classical problem! Namely, for a non-zero $\tilde{L}$, the "centrifugal barrier" of $\tilde{V}_{\rm eff}$ is infinitely high, so it cannot be flown overflown with any finite energy. Therefore, in the Newtonian case, the particle cannot be captured by the centre in any other way than by directly hitting its surface (which would only be an option if the surface lay higher than at $r=0$ and the diagram only held above the surface). For a *point-like* centre it simply implies that every particle which is arriving from infinity again flies away to infinity.

- **1.** This case corresponds to the Newtonian hyperbolic trajectories, plus, in addition, it includes the motions below the potential maximum, within the range $0[=\tilde{V}(r=2M)]<\tilde{E}\leqslant\tilde{V}_{\rm max}$. This new feature also indicates that the relativistic centre is stronger, because in the Newtonian case *every* particle with $\tilde{E}\geqslant 0$ is able to escape.

- **2.** This case corresponds to the Newtonian elliptic orbits.

[5] More specifically, it is the situation with $\tilde{L}>2\sqrt{3}\,M$, as we will see in a minute.

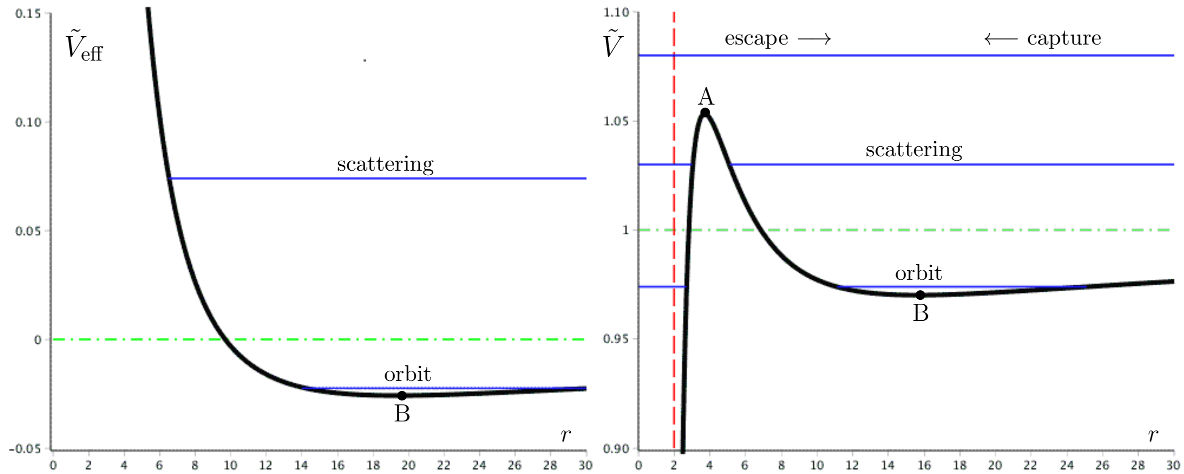

Figure 12.3 *Typical course of the Newtonian (left) and relativistic (right) effective potential ($\tilde{V}_{\rm eff}$ and $\tilde{V}$, respectively) for radial motion of a free massive test particle in the field of a spherically symmetric centre. The angular-momentum value $\tilde{L} = 4.4M$ is chosen. Indicated in blue are the main types of motion discussed in the text (in particular, bullets mark the circular orbits ... A unstable, B stable), and also the "parabolic" case (horizontal green line at $\tilde{E} = 0$, resp. $\tilde{E} = 1$). In the right-hand plot, also drawn is the black-hole horizon (vertical red line at $r = 2M$).*

Let us notice, in particular, the options for circular orbits, i.e. for such motions in which $\tilde{E} = \tilde{V}$ permanently. Again there is a difference from the Newtonian treatment: there, only the circular orbits of type B can exist – the limit case of the 2nd (elliptic) type of trajectories. From their position at the graph of $\tilde{V}(\tilde{L}; r)$ it is clear that they are bound energetically ($\tilde{E} < 1$) and stable (they rest at the potential minimum, $\partial^2\tilde{V}/\partial r^2 > 0$). Around the relativistic centre, there exist type-A circular orbits in addition, which are all unstable, because they lie on the potential maximum ($\partial^2\tilde{V}/\partial r^2 < 0$); according to the value of the maximum ($\tilde{E} < 1$ or $\tilde{E} > 1$), they are either bound or unbound. These orbits have no Newtonian analogue, since the potential $\tilde{V}_{\rm eff}$ does not have any *local* maximum.

Effective-potential shapes and properties of radial motion

Figure (12.4) shows how $\tilde{V}(\tilde{L}; r)$ depends on the value of $\tilde{L}$. Next we summarize what this implies for the radial motion:

- The orbits having pericentres at $r \gg M$ are very "Keplerian", except for the effect of the azimuthal pericentre shift (see section 17.1.2). The orbits reaching below $r \sim 10M$ may be rather different, on the contrary:
- Circular orbits (sitting on the extreme of $\tilde{V}$) may exist in the whole region $r > 3M$, but they are only stable at $r > 6M$ where the extreme is a minimum. The circular orbits on radii $4M < r < 6M$ correspond to $\tilde{E} = \tilde{V}_{\max} < 1$, so they are bound (though unstable): subject to an "inward" perturbation, they change to a spiral trajectory which falls to the centre; if

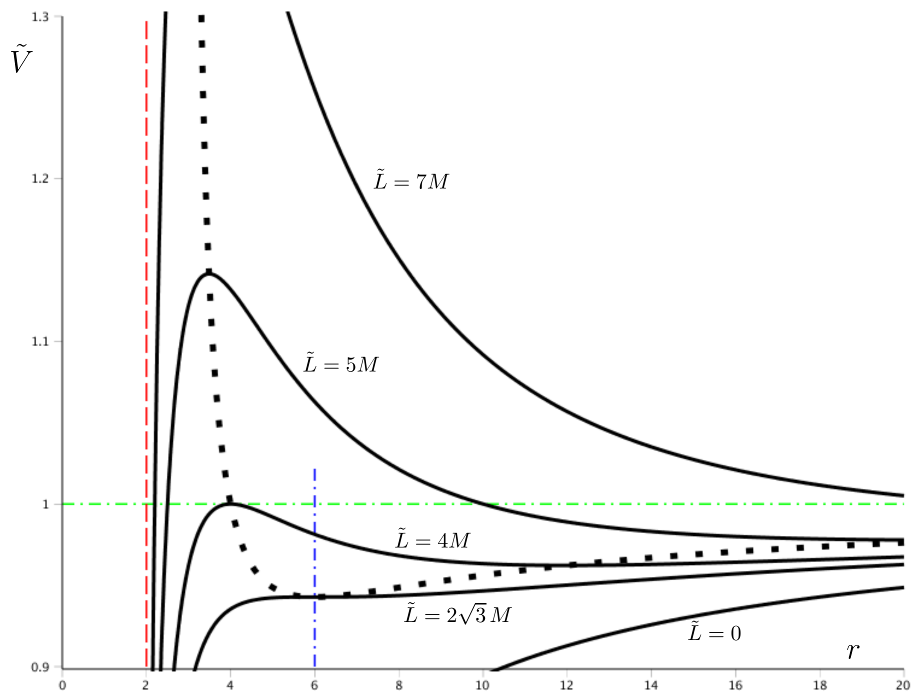

Figure 12.4 *Radial course of the effective potential $\tilde{V}$ in dependence on the value of the angular momentum $\tilde{L}$. For $\tilde{L} < 2\sqrt{3}\,M$ the potential has no local extremes (it grows monotonously with r); for $\tilde{L} = 2\sqrt{3}\,M$ an inflection point occurs at $r = 6M$, $\tilde{V} = 2\sqrt{2}/3$; when further increasing $\tilde{L}$, a local maximum rises to the left from that point and a local minimum rises to the right; while shifting down, the maximum rises and for $\tilde{L} = 4M$ it reaches and then exceeds the value $\tilde{V}(r \to \infty) = 1$, so it becomes a* global *maximum; for still larger $\tilde{L}$ (than $4M$) the maximum shifts from $r = 4M$ to $r = 3M$ while increasing further (finally to infinity for $\tilde{L} \to \infty$); the minimum increases as well (towards zero) and shifts from $r = 12M$ (for $\tilde{L} = 4M$) to infinity. The dotted curve shows the positions of local extremes of $\tilde{V}$ (to the left/right of $r = 6M$ these are maxima/minima), and also indicated by vertical lines are the important radii $r = 6M$ and $r = 2M$. Radius is in the units of M.*

perturbed "outwards", it either ends in the same way (as, for example, if $\delta\tilde{E} > 0$, $\delta\tilde{L} = 0$), or the particle settles to an elliptic orbit to the right of the original circular one (as, for example, if $\delta\tilde{E} < 0$, $\delta\tilde{L} = 0$). The circular orbits on radii $3M < r < 4M$ have $\tilde{E} = \tilde{V}_{\max} > 1$, so they are unstable and unbound, albeit spatially bounded; if perturbed "inwards", their particles fall to the centre, whereas if perturbed "outwards", they fly away to radial infinity. Under $r = 3M$ there exist no circular orbits, because $\tilde{V}$ never has a local extreme there.

- No trajectory can have pericentre at $r < 3M$ – any trajectory which reaches there from above is unavoidably captured by the centre. The trajectories with $\tilde{L} < 2\sqrt{3}\,M$ do not *at all* have a pericentre – if they are "ingoing", they inevitably fall to $r = 2M$. On the other hand, the particles with $\tilde{E} \geqslant 1$ can only have apocentre at $r \leqslant 4M$.

- A particle arriving from infinity is captured by the centre if $\tilde{E} > \tilde{V}_{\max}$; it winds up to a (unbound and unstable) circular orbit if $\tilde{E}=\tilde{V}_{\max}$; and it is reflected back to infinity if $\tilde{E}<\tilde{V}_{\max}$. For particles falling from infinity *from rest* (thus having $\tilde{E}=1$), the first/second/third option happens if $\tilde{L}<4M$, $\tilde{L}=4M$ and $\tilde{L}>4M$, respectively. A *generic* particle coming from infinity must have $\tilde{E}\geqslant 1$, so if $\tilde{L}<4M$, it is captured *in any case*.

- The most strongly bound "permanent" orbit is the marginally stable circular orbit on $r=6M$ which is given by constants $\tilde{L}=2\sqrt{3}\,M$, $\tilde{E}=\tilde{V}=2\sqrt{2}/3$, so the corresponding binding energy (per unit mass) is $\tilde{E}-1\doteq -0.0572$.
- Nowhere (outside the horizon) can exist a particle with negative energy E with respect to infinity. (This justifies that we have not been considering the $\tilde{E}<0$ option when square-rooting E^2.) In the field of *rotating* black holes, this will be different!

Expressing constants of the motion in terms of locally measured velocity

The meaning of the constants of motion is clear, yet infinity is not the only place to perform physical measurements! One would like to link E and L with quantities measured *anywhere* along a given world-line. Let's turn to the class of **static observers** – those who do not move relative to observers resting at infinity, which means those who have no spatial motion in Schwarzschild coordinates. We will denote by hat the quantities relating to these observers (for instance, $\hat{\tau}$ will stand for their proper time), in order to distinguish them from the characteristics of the followed particle. The world-lines of static observers have as tangents the four-velocity field

$$\hat{u}^\mu=\frac{\mathrm{d}x^\mu}{\mathrm{d}\hat{\tau}}=\frac{t^\mu}{|g_{\rho\sigma}t^\rho t^\sigma|^{1/2}}=\frac{t^\mu}{\sqrt{-g_{tt}}}$$

which has Schwarzschild components

$$\hat{u}^\mu=\left(\frac{1}{\sqrt{-g_{tt}}},0,0,0\right). \tag{12.29}$$

Denote by $\hat{v}$ the three-velocity of the particle with respect to the local static observer.[6] Naturally, its coordinate components we denote by $\hat{v}^i$, while its "triad" components (actually measured by the observer using the locally Cartesian rulers laid along the directions r, θ and ϕ) we denote by $\hat{v}^{\hat{\imath}}$; they read

$$\hat{v}^i=\frac{\mathrm{d}x^i}{\mathrm{d}\hat{\tau}}\,,\qquad \hat{v}^{\hat{\imath}}=\frac{\sqrt{g_{ii}}\,\mathrm{d}x^i}{\mathrm{d}\hat{\tau}}=\sqrt{g_{ii}}\,\hat{v}^i\,, \tag{12.30}$$

where *no summation* is to be performed over i! (the term $\sqrt{g_{ii}}\,\mathrm{d}x^i$ represents an element of proper distance in the direction of x^i). Now we express

$$\hat{v}^{\hat{\imath}}=\frac{\sqrt{g_{ii}}\,\mathrm{d}x^i}{\mathrm{d}\hat{\tau}}=\sqrt{g_{ii}}\,\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\hat{\tau}}=\sqrt{-g_{tt}g_{ii}}\,\frac{u^i}{\tilde{E}}\,,$$

[6] Here "local" means "at the same point where the particle is at a given instant".

where we have substituted

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} \equiv \frac{u^i}{u^t}\,, \qquad u^t = \frac{\tilde{E}}{-g_{tt}} \quad \text{from (12.21)}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\hat{\tau}} \equiv \hat{u}^t = \frac{1}{\sqrt{-g_{tt}}} \quad \text{from (12.29)}\,.$$

Expressing, in the $\hat{v}^{\hat{\imath}}$ above, $(u^r)^2$ from (12.24) and $(u^\phi)^2$ from (12.21), plus the respective metric components, we thus arrive at

$$\left(\hat{v}^{\hat{r}}\right)^2 = \frac{(u^r)^2}{\tilde{E}^2} = 1 - \frac{\tilde{V}^2}{\tilde{E}^2}\,, \qquad \left(\hat{v}^{\hat{\phi}}\right)^2 = -\frac{g_{tt}}{g_{\phi\phi}}\frac{\tilde{L}^2}{\tilde{E}^2} = \frac{\ell^2}{r^2}\left(1 - \frac{2M}{r}\right), \tag{12.31}$$

with $\ell := \tilde{L}/\tilde{E} \equiv L/E$.

Circular orbits

We have seen that neither the unbound and unstable circular orbits can lie arbitrarily close to the Schwarzschild black hole – the innermost one lies on $r = 3M$ which is not on the very horizon. What is its physical nature? Circular orbits are given by two conditions,

$$u^r = 0 \quad \Longrightarrow \quad \tilde{E} = \tilde{V}\,, \qquad \frac{\partial \tilde{V}}{\partial r} = 0 \quad \Longrightarrow \quad \tilde{L}^2 = \frac{Mr^2}{r - 3M} \tag{12.32}$$

(just differentiate the relation (12.25)). Substitution of this $\tilde{L}^2$ into $\tilde{E}^2 = \tilde{V}^2$ yields

$$\tilde{E}^2 = \frac{(r - 2M)^2}{r(r - 3M)}\,. \tag{12.33}$$

Now both the above expressions for $\tilde{L}^2$ and $\tilde{E}^2$ (valid for circular orbits) can be substituted into (12.31), in order to find the linear velocity with respect to static observers on a given radius r, as given by $\hat{v}^2 \equiv (\hat{v}^{\hat{\phi}})^2$ (for circular orbits, it only has the ϕ component):

$$\hat{v}^2 = \left(\hat{v}^{\hat{\phi}}\right)^2 = \frac{M}{r - 2M}\,. \tag{12.34}$$

Far from the centre ($r \gg M$) the relation gives $(\hat{v}^{\hat{\phi}})^2 = -\Phi$ as in the Newtonian case. With decreasing radius, the orbital speed grows quicker than according to the Newtonian limit, reaching a maximum $(\hat{v}^{\hat{\phi}})^2 = 1$ on the innermost existing orbit at $r = 3M$. Therefore, *the last circular orbit is the orbit of light*. It is, of course, energetically unbound ($\tilde{E} \to \infty$) and unstable ($\tilde{V} = \tilde{V}_{\max} \to \infty$). At $r < 3M$, nothing can however keep on circular orbit, the centre's attraction is too strong there.

12.3.4 Radial motion of free massless test particles

For particles having $m = 0$ ("photons"),[7] one naturally does not introduce E/m a L/m; however, otherwise the treatment is similar as for massive particles. From (12.23), we have in this case

$$(p^r)^2 = E^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = E^2\left(1 - \frac{\ell^2}{\lambda^2}\right), \tag{12.35}$$

[7] ... though it would be especially *attractive* to also mention gravitons here, like [29] do in section 25.6.

where we have denoted

$$\lambda := \sqrt{\frac{g_{\phi\phi}}{-g_{tt}}} = \frac{r}{\sqrt{1-\frac{2M}{r}}} \tag{12.36}$$

the function which plays the role of an inverse effective potential (the motion can only happen *below* its graph), and

$$\ell := \frac{L}{E} \tag{12.37}$$

is a constant of the motion which we already introduced for massive particles. In contrast to those, however, for photons the ℓ ratio completely determines the motion – it is *the only* relevant constant.

Let us remark that if we wished to use locally measured velocity for discussion, it would be best to express $(\hat{v}^{\hat{\phi}})^2$ from (12.31) and then just consider that for photons $(\hat{v}^{\hat{r}})^2+(\hat{v}^{\hat{\phi}})^2 = 1$:

$$(\hat{v}^{\hat{\phi}})^2 = \frac{\ell^2}{r^2}\left(1-\frac{2M}{r}\right) = \frac{\ell^2}{\lambda^2}\,, \qquad (\hat{v}^{\hat{r}})^2 = 1-(\hat{v}^{\hat{\phi}})^2 = 1-\frac{\ell^2}{\lambda^2}\,. \tag{12.38}$$

Interpretation of $\ell = L/E$

The meaning of the constant ℓ can be understood at large distances from the centre. From the above relation for $\hat{v}^{\hat{\phi}}$, we have

$$\ell = \frac{r\hat{v}^{\hat{\phi}}(r)}{\sqrt{1-\frac{2M}{r}}} \stackrel{r\to\infty}{\longrightarrow} r\hat{v}^{\hat{\phi}}(\infty)\,, \tag{12.39}$$

i.e. $\frac{\ell}{r} = \frac{\hat{v}^{\hat{\phi}}(\infty)}{1}$ (good to realize that the unity is in fact c in standard units). Sketching the situation “at infinity” (right-hand part of figure 12.5), the obtained relation follows from similarity of triangles, but mainly we can recognize the sense of ℓ: it represents the smallest distance at which the photon would fly by the centre if it followed, from infinity, a straight line in a fictive Euclidean space – such a measure is being called the **impact parameter** in the scattering theory.

Radial motion under the effective potential λ

For massive particles, the effective potential depended on their angular momentum. The (inverse) effective potential λ for photons does not depend on any characteristics of the particle – it is the same for all photons. And also its behaviour is very simple: it goes to infinity for $r\to 2M$ and for $r\to\infty$, while at $r=3M$ it has a global minimum. The expression (12.35) tells that photons can only move *below* the radial graph of λ, along a straight line given by the value of ℓ. Hence, from the left-hand plot of figure 12.5 it is seen that:

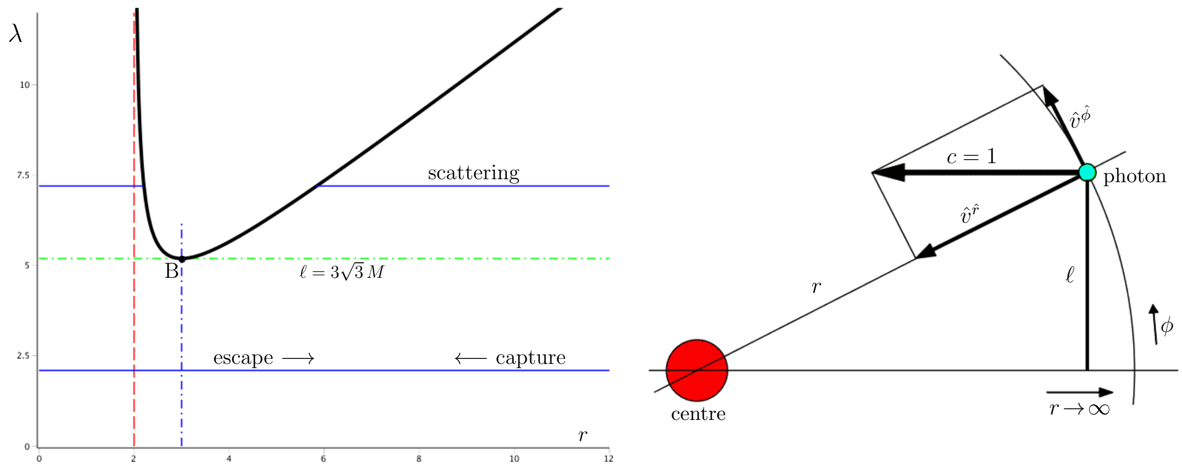

Figure 12.5 *Left:* *The course of the relativistic (inverse) effective potential λ for the motion of massless test particles in the field of a spherically symmetric centre. Indicated are the major types of motion discussed in the main text, in particular the photon circular orbit (B) and its radius $r = 3M$, and also the corresponding limit case of the motion "tangent" to the minimum of the potential (horizontal straight line $|\ell| = 3\sqrt{3}\,M$) and the black-hole horizon (vertical red line $r = 2M$). Right: Components of the photon velocity at large distance from the centre and the meaning of ℓ.*

- There exists exactly one light-like circular orbit; it lies on $r = 3M$ and corresponds to $|\ell| = \lambda_{\min} = 3\sqrt{3}\,M$. It is unstable – if perturbed, the photon either falls to the centre or flies away to infinity.

- No pericentre can occur at $r < 3M$ – any trajectory which enters that region from outside is unavoidably captured by the centre. On the other hand, apocentre can only lie at $r < 3M$; specifically, it applies to photons with $|\ell| > 3\sqrt{3}\,M$.

- A photon coming from infinity is captured by the centre if having $|\ell| < 3\sqrt{3}\,M$, winds up to a circular orbit at $r = 3M$ if having $|\ell| = 3\sqrt{3}\,M$, and reflects (in fact "bends") back to infinity if having $|\ell| > 3\sqrt{3}\,M$. A photon *outgoing* from the centre escapes to infinity if and only if it has $|\ell| < 3\sqrt{3}\,M$.

Photon escape cones

We have learnt that the Schwarzschild centre can capture particles more efficiently than the Newtonian one. In fact the latter can only capture the particle with non-zero angular momentum if it is extended (has non-zero size) and the particle just hits it. Around the relativistic centre, there exists a certain region from where particles are captured irrespectively of how they entered it, and that is even true in the point-like limit (which actually means the black-hole limit, because there necessarily occurs a horizon then). In more sophisticated terms, the effective cross section of the Newtonian centre equals its geometric cross section, whereas for the relativistic centre it is larger. This is nicely illustrated by a radial behaviour of spatial angles within which photons escape, from a given location, to infinity. The complements of these angles are generated by directions in which photons end in/on the centre; in the opposite sense, these are the directions in which no photons can reach the given point, because such photons could only come from the centre – but the centre is a black hole (therefore, the complements of the photon escape angles delimit directions in which the centre is seen from the respective radii).

The photon escape cones we will determine with respect to the privileged class of static observers, similarly as we did with three-velocities in the preceding discussion. To collect all photons escaping from a given radius, one has to take i) the photons which start from $r < 3M$ in the outward direction while having sufficiently small ℓ not to fall back, plus ii) the photons starting from $r > 3M$ – those which start in the outward direction and those which start in the *inward* direction but have so large ℓ that they do not hit the region $r \leqslant 3M$ and turn back towards infinity.

- From $r < 3M$, a photon escapes to infinity if ($\hat{v}^{\hat{r}} > 0 \wedge |\ell| < 3\sqrt{3}\,M$). Substituting from (12.37), the second constraint yields
$$\left|\hat{v}^{\hat{\phi}}\right| < 3\sqrt{3}\,\frac{M}{r}\sqrt{1-\frac{2M}{r}}\,.$$
For the radial velocity, this means
$$\hat{v}^{\hat{r}} = \sqrt{1-(\hat{v}^{\hat{\phi}})^2} > r^{-3/2}[r^3 - 27M^2(r-2M)]^{1/2} = \left|1-\frac{3M}{r}\right|\sqrt{1+\frac{6M}{r}}\,.$$
The ratio $\frac{\hat{v}^{\hat{r}}}{|\hat{v}^{\hat{\phi}}|}$ represents cotangent of the angle (α) by which the direction of the photon turns away from radial outward direction (from $\partial/\partial r$), as taken in the space of the local static observer, so the condition can be written
$$\cot\hat{\alpha}|_{r<3M} = \frac{\hat{v}^{\hat{r}}}{\left|\hat{v}^{\hat{\phi}}\right|} > \frac{|r-3M|}{3\sqrt{3}\,M}\sqrt{\frac{r+6M}{r-2M}}\,.$$
- From $r > 3M$, a photon escapes if either $\hat{v}^{\hat{r}} \geqslant 0$ or ($\hat{v}^{\hat{r}} < 0 \wedge |\ell| > 3\sqrt{3}\,M$). The first option yields the whole half-space lying "away from the centre" while the second option adds a certain piece of directions towards the centre, as far as
$$0 > \cot\hat{\alpha}|_{r>3M} = \frac{\hat{v}^{\hat{r}}}{\left|\hat{v}^{\hat{\phi}}\right|} > -\frac{|r-3M|}{3\sqrt{3}\,M}\sqrt{\frac{r+6M}{r-2M}}\,.$$
The obtained constraints can be joined on $r = 3M$,
$$\cot\hat{\alpha} > -\frac{r-3M}{3\sqrt{3}\,M}\sqrt{\frac{r+6M}{r-2M}}\,, \qquad \text{i.e.} \quad \cos\hat{\alpha} > -\left(1-\frac{3M}{r}\right)\sqrt{1+\frac{6M}{r}}\,. \quad (12.40)$$

Radial behaviour of the escape-sector boundary is drawn in figure 12.6. It reveals a rather big difference from the Newtonian centre which cannot capture the orbiter *gravitationally* (so the limit value would everywhere be $180°$ in such a graph). Recall once again that the complement of the escape cone defines how big the black hole is in the viewing field of the static observer.

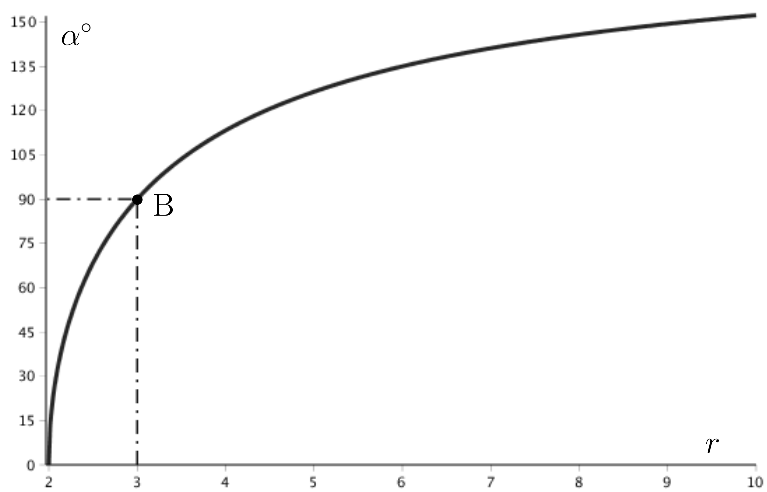

Figure 12.6 *Radial dependence of the spatial angle within which photons escape to infinity from a given location in the field of the Schwarzschild black hole. The borders of this "escape cone" are defined with respect to the static observer at the given location. Indicated is the important value 90° corresponding to the circular photon orbit on $r = 3M$; for the static observer at $r = 3M$ it means that the black hole fills just half of the sky.*

CHAPTER 13

Cosmology: homogeneous and isotropic models

On the 4th February 1917, Einstein writes to P. Ehrenfest: "... I have perpetrated something again as well in gravitation theory, which exposes me a bit to the danger of being committed to a madhouse. I hope there are none over there in Leyden, so that I can visit you again safely. ..." At the end of 1916, he introduced a cosmological term into the field equations – and, thanks to it, he was able to find a static solution for the Universe. It was filled, in a homogeneous and isotropic way, with matter whose mutual attraction was compensated by a repulsive effect of the newly postulated term; this was the first cosmological model. The tone of Einstein's letter is understandable, because there was no other "justification" for the new term than the assumption of staticity. To tell the truth, the solution mainly impressed Einstein by something else: its spatial geometry is that of a 3D sphere, so it is closed, having no boundaries, hence no boundary conditions are needed – in particular, in such a universe, the inertia of bodies can be interpreted in a totally "Machian" way, as a consequence of mutual interaction of all the matter present. Besides, the 3D sphere has a positive curvature everywhere, which well corresponds to the attractive nature of gravitation. However, Einstein's theoretical colleagues who believed in mathematical aesthetics did not see much rationale for "putting Λ by hand" to the theory. W. de Sitter, the director of the Leiden observatory, was the strongest opponent.

Quite ironically, just days after expressing his reservations, on 20th March 1917, de Sitter sent to Einstein his famous solution, describing a space-time containing *only* the cosmological constant, otherwise vacuum (thus governed by equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$). Positive Λ makes it expand (free test particles are accelerated away from each other). After Schwarzschild's and Reissner's (later also Nordström's) solutions and after Einstein's static cosmological model, de Sitter's universe was historically the 4th exact solution of the Einstein equations. The de Sitter solution is very important cosmologically, since the cosmological term dominates on the largest scales, making the de Sitter model a default asymptotic state of any expanding universe with positive Λ. In 1917, however, it primarily surprised Einstein conceptually, because it showed that the Minkowski solution is not the only vacuum possibility – that the space-time may be curved "in itself", not necessarily by matter only.

This was later important for the acceptance of the picture of gravitational waves, but at the given moment it mainly was another counter-example to Einstein's Machian faith [to de Sitter, 24th March]: "In my opinion, it would be unsatisfactory if a world without matter were possible. Rather, the $g^{\mu\nu}$-field should *be fully determined by matter and not be able to exist without the matter*. This is the core of what I mean by the requirement of the relativity of inertia. One could just as well speak of the 'matter conditioning geometry'. To me, as long as this requirement had not been fulfilled, the goal of general relativity was not yet completely achieved. This only came about with the Λ term." A week later (1st April), de Sitter explicitly protested against Einstein's satisfaction with the static model: "I must emphatically contest your assumption that the world is mechanically quasi-stationary. We only have a snapshot of the world, and we cannot and *must not* conclude from the fact that we do not see any large changes on this photograph that everything will always remain as at that instant when the picture was taken."

At the time when K. Schwarzschild was acting as a volunteer on the German side of the Eastern front, on its opposite side another volunteer, A. A. Friedmann, was working in an aeronautic navigation service; later he became professor of mathematics, physics and meteorology at several Russian universities. In 1922, he wrote a considerable paper on possible solutions of the field equations containing homogeneous and isotropic incoherent dust (no pressure) and the cosmological constant. He showed the Einstein and the de Sitter models are but limits of a wide range of dynamical possibilities with constant positive curvature of space. He introduced "cosmic time" as "the time since the creation of the world", pointing out that such a time may actually be both finite and infinite. And at the end he remarks: "It is left to remark that the 'cosmological' quantity Λ remains undetermined in our formulae, since it is an extra constant in the problem; possibly electrodynamical considerations can lead to its evaluation. If we set $\Lambda = 0$ and $M = 5 \cdot 10^{21}$ solar masses, then the world period becomes of the order of 10 billion years." In another paper (1924), he added a similar discussion of the constant-negative-curvature models. Friedmann's papers basically founded the current theoretical cosmology. In 1927, G. Lemaître confirmed the result by deriving the solution with matter, radiation (including pressure) and cosmological constant.[1]

In the meantime, statistics of spectra of "spiral nebulae" were more and more favouring redshifts, thus recession. V. Slipher actually started with the measurements about 1912, but only in 1927 G. Lemaître clearly interpreted the data as evidencing the cosmic expansion. In 1928-29, E. P. Hubble confirmed that conclusion. In the meantime (1924), he also found that rather than belonging to our Galaxy, the "nebulae" were in fact other galaxies, similar to our one. The data thus showed that A. Friedmann was right – our Universe is expanding. Einstein admitted the introduction of Λ (actually, insisting on the closed static universe) had been "the biggest blunder of my life" and accepted the expanding picture.

[1] Although he was a catholic priest, he concludes by: "It remains to find the cause of the expansion of the universe."

13.1 Special nature of cosmology

Cosmology deals with the structure and evolution of the Universe as a whole. This is not a minor plan (K. Popper remarked that "all science is cosmology" in fact), though it focuses on an object which – at least "the our one" – exists in one issue only. One cannot experiment with the Universe as with other physical objects, namely to study it under various initial and boundary conditions. It is even not at all possible to leave it and study it *from outside*; actually, our perspective is very much fixed – due to the immense cosmic distances and time intervals, we can only observe the Universe from a tiny region around the Earth and within only a negligible interval of its history. Inevitable extrapolations may seem hardly digestible, yet still, cosmology is a standard scientific field: on the basis of current best knowledge and of certain assumptions concerning symmetry and/or simplicity, it creates models whose predictions can be confronted with observations. After all, the extrapolations made in cosmology are only far-reaching in time and space which need not be the most important respect. And, besides, consider how they compare to those necessary in the Planck-scale physics: taking $1\mathtt{m} = 10^0\mathtt{m}$ or $10^{-1}\mathtt{m}$ as the scale we have more immediate experience with, the scale of "mega-world" is 1 megaparsec = about $10^{22}\mathtt{m}$, and the whole observable Universe has the characteristic size of about $10^{26}\mathtt{m}$. That is a large difference, if regarding how different conditions exist in microworld where $1\,\mathtt{angstrom} = 10^{-10}\mathtt{m}$ (atomic scale) or $1\,\mathtt{fermi} = 10^{-15}\mathtt{m}$ (nuclear scale) are used as adequate units. By far the most remote, however, is the Planck scale, $\sim 10^{-35}\mathtt{m}$.

From the mega-scales we have yet much more *observations* than from the Planck scale: and the radiation we receive from various astronomical sources show exactly the same parameters we are used to register on radiation from the neighbouring galaxies, from the Sun or from a candle. What we see on the sky thus evinces that even at times and places which are unthinkably remote from us the matter had apparently been subjected to the same physics we know from now and here. This observation may indeed be the most compelling footing for seeking the "laws of Nature".

Entering some new scale of phenomena, extrapolations from the known realm are natural, yet not the only option. Actually, the Universe as a whole might be also governed by some specific, "cosmic" laws, which have negligible effect on local scale; one of such "laws" effectively is the cosmological term in Einstein's equations. Or, there may be no (other) such laws, while there also act – besides the celebrated extrapolations – the opposite implications: the local physics might be influenced by properties of the mega-scale Universe. Mach's principle is an example of such a statement – according to it, the locally acting property of inertia is determined by interaction between all the matter in the Universe.

A reasonable cosmological model obviously cannot be found among the asymptotically flat solutions of Einstein equations, describing *isolated* sources (e.g. black holes), because, in the observable Universe, matter seems to exist "everywhere". On largest scales, it seems to be distributed uniformly, so a certain "principle of uniformity" is being accepted as the starting idea. One of such principles – the Copernican principle, saying that our location in the Universe is completely general (not exceptional) – accompanied the dawn of modern science. Today, a stronger, so-called **cosmological principle** is being stated as the starting point of "standard" cosmological models; it claims that the large-scale world is *homogeneous and isotropic*.

13.2 Basic observations

Stars live either alone (as e.g. our Sun), or in multiple, gravitationally bound systems; if the latter contain many components, they are called star clusters. However, the most distinct structures in the Universe are **galaxies** – much larger gatherings containing some $10^{7\div 12}$ stars. Galaxies have about $10^{4\div 6}$ light years across and are typically more than $10^{6\div 7}$ light years from each other. In spite of great differences in size and structure it holds for almost all galaxies that most of their angular momentum is borne by a central disc component, within which stars as well as interstellar matter strongly concentrate towards the centre – the galactic nucleus. The galaxies in turn form bigger associations – local groups of galaxies, clusters, superclusters... This hierarchy seems to have upper limit on the scale of several hundreds of megaparsecs (Mpc). The largest recognizable structure appears to be like network of filaments with (super)clusters at its vortices. In the observable Universe, there exist some 10^{12} galaxies.

13.2.1 Olbers' paradox

If you believe that the relativistic theory is only necessary in situations where something is moving very fast or where gravitational field is very strong, try this: throw some thing and watch how it falls. If you wanted to describe the fall by Newton's theory of gravity and the travel of photons to your eyes by Maxwell's electrodynamics, you would get into trouble, because these two theories are not compatible. Another similarly simple experiment indicates very important property of the Universe. If your neck is totally blocked from monitor, it may not be easy, however it does not do any harm at times: go out to the night and look at the sky. – Nothing? – Well, it's *dark*!

Such an experiment was being made long before general relativity. E. Halley (1720), P. L. de Chéseaux (1744), and mainly H. W. M. Olbers (1826), considered whether its result is obvious. They assumed the universe to be Euclidean and static, infinite in both time and space, with stars distributed roughly uniformly and shining for ever. If the stars were point-like, each having luminosity $L > 0$, a static observer at location $r = 0$ would measure, from each star (lying r apart), the intensity $\frac{L}{4\pi r^2}$, so from all stars in the universe it would yield

$$\int\limits_{r=0}^{\infty} \frac{L}{4\pi r^2}\, 4\pi r^2 n \, \mathrm{d}r \simeq Ln \int\limits_{r=0}^{\infty} \mathrm{d}r = \infty \,,$$

$n \simeq \mathrm{const}$ being the number density of stars in space.

In reality, the stars are not point-like, with the nearer ones obscuring those at larger distance. Hence, the upper limit of the integral is effectively finite and, consequently, also finite is the result. For extended stars (with same L and size), it holds that a closer star causes higher intensity, but, on the other hand, it obscures more stars lying behind. Both these tendencies fall off with r^2, so they just cancel mutually. All in all, the intensity from all stars the observer sees is the same as the intensity from stars placed side by side on a sphere of some (*arbitrary*) radius R – and that of course equals the intensity at the star surface! (Actually, inside a sphere of given brightness, the intensity is independent of the sphere's radius, because the total radiated power grows with R^2, whereas the intensity *decreases* with R^2 on the contrary. In passing, the limit $R \to 0^+$ corresponds to an observer just between two touching stars.)

A thermodynamical consideration leads to a similar result: an infinite static universe is undoubtedly in the state of thermodynamic equilibrium, that is, every its star emits the same power as it absorbs. This argument eliminates the explanation suggested by Olbers himself, namely that the light of stars is being absorbed by an interlaying gas. (The gas would have to already be in equilibrium as well.) The paradox arises too in a non-Euclidean static universe, and diminishing the universe to finite spatial size only slightly modifies the above result (the observer might see less stars than would fit to the celestial sphere, which would decrease the received intensity accordingly). Neither offers a solution to reduce the stellar lifetime to a finite value, at least if we allow stars to form in the same rate as they die. Finally, the assumption of homogeneity of star distribution on large scales is well supported by observations.

So how on Earth it can be dark at night? Very probably, it is because the Universe *is not static*. Actually, if stars recede from (any) observer (the faster the farther they are), the received flux of energy is lower. (And, still better if the Universe were not infinitely old...)

13.2.2 The Hubble-Lemaître expansion and the "day without yesterday"

By 1929, E. P. Hubble confirmed convincingly that the farther the galaxies, the more their spectra are redshifted. Lemaître interpreted that as evidencing the expansion of the Universe. The distance l and the speed of recession v (derived in a Dopplerian way) were found to satisfy the Hubble-Lemaître relation $v = Hl$, where H is being called the **Hubble-Lemaître constant**. The relation mainly holds accurately for distant galaxies, since for such their "peculiar motion" is relatively less important. Nearby galaxies may in fact move towards each other, mainly if they are directly gravitationally bound (as e.g. our Galaxy with the well known galaxy M31 in Andromeda). On the other hand, the relation also does not work accurately for very large distances, since there it is necessary to employ a general relativistic formula.

Friedmann interpreted the time parameter t of his equations as "the time since the creation of the world". Actually, if playing the cosmic-expansion film backwards, the galaxies would approach each other. If some two galaxies are now l from each other and are receding with speed v, then before some l/v they must have been very close. According to the Hubble-Lemaître relation, this equals $1/H$, which, if taking $H \doteq 70\,\mathtt{km/s/Mpc}$, yields the "Hubble time" of about 14 billion years. The actual age of the Universe may be somewhat different, because the cosmic-expansion rate need not have been constant (the Hubble-Lemaître "constant" may change in time). Actually, the mutual attraction between all the matter must slow down the expansion, and even more pronounced may be the effect of the cosmological constant. In the 1930s, Eddington expressed his aesthetic disgust by the finite-age world, but his postdoc Lemaître replied [beginning of his "primeval atom" hypothesis published in Nature, 1931]: "Sir Arthur Eddington states that, philosophically, the notion of a beginning of the present order of Nature is repugnant to him. I would rather be inclined to think that the present state of quantum theory suggests a beginning of the world very different from the present order of Nature. Thermodynamical principles from the point of view of quantum theory may be stated as follows: (1) Energy of constant total amount is distributed in discrete quanta. (2) The number of distinct quanta is ever increasing. If we go back in the course of time we must find fewer and fewer quanta, until we find all the energy of the universe packed in a few or even in a unique quantum." Also interesting, in connection with the issue of determinism vs. free will, is the end of Lemaître's note: "Clearly the initial quantum could not conceal in itself the whole course of evolution; but, according to the principle of indeterminacy, that is not necessary. Our world is now understood to be a world where something really happens; the whole story of the world need not have been written down in the first quantum like a song on a disc of a phonograph. The whole matter of the world must have been present at the beginning, but the story it has to tell may be written step by step."

Within the Newtonian theory, the Lemaître's "day without yesterday" would be quite a daring retrodiction, since the galaxies do not move away from each other in an exactly radial directions, so they might pass by rather than collide in the backward video. In GR, the picture is very different: the galaxies do not move in any *background*, pre-existing space – in fact they are "at rest" (except their peculiar motions) while the whole space expands or contracts. Hence, if the space itself collapses, all the matter necessarily gets cumulated (irrespectively of its own, "peculiar motion"). We will see that with a repulsive (= positive) cosmological constant, the past is not that certain, because – in dependence on its value (possibly changing in time?) and on the values of other parameters – the Universe did not have to start in a singular way. Yet the existence of the **cosmic microwave background radiation** evidences that the very hot phase in which matter and radiation were in equilibrium did really happen.

13.2.3 Homogeneity and isotropy

On very large scale (of several hundred of megaparsecs), the Universe appears homogeneous and isotropic. On smaller scales it is not so, but if one divided it into cells of about $250\,\mathtt{Mpc}$ size and averaged physical quantities (mainly density and pressure) over them, the result would be approximately the same for all the cells. The above size is *not* small, but still enough such cells fit into the observable Universe for statistical purposes. Actually, the observable-Universe "radius" is roughly $\frac{c}{H} \doteq 4.5\,\mathtt{Gpc}$ ($\doteq 1.37\cdot 10^{28}\,\mathtt{cm}$), so its volume is about $370\,\mathtt{Gpc}^3$. The above "pixels" of $250\,\mathtt{Mpc}$ size have volume $0.016\,\mathtt{Gpc}^3$, so some 23000 of them fit into the observable Universe.[2]

The observation of large-scale homogeneity and isotropy, extended to *the whole Universe* (including regions which are unobservable, even in principle), is the fundamental starting point of "standard" cosmological models. It leads to a considerable simplification of the metric already before that is subjected to Einstein's equations. First, however, it will be necessary to make the statements about homogeneity and isotropy more rigorous.

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[2] As opposed to ordinary life, in cosmology the speed of light seems *very small*. Since the Universe expands, it is good to bear in mind that everything we see we see in the past, so the "instantaneous" location and state of any source is different than how it is given by the observed radiation. The study of the early Universe thus requires to look as far as possible. -Kind of trivial remark, but it can make things surprising – in particular, the radiation we are receiving from very distant objects was emitted when the Solar system did not at all exist, while some of those objects need not exist "now" any more (or they may be very different).

Let us remark that whereas the isotropy is generally accepted (also thanks to the isotropy of the CMBR), there is still discussion about homogeneity and its scale. Namely, several quasar or galaxy structures seem to be larger than the above value. A debate is also going on whether the inhomogeneities on smaller scales do not significantly affect the cosmic dynamics as an effective extra source.

13.2.4 Be careful with "relative velocities"

You certainly understand how delicate it is, in a curved space, to try to compare quantities defined at different points; we notably stressed this before deriving parallel transport. Still we beg to emphasize it once more here in cosmology, in order for you not to be surprised when asked: "And so the cosmic expansion is superluminal?"

Yes, it is. But what does it mean "relative velocity", "distance"? In a curved space(-time), the proper distance depends on curve along which it is taken. Imagine high mountains and two expeditions separated by five complicated valleys. What is their distance and what is their relative velocity? Imagine someone travelling from Fiji to Tahiti. What is their distance and velocity with respect to someone in Prague? Are they approaching or receding? And in GR, the geometry may be dynamical, in addition. In cosmology, one standardly says "galaxies are receding from each other", but in fact it is space in-between which is expanding; the galaxies may well "be at rest" in a good sense, like dots marked on a balloon which is being inflated. In such a case, receding (or approaching) may easily be "superluminal". Actually, redshifts greater than unity are commonly being observed at distant cosmic sources, which formally corresponds to a superluminal recession. This is *not* in contradiction with the relativity principles, because the speed of light is defined with respect to an *inertial system* – a global one in special relativity, but only a local one in general relativity. In general relativity, one cannot take (four-)velocity there and (four-)velocity here and subtract them. One might try to do this carefully and to first transport the vectors to the same point, but how? –Which transport is "correct" and along which curve? In GR, the only sensible and unique notion of relative velocity is the velocity between two (point-like) objects located, at the infinitesimal moment of measurement, at the same space-time point (at the same event). *This* velocity must have magnitude smaller than c. Imagine, finally, a one-dimensional equivalent of what happens in cosmology – a circle whose radius is growing. In the Euclidean case, the circumference $o = 2\pi r$ evolves according to the equation $\frac{\mathrm{d}o}{\mathrm{d}t} = 2\pi \frac{\mathrm{d}r}{\mathrm{d}t}$, which for $\frac{\mathrm{d}r}{\mathrm{d}t}$ sufficiently close to c may naturally be superluminal.

13.3 Cosmological principle and the FLRW metric

Vaguely speaking, by homogeneity and isotropy one wishes to say that at any given instant of time, the Universe is everywhere the same and in all spatial directions the same. However, we do not have any time coordinate yet, and both homogeneity and isotropy are relative properties – they depend on an observer. Let us define them more carefully – and geometrically.

Before doing so, it is worth to realize how to formulate the basic coordinate pictures in a covariant manner.

- "Time" translates into a parameter (t) such that $\partial x^\mu / \partial t$ is a time-like vector (in a given region).
- "Constant-time slice" translates to a hypersurface (or at least a local hyperplane) which is orthogonal to some time-like vector-field (or at least vector).
- The notion "time component of something (of a vector)" translates to "projection of that quantity (of that vector) to some time-like vector" (either arbitrary or at least one).
- Similarly, "spatial part of something (of a vector)" translates to "projection of that quantity (of that vector) to some space-like vector" (either arbitrary or at least one).

13.3.1 Symmetries of the large-scale Universe – homogeneity and isotropy

Homogeneity of the Universe Through every event in the Universe there passes, uniquely, such a global space-like hypersurface on which all physical properties are everywhere the same. It is called the **hypersurface of homogeneity**. Constancy of physical parameters means invariance with respect to translations over any given hypersurface of homogeneity, so homogeneity corresponds to the existence of 3 space-like Killing vector fields.

Isotropy of the Universe There exists, globally, such a family of observers for whom all spatial directions are equivalent. Isotropy means the existence of 3 space-like Killing vector fields, which at every point represent generators of rotations about 3 spatial axes.

Immediate corollaries First, since the real Universe is filled with matter and radiation, the only observers who *may* see the Universe isotropic are those who move together with matter (otherwise the relative velocity would fix a privileged direction). Such observers are thus called the **comoving observers**, sometimes also the **fundamental observers**. Second, radiation (thus) has to be isotropic in the rest system of matter. Third, the four-velocity of the above observers (thus of cosmic matter) has to be always and everywhere orthogonal to the hypersurfaces of homogeneity (otherwise a privileged direction could be obtained by projection of the four-velocity on those hypersurfaces).

Worth to realize that isotropy *seen from one point* and homogeneity are independent properties, but isotropy *valid for all points* implies homogeneity. But exactly such an isotropy we are postulating here: the existence of such a class of observers that *for any of them* and *at any instant of their time*, the space is isotropic to an arbitrary distance. In other words, to the local three-spaces of comoving observers, there must exist integral hypersurfaces whose all points have to be equivalent ... and these exactly are the hypersurfaces of homogeneity.

The observationally motivated assumptions of homogeneity and isotropy of the large-scale Universe – the so-called **cosmological principle** – forms a fundamental principle of the "standard", Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models.

13.3.2 Synchronous coordinate system

Similarly as in derivation of the Schwarzschild solution, the second step after statements about symmetry is to introduce such coordinates in which the metric may be expected to

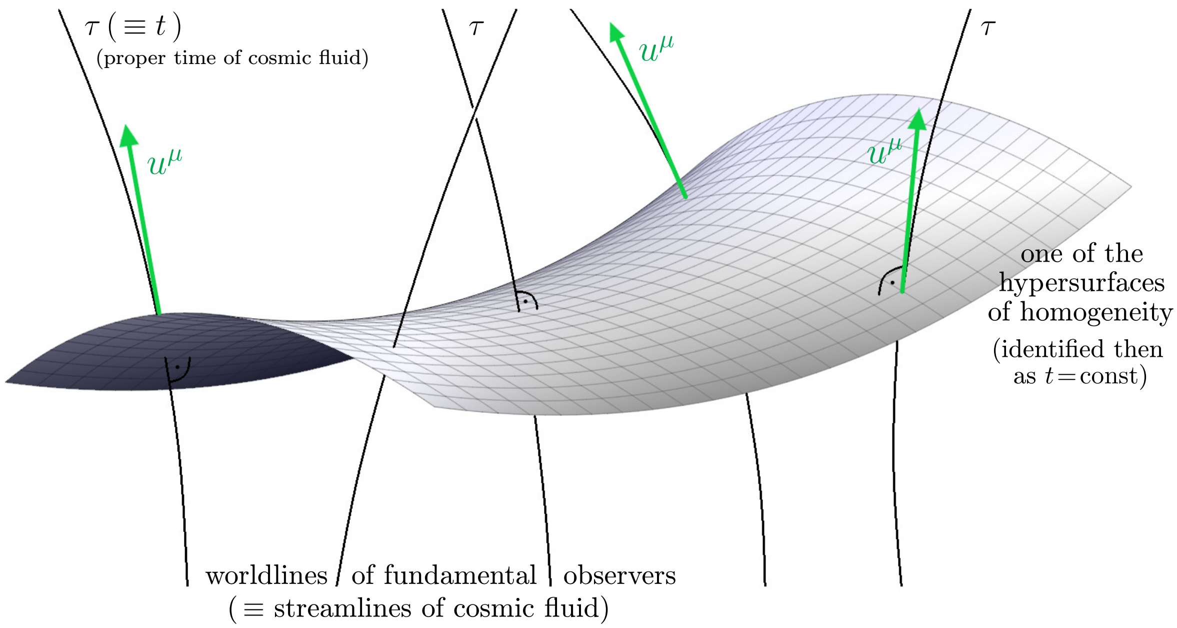

Figure 13.1 *Geometric implications of homogeneity and isotropy: i) The large-scale Universe is claimed to be foliated by* ***hypersurfaces of homogeneity****, i.e. by a smooth distribution of integral space-like hypersurfaces which, individually, have the same curvature everywhere (see Section 24.4 and Chapter 25 for a more precise description of a foliation). ii) A family of* ***fundamental observers*** *is claimed to exist who find the large-scale Universe spatially isotropic; their world-lines have to be everywhere orthogonal to the hypersurfaces of homogeneity (in order that their four-velocity u^μ does not induce any privileged spatial direction).*

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have simple form. No hesitation how to introduce the time coordinate in the cosmic case: it will be exactly the proper time of fundamental observers (thus of the matter itself),

$$t = t(\tau_0) + \int_{\tau_0}^{\tau} \mathrm{d}\tau \qquad (\text{so} \quad \mathrm{d}t = \mathrm{d}\tau)\;. \tag{13.1}$$

Such a t is being called the **cosmic time**; the corresponding metric component is $g_{tt} = -1$. Let us appreciate, once more, that the possibility to introduce such a privileged global time (even having perfect physical sense) follows from the postulated existence of the global, integral hypersurfaces to the local three-sections orthogonal to the world-lines of fundamental observers. Since all these observers have to be equivalent, their clocks tick at the same pace and can be synchronized over the hypersurfaces of homogeneity. These hypersurfaces thus correspond to $t = \mathrm{const}$.

Now, let the hypersurfaces of homogeneity be covered by some (yet arbitrary) spatial coordinates x^i. Imagine, at any point, the four-velocity of the local fundamental observer u^μ and an arbitrary four-vector s^μ orthogonal to u^μ (thus tangent to the local hypersurface of homogeneity). In the (t, x^i) coordinates, these two vectors read

$$u^\mu = (1, 0, 0, 0)\,, \qquad s^\mu = (0, s^i)\,.$$

Then, however, from their orthogonality it follows

$$0 = g_{\mu\nu} u^\mu s^\nu = g_{ti} u^t s^i \qquad \Longrightarrow \qquad g_{ti} = 0\,. \tag{13.2}$$

The metric thus assumes a rather simple form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\sigma^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i \mathrm{d}x^j\,, \tag{13.3}$$

where $\mathrm{d}\sigma^2$ describes the geometry of hypersurfaces of homogeneity.

13.3.3 Metric of the hypersurfaces of homogeneity

By definition, on the hypersurfaces of homogeneity, the Universe has everywhere the same properties. Hence, these hypersurfaces have themselves to be everywhere the same geometrically: their *curvature* has to be everywhere the same. The **manifolds of constant curvature** represent, at the same time, the so-called **maximally symmetric spaces**, because they host a maximal possible number of independent symmetries (Killing fields). In the 3D case, this number is 6. Sure, we know from above we have 3 translations due to the homogeneity and 3 rotations due to the isotropy. Now "almost everything is clear", since there are only 3 types of 3D manifolds with constant curvature: those with positive, negative and zero curvature (i.e., the 3D sphere, hyperboloid and plane). Within each of these subclasses, the manifolds differ only by "size", as it is easy to fathom on the 2D case: there, for instance, the manifolds of constant positive curvature are spheres which only differ in radius.

The 3D Euclidean space has in the Cartesian and spherical coordinates the well known metric

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{13.4}$$

The 3D sphere/hyperboloid can be embedded in the 4D space ($\mathbb{R}^4$) with Euclidean/Lorenzian metric, respectively (the hypersurfaces with negative curvature can only be represented locally in $\mathbb{R}^4$). Let us first write the metrics of these 4D spaces in Cartesian coordinates (x^1, x^2, x^3, x^4),

$$\mathrm{d}\Sigma^2 = \sum_{i=1}^{3}(\mathrm{d}x^i)^2 \pm (\mathrm{d}x^4)^2\,,$$

where the upper/lower sign correspond to the Euclidean/Lorentzian metric. The embeddings of the 3D sphere/hyperboloid in the above spaces are given by equations

$$(x^4)^2 \pm \sum_{i=1}^{3}(x^i)^2 = a^2 \qquad (a = \text{const} > 0)\,, \tag{13.5}$$

with the upper/lower sign corresponding to sphere/hyperboloid. The metric describing the 3-sphere/3-hyperboloid is obtained by constraining the elements $\mathrm{d}x$ of the above 4D metrics $\mathrm{d}\Sigma^2$ by the embedding equations (13.5), so by substituting to the metric from the differential of the equations.

In order to easily compare the two curved metrics with the Euclidean one (13.4), it is natural to work in spherical coordinates introduced in the (x^1,x^2,x^3) subspace by

$$x^1 = r\sin\theta\cos\phi, \quad x^2 = r\sin\theta\sin\phi, \quad x^3 = r\cos\theta.$$

In them, the metrics of the embedding spaces read

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \pm (\mathrm{d}x^4)^2$$

and the equations for the 3D sphere/hyperboloid are

$$(x^4)^2 \pm r^2 = a^2\,.$$

Differentiating the latter and taking the second power, one has

$$r^2\mathrm{d}r^2 = (x^4)^2(\mathrm{d}x^4)^2 = (a^2 \mp r^2)(\mathrm{d}x^4)^2 \qquad \Longrightarrow \qquad (\mathrm{d}x^4)^2 = \frac{r^2\mathrm{d}r^2}{a^2 \mp r^2}\,.$$

Substituting this $(\mathrm{d}x^4)^2$ to the above $\mathrm{d}\Sigma^2$, we obtain

$$\mathrm{d}\sigma^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \pm \frac{r^2\mathrm{d}r^2}{a^2 \mp r^2} = \frac{a^2\mathrm{d}r^2}{a^2 \mp r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

Hence, all the three 3D metrics (including the flat space) can in the spherical coordinates be expressed by a single formula

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - K\frac{r^2}{a^2}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{13.6}$$

where the *curvature parameter* K has been introduced by

$$K := \begin{cases} +1 & \text{for 3-sphere} \\ \ \ 0 & \text{for 3-plane} \\ -1 & \text{for 3-hyperboloid} \end{cases}\,. \tag{13.7}$$

The final step is to scale the radial coordinate by the length parameter a,

$$r = a\Sigma\,, \qquad \text{where} \quad \Sigma := \begin{cases} \sin\chi & (0 \leqslant \chi \leqslant \pi) & \text{for} \ \ K = +1 \\ \chi & (0 \leqslant \chi < \infty) & \text{for} \ \ K = \ \ 0 \\ \sinh\chi & (0 \leqslant \chi < \infty) & \text{for} \ \ K = -1 \end{cases}\,, \tag{13.8}$$

after which the metric of the hypersurfaces of homogeneity assumes the form

$$\mathrm{d}\sigma^2 = a^2\left[\mathrm{d}\chi^2 + \Sigma^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]. \tag{13.9}$$

Circumference, area and volume on the hypersurfaces of homogeneity

Consider, on some hypersurface of homogeneity, a equatorial ($\theta = \pi/2$) circle $\chi = \chi_0$. Its

$$\text{proper radius} = \int_0^{\chi_0} \sqrt{g_{\chi\chi}}\,\mathrm{d}\chi = \int_0^{\chi_0} a\,\mathrm{d}\chi = a\chi_0\,,$$

$$\text{proper circumference} = \int_0^{2\pi} \sqrt{g_{\phi\phi}(\chi = \chi_0, \theta = \pi/2)}\,\mathrm{d}\phi = \int_0^{2\pi} a\Sigma(\chi_0)\,\mathrm{d}\phi = 2\pi a\Sigma(\chi_0)$$

are in relation typical for curved manifolds: for $K = +1$ the circumference ($2\pi a \sin\chi_0$) is smaller then ($2\pi \cdot \text{radius}$), whereas for $K = -1$ the circumference ($2\pi a \sinh\chi_0$) is greater then ($2\pi \cdot \text{radius}$); in the flat case $K = 0$ the relation is Euclidean of course. Similarly behaves the proper area of the sphere $\chi = \chi_0$:

$$\text{proper area} = \int_0^{2\pi}\int_0^{\pi} \sqrt{(g_{\theta\theta}g_{\phi\phi})_{\chi=\chi_0}}\,\mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi} a^2\Sigma^2(\chi_0)\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = 4\pi a^2\Sigma^2(\chi_0)\,.$$

Note in particular that if χ increases above $\pi/2$ in the $K = +1$ case, both the circumference and the area *decrease* (see Section 6.1).

We will also compute the volume enclosed by the sphere $\chi = \chi_0$,

$$\text{proper volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_0} \sqrt{g_{\chi\chi}g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_0} a^3\Sigma^2\sin\theta\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi =$$
$$= 4\pi a^3 \int_0^{\chi_0} \Sigma^2\mathrm{d}\chi\,. \tag{13.10}$$

For the volume of the whole universe (of the whole hypersurface of homogeneity, $\chi_0 \to \chi_{\max}$), we thus obtain, as expected, finite value for the 3-sphere (this universe is “closed”), whereas infinite value for the 3-plane and 3-hyperboloid (these are “open” universes):

$$V = \begin{cases} 4\pi a^3 \int_0^{\pi} \sin^2\chi\,\mathrm{d}\chi = 2\pi^2 a^3 & \text{for } K = +1 \quad (\chi_{\max} = \pi) \\ 4\pi a^3 \int_0^{\infty} \chi^2\,\mathrm{d}\chi = \infty & \text{for } K = \ \ 0 \quad (\chi_{\max} = \infty) \\ 4\pi a^3 \int_0^{\infty} \sinh^2\chi\,\mathrm{d}\chi = \infty & \text{for } K = -1 \quad (\chi_{\max} = \infty) \end{cases}\,. \tag{13.11}$$

13.3.4 Friedmann-Lemaître-Robertson-Walker metric

Friedmann-Lemaître-Robertson-Walker (FLRW) metric is the final metric obtained by substituting the spatial part (13.9) into (13.3),

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + \Sigma^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]. \tag{13.12}$$

It describes the homogeneous and isotropic universe in the **comoving hyperspherical coordinates** (t, χ, θ, ϕ). It involves one important novelty: the scale factor a we have written as *depending on time*, $a = a(t)$. In order to allow for dynamical solutions, one *has to* allow a to be time-dependent, since otherwise nothing in the metric can be such. The parameter a is thus called the **expansion factor**. One should make oneself sure that its dependence on time by no means breaks the assumptions of homogeneity and isotropy: indeed, the factor a stands in front of the whole spatial metric, isotropically, so it only fixes the overall scale. It may change with time in an arbitrary way, and the spatial submanifold will still keep constant curvature everywhere (although the curvature will change in time). In other words, the cosmological principle does not fix the "size" of the hypersurfaces of homogeneity, and allows for time variability of this size. *The dynamics of FLRW models is fully described by the dependence* $a = a(t)$ since there is no other time-depending quantity in the metric. In passing, the dependence of a on time could *not* have been permitted in the preceding form of the metric (before introducing the angular radial coordinate χ)

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - K\frac{r^2}{a^2}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

because there it did *not* appear isotropically. Actually, in differentiating the transformation $r = a(t)\Sigma$, one would have obtained in such a case $\mathrm{d}r = a_{,t}\Sigma\,\mathrm{d}t + a\,\mathrm{d}\Sigma$, which would have brought $a_{,t}$ into the metric.

Let us notice, finally, that the translational symmetry of the FLRW cosmological models, namely the invariance with respect to shift along any hypersurface of homogeneity, does not simply imply that the metric would be independent of χ, θ, ϕ. In particular, the metric does depend on χ, and the vector field $\frac{\partial x^\mu}{\partial\chi}$ indeed is *not* Killing.

13.4 Description of sources: the cosmic fluid

In standard cosmological models, the content of the Universe is being described as an ideal fluid, i.e. by the energy-momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} \tag{13.13}$$

which we already studied in Section (7.4.2). Recall that u^μ denotes four-velocity of the fluid, and ρ and P are mass density and pressure measured in the rest system of the fluid (thus by observers with four-velocity u^μ). Besides (baryon) matter, also present in the Universe is EM radiation and EM field. The EM field's energy density is negligible with respect to that of matter, perhaps except for very early Universe. Of (EM) radiation, the most important cosmologically is the relict radiation (CMBR) – its average energy density is about 10 times greater than that of radiation of all the discrete sources (stars) which has yet lived in the Universe. The CMBR is very precisely isotropic in the system comoving with the cosmic matter, so it can be included in the same $T^{\mu\nu}$ describing the matter just by adding its contribution to ρ and P, i.e. by taking

$$\rho = \rho_{\mathrm{mat}} + \rho_{\mathrm{rad}}\,, \qquad P = \cancel{P_{\mathrm{mat}}} + P_{\mathrm{rad}} = P_{\mathrm{rad}} = \frac{\rho_{\mathrm{rad}}}{3}\,. \tag{13.14}$$

To more understand the above, imagine that the quantities are measured by an observer who is comoving with the centre of mass of come cosmic “cell” (of some hundreds megaparsecs in size). They adds all the mass-energy present in a given element of their proper volume, i.e. (i) rest energy of the matter particles, (ii) kinetic energy of those particles, and (iii) energy of the present radiation. With pressure it is similar, however, the matter pressure between the adjacent cosmic “cells” being negligible. Even the matter pressure *within* each of the cells is unimportant as estimated from the so-called peculiar velocities of the galaxies (= their “proper”, local velocities *on the top* of the average cosmological flow given by u^μ) which are observed not to exceed $\langle v\rangle \sim 300\,\mathtt{km/s} \sim 10^{-3}c$. Then, from the equipartition theorem, one has (in standard units)

$$\frac{P}{\rho c^2} \sim \frac{\frac{3}{2}\rho\,\langle v\rangle^2}{\rho c^2} \sim 10^{-6}.$$

The pressure of every isotropic radiation is one third of its energy density, which for the relict radiation comes out about $3\cdot 10^{-4}$ of the matter energy density (hence, the radiation pressure is about 100 times more important than that of matter). Neutrinos, though having tiny non-zero rest mass, are usually included to the radiation part, with contribution to ρ of similar order (but none to pressure, since they practically do not interact). Finally, there is also some energy density in gravitational waves, but that is estimated to be only about $5\cdot 10^{-7}$ of the matter energy density.

13.4.1 Role of matter and radiation in the cosmic history

In Section 7.4.2, the component of $T^{\mu\nu}{}_{;\nu}=0$ parallel to u^μ provided the continuity equation (7.32),

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + (\rho+P)u^\nu{}_{;\nu} = 0\,.$$

Here in cosmology, we are using the comoving coordinate system in which proper time of the fluid is the time coordinate ($\tau\equiv t$) and the fluid four-velocity only has the time component, $u^t = \frac{1}{\sqrt{-g_{tt}}} = 1$, and the only time-dependent quantity is the expansion factor a, so the $u^\nu{}_{;\nu}$ term becomes

$$u^\nu{}_{;\nu} = \Gamma^\nu{}_{\nu t} = \frac{1}{2}\,g^{\nu\iota}(g_{\iota\nu,t} + \cancel{g_{t\iota,\nu}} - \cancel{g_{\nu t,\iota}}) = \frac{1}{2}\left(\frac{g_{\chi\chi,t}}{g_{\chi\chi}} + \frac{g_{\theta\theta,t}}{g_{\theta\theta}} + \frac{g_{\phi\phi,t}}{g_{\phi\phi}}\right) = \frac{3a_{,t}}{a} = 3H\ , \tag{13.15}$$

where H is the Hubble-Lemaître “constant”. Actually, considering two galaxies, fixed to the cosmic stage at $\chi=0$ and $\chi>0$, say (and other coordinates same), their relative velocity (the one solely given by cosmological expansion) is given by the change of their proper distance

$$l(0,\chi) = \int\limits_0^\chi \sqrt{g_{\chi\chi}}\,\mathrm{d}\chi = \int\limits_0^\chi a\,\mathrm{d}\chi = a\chi$$

in proper time, so

$$v = \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\mathrm{d}(a\chi)}{\mathrm{d}t} = a_{,t}\,\chi = \frac{a_{,t}}{a}\,a\chi = \frac{a_{,t}}{a}\,l\,,$$

which implies that $\frac{a_{,t}}{a} \equiv H$ when compared with the Hubble-Lemaître law $v = Hl$.

Therefore, the continuity equation assumes, in the FLRW cosmology, the form

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + P) = 0}\,. \tag{13.16}$$

Multiplying by V, we have

$$V\,\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho V_{,t} + PV_{,t} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}(\rho V)}{\mathrm{d}t} = -P\,\frac{\mathrm{d}V}{\mathrm{d}t}\,,$$

which is the 1st law of thermodynamics in an adiabatic case.

Now we decompose ρ and P according to (13.14) and thus obtain equation

$$\frac{\mathrm{d}(\rho_{\mathrm{mat}}V)}{\mathrm{d}t} + \frac{\mathrm{d}(\rho_{\mathrm{rad}}V)}{\mathrm{d}t} = -\frac{\rho_{\mathrm{rad}}}{3}\frac{\mathrm{d}V}{\mathrm{d}t}\,. \tag{13.17}$$

This can only be integrated if one knows the dependence between ρ_{mat} and ρ_{rad} as given by energy exchange between matter and radiation. Such an exchange was definitely important at the beginning of the cosmic history (when the Universe was hot), but today it is negligible (with respect to the values ρ_{mat} and ρ_{rad} themselves). In such a *decoupled* case, one can simply divide the equation into a one for matter and one for radiation, without any mutual dependence or interaction terms,

$$\frac{\mathrm{d}\left(\rho_{\mathrm{mat}}V\right)}{\mathrm{d}t} = 0\,, \tag{13.18}$$

$$\frac{\mathrm{d}\left(\rho_{\mathrm{rad}}V\right)}{\mathrm{d}t} = -\frac{\rho_{\mathrm{rad}}}{3}\frac{\mathrm{d}V}{\mathrm{d}t} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\left(\rho_{\mathrm{rad}}V^{4/3}\right)}{\mathrm{d}t} = 0\,. \tag{13.19}$$

The first equation says that

$$\rho_{\mathrm{mat}}V = \mathrm{const}\,, \qquad \text{i.e.} \qquad \rho_{\mathrm{mat}}a^3 = \mathrm{const}\,,$$

while the second says that

$$\rho_{\mathrm{rad}}V^{4/3} = \mathrm{const}\,, \qquad \text{i.e.} \qquad \rho_{\mathrm{rad}}a^4 = \mathrm{const}\,.$$

Therefore, the density and pressure evolve according to

$$\rho(t) = \rho_{\mathrm{mat}} + \rho_{\mathrm{rad}} = \rho_{\mathrm{mat}}(t_0)\left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{\mathrm{rad}}(t_0)\left[\frac{a(t_0)}{a(t)}\right]^4\,, \tag{13.20}$$

$$P(t) = P_{\mathrm{rad}} = \frac{1}{3}\,\rho_{\mathrm{rad}}(t_0)\left[\frac{a(t_0)}{a(t)}\right]^4\,, \tag{13.21}$$

where t_0 is an arbitrary specific value of t (usually “now”).

Two immediate implications:

- Since today $\rho_{\rm rad}(t_0) > 0$, there must have been a certain initial period during which radiation was more important than matter ... the **radiation era**.
- Since $\rho_{\rm mat}(t_0) > 0$, the radiation era must end at a certain moment (actually, it ended after some 50 thousand years), and then matter dominates radiation for ever already (**matter era**). Well, not for ever if the Universe will happen to shrink back in the future...

13.4.2 The cosmic-fluid flow is geodesic, vorticity-free and shear-free

It is to be expected that the cosmic fluid is free, because pressure is homogeneous in every its instantaneous three-space (no gradient of pressure). In other words, each and every flow-line is a centre of spherical symmetry, so its history must be a geodesic. One could also guess from symmetry that the cosmic flow should have zero vorticity and shear. We will check it in the comoving coordinates in which $g_{tt} = -1$, $g_{ti} = 0$, and four-velocity has but time component, $u^\mu = (1,0,0,0)$ and $u_\mu = (-1,0,0,0)$. First we compute

$$u_{\mu;\nu} = u_{\mu,\nu} - \Gamma^\sigma{}_{\mu\nu} u_\sigma = -\Gamma_{t\mu\nu} = -\frac{1}{2}(\cancel{g_{t\mu,\nu}} + \cancel{g_{\nu t,\mu}} - g_{\mu\nu,t}) = \frac{1}{2} g_{\mu\nu,t} =$$
$$= \frac{a_{,t}}{a}(g_{\mu\nu} + u_\mu u_\nu) \equiv H h_{\mu\nu}\,, \tag{13.22}$$

where we have substituted $g_{ij,t} = 2H g_{ij}$, while $g_{tt,t} = 0$. From here, one confirms the result (13.15),

$$(\Theta \equiv)\; u^\nu{}_{;\nu} = 3H\;, \tag{13.23}$$

so the expansion scalar is non-zero (positive) as expected, and by substituting into the definitions of four-acceleration, of the vorticity tensor $\omega_{\mu\nu}$ and of the shear tensor $\sigma_{\mu\nu}$ (see Section 24.1), one finds

$$a_\mu \equiv u_{\mu;\nu} u^\nu = 0\,, \tag{13.24}$$
$$\omega_{\mu\nu} \equiv u_{[\mu;\nu]} + a_{[\mu} u_{\nu]} = 0\,, \tag{13.25}$$
$$\sigma_{\mu\nu} \equiv u_{(\mu;\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{3} u^\iota{}_{;\iota} h_{\mu\nu} = H h_{\mu\nu} - H h_{\mu\nu} = 0\,. \tag{13.26}$$

We will see in Section 24.4 that it is the property $\omega_{\mu\nu} = 0$ which ensures that there exist integral submanifolds (the hypersurfaces of homogeneity) to the local three-spaces orthogonal to u^μ. Well yes, our implication was just opposite actually: we *assumed* the foliation by hypersurfaces of homogeneity exists, which implies that the normal flow has to be vorticity-free. (Very simple argument: normal flow is orthogonal to hypersurfaces, so it is proportional to a gradient of some scalar function – t in our case. And rotation of a gradient is zero.)

13.5 Geodesic motion in the FLRW metric

Consider now a *generic* motion of free *test*[3] particles in the FLRW space-time. To grasp it intuitively, imagine a 2D case: on a sphere, geodesics are main circles. Sphere is a manifold

[3] We beg to stress that the cosmic fluid is *not* test. Sorry...

of constant curvature, so all its points are geometrically equivalent. Hence, if covering the sphere with coordinates (obviously by angles θ and ϕ), their origin $\theta=0$ can be chosen at an arbitrary point. If interested in a certain particular geodesic, it is natural to choose the origin so that it lies on it ... namely, then $\mathrm{d}\phi/\mathrm{d}p=0$ will have to hold along the whole geodesic.

Let us now do it similarly in our 3D manifolds of constant curvature (hypersurfaces of homogeneity). Since the metric (13.12) is spatially homogeneous and isotropic, all their points are equivalent, so the origin of hyperspherical coordinates $\chi=0$ can be located arbitrarily. Choosing it on the studied geodesic, one easily infers one constant of the motion. Namely, as the FLRW metric (13.12) does not depend on ϕ, the corresponding angular momentum is constant,

$$u_\phi = g_{\phi\phi}u^\phi = a^2\Sigma^2\sin^2\theta\, u^\phi = \mathrm{const}\,.$$

However, at $\chi=0$ one has $\Sigma=0$, so $u_\phi(\chi)=u_\phi(\chi=0)=0$ and, consequently, $u^\phi=0$ (we suppose $\sin\theta\neq 0$).

Also simple then is the θ-component:

$$\frac{\mathrm{d}u_\theta}{\mathrm{d}p} = \frac{1}{2}g_{\kappa\lambda,\theta}u^\kappa u^\lambda = \frac{1}{2}g_{\phi\phi,\theta}(u^\phi)^2 = 0\,,$$

hence

$$u_\theta = g_{\theta\theta}u^\theta = a^2\Sigma^2 u^\theta = \mathrm{const}\,.$$

This constant is zero too thanks to the vanishing at $\chi=0$, so again $u^\theta=0$ as well. (Now it is clear that neither the case $\sin\theta=0$ omitted above brings any problems – it is just a special subcase of $\theta=\mathrm{const.}$)

Owing to $u^\phi=0$ and $u^\theta=0$, we also find

$$\frac{\mathrm{d}u_\chi}{\mathrm{d}p} = \frac{1}{2}g_{\kappa\lambda,\chi}u^\kappa u^\lambda = \frac{1}{2}\left[g_{\theta\theta,\chi}(u^\theta)^2 + g_{\phi\phi,\chi}(u^\phi)^2\right] = 0\,,$$

so neither the covariant radial component is changing,

$$u_\chi = g_{\chi\chi}u^\chi = a^2u^\chi = \mathrm{const}\,.$$

Finally, the time component is fixed from normalization:

$$\begin{aligned}
&\text{for massive particles}: \quad g_{\mu\nu}u^\mu u^\nu = -(u^t)^2 + a^2(u^\chi)^2 = -1\,,\\
&\text{for massless particles}: \quad g_{\mu\nu}p^\mu p^\nu = -(p^t)^2 + a^2(p^\chi)^2 = 0\,.
\end{aligned}$$

To summarize, the geodesic motion in the FLRW space-times proceeds, in the hyperspherical coordinates, by equations $\phi=\mathrm{const}$, $\theta=\mathrm{const}$, and

$$a^2u^\chi = \mathrm{const}\,, \quad (u^t)^2 = 1 + a^2(u^\chi)^2 = 1 + \frac{(\mathrm{const})^2}{a^2} \qquad \ldots \quad \text{for} \quad m\neq 0\,, \tag{13.27}$$

$$a^2p^\chi = \mathrm{const}\,, \quad (p^t)^2 = a^2(p^\chi)^2 = \frac{(\mathrm{const})^2}{a^2} \qquad \ldots \quad \text{for} \quad m = 0\,. \tag{13.28}$$

One may observe that due to the triviality of the angular components of geodesic motion (without any loss of generality, any geodesic may be viewed as coming from the "pole", i.e. as purely radial), the motion of the cosmic fluid itself is in fact only special in having $u^\chi=0$ (and thus $u^t=1$).

13.6 Cosmological redshift of radiation

In a cosmological jargon, “the farther the galaxy, the faster it recedes from us and thus the more redshifted its light”. However, it should be clear now that the galaxies do not recede from each other in an ordinary sense – they are kind-of attached to basically fixed locations but in space which is itself expanding. We saw that in such a case the “recession speed” inferred from the Dopplerian logic may well be “superluminal” (the corresponding redshift may be above unity). Hence, the cosmological-redshift formula should be derived in a proper way.

Imagine two fundamental observers (comoving with the cosmic fluid), one at $\chi_{\rm em} = \mathrm{const} > 0$ and the other at $\chi_{\rm obs} = \mathrm{const} = 0$ (this is without any loss of generality, because the origin of χ can be chosen arbitrarily). From the emitter to the observer, the radiation travels exactly radially (due to symmetry), so the element along its world-line reads

$$0 = \mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2\mathrm{d}\chi^2 \quad\Longrightarrow\quad \mathrm{d}\chi = -\frac{c\,\mathrm{d}t}{a} \quad\Longrightarrow\quad \chi_{\rm obs} = \chi_{\rm em} - \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)}\,. \tag{13.29}$$

The locations $\chi_{\rm em}$, $\chi_{\rm obs}$ do not change in time, so if we write the above relation for two successive wave maxima (say), (1) and (2), and subtract the equalities, we obtain

$$0 = \int_{t^{(2)}_{\rm em}}^{t^{(2)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)} - \int_{t^{(1)}_{\rm em}}^{t^{(1)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)}\,. \tag{13.30}$$

The course of $a(t)$ seems to be needed in order to proceed with the integrals (if possible), but in fact it is not necessary. Namely, the experiment only has sense as a global one, i.e., the radiation has to spend a considerable time “on the road”;[4] in such a case, the integration limits satisfy

$$t^{(1)}_{\rm em} < t^{(2)}_{\rm em} = t^{(1)}_{\rm em} + \mathrm{d}t_{\rm em} \;\ll\; t^{(1)}_{\rm obs} < t^{(2)}_{\rm obs} = t^{(1)}_{\rm obs} + \mathrm{d}t_{\rm obs}\,.$$

It is thus advantageous to decompose the integrals as

$$\int_{t^{(2)}_{\rm em}}^{t^{(1)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)} + \int_{t^{(1)}_{\rm obs}}^{t^{(2)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)} = \int_{t^{(1)}_{\rm em}}^{t^{(2)}_{\rm em}} \frac{c\,\mathrm{d}t}{a(t)} + \int_{t^{(2)}_{\rm em}}^{t^{(1)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)}\,,$$

because the contributions from the common part $\langle t^{(2)}_{\rm em}, t^{(1)}_{\rm obs}\rangle$ cancel out (integrand is the same!) and one is left with

$$\int_{t^{(1)}_{\rm obs}}^{t^{(2)}_{\rm obs}} \frac{c\,\mathrm{d}t}{a(t)} = \int_{t^{(1)}_{\rm em}}^{t^{(2)}_{\rm em}} \frac{c\,\mathrm{d}t}{a(t)}\,,$$

[4] Remember that the cosmological principle applies for scales above some 250 Mpc, which corresponds to some 800 million light years; hence, we are typically speaking about light travelling for hundred millions to billions of years.

which is already very easy to evaluate, because now the integrations are only performed over very small elements ($\mathrm{d}t_{\rm em}$ and $\mathrm{d}t_{\rm obs}$) during which the universe changes only negligibly. Hence, it is possible to represent the function $a(t)$ in the integrands by mean values $a(t_{\rm obs})$ and $a(t_{\rm em})$, respectively. The equation thus assumes the form

$$\frac{c(t^{(2)}_{\rm obs}-t^{(1)}_{\rm obs})}{a(t_{\rm obs})}=\frac{c(t^{(2)}_{\rm em}-t^{(1)}_{\rm em})}{a(t_{\rm em})} \quad\Longleftrightarrow\quad \frac{\lambda(t_{\rm obs})}{a(t_{\rm obs})}=\frac{\lambda(t_{\rm em})}{a(t_{\rm em})} \quad\Longleftrightarrow$$
$$\Longleftrightarrow\quad (\nu a)_{t_{\rm em}}=(\nu a)_{t_{\rm obs}} \quad\Longleftrightarrow\quad \boxed{\nu a=\mathrm{const}}\,. \tag{13.31}$$

The above result could even be reached easier, using the preceding part 13.5 about FLRW geodesics and the generic formula for frequency shift (4.15),

$$\frac{\hat\nu_{\rm B}}{\hat\nu_{\rm A}}=\frac{(p^\mu\hat u_\mu)_{\rm B}}{(p^\mu\hat u_\mu)_{\rm A}}\,.$$

In our case the observers $\hat u^\mu$ have, in the comoving coordinates, the four-velocity $\hat u_\mu=(-1,0,0,0)$, so the formula reduces to $\frac{\hat\nu_{\rm B}}{\hat\nu_{\rm A}}=\frac{(p^t)_{\rm B}}{(p^t)_{\rm A}}$. But we found in (13.28) that in the FLRW models $p^t=\mathrm{const}/a$ holds along light-like geodesics, so we can conclude by

$$\frac{\hat\nu_{\rm B}}{\hat\nu_{\rm A}}=\frac{a(t_{\rm A})}{a(t_{\rm B})}\,,\qquad \text{i.e.}\quad \nu a=\mathrm{const}\,. \tag{13.32}$$

Now we also fully understand the result of Section 13.4.1 that $\rho_{\rm rad}$ behaves as $1/a^4$: the *number* density of photons is proportional to $1/a^3$ as usual, but, in addition, each photon loses (or gains) energy according to $\nu\sim 1/a$. Regarding that $h\nu\sim kT$, one also checks that the above result agrees with the Stefan-Boltzmann law $\rho_{\rm rad}\sim T^4$. Such a behaviour is often being illustrated on that of a wave drawn on a ball whose radius is changing.

13.6.1 Astronomical redshift: z

"Redshift" means the effect itself, but it also represents a particular quantity which in cosmology is one of the most important: z. The latter is defined as the **relative change of wavelength**, i.e.

$$z:=\frac{\lambda_{\rm obs}-\lambda_{\rm em}}{\lambda_{\rm em}}=\frac{\lambda_{\rm obs}}{\lambda_{\rm em}}-1\left(=\frac{\nu_{\rm em}}{\nu_{\rm obs}}-1\right)=\frac{a(t_{\rm obs})}{a(t_{\rm em})}-1\,. \tag{13.33}$$

In observational cosmology, $t_{\rm obs}\equiv t_0$ always stands for "today" (t_0 is the current age of the Universe), so the formula represents the dependence $z=z(t)=\frac{a(t_0)}{a(t)}-1$, where, naturally, $t\equiv t_{\rm em}$. The highest redshifts are being detected at distant galaxies, quasars and some gamma-ray burst progenitors (the current record being $z=11.09$).

Redshift is a very useful quantity since it is directly measurable. It is thus natural to express $a(t_0)/a(t)$ as $(1+z)$ everywhere (the expansion factor a is not measurable). Finally, let us add a useful relation obtained by differentiation of $1+z=\frac{a_0}{a}$:

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-\frac{a_0}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}=-(1+z)H\,. \tag{13.34}$$

13.7 Friedmann equation and possible histories of the FLRW universes

Hope you noticed we have not yet employed Einstein's equations! Really, the above considerations were purely geometrical, not relying on any particular theory of gravitation. However, being fans of GR, let us substitute now the FLRW metric (13.12) and the ideal-fluid energy-momentum tensor (7.27) to the field equations (8.4). Two non-trivial equations arise,[5]

$$\boxed{H^2 + \frac{Kc^2}{a^2} = \frac{\Lambda c^2}{3} + \frac{8\pi G}{3}\rho}\,, \tag{13.35}$$

$$-2qH^2 + H^2 + \frac{Kc^2}{a^2} = \Lambda c^2 - \frac{8\pi G}{c^2}P\,, \tag{13.36}$$

where $K = +1, 0, -1$ again distinguishes between the spherical/flat/hyperboloidal cases, $H \equiv a_{,t}/a$ is the Hubble-Lemaître constant as above, and q is the **deceleration parameter** defined by

$$q \equiv -\frac{a_{,tt}a}{(a_{,t})^2} = -\frac{a_{,tt}}{a}\frac{1}{H^2} = -\frac{1}{H^2}\frac{\mathrm{d}H}{\mathrm{d}t} - 1 = \frac{\mathrm{d}(H^{-1})}{\mathrm{d}t} - 1\ . \tag{13.37}$$

We have only framed the first equation, called **the Friedmann equation**, because the deceleration equation can actually be obtained from it if also using the conservation laws $T^{\mu\nu}{}_{;\nu} = 0$, more specifically, using the continuity equation (13.16)

$$\rho_{,t} + 3H\left(\rho + \frac{P}{c^2}\right) = 0\ . \tag{13.38}$$

Indeed, by differentiating equation (13.35) and using the relation $H_{,t} = -qH^2 - H^2$ from (13.37), one obtains easily

$$-2qH^3 - 2H^3 - \frac{2Kc^2}{a^2}\,H = \frac{8}{3}\pi G\rho_{,t}\ ,$$

which after substitution for $\rho_{,t}$ from (13.38) and division by H reads

$$-2qH^2 - 2H^2 - \frac{2Kc^2}{a^2} = -8\pi G\left(\rho + \frac{P}{c^2}\right).$$

Substituting now to the right-hand side from (13.35) for

$$-8\pi G\rho = \Lambda c^2 - 3H^2 - \frac{3Kc^2}{a^2}\,,$$

one exactly reaches the deceleration equation (13.36). Note that the opposite reasoning also works – from (13.35) and (13.36), it is possible to *derive* the continuity equation (13.38).

[5] We will write the equations in standard units (rather than in the geometrized ones) as it is quite common in cosmology.

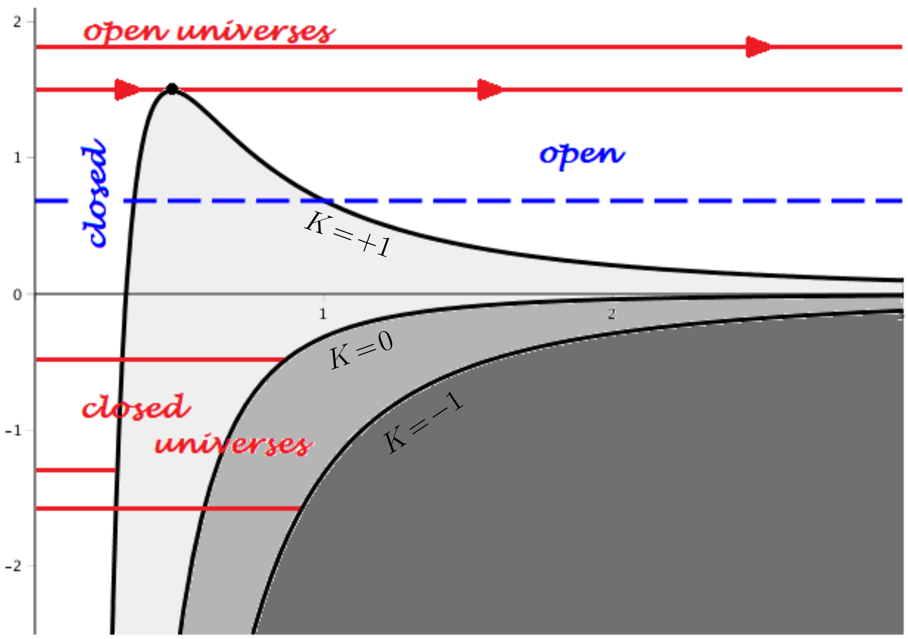


Figure 13.2 *Graphical form of the Friedmann equation (13.39). Potential $V(a)$ is drawn for all the three possible values of K, with several typical values of $\Lambda c^2/3$ shown to indicate the respective evolutions. For a_0 we have chosen the radius of the cosmological horizon, $a_0 \simeq c/H$ (a is not an observable quantity anyway). The dashed blue line corresponds to the case with all the parameters set at values following from observations (data mainly from the Planck CMBR mission, 2018/20), in particular, $H_0 \doteq 67.4\,\mathtt{km/s/Mpc}$, $\rho_{\rm crit} \doteq 0.85{\cdot}10^{-29}\,\mathtt{g/cm}^3$, $\Lambda \doteq 0.684\,\frac{8\pi G\rho_{\rm crit}}{c^2} \doteq 1.09{\cdot}10^{-56}\,/\mathtt{cm}^2$, $\rho_{\rm mat}(t_0) \doteq 0.315\,\rho_{\rm crit}$, $\rho_{\rm rad}(t_0) \doteq 8{\cdot}10^{-5}\rho_{\rm crit}$. In order that the axis values be of the order of unity, a is given in the units of a_0 and the equation (13.39) has been multiplied by $\frac{a_0^2}{c^2}$, therefore, to be precise, the vertical axis represents the (dimensionless) value of $\frac{a_0^2}{c^2}\,V(a) = K\left(\frac{a_0}{a}\right)^2 - \frac{8\pi G a_0^2}{3c^2}$, with the horizontal Λ-line lying at $a_0^2\Lambda/3$.*

Friedmann equation (13.35) is thus the "master" equation of standard cosmological models. It describes the cosmological dynamics by fixing the time behaviour of the expansion factor, $a = a(t)$. Before embarking on any specific solution, it is most useful to look how in principle such a behaviour may look like. Best to do so using the same method we employed when studying the properties of geodesic motion in the Schwarzschild field – the method of effective potential. Though describing different physics, the picture is really very similar to equation (12.24), i.e. $(u^r)^2 = \tilde{E}^2 - \tilde{V}^2$, only with H^2 now playing the role of $(u^r)^2$ and with $\Lambda c^2/3$ now playing the role of "constant of the motion" ($\tilde{E}^2$); the rest is the effective potential:

$$H^2 = \frac{\Lambda c^2}{3} - V(a)\,, \qquad \text{where} \tag{13.39}$$

$$V(a) := \frac{Kc^2}{a^2} - \frac{8\pi G}{3}\rho = \frac{Kc^2}{a^2} - \frac{8\pi G}{3}\left[\rho_{\rm mat}(t_0)\left(\frac{a_0}{a}\right)^3 + \rho_{\rm rad}(t_0)\left(\frac{a_0}{a}\right)^4\right]; \tag{13.40}$$

in the second expression, we substituted for ρ the behaviours (13.20). Obviously, for any specific value of a, the effective potential $V(a)$ represents a minimal value of $\Lambda c^2/3$ for which a universe of such a size (a) may exist. According to which of the terms on the right-hand side of (13.39) dominates, one speaks of a universe dominated by cosmological constant, by curvature, by matter, or by radiation, respectively. Since the cosmological term is constant, curvature going as a^{-2}, matter term as a^{-3} and radiation term as a^{-4}, it is clear that the universes which expand from a "big bang" are typically first dominated by radiation, then by matter, later by curvature and finally by Λ. (Naturally, this only holds if they live sufficiently long to experience all the epochs – we will see that sooner or later they may shrink back instead.)

Equation (13.39) is graphically represented by a graph with $V(a)$ and $\Lambda c^2/3$ on the vertical axis and with a on the horizontal one – see Figure 13.2. The graph works in exactly the same manner as we experienced in geodesic motion in Schwarzschild, just with the cosmic physics behind now:

- A universe with given Λ moves, above its curve of effective potential $V(a)$ (given by the values of K, $\rho_{\rm mat}(t_0)$, $\rho_{\rm rad}(t_0)$ and a_0), along a horizontal straight line given by the value of $\Lambda c^2/3$, with the "velocity" squared H^2 given by how deep below $\Lambda c^2/3$ the potential $V(a)$ momentarily lies. Where the line $\Lambda c^2/3$ hits the curve of $V(a)$, the H^2 vanishes – it is a turning point where expansion changes to contraction or vice versa. If, in addition, $\frac{{\rm d}V}{{\rm d}a}=0$ at such a point, it is an "eternal turning point" – a stationary point where the evolution may stall for arbitrarily long. (We do not care about stability now.)

- For an empty ($\rho = 0$) and flat ($K = 0$) universe, the potential is identically zero. In all non-trivial cases it has a global extreme at $a=0$; this extreme is $-\infty$, with the exception of the empty case with $K = +1$ when it is $+\infty$. For $a \to \infty$ the potential goes to zero. For $K = -1, 0$ the potential grows monotonously with a (it has no local extreme and is everywhere negative), whereas for $K = +1$ it has a positive maximum "on the way" and then *decreases* to zero. The stationary point we mentioned above is thus unstable. It is the only possible stationary "equilibrium" (between attractive gravity of the cosmic fluid and repulsive cosmological term) – it corresponds to the famous **Einstein static universe**. The scenarios which have it as their asymptotic ($t\to\pm\infty$) state are called the **Eddington-Lemaître universes**.

- The cosmic evolution has either one turning point (if $\frac{\Lambda c^2}{3} < V_{\rm max}$) or none (if opposite holds).

- Very small universe is always dominated by radiation (if $\rho_{\rm rad}>0$ today, of course – which apparently is the case), whereas curvature has a negligible role in it. Very large universe is either dominated by the cosmological constant (if $\Lambda\neq 0$), or by curvature (if $\Lambda=0$, $K\neq 0$), or by matter (if $\Lambda=0$ and $K=0$); radiation is negligible in it.

In the actual Universe, both matter and radiation as well as cosmological term are present. However, in order to get some idea of how the individual terms affect the dynamics, it is illustrative to solve the Friedmann equation (13.39) with each of them present exclusively:

- for pure radiation ($\rho = \rho_{\rm rad}$),

$$H^2 = \frac{8\pi G a_0^4}{3a^4}\rho_0 \quad \Rightarrow \quad a\,{\rm d}a = \sqrt{\frac{8\pi G a_0^4}{3}\rho_0}\,{\rm d}t \quad \Rightarrow \quad \frac{a}{a_0} = 2\left(\frac{2\pi G}{3}\rho_0\right)^{1/4} t^{1/2} \tag{13.41}$$

- for pure matter ($\rho = \rho_{\rm mat}$),

$$H^2 = \frac{8\pi G a_0^3}{3a^3}\rho_0 \quad \Rightarrow \quad \sqrt{a}\,{\rm d}a = \sqrt{\frac{8\pi G a_0^3}{3}\rho_0}\,{\rm d}t \quad \Rightarrow \quad \frac{a}{a_0} = (6\pi G\rho_0)^{1/3} t^{2/3} \tag{13.42}$$

(called the Einstein-de Sitter universe)

- for pure curvature term,

$$H^2 = -\frac{Kc^2}{a^2} \quad \Rightarrow \quad \begin{cases} \text{impossible} & \text{for } K = +1 \\ a = \text{const} & \text{for } K = 0 \\ a = ct & \text{for } K = -1 \end{cases} \tag{13.43}$$

- for pure cosmological term (if alone, it has to be non-negative, $\Lambda \geqslant 0$),

$$H^2 = \frac{\Lambda c^2}{3} \quad \Rightarrow \quad \frac{{\rm d}a}{a} = \sqrt{\frac{\Lambda}{3}}\,c\,{\rm d}t \quad \Rightarrow \quad a = \text{const}\cdot\exp\left(\sqrt{\frac{\Lambda}{3}}\,ct\right). \tag{13.44}$$

(de Sitter universe).

We have everywhere chosen positive square root of H^2, because we are interested in *expanding* scenarios. Obviously, radiation and matter decelerate the expansion, whereas positive Λ accelerates it. Note that even the universe with $\Lambda > 0$ need not live to the era of exponential expansion, because if $K = +1$, it need not make it over the potential maximum (may be reflected back). However, if it does overcome the maximum, it will expand indefinitely. Namely, the matter and radiation density quickly falls off in expansion and curvature decreases as well (although the curvature *parameter* K stays constant, of course), whereas the density of vacuum energy does *not* decay – it is a true cosmological *constant*.

Note that in a vacuum case ($\rho = 0$), the only realizable options are $\Lambda > 0$ (so called de Sitter models; K may assume any of the three values), or $\Lambda < 0$ and $K = -1$ (anti-de Sitter model), or $\Lambda = 0$ and $K = 0, -1$. One also sees that the universe with $\Lambda < 0$ cannot be arbitrarily large (whatever values the other parameters may have).

Histories of the cosmic evolution – i.e. possible courses $a(t)$ obtained by solution of the Friedmann equation – are qualitatively clear from Figure 13.2. Naturally, one only focuses on those which include an expanding phase.

13.7.1 Dust models with zero Lambda

Before it began to turn out, in 1990s, that on the largest scale the cosmological constant is far from negligible, the simplest ideal of a cosmological model was the universe containing only an incoherent dust and nothing else. The Friedmann equation reduces in that case to

$$\frac{(a_{,t})^2}{a^2} + \frac{Kc^2}{a^2} = \frac{8\pi G}{3a^3}\,\rho_0 a_0^3\,, \qquad \text{i.e.} \qquad (a_{,t})^2 + Kc^2 = \frac{8\pi G}{3a}\,\rho_0 a_0^3\,.$$

We already solved this equation in the flat, $K = 0$ case in (13.42) (the Einstein-de Sitter universe), but let us also solve it in the remaining, $K = \pm 1$ cases. It is advantageous to first introduce a conformal time η by equation

$$c\,\mathrm{d}t = a\,\mathrm{d}\eta\,.$$

This makes the metric (13.12) appear

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \Sigma^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right],$$

so it has become conformally static (the time dependence only appears in the conformal factor a^2); and worth to add that in the $K = 0$ case, when $\Sigma = \chi$, the metric is even conformally flat (any possible curvature is entirely rendered by the expansion factor).

In terms of η, the Friedmann equation becomes [use $a_{,t} = (c/a)\,a_{,\eta}$ and multiply the equation by a^2/c^2]

$$(a_{,\eta})^2 + Ka^2 = 2Aa\,, \qquad \text{where} \quad A := \frac{4\pi G}{3c^2}\,\rho_0 a_0^3\,.$$

Notice that A is the mass of a Euclidean ball with radius a_0 and density ρ_0, given in geometrized form, i.e. multiplied by G/c^2 (in the closed, $K = 1$ case, in which the spatial volume of the universe amounts to $2\pi^2 a^3$, it is $2/(3\pi)$-multiple of the total mass). Denoting

$$C_K(\eta) := \cos\eta\,, \qquad S_K(\eta) := \sin\eta \qquad \text{for } K = +1\,, \tag{13.45}$$

$$C_K(\eta) := \cosh\eta\,, \qquad S_K(\eta) := \sinh\eta \qquad \text{for } K = -1\,, \tag{13.46}$$

the solution can be written as

$$a(\eta) = KA\left[1 - C_K(\eta)\right] \quad \Longrightarrow \quad t(\eta) = \frac{1}{c}\int^{\eta} a(\eta')\mathrm{d}\eta' = \frac{KA}{c}\left[\eta - S_K(\eta)\right]. \tag{13.47}$$

- In the $K = +1$ case, this yields a cycloid,

$$a(\eta) = \frac{a_{\max}}{2}\,(1 - \cos\eta), \qquad t(\eta) = \frac{a_{\max}}{2c}\,(\eta - \sin\eta)\,. \tag{13.48}$$

  The universe starts from $a = 0$ ("big bang") at $\eta = 0$, $t = 0$, it expands to reach $a = 2A =: a_{\max}$ at $\eta = \pi$, $t = \pi A/c$, and then symmetrically contracts back to singularity ("big crunch") at $\eta = 2\pi$, $t = 2\pi A/c$. This is thus a *closed* model – it has finite volume, and it is finite in time as well.

- In the $K = -1$ case, the solution reads

$$a(\eta) = A(\cosh\eta - 1), \qquad t(\eta) = \frac{A}{c}(\sinh\eta - \eta)\,. \tag{13.49}$$

  Asymptotically, for $\eta \to \infty$, one has $\begin{matrix}\cosh\\ \sinh\end{matrix}\;\eta = \frac{1}{2}(e^\eta \pm e^{-\eta}) \to \frac{e^\eta}{2}$, hence $a(\eta) \to \frac{A}{2}\,e^\eta$, $t(\eta) \to \frac{A}{2c}\,e^\eta$, and so $a(t) \sim ct$ ... sure, for large a the curvature term dominates over the matter term, so one approaches the third case of (13.43). Therefore, this is an *open* model – it has infinite volume, and it lasts infinitely long as well (similarly as the $K = 0$ model).

13.8 FLRW models in the language of Omega-factors

In standard units, the value $H_0 \doteq 67.4\,\mathtt{km/s/Mpc}$ of the Hubble-Lemaître constant reads $H_0 \doteq 2.18\cdot 10^{-18}/\mathtt{s}$. Hence, the typical magnitude of the Friedmann-equation terms is 10^{-36} (in the units of $1/\mathtt{s}^2$), which is not very practical. The best for discussion is the order of unity – and that can simply be reached by dividing the equation by H^2. The Friedmann and the deceleration equations, (13.35) and (13.36), thus assume the form

$$1 + \Omega_K = \Omega_\Lambda + \Omega_{\rm mat} + \Omega_{\rm rad}\,, \tag{13.50}$$
$$-2q + 1 + \Omega_K = 3\Omega_\Lambda - \Omega_{\rm rad}\,, \tag{13.51}$$

where we have already inserted $P = P_{\rm rad} = \rho_{\rm rad}c^2/3$. The Ω factors are just the individual terms of the equations divided by H^2, i.e.

$$\Omega_{\rm mat} := \frac{8\pi G\rho_{\rm mat}}{3H^2} = \frac{\rho_{\rm mat}}{\rho_{\rm crit}}\,, \tag{13.52}$$
$$\Omega_{\rm rad} := \frac{8\pi G\rho_{\rm rad}}{3H^2} = \frac{\rho_{\rm rad}}{\rho_{\rm crit}}\,, \tag{13.53}$$
$$\Omega_\Lambda := \frac{\Lambda c^2}{3H^2} = \frac{\Lambda c^2}{8\pi G\rho_{\rm crit}}\,, \tag{13.54}$$
$$\Omega_K := \frac{Kc^2}{H^2a^2}\,, \tag{13.55}$$

where we have also introduced the **critical density**

$$\rho_{\rm crit} := \frac{3H^2}{8\pi G} \doteq 0.85\cdot 10^{-29}\,\mathtt{g/cm^3}\,. \tag{13.56}$$

This density, used as a reference standard, would correspond to the $\Lambda = 0$ and $K = 0$ model, i.e. to a flat universe solely filled with fluid.

The two dynamical equations contain 7 quantities: four of them are measurable in principle – H, q, $\rho_{\rm mat}$ and $\rho_{\rm rad}$; the other 3 are not directly accessible to observation – Λ, K, a. Translated into the Ω-language, it means that $\Omega_{\rm mat}$, $\Omega_{\rm rad}$ and q are measurable, whereas Ω_Λ and Ω_K are not. Each of the equations (13.50), (13.51) contains two non-measurable quantities. If, however, substituting for $1+\Omega_K$ from the former to the latter, one obtains equation

$$\boxed{q + \Omega_\Lambda = \frac{1}{2}\Omega_{\rm mat} + \Omega_{\rm rad}} \tag{13.57}$$

in which only the term Ω_Λ is not measurable. Another such relation (with only one "unknown") is obtained by adding (13.51) + $3\times(13.57)$:

$$1 + q + \Omega_K = \frac{3}{2}\Omega_{\rm mat} + 2\Omega_{\rm rad}\,. \tag{13.58}$$

13.8.1 Values inferred from current observations

Let us add what observations say about the cosmological parameters (in 2020).[6] Despite some 10% discrepancy still existing between different measurements, the current value of the Hubble-Lemaître constant is about $70\,\mathtt{km/s/Mpc}$. Let us give the value $H \doteq 67.4\,\mathtt{km/s/Mpc} \doteq 2.18\cdot 10^{-18}/\mathtt{s}$ inferred from the studies of relict radiation (the Planck project), and the value of the deceleration parameter $q \doteq -0.53$. The energy-density components are

$$\Omega_{\rm mat} \doteq 0.315, \quad \Omega_{\rm rad} \doteq 8\cdot 10^{-5} \quad \Longrightarrow \quad \Omega_\Lambda \doteq 0.684 \quad \Longrightarrow \quad \Omega_K \simeq 0\,. \tag{13.59}$$

Two main messages arise immediately:

- The deceleration parameter is actually an *acceleration* one... Certainly, it was natural to expect that the cosmic expansion decelerates due to the attraction acting within the cosmic material. However, from the 1990s it has been turning out that the expansion of our Universe is speeding up rather than slowing down. From there it follows that the cosmological term, up to the 1990s standardly neglected in physical cosmology, must be there, and must be positive ("repulsive"); it is actually a dominant term of the Friedmann equation.

- The effective total mass-energy density in the Universe is very close to the critical density, which implies that the curvature term is very small. This may either mean that the large-scale curvature of the Universe is zero ($K=0$), or that the Universe is very large (very large a makes $\Omega_K \equiv \frac{Kc^2}{H^2a^2}$ very small even for non-zero K). Such an option has already before precise measurements been preferred by theorists – surely for aesthetic reasons, but mainly because of the inflationary models from 1980s which seem to "converge" to that case.

In the first version of these lecture notes (from 2002) we wrote: "During the turn of the 20th and 21st centuries, the knowledge about the large-scale Universe has *expanded* considerably, making the cosmological textbooks ageing faster than ever since the 1960s, if not from the 1920s. In particular, much progress has been achieved in the mapping of distribution and motions of galaxies and their clusters, and in measurements of the properties of CMBR." In recent years, the quality of current cosmological data is even being stressed by the term "era of *precision cosmology*".

In cosmological circles, the Λ term is usually being called the **dark energy**. However, it is still possible that it does not have a character of *source*, it may just be an independent constant characterizing gravitation (besides G). Anyway, even more of the Universe is dark. Of the matter ($\Omega_{\rm mat} \doteq 0.315$), only $\Omega_{\rm bar} \doteq 0.05$ is allotted to baryons (light-emitting matter), while about $\Omega_{\rm dark} \doteq 0.265$ ($\sim 84\%$ of all matter) goes to the **dark matter** – a matter which has a gravitational effect but practically does not interact otherwise (and is thus invisible). The nature of this matter is unclear, but it is mostly assumed to be non-baryonic, perhaps made of some yet unknown "weakly interacting massive particles" (WIMPs) such as those predicted by certain theories going beyond standard model of particle physics. According to

[6] The following values are *current* ones, so they should be denoted by nought, like H_0 etc., but we will omit it this time.

how energetic ("hot") these particles are, the dark matter is being called "cold", "warm" or "hot" (it is important in how structures form within such a matter).

Consider, in addition, that about 74% of mass of the cosmic baryon matter is in hydrogen, about 24% in helium,[7] 1% in oxygen and 0.5% in carbon. Other elements together make just some 6 permille of baryon matter. In terms of the Omega factor, it means

$$\Omega_{\rm H} \doteq 0.037, \qquad \Omega_{\rm He} \doteq 0.012, \qquad \Omega_{>{\rm He}} \doteq 0.001.$$

It is possible to estimate how many photons per 1 baryon exist in the Universe. The radiation energy density is mainly due to the relict photons, and these have Planck spectrum with temperature $2.725\,\mathtt{K}$ ($\sim$ average energy $10^{-3}\mathtt{eV}$), while energy of one baryon can be approximated by the proton rest energy $m_{\rm p}c^2 \sim 10^9\mathtt{eV}$, so one has

$$\frac{\Omega_{\rm rad}}{h\,\langle\nu_{\rm CMBR}\rangle}\frac{m_{\rm p}c^2}{\Omega_{\rm bar}} \sim 10^{12}\,\frac{\Omega_{\rm rad}}{\Omega_{\rm bar}} \sim 10^9 \text{ photons per baryon}\,. \tag{13.60}$$

Current "concordance scenario" of cosmology is embodied by the so-called ΛCDM (Lambda + Cold Dark Matter) model which relies on GR and includes non-zero cosmological constant, (cold) dark matter, ordinary matter and radiation. A possible phase of exponential expansion in the very early Universe (the so-called **cosmic inflation**) is often included. Inflation is being discussed since 1980s by particle theorists as a possible consequence of the transition of a "false vacuum" of some cosmic field (a metastable state which is just local minimum of energy, not the global one) to its "true vacuum". The Higgs field is being referred to in particular in this respect. Inflation may help to solve several queries of the standard ΛCDM scenario.

[7] Helium cache in the cryogenic pavilion of our Trója campus has been included.

CHAPTER 14

Schwarzschild space-time: analytic extension

The main novel feature we met in the Schwarzschild solution is the possible appearance of the horizon and (thus) of its mysterious interior region – the black hole. Various properties become strange close to the black hole, not speaking about its interior. However, only some of these are *really* strange (in some invariant sense), whereas others turn out to be a consequence of the *choice of coordinates*. Actually, though the Schwarzschild coordinates are most natural for description of the black-hole exterior, they fail at the horizon (and switch their roles below). In this chapter, we demonstrate the "cons" of the Schwarzschild coordinates, thus collecting the main reasons why to look for better ones. In particular, we will see – surprisingly at first sight – that the Schwarzschild coordinates do not cover the whole Schwarzschild manifold. All the "cons" will be remedied by transformation to Kruskal-Szekeres coordinates, possibly followed by compactification of the resulting picture in the Penrose-Carter conformal diagram.

Let us remind once more that the Schwarzschild solution describes, in GR, *any* spherically symmetric vacuum region. At large radii ($r \gg M$), the difference from the Newtonian central field $\Phi = -M/r$ is only negligible; note that in the case of an extended spherically symmetric body (a "star"), $r \gg M$ typically holds *everywhere* outside it. The more compact the source is, the stronger is the field in its vicinity – and the more its relativistic shape deviates from the Newtonian one. Extreme is the case when the whole body (of mass M) is concentrated below its Schwarzschild radius $r = 2M$; the region $r \leqslant 2M$ then becomes a black hole – it encloses by a horizon and causally separates from the "domain of outer communications" $r > 2M$; and the light cones prevent the body from assuming any stationary configuration below $r = 2M$ – its only possible fate becomes a collapse to the point $r = 0$.

14.1 Invariant and coordinate properties of the Schwarzschild space-time

14.1.1 Measuring the radial distance

Distances can be measured in two basic ways – by a meter (**proper distance**) or by a light signal (**radar distance**).

- In the radial direction of the Schwarzschild space-time, the proper distance between some radii r_A and $r_B > r_A$ computes as

$$l(r_A, r_B) = \int_{r_A}^{r_B} \frac{\mathrm{d}r}{\sqrt{1-\frac{2M}{r}}} = \left[\sqrt{r(r-2M)} + 2M \ln\left(\sqrt{\frac{r}{2M}-1} + \sqrt{\frac{r}{2M}}\right)\right]_{r_A}^{r_B}. \quad (14.1)$$

- The radial radar distance follows from the coordinate time in which light makes it from r_A to $r_B > r_A$ (and back), or vice versa. As in determining the light cones, we obtain from the relation

$$0 = \mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2M}{r}}$$

that $\mathrm{d}t = \pm\frac{\mathrm{d}r}{1-\frac{2M}{r}} =: \pm\mathrm{d}r^*$, so

$$t(r_A, r_B) = \int_{r_A}^{r_B} \frac{\mathrm{d}r}{1-\frac{2M}{r}} = \int_{r_A}^{r_B} \mathrm{d}r^* = |r^*_B - r^*_A| = \left[r + 2M\ln\left(\frac{r}{2M}-1\right)\right]_{r_A}^{r_B}. \quad (14.2)$$

Clearly the result is different than the one found by the meter. However, the coordinate time is not the final result. Practically, the measurement would proceed in one of two ways: either an observer would stay at r_A and place a static mirror at r_B, or vice versa. For a proper time of any static observer (the one with $\hat{u}^i = 0$) we have $\mathrm{d}s^2 = -g_{tt}\mathrm{d}t^2 = -\mathrm{d}\hat{\tau}^2$, hence, in the first/second case (observer/mirror at r_A, mirror/observer at r_B) the light returns after

$$\hat{\tau}(r_A, r_B) = 2\sqrt{-g_{tt}(r_A)}\, t(r_A, r_B)\,, \qquad \hat{\tau}(r_B, r_A) = 2\sqrt{-g_{tt}(r_B)}\, t(r_A, r_B)\,, \quad (14.3)$$

respectively.

Far from the centre the above distances almost coincide, but closer they more and more differ – and in the limit when one of the radii approaches the horizon, they even differ infinitely! Actually, the proper radial distance of a generic location $r_B \equiv r > 2M$ from the horizon ($r_A \to 2M$) is

$$l(2M, r) = \sqrt{r(r-2M)} + 2M\ln\left(\sqrt{\frac{r}{2M}-1} + \sqrt{\frac{r}{2M}}\right),$$

which is finite, while the radar distance goes to zero in the first case (observer approaches the horizon) whereas it goes to infinity in the second case (mirror approaches the horizon). Namely, the coordinate time (14.2) diverges in any case, but in the first case the dilation factor $\sqrt{-g_{tt}(r_A)}$ goes to zero faster than the logarithm in (14.2) diverges, whereas in the second case the dilation factor $\sqrt{-g_{tt}(r_B)}$ is finite and does not alter the infinity of (14.2).

14.1.2 Embedding of the equatorial plane in 3D Euclidean space

In GR, intuition is more intricate due to the space-time curvature. If a d-dimensional manifold is curved, it is curved "to the $(d+1)$-th dimension" (at least), so, in order to depict the curvature of its small region at least, one has to embed that region in a $(d+1)$-dimensional Euclidean space.[1] Should the embedding Euclidean space be at most 3D, one can maximally embed 2D sections, i.e. surfaces. A suitably chosen section can however illustrate the space-time geometry very well. Below, we try to elucidate the curvature of Schwarzschild space-time by embedding, in $\mathbb{E}^3$, its equatorial plane $\{t=\mathrm{const}, \theta=\pi/2\}$ (thus in fact *any* main plane).

At any constant time t, the equatorial plane has metric

$$\mathrm{d}s^2(t\!=\!\mathrm{const}, \theta\!=\!\pi/2) = \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2\mathrm{d}\phi^2\,. \tag{14.4}$$

This differs from the Euclidean one in that in the radial direction the proper distance changes faster than the Euclidean radius r of a circle with proper circumference

$$\int_0^{2\pi} \sqrt{g_{\phi\phi}}\,\mathrm{d}\phi = \int_0^{2\pi} r\,\mathrm{d}\phi = 2\pi r\,,$$

namely, the radial distance element is $\frac{\mathrm{d}r}{\sqrt{1-\frac{2M}{r}}}$ rather than $\mathrm{d}r$. Adding the axial symmetry, one can well imagine how such a surface has to look when represented in the (r,ϕ,z) flat cylindrical coordinates, including the fact that at the horizon $r=2M$ the ratio $\sqrt{g_{rr}}$ between the element of proper distance and its "projection" $\mathrm{d}r$ increases without limit. Vaguely, it should look like a gradually opening/narrowing funnel.

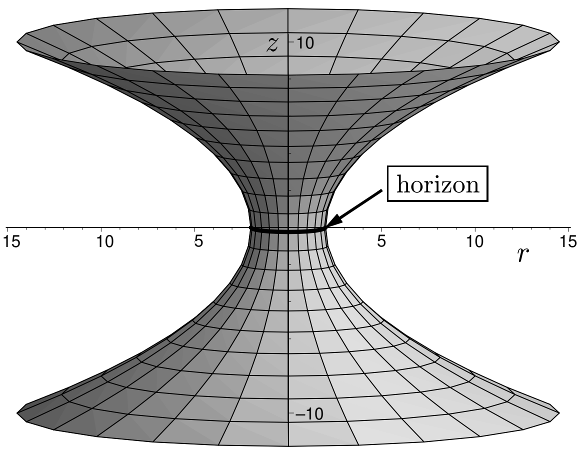
Let us make it more precise – let us embed the equatorial plane in $\mathbb{E}^3$ in such a way that both the proper azimuthal circumference $\int_0^{2\pi}\sqrt{g_{\phi\phi}(\theta\!=\!\pi/2)}\,\mathrm{d}\phi = 2\pi r$ (thus the corresponding circumferential radius r) and the proper radial distance $l(r_A, r_B)$ be represented faithfully (isometrically). The metric of $\mathbb{E}^3$ in cylindrical coordinates (r,ϕ,z) reads

$$\mathrm{d}\sigma^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2\,.$$

An axially symmetric surface is given by some function $z=z(r)$ (no dependence on ϕ due to the axisymmetry), so its embedding is given by substituting for $\mathrm{d}z$ the specific $\mathrm{d}z = \frac{\mathrm{d}z(r)}{\mathrm{d}r}\mathrm{d}r$ given by that $z(r)$ function. Comparing the result with what should come out, i.e. with (14.4), we thus have

$$\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \left(\frac{\mathrm{d}z(r)}{\mathrm{d}r}\right)^2\mathrm{d}r^2 = \left[1+\left(\frac{\mathrm{d}z(r)}{\mathrm{d}r}\right)^2\right]\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \quad \ldots \quad = g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\,,$$

[1] To explain "at least" and "small region" (at least): in order to be able to *globally* embed a smooth d-dimensional real manifold into a Euclidean space $\mathbb{E}^n$, the dimension of the latter has to be in the least favourable case $n=2d$ [*Whitney's embedding theorem*]. Depending on the manifold, less dimensions may be sufficient (down to $n=d+1$), but in general the embedding is only secure for $n=2d$. Note just loosely that **embedding** is such a representation of a manifold within some larger manifold that it has no irregularities (edges, corners, spikes; self-intersections). If self-intersections are allowed (but none of the sharp features), the map is called an **immersion**.



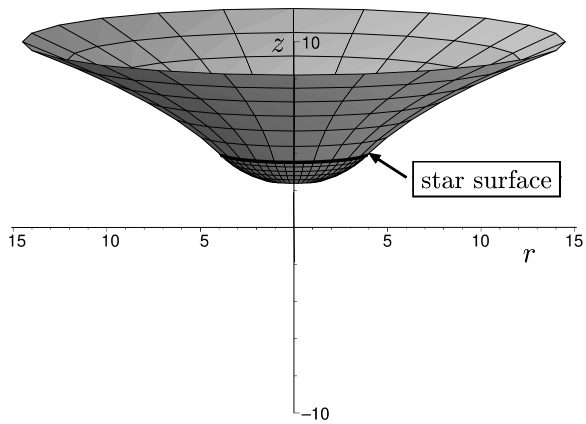


Figure 14.1 *Left:* *Embedding to $\mathbb{E}^3$ of the equatorial plane around the Schwarzschild black hole has the shape of rotational paraboloid (14.5). If the source is an extended spherical body (bigger than its Schwarzschild radius) rather than a black hole, the paraboloid ends on its surface and is capped there with a kind-of dish which represents embedding of the equatorial plane inside the body. Right: Illustration of such a matching for a homogeneous body for which the interior embedding is part of a sphere. For the constant density ρ, the total mass is $M = \frac{4}{3}\pi R^3\rho$, from where we can express $\frac{3}{8\pi\rho} = \frac{R^3}{2M}$ (R denotes here the Schwarzschild radius of the body, not the isotropic radial coordinate as in the main text!). In order that the paraboloid and the spherical cap touch, we set the constant occurring in the interior embedding (14.7) to $(R+4M)\sqrt{\frac{R}{2M}-1} - R\sqrt{\frac{R}{2M}}$, so the interior-embedding spherical surface reads $r^2 + \left[z - (R+4M)\sqrt{\frac{R}{2M}-1}\right]^2 = \frac{R^3}{2M}$. In the plot, we have specifically chosen $R = 4M$, so the interior is described by $r^2 + (z-8M)^2 = 32M^2$ and the border circle is at $z = 4M$. The values along both axes are given in the units of M.*

from where we obtain the "true shape" of the equatorial plane,

$$
\begin{aligned}
&\left(\frac{\mathrm{d}z(r)}{\mathrm{d}r}\right)^2 = g_{rr} - 1 = \frac{2M}{r-2M} \\
\Longrightarrow \quad & z(r) = \pm \int \sqrt{\frac{2M}{r-2M}}\,\mathrm{d}r = \pm\sqrt{8M(r-2M)} \;\; (+\cancel{\text{const}}) \\
\Longrightarrow \quad & r(z) = 2M + \frac{z^2}{8M}\,. \qquad (14.5)
\end{aligned}
$$

After rotating this in ϕ, we have a rotational paraboloid which is asymptotically flat and has the circle $r = 2M$ as its "throat" (this corresponds to the horizon, see left plot in Figure 14.1). Each half of the paraboloid is covered by one set of Schwarzschild coordinates (r, ϕ).

Were the source a spherically symmetric star (with surface on $r = R > 2M$) rather than a black hole, the above paraboloid would only apply outside the star (at $r > R$). Inside the star, the equatorial plane is described by the metric

$$\mathrm{d}s^2(t\!=\!\mathrm{const}, \theta\!=\!\pi/2; r\!<\!R) = \frac{\mathrm{d}r^2}{1-\frac{2m(r)}{r}} + r^2\mathrm{d}\phi^2\,,$$

where $m(r)$ is a mass contained inside the sphere of radius r (see Section 20.3.1 later). Similarly as above, one obtains the embedding

$$\left(\frac{\mathrm{d}z(r)}{\mathrm{d}r}\right)^2 = g_{rr} - 1 = \frac{2m(r)}{r-2m(r)} \quad\Longrightarrow\quad z(r) = \pm\int\sqrt{\frac{2m(r)}{r-2m(r)}}\,\mathrm{d}r \tag{14.6}$$

which on $r\!=\!R$ matches the respective part of the outer paraboloid and whose exact shape is given by the function $m(r)$, so by the radial profile of density $\rho(r)$.[2]

As an example, consider a star with constant density ρ (in a general case, this at least holds close to the star centre). In such a case, one has $m(r)\!=\!\frac{4}{3}\pi r^3\rho$ (Section 20.3.1) and the integral can be evaluated (with upper sign) to

$$z(r) = \sqrt{\frac{3}{8\pi\rho}} - \sqrt{\frac{3}{8\pi\rho} - r^2}\;(+\mathrm{const})\,,$$

or

$$r^2 + \left[z - \sqrt{\frac{3}{8\pi\rho}}\;(-\mathrm{const})\right]^2 = \frac{3}{8\pi\rho}\,. \tag{14.7}$$

This is a sphere with radius $\sqrt{\frac{3}{8\pi\rho}}$ centred at $r = 0$, $z = \sqrt{\frac{3}{8\pi\rho}}\,(+\mathrm{const})$. (The constant has to be chosen so that the exterior and interior embeddings reach the same z and touch there.) Were the density constant within the whole star, the embedding of its whole interior would be of that character – it would form a spherical cap which would "close" the outer, Schwarzschildian paraboloid (see the right part of Figure 14.1). In reality, the star is not homogeneous, so in general its interior embeds as kind-of "dish" along which $z(r)$ grows monotonously with r and which matches the outer paraboloid on the stellar surface.

14.1.3 Isotropic coordinates

Although this is just a remark to the above embedding, it is worth to present it as subsection, since it indicates, for the first time, several important points which are central to this chapter.

- First, we saw the complete paraboloid of the equatorial-plane embedding has to be covered by *two* symmetric sets of Schwarzschild coordinates. This indicates that the Schwarzschild manifold probably is (twice?) larger than how it appears in the Schwarzschild coordinates.

[2] Equation (14.6) is a generic shape of the embedding actually. Since $m(r = R) \equiv M$ and $\mathrm{d}z(r)/\mathrm{d}r$ is continuous there for any density profile, the interior and exterior embeddings match on the surface.

- However, one can also cover the paraboloid by just one set of coordinates. Actually, if transforming to the so-called **isotropic coordinates** by a simple, purely radial transformation

$$r = R\left(1 + \frac{M}{2R}\right)^2, \tag{14.8}$$

we see that the central circle ($r = 2M$) corresponds to the isotropic radius $R = M/2$, with R decreasing (to zero) towards one asymptotically flat region while increasing (to infinity) towards the other one. (It is easy to check that both the cases $R \to 0$ and $R \to \infty$ really correspond to $r \to \infty$.) This indicates that even if the Schwarzschild manifold is larger, it should be possible to cover it by just one set of reasonable coordinates.

- Let us write down the metric in the new coordinates. The standard transformation rule $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}$ implies but one non-trivial relation,

$$g_{RR} = \frac{\partial x^\alpha}{\partial R}\frac{\partial x^\beta}{\partial R}\, g_{\alpha\beta} = \left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2 g_{rr} = \left(1 + \frac{M}{2R}\right)^4,$$

where we have used

$$\frac{\mathrm{d}r}{\mathrm{d}R} = \frac{(2R+M)(2R-M)}{4R^2}\,.$$

Otherwise it is sufficient to just substitute

$$1 - \frac{2M}{r} = \frac{(2R-M)^2}{(2R+M)^2}$$

into the Schwarzschild metric, and thus obtain

$$\mathrm{d}s^2 = -\left(\frac{2R-M}{2R+M}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{M}{2R}\right)^4 \left[\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{14.9}$$

Now we see why the coordinates (t, R, θ, ϕ) are called isotropic: both the radial and the angular part of the spatial metric share the same coefficient (in contrast to the original Schwarzschild metric).

Notice an important feature: at the horizon ($R = M/2$) the spatial part of the metric is regular! (Of course, the coefficient g_{tt} is still zero on the horizon, but the metric shows no divergence there.) This indicates, for the first time, that the horizon is in fact a regular region, not a singularity as it appears in the Schwarzschild coordinates.

- Isotropic coordinates still have one disadvantage: they do not at all describe the black-hole interior. Actually, no value of R makes the Schwarzschild radius r less than $2M$, so one only covers by isotropic coordinates the region (actually *two* regions) $r \geqslant 2M$. In particular, it is thus not possible to say anything about the central singularity at $r = 0$.

14.1.4 Nature of the Schwarzschild-metric singularities

Actually, we already know how it is: with the Kretschmann invariant reading $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{48M^2}{r^6}$ it is clear that $r=0$ is the true space-time pathology. On the other hand, the horizon $r=2M$ is only a coordinate singularity, as we have just inferred from its regularity in the isotropic coordinates. However, let us go somewhat more into this issue – let us check how a test particle crosses the horizon, and what is the dimension (proper volume) of various parts of the horizon.

Radial free fall of a test particle

Let a particle with $m \neq 0$ be dropped from rest from some initial radius $r_{\rm in} > 2M$, to fall freely to the black hole. We know its energy at infinity

$$\tilde{E} \equiv -u_t = -g_{tt}u^t$$

is constant during the fall, so it may be evaluated at any point of the geodesic. Natural is to evaluate it at the starting point, because we know $u^i_{\rm in}=0$ and thus $u^t_{\rm in} = \frac{1}{\sqrt{-g_{tt}(r_{\rm in})}}$ there (from normalization). Hence,

$$\tilde{E} = (-g_{tt}u^t)_{r=r_{\rm in}} = \sqrt{-g_{tt}(r_{\rm in})} = \sqrt{1-\frac{2M}{r_{\rm in}}}\,.$$

Substituting this into the first equation yields

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \equiv (u^t)^2 = \frac{\tilde{E}^2}{(g_{tt})^2} = \frac{1-\frac{2M}{r_{\rm in}}}{\left(1-\frac{2M}{r}\right)^2}\,. \tag{14.10}$$

The second coordinate which changes in the radial fall is the radius. The pertinent differential equation we obtain from the generic radial-motion equation (12.24) by using the above $\tilde{E}$ and by realizing that purely radial motion corresponds to zero angular momentum, $\tilde{L}=0$, which reduces the effective potential (12.25) to $\tilde{V}^2 = 1-\frac{2M}{r}$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \equiv (u^r)^2 = \tilde{E}^2 - \tilde{V}^2 = \frac{2M}{r} - \frac{2M}{r_{\rm in}}\,. \tag{14.11}$$

The solution of the above equations (14.10) and (14.11) is usually being expressed in a parametric form (the parameter η being called conformal time),

$$r = \frac{r_{\rm in}}{2}(1+\cos\eta) = r_{\rm in}\cos^2\frac{\eta}{2}\,, \tag{14.12}$$

$$\tau = \sqrt{\frac{r_{\rm in}^3}{8M}}\,(\eta+\sin\eta)\,, \tag{14.13}$$

$$t = 2M\ln\left|\frac{\sqrt{\frac{r_{\rm in}}{2M}-1}+\tan\frac{\eta}{2}}{\sqrt{\frac{r_{\rm in}}{2M}-1}-\tan\frac{\eta}{2}}\right| + 2M\sqrt{\frac{r_{\rm in}}{2M}-1}\left[\eta+\frac{r_{\rm in}}{4M}(\eta+\sin\eta)\right]. \tag{14.14}$$

The radial geodesic is plotted, for $r_{\rm in}=3M$ and in dependence on both τ and t, in the right-hand part of Figure 12.2. Let us check significant locations:

- Start: $r = r_{\text{in}}$ requires $\cos\eta = 1$ which means $\eta = 0$. This in turn implies $\tau = 0$ and $t = 0$. Therefore, all the times are adjusted in a natural way.

- Crossing the horizon: $r = 2M$ requires $\cos^2\frac{\eta}{2} = \frac{2M}{r_{\text{in}}}$, which in turn implies

$$\tau = \sqrt{\frac{r_{\text{in}}^3}{8M}}\left(\eta + 2\sin\frac{\eta}{2}\cos\frac{\eta}{2}\right) = \sqrt{\frac{r_{\text{in}}^3}{2M}}\arccos\sqrt{\frac{2M}{r_{\text{in}}}} + r_{\text{in}}\sqrt{1 - \frac{2M}{r_{\text{in}}}}\ .$$

To compute the corresponding t, it is crucial to compute

$$\tan\frac{\eta}{2} = \frac{\sin\frac{\eta}{2}}{\cos\frac{\eta}{2}} = \frac{\sqrt{1 - \frac{2M}{r_{\text{in}}}}}{\sqrt{\frac{2M}{r_{\text{in}}}}} = \sqrt{\frac{r_{\text{in}}}{2M} - 1}\ .$$

Plugging this into (14.14) makes the argument of the logarithm infinite, so the logarithm itself is infinite as well. The second term of t is finite, so it does not change anything on that divergence.

- Reaching the singularity: $r = 0$ requires $\cos\eta = -1$ which means $\eta = \pi$. This in turn implies

$$\tau = \pi\sqrt{\frac{r_{\text{in}}^3}{8M}}\ , \qquad t = 2\pi M\sqrt{\frac{r_{\text{in}}}{2M} - 1}\left(1 + \frac{r_{\text{in}}}{4M}\right)$$

(the logarithmic term in t vanishes).

Important is the finding that t goes from zero to infinity at the horizon and than decreases back to some finite value at the singularity. We suspected that from the behaviour of light cones already, the left-hand plot of Figure 12.2.[3] Anyway, what is important for anything is *its* proper time – and that behaves very reasonably: it only assumes finite positive values; in fact one easily checks that τ increases monotonously during the fall,

$$\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \sqrt{\frac{r_{\text{in}}^3}{8M}}\,(1 + \cos\eta) = r\sqrt{\frac{r_{\text{in}}}{2M}} > 0\ ,$$

where we have used (14.12). Cf. the right-hand plot of Figure 12.2.

The above (infinitely different) representation of the fall in terms of τ and t is popularly known as the effect of "relativity of immortality". Its reason is obvious: the clock falling together with the particle (measuring its proper time) ticks "twice infinitely slower" than the clock standing at radial infinity – one infinity is because of the gravitational redshift and the other is because of the Doppler redshift, both diverging on the horizon.

One more remark is worth: comparison with the Newtonian free fall. There, equations for free radial motion, (12.27) and (12.28), reduce to

$$\tilde{L} = 0, \quad \tilde{E} = -\frac{M}{r_{\text{in}}}\,, \quad \tilde{V}_{\text{eff}} = -\frac{M}{r}\,, \qquad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \equiv (v^r)^2 = 2(\tilde{E} - \tilde{V}_{\text{eff}}) = \frac{2M}{r} - \frac{2M}{r_{\text{in}}}\ .$$

[3] And remember that t behaves similarly for light as well, as we have seen in Section 14.1.1 – well, we rather did not *see* anything, because light only returned from the horizon after infinitely long...

Comparing this with equation (14.11), we see that the relativistic fall proceeds, when parametrized by proper time τ, in exactly the same manner as the Newtonian fall (in terms of absolute time t). Hence, in particular, the total time of fall is the same.

Dimensionality of the horizon

As enforced by the causal structure itself, all the time-like and light-like world-lines cross the horizon at infinite values of the Schwarzschild time t. (It even applies to most space-like world-lines.) One thus suspects that the regions $\{r = 2M, |t| < \infty\}$ which are not "hit" by almost any physics are in fact "small" (in terms of dimensionality), whereas the regions $\{r = 2M, |t| = \infty\}$ are "large". In the Schwarzschild coordinates, however, they are represented in exactly an opposite way: any finite-time segment of the horizon $\{r = 2M\}$ is 3D (because it is part of a straight-line in the (t, r) plane), whereas the regions $\{r = 2M, |t| = \infty\}$ are only 2D (because they are just points there). In order to prove the first part of the above suspicion, let us compute the 3D "volume" of a generic $\{r = 2M, |t| < \infty\}$ region:

$$V\{r = 2M, |t| < \infty\} = \int_{r=2M,\,|t|<\infty} (-\underbrace{g_{tt}}_{=0}\, g_{\theta\theta} g_{\phi\phi})^{1/2}\, \mathrm{d}t\mathrm{d}\theta\mathrm{d}\phi = 0\,.$$

The second part of the suspicion is also well justified, but will only be confirmed clearly within the analytic extension of the metric.

In Schwarzschild coordinates, the manifold is not geodesically maximal

The following observation will be crucial for our plan of analytic extension. Let us start by definitions, however:

- A manifold is said to be **geodesically complete** if every its geodesic can be extended to infinite values of the affine parameter. Practically, geodesically complete is the manifold which does not contain space-time singularities. Clearly the geodesic completeness is an invariant, coordinate-independent notion.

- A manifold is said to be **geodesically maximal** if every its geodesic can either be extended to infinite values of the affine parameter or it ends at a space-time singularity. Loosely speaking, geodesically maximal is the manifold in which no geodesic ends "without good reason". Typically, geodesic non-maximality indicates that the manifold is not fully covered by a given coordinate patch – some geodesics then simply end (at finite values of their affine parameters) because the coordinate coverage ends (while the manifold itself continues). The geodesic maximality is thus a coordinate-dependent notion. When seeing that a manifold is not geodesically maximal, one naturally asks about "better" coordinates to cover the whole manifold (or at least the part which in the original coordinates was not covered).

The Schwarzschild manifold is *of course* not geodesically complete since it does contain a singularity ($r = 0$). Actually, we explicitly treated the radial geodesic fall and saw that the singularity is reached in finite value of the proper time. However, in Schwarzschild coordinates the manifold is neither geodesically maximal. Actually, imagine any geodesic $t = \mathrm{const}$ in the region $r > 2M$. Such a geodesic cannot be extended below $r = 2M$. Let us think it over in detail:

- Consider a purely radial geodesic, we mean the one whose tangent vector has only radial component in the Schwarzschild coordinates, $\frac{\mathrm{d}x^\mu}{\mathrm{d}p} = \delta^\mu_r \frac{\mathrm{d}r}{\mathrm{d}p}$. The geodesic equation thus reduces to
$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}p^2} + \Gamma^\mu{}_{rr} \left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 = 0\,.$$
The Christoffel symbols
$$\Gamma^\mu{}_{rr} = \frac{1}{2}\, g^{\mu\nu}(g_{\nu r,r} + g_{r\nu,r} - g_{rr,\nu})$$
are clearly zero, except
$$\Gamma^r{}_{rr} = \frac{1}{2}\, g^{rr} g_{rr,r} = \frac{1}{2}\,\frac{g_{rr,r}}{g_{rr}}\ .$$
Normalizing the tangent vector as
$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\frac{\mathrm{d}x^\nu}{\mathrm{d}p} = g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 = 1 \quad\Longrightarrow\quad \frac{\mathrm{d}p}{\mathrm{d}r} = \pm\sqrt{g_{rr}}$$
means that the affine parameter p represents arc length (proper length along the geodesic). Substituting the above with
$$g_{rr,r} = \left(\frac{1}{1-\frac{2M}{r}}\right)_{,r} = \frac{-\frac{2M}{r^2}}{\left(1-\frac{2M}{r}\right)^2} = -\frac{2M}{r^2}\,(g_{rr})^2$$
to the non-trivial, radial component of the geodesic equation, we have
$$\frac{\mathrm{d}^2 r}{\mathrm{d}p^2} = -\Gamma^r{}_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 = -\frac{1}{2}\,\frac{g_{rr,r}}{(g_{rr})^2} = \frac{M}{r^2}\ .$$
Hence, there do exist non-trivial geodesics $t = \mathrm{const}$.

- Since the affine parameter p has the meaning of radial proper length, it does *not* reach infinite value at the horizon as we know from Section 14.1.1.

- Finally, why such a geodesic cannot be extended below $r = 2M$: above that radius, $t = \mathrm{const}$ is a space-like curve, whereas below that radius it would be time-like (remember how the light cones look like). But such a behaviour is impossible for a geodesic. Actually, recall that geodesic is a curve along which its tangent vector transports parallelly. And parallel transport conserves scalar product, so, in particular, it keeps the space-time character of the transported vector.

Needless to say, any geodesic $t = \mathrm{const}$ existing *below* $r = 2M$, similarly, cannot be extended *above* the horizon.

Conclusion: In Schwarzschild coordinates, the Schwarzschild manifold is not geodesically maximal, so these coordinates apparently do not cover the whole manifold.

14.2 Analytic extension of the Schwarzschild metric

Let us summarize first which features we wish to remedy by finding "better" coordinates:

- The metric singularity on the horizon (we have seen this singularity is just a coordinate one).
- Weird behaviour of world-lines in crossing the horizon (passing via $|t|=\infty$).
- Wrong dimensionality of the horizon subsets: $\{r=2M, |t|<\infty\}$ should be 2D, whereas $\{r=2M, |t|=\infty\}$ should likely be 3D.
- Geodesic non-maximality of the manifold (Schwarzschild coordinates almost certainly do not cover the whole manifold).

Several coordinate systems have been found in which metric is not singular on the horizon and in which possibly also some other of the above requirements are satisfied. The best turned out to be the **Kruskal-Szekeres system** in which *all* the above issues are remedied. Below, we go over to that system in several steps. The whole transformation will only concern the time-radial part of the metric, so we will, for simplicity, denote the (untouched) angular part as $\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 =: \mathrm{d}\Omega^2$.

- First the so-called *tortoise radial coordinate* is introduced by

$$\mathrm{d}r^* := g_{rr}\mathrm{d}r \qquad \Longrightarrow \qquad r^* = r + 2M\ln\left|\frac{r}{2M} - 1\right|. \tag{14.15}$$

- The advanced and retarded times (also called Eddington-Finkelstein coordinates) are introduced then by $t^{\pm} := t \pm r^*$. Since

$$\mathrm{d}t^+\mathrm{d}t^- = \mathrm{d}t^2 - (\mathrm{d}r^*)^2 = \mathrm{d}t^2 - (g_{rr})^2\mathrm{d}r^2$$

and $g_{tt}g_{rr}=-1$, the metric now reads

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^+\mathrm{d}t^- + r^2\mathrm{d}\Omega^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^+\mathrm{d}t^- + r^2\mathrm{d}\Omega^2\,. \tag{14.16}$$

The $t^{\pm}$ times are clearly *light (null) coordinates* – $t^{\pm} = \mathrm{const}$ correspond to radially ingoing/outgoing light-like world-lines. (This step is obviously motivated by introduction of $t^{\pm} = t \pm r$ in special relativity, where it brings the metric into the form $\mathrm{d}s^2 = -\mathrm{d}t^+\mathrm{d}t^- + r^2\mathrm{d}\Omega^2$.)

- Crucial step is the introduction of the light Kruskal-Szekeres coordinates,

$$u := -\epsilon\,\exp\left(-\frac{t^-}{4M}\right), \qquad v := \exp\left(\frac{t^+}{4M}\right), \tag{14.17}$$

where $\epsilon := \mathrm{sign}\left(\frac{r}{2M} - 1\right)$ just tells whether one is above or below horizon. Writing out

$$\exp\frac{r^*}{4M} = \exp\left(\frac{r}{4M} + \frac{1}{2}\ln\left|\frac{r}{2M} - 1\right|\right) = \sqrt{\left|\frac{r}{2M} - 1\right|}\;\exp\frac{r}{4M}\;, \tag{14.18}$$

we can express the transformation explicitly,

$$u = -\epsilon\sqrt{\left|\frac{r}{2M}-1\right|}\ \exp\frac{r-t}{4M}\,, \qquad v = \sqrt{\left|\frac{r}{2M}-1\right|}\ \exp\frac{r+t}{4M}\,. \tag{14.19}$$

Let us also prepare

$$\begin{aligned} -uv &= \epsilon\,\exp\frac{t^+-t^-}{4M} = \epsilon\,\exp\frac{r^*}{2M} = \epsilon\,\left|\frac{r}{2M}-1\right|\exp\frac{r}{2M} = \\ &= \left(\frac{r}{2M}-1\right)\exp\frac{r}{2M} = \frac{r}{2M}\left(1-\frac{2M}{r}\right)\exp\frac{r}{2M} \end{aligned} \tag{14.20}$$

which will be useful at several places. The first of them is the computation of

$$\mathrm{d}u\,\mathrm{d}v = \frac{-uv}{16M^2}\,\mathrm{d}t^+\mathrm{d}t^- = \frac{r}{32M^3}\left(1-\frac{2M}{r}\right)\exp\frac{r}{2M}\,\mathrm{d}t^+\mathrm{d}t^- \tag{14.21}$$

which follows easily from (14.17). From it, one expresses $\mathrm{d}t^+\mathrm{d}t^-$ and substitutes into the metric (14.16),

$$\mathrm{d}s^2 = -\frac{32M^3}{r}\,\exp\left(-\frac{r}{2M}\right)\,\mathrm{d}u\,\mathrm{d}v + r^2\mathrm{d}\Omega^2. \tag{14.22}$$

This has been the celebrated moment when the terms $\left(1-\frac{2M}{r}\right)$ just cancel out and thus the horizon singularity completely disappears from the metric.

- Finally, one returns to the "normal"-type coordinates – the Kruskal-Szekeres (V, U):

$$v \equiv V+U, \quad u \equiv V-U \quad \Longrightarrow \quad V = \frac{1}{2}(v+u), \quad U = \frac{1}{2}(v-u)\,. \tag{14.23}$$

Just a trivial change in the metric,

$$\boxed{\mathrm{d}s^2 = \frac{32M^3}{r}\,\exp\left(-\frac{r}{2M}\right)\,(-\mathrm{d}V^2+\mathrm{d}U^2) + r^2\mathrm{d}\Omega^2}\,; \tag{14.24}$$

this is its ultimate form, valid both above and below horizon.[4] Obviously nothing strange happens on the horizon. Since $g_{UU} = -g_{VV} > 0$ everywhere, V and U everywhere have the character of time and radial coordinates, respectively. The final transformation reads, explicitly,

$$V = \sqrt{\frac{r}{2M}-1}\ \exp\frac{r}{4M}\ \sinh\frac{t}{4M}\,, \quad U = \sqrt{\frac{r}{2M}-1}\ \exp\frac{r}{4M}\ \cosh\frac{t}{4M} \tag{14.25}$$

in the region $r > 2M$, while

$$V = \sqrt{1-\frac{r}{2M}}\ \exp\frac{r}{4M}\ \cosh\frac{t}{4M}\,, \quad U = \sqrt{1-\frac{r}{2M}}\ \exp\frac{r}{4M}\ \sinh\frac{t}{4M} \tag{14.26}$$

in the region $r < 2M$.

[4] This "hybrid" form is indeed much more lucid than if we tried to express r in terms of V a U – see below equation (14.27).

- From the above expressions for V a U follows the *inverse transformation*, that is, the relations which determine t and r as functions of V and U:

$$\frac{V}{U} = \begin{cases} \tanh\frac{t}{4M} & \text{for } \; r > 2M \\ \coth\frac{t}{4M} & \text{for } \; r < 2M \end{cases} ,$$

$$U^2 - V^2 = -uv = \left(\frac{r}{2M} - 1\right) \exp\frac{r}{2M} = \epsilon \, \exp\frac{r^*}{2M} \,. \tag{14.27}$$

  This is actually the simplest representation of the whole transformation. Its basic information is that, in the (V, U) plane, $r = \mathrm{const}$ are hyperbolas with asymptotes $V = \pm U$ (which correspond to $r = 2M$) while $t = \mathrm{const}$ are straight lines passing through the origin.

- Attention! So the horizon is represented by *two* diagonals $V = \pm U$, i.e., there are in fact *two* horizons. Similarly, there are *two* outer regions $r > 2M$, given by $U^2 - V^2 > 0$, i.e. $U > |V|$ and $U < -|V|$, and there are also *two* inner regions $r < 2M$; in particular, there are *two* singularities $r = 0$ – these lie where $V^2 = 1 + U^2$, i.e. at $V = \pm\sqrt{1 + U^2}$.

  Therefore, in Kruskal-Szekeres coordinates there appear *two new regions of the manifold* which were not covered by the original Schwarzschild coordinates: the second "outer" region $r > 2M$ and the second "inner" region $r < 2M$. (The two new regions could be covered by a second set of Schwarzschild coordinates, but just one set does not suffice for the whole manifold.)

- The new regions are clearly symmetric about the origin $(V = 0, U = 0)$ with respect to the "old" ones, so it is simple to add how the final transformations look there: they are the same as (14.25), (14.26), just with minuses. In total, one thus covers the manifold by 4 sets of transformations $(t, r) \to (V, U)$. Nicely enough, the inverse transformation (14.27) works everywhere.

- One of the most favourable features of the Kruskal diagram (= space-time diagram in the Kruskal-Szekeres coordinates) is how it represents the causal structure: light cones are everywhere "45°" like in special relativity. This is clear from the metric (14.24) – setting the interval to zero, one obtains

$$\mathrm{d}V^2 = \mathrm{d}U^2 + \frac{r^3}{32M^3} \, \exp\frac{r}{2M} \, \mathrm{d}\Omega^2 \geqslant \mathrm{d}U^2 \qquad \Longrightarrow \qquad \left|\frac{\mathrm{d}V}{\mathrm{d}U}\right| \geqslant 1. \tag{14.28}$$

  Hence, the light cones are 45° or narrower, with 45° applying to purely radial motion.

- Is the resulting metric (14.24) indeed *analytic* in the new coordinates? It is (except the true singularities $r = 0$ of course). Actually, it is much "nicer" than the original, Schwarzschild metric, and it is also much nicer than the transformations we employed to make it nice. Naturally, in order to remove some singularity, one has to perform a transformation which itself is singular. Also "suspicious" might be that we had to cover the manifold by 4 new coordinate maps and match them along the horizons. However, since everything suspicious has been related to the horizons and no other location, it is sufficient to check the analyticity there – and that holds.

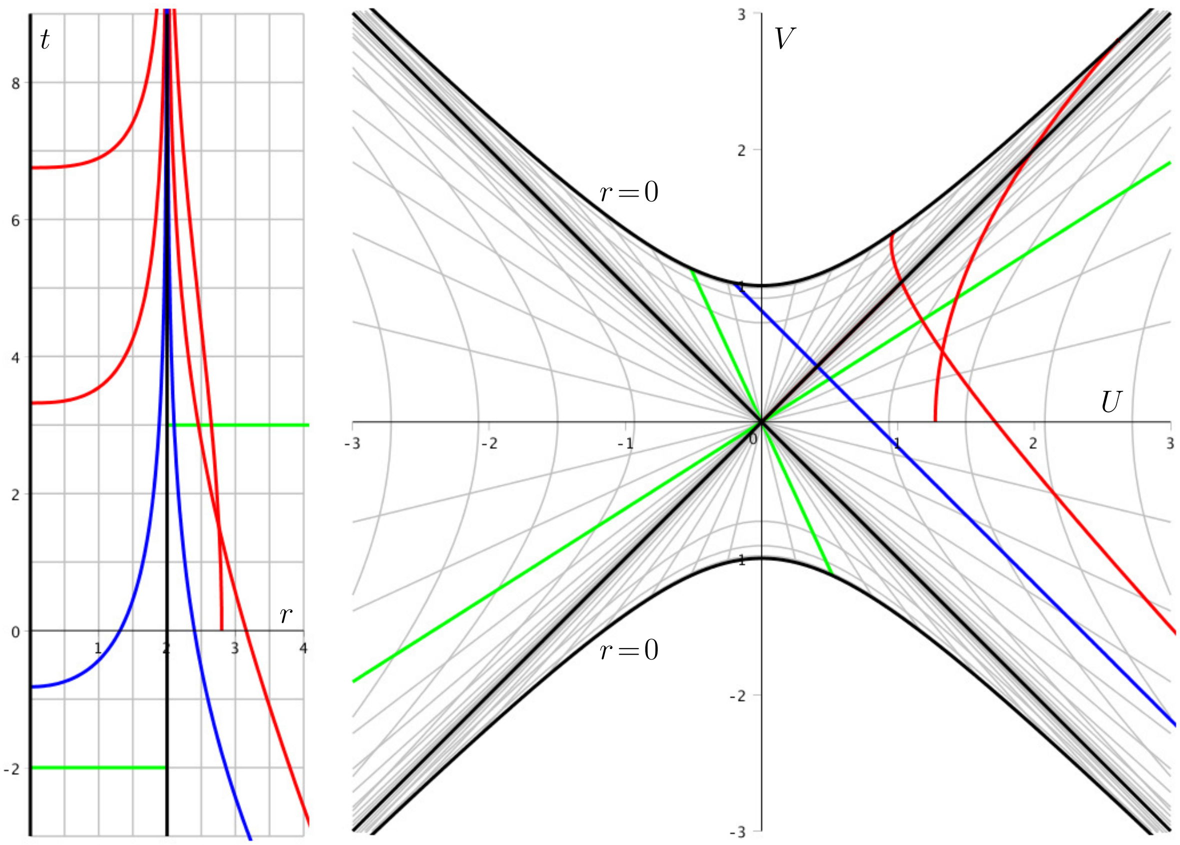


Figure 14.2 *Left:* Passage through the horizon of three selected geodesics, plotted in Schwarzschild coordinates (axes are in the units of M). Red are two massive particles – one is freely falling from rest from ($t=0$, $r_{\rm in}=2.8M$); it hits $r=0$ at $t\doteq 6.76M$. Shortly after start, its trajectory is crossed by a massive particle freely falling from a large radius; the latter ends at $r=0$ at $t\doteq 3.32M$. Blue trajectory belongs to a radial photon; this ends at $r=0$ at $t\doteq -0.82M$. The horizon $r=2M$ and the singularity $r=0$ are indicated in bold. Also drawn (in green) are two geodesics $t=\mathrm{const}$, one ($t=3M$) above the horizon and the other ($t=-2M$) below; in Schwarzschild coordinates, neither of them can be extended over the horizon. *Right:* The same geodesics plotted in the Kruskal-Szekeres coordinates; they are easily identifiable. Bold are horizons on diagonals $V=\pm U$ and singularities on hyperbolas $V=\pm\sqrt{1+U^2}$. The Schwarzschild-mesh structure follows from the inverse transformation (14.27): $t=\mathrm{const}$ are straight lines intersecting the origin, with $t=0$ on the axes while $t=\pm\infty$ on the horizons $V=\pm U$; between axes and horizons t increases or decreases monotonously. It is clear now why the (green) geodesics $t=\mathrm{const}$ could not be extended over the horizon: they enter there the second half of the manifold which is not covered by the given Schwarzschild coordinates. The mesh of $r=\mathrm{const}$ is represented by hyperbolas; specifically, $r=0.5M$, $1M$, $1.5M$, ... are shown. The main weird feature of the left diagram – that almost all ingoing geodesics cross the horizon at $t=\infty$ – is "corrected" in the Kruskal diagram, thanks to the fact that the "points" ($|t|=\infty, r=2M$) have been stretched to the whole diagonals $V=\pm U$, whereas the "straight lines" ($|t|<\infty, r=2M$) have been compressed to the origin $U=V=0$. In the Kruskal-Szekeres coordinates the Schwarzschild manifold is geodesically maximal.

14.2.1 Kruskal diagram and Penrose-Carter conformal diagram

The Kruskal-Szekeres coordinates satisfy all the points we wished. Metric is regular at the horizon, and the dimensionality of the regions $\{|t| < \infty, r = 2M\}$ and $\{|t| = \infty, r = 2M\}$ is just opposite to how it was in Schwarzschild coordinates. Actually, $|t| = \infty$ is stretched to the whole diagonals $V^2 = U^2$, while $|t| = \mathrm{const} < \infty$ lines only intersect the origin of the diagram. Thanks to this, also the world-lines which cross the horizons (at infinite times) are rendered in a natural and smooth way. And, in the new coordinates the manifold is geodesically maximal – all geodesics which do not end or start at singularities are infinite; in particular, the geodesics $t = \mathrm{const}$ can now be prolonged across the horizon (while it is seen that in the Schwarzschild map this was simply not possible because the regions which such geodesics enter are not covered by the Schwarzschild coordinates).

In addition to this, there is the convenient property of the everywhere "$45°$" form of the radial-motion light cones. This makes causal arguments very easy. Particularly clear is the null character of the horizons and the space-like character of the singularities. Through the future horizon (that at $V > 0$), causal motions can only cross in the "inward" sense, and there they have no other option than to fall towards the future singularity and to end there. On the contrary, the past horizon (the one at $V < 0$) can only be crossed in the "outward" direction, and all the causal motions which do so must have started at the past singularity. The region below the past horizon, representing thus causal reverse of the black hole, is often called the **white hole**. Note that the two domains of outer communications ($r > 2M$) are causally disconnected – the causal world-lines starting on opposite sides ($U > 0$ and $U < 0$) can only meet inside the future black hole.

The Kruskal diagram of the Schwarzschild manifold is shown in Figure 14.2. For illustration, three ingoing geodesics are drawn and their course compared with how they appear in the Schwarzschild map covering the "top right" half of the manifold.

The Kruskal diagram only has one serious disadvantage: it is infinite. Hence totally unsuitable for discussion of global properties of the manifold, for the behaviour of things at infinities. Exactly one week ago (with respect to when I am writing this part), R. Penrose was awarded the 2020' Nobel prize for physics. One of the real gurus of mathematical relativity and geometry. In 1962, he proposed "a new technique for studying asymptotic questions in (special or) general relativity" ... a conformal transformation after which the manifold's infinity becomes a three-dimensional boundary to a finite conformal region. Conformal mappings (naturally) do not preserve lengths, but they do preserve angles, hence, in the conformal space-time, the light cones remain the same as in the original one. In the case of the Kruskal picture of the Schwarzschild manifold, the conformal transformation reads

$$\begin{aligned}\psi &= \arctan(V+U) + \arctan(V-U) = \arctan v + \arctan u\,,\\ \xi &= \arctan(V+U) - \arctan(V-U) = \arctan v - \arctan u\,,\end{aligned} \tag{14.29}$$

shrinking the infinite extent of (u, v) to $-\pi < \xi < +\pi$, $-\pi/2 < \psi < +\pi/2$. How these values arise: first, according to the inverse transformation (14.27), the singularities $r = 0$ correspond to $uv = 1$, with $u = \pm\exp(-t/4M)$ and $v = \pm\exp(+t/4M) = 1/u$ (for future/past singularity). And there exists the formula $\arctan(1/x) = \pm\pi/2 - \arctan(x)$, valid for $x > 0$ and $x < 0$, respectively. Hence, the singularities occur at $\psi = \pm\pi/2$. Second, consider that the

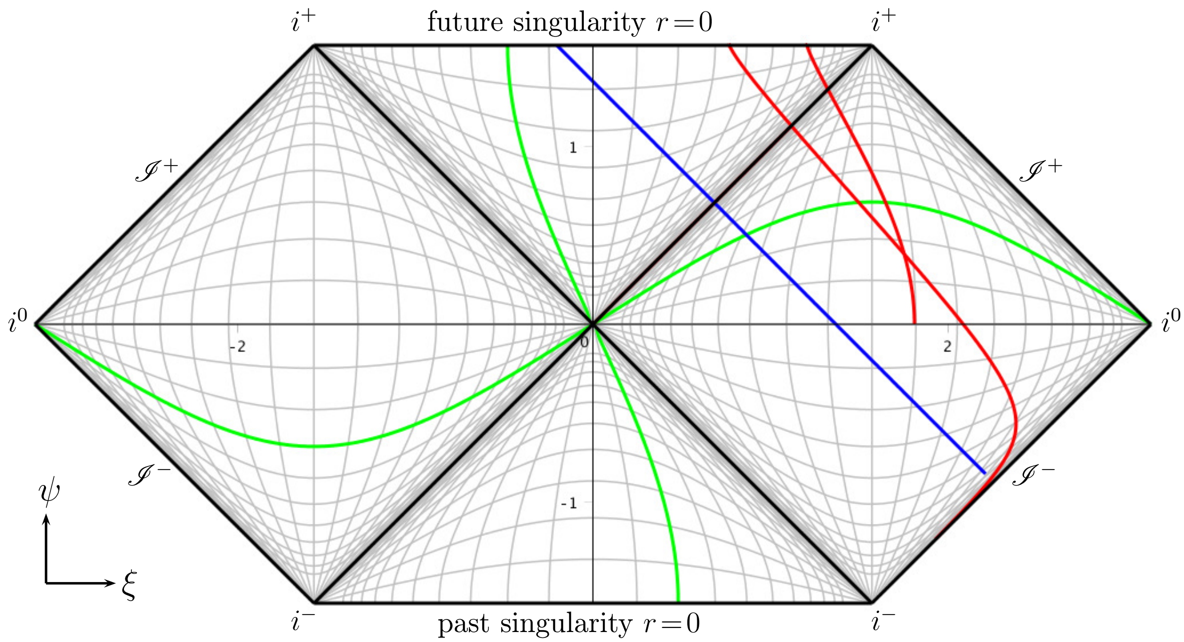


Figure 14.3 *Penrose-Carter diagram of the Schwarzschild space-time. The figure corresponds to the Kruskal diagram 14.2, with the same three geodesics depicted as there (a massive particle falling from rest from $r_{\text{in}} = 2.8M$, $t = 0$, a massive particle falling from infinity, and a radially ingoing photon). Thanks to their null character, the horizons remained on diagonals, and the space-time is "covered" by singularities. Again in grey is the Schwarzschild (t, r)-mesh, specifically, we have drawn the lines $t = -10M$, $-9M$, $-8M$, $\ldots$, $8M$, $9M$, $10M$ (with $t = 3M$ and $t = -2M$ emphasized again) and the "dual" lines $r^* = -10M$, $-9M$, $-8M$, $\ldots$, $8M$, $9M$, $10M$. Future/past time infinities are denoted by $i^\pm$, radial infinities by i^0, and future/past light infinities by $\mathscr{I}^\pm$. The diagram covers the intervals $(-\pi, +\pi)$ on the horizontal axis ξ and $(-\pi/2, +\pi/2)$ on the vertical axis. The time infinities are singular points of the conformal transformation.*

left and right $\mathscr{I}^+$ (see Figure 14.3) are represented by $\psi \mp \xi = \pi$, i.e. by $\arctan u = \pi/2$ and $\arctan v = \pi/2$, respectively. These really correspond to $u \to \infty$ and $v \to \infty$, respectively. And similarly for $\mathscr{I}^-$.

The inverse conformal transformation being

$$V = \frac{\sin\psi}{\cos\psi + \cos\xi}\,, \qquad U = \frac{\sin\xi}{\cos\psi + \cos\xi}\,, \tag{14.30}$$

one can substitute to (14.27)

$$U^2 - V^2 = -uv = \frac{\cos\psi - \cos\xi}{\cos\psi + \cos\xi}\,, \qquad \frac{V}{U} = \frac{\sin\psi}{\sin\xi} \tag{14.31}$$

and obtain simple relations for the shape of the original Schwarzschild coordinates in the conformal (ψ, ξ) plot:

$$\frac{\cos\psi}{\cos\xi} = \begin{cases} -\coth\frac{r^*}{4M} \\ -\tanh\frac{r^*}{4M} \end{cases}, \qquad \frac{\sin\psi}{\sin\xi} = \begin{cases} \tanh\frac{t}{4M} & \text{for} \quad r > 2M \\ \coth\frac{t}{4M} & \text{for} \quad r < 2M \end{cases}. \tag{14.32}$$

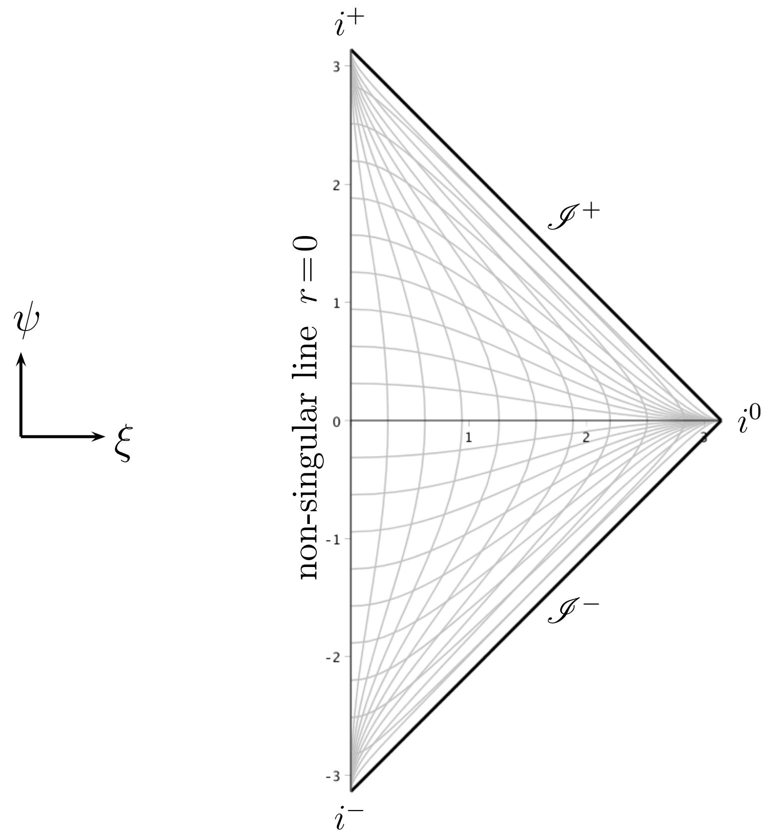


Figure 14.4 *Penrose-Carter diagram of the flat space-time. The t, r mesh is grey, with $t/M = -\tan(9\pi/20)$, $-\tan(8\pi/20)$, $-\tan(7\pi/20)$, $\ldots$, $\tan(7\pi/20)$, $\tan(8\pi/20)$, $\tan(9\pi/20)$ (rather "horizontal" lines) and, similarly, $r/M = \tan(1\pi/20)$, $\tan(2\pi/20)$, $\tan(3\pi/20)$, $\ldots$, $\tan(9\pi/20)$ (rather "vertical" lines) shown. Light cones are everywhere 45° again. The history of the origin $r=0$ is time-like here and it is not singular of course. Horizontal axis goes from 0 to π, vertical axis goes from $-\pi$ to $+\pi$.*

And one may check that the light cones really stay 45° by computing

$$-\mathrm{d}V^2 + \mathrm{d}U^2 = -\mathrm{d}u\,\mathrm{d}v = \frac{-\mathrm{d}\psi^2 + \mathrm{d}\xi^2}{(\cos\psi + \cos\xi)^2}\,. \tag{14.33}$$

Conformal diagram of the Schwarzschild manifold is shown in Figure 14.3. Of all the graphs, it obviously is the most clear, and the illustration geodesics have in it very calm shape. We see the asymptotic regions are of three types: future/past **time infinities** $i^\pm$, **spatial (radial) infinities** and future/past **light (null) infinities** $\mathscr{I}^\pm$:

$$\begin{aligned}
i^\pm &= \{t \to \pm\infty,\, r \text{ finite},\, \theta,\, \phi\}\,,\\
i^0 &= \{t \text{ finite},\, r \to \infty,\, \theta,\, \phi\}\,,\\
\mathscr{I}^\pm &= \{t \pm r \to \pm\infty,\, t \mp r \text{ finite},\, \theta,\, \phi\}\,.
\end{aligned} \tag{14.34}$$

Since being denoted by “script I”, the light infinities are commonly called just *“scri”*; in every discussion with a mathematical relativist, you should make this sound at least once. On the diagram it can also be seen that

- at i^+, all time-like world-lines end, *except* (i) those which enter the future black hole, and necessarily end at the future singularity, and (ii) those which are accelerated in the radially outgoing direction as far as $t \to \infty$ – those end at $\mathscr{I}^+$ (generalization of the hyperbolic motion from special relativity)
- at $\mathscr{I}^+$ – besides the outgoing “hyperbolic motions” – end outgoing photons, *except* (i) those which enter the future black hole and thus end at the future singularity, (ii) generators of the future horizon, and (iii) photons on circular orbit at $r = 3M$ (those end at i^+)
- for i^- and $\mathscr{I}^-$ naturally hold time-inversions of the above
- at i^0 end the hypersurfaces $t = \mathrm{const}$, except those which stretch between the singularities.

It may be illustrative to compare the conformal diagram of Schwarzschild with that of Minkowski. In flat space-time, the counterpart of Schwarzschild coordinates are spherical coordinates in which the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

They cover *the whole* manifold (in them, flat space-time is geodesically maximal – and it is even complete, needless to say), so it is right away possible to compactify (rather than to introduce Kruskal-type coordinates first),

$$\begin{aligned} \psi &= \arctan\left(\frac{t+r}{M}\right) + \arctan\left(\frac{t-r}{M}\right) = \arctan\frac{t^+}{M} + \arctan\frac{t^-}{M}\,, \\ \xi &= \arctan\left(\frac{t+r}{M}\right) - \arctan\left(\frac{t-r}{M}\right) = \arctan\frac{t^+}{M} - \arctan\frac{t^-}{M}\,, \end{aligned} \tag{14.35}$$

where M does not represent any mass, it is just *some* constant of the dimension of length/time which we put there in order to make the arguments dimensionless. The conformal diagram of the flat space-time is in Figure 14.4.

Bifurcate horizon

For obvious reason, such an arrangement of horizons as in the above diagrams is called the **bifurcate horizon**. The origin of the diagram – the vertex of the horizon light-cone – represents a two-sphere which is called the **bifurcation two-sphere**. Its basic characteristic property can be seen by computing how the time Killing vector field $t^\mu = \partial x^\mu/\partial t$ looks in the (V, U) coordinates. Differentiating (14.25) and (14.26) accordingly, one obtains the same result both above and below the horizons,

$$t^V = \frac{\partial V}{\partial t} = \frac{U}{4M}\,, \qquad t^U = \frac{\partial U}{\partial t} = \frac{V}{4M}\,. \tag{14.36}$$

It implies *vanishing* of the Killing field at the bifurcation sphere $(V = 0, U = 0)$.

14.2.2 Dynamics of the Schwarzschild manifold in the time coordinate V

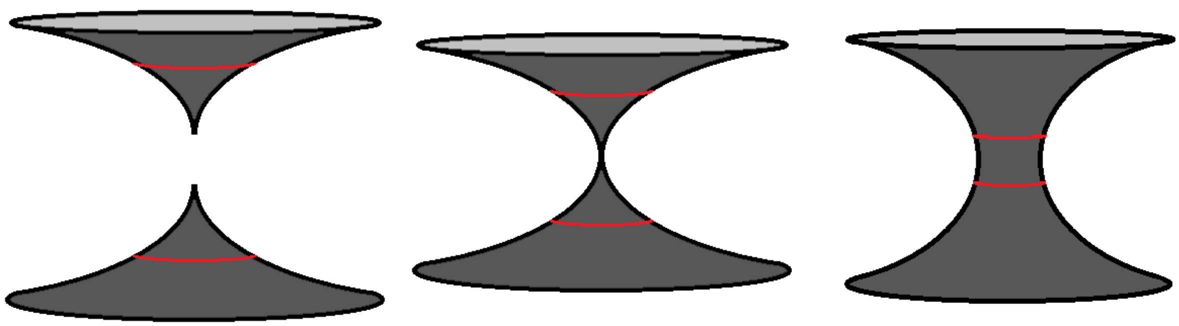


Figure 14.5 *A qualitative sketch of embeddings of the surfaces* $\{V = \mathrm{const}, \theta = \pi/2\}$*, for* $|V| > 1$ *(left),* $|V| = 1$ *(middle) and* $|V| < 1$ *(right). With* V *growing from* $V < -1$ *to* $V > +1$*, the geometry changes from the left to the right surface and back, that is, the two asymptotically flat regions interconnect via a throat for a certain (short) time. Red are horizons.*

The Schwarzschild manifold is stationary (even static) above the horizon and dynamical below. This is an invariant fact – the Killing field $t = \partial x^\mu/\partial t$ is time-like above the horizon and space-like below. This fact is very well reflected in Schwarzschild coordinates, simply because the time t just corresponds to the parameter of this asymptotically time-like symmetry (it is the "Killing time"). However, we have just finished the chapter where we showed that there exist coordinates which in many respects are more natural than the Schwarzschild ones. In the Kruskal time V, the metric is *not* stationary (it depends on r which in turn depends on both V and U). It is thus worth to learn how the space-time *evolves* in terms of this time.

It is natural to follow, in particular, how the geometry evolves of the equatorial plane $\{V = \mathrm{const}, \theta = \pi/2\}$. Imagine such a section on the Kruskal diagram (see Figure 14.5 for a scheme): if $V < -1$, the section consists of two disconnected identical regions, each having a singularity and a horizon at its centre and an asymptotically flat region around. At $V = -1$, these two regions touch at their singularities. With V growing above -1, the now common central singularity opens into a regular funnel, on which the two horizons are getting closer and finally pass through each other at $V = 0$. At this moment, the embedding exactly corresponds to the embedding of the equatorial plane $\{t = \mathrm{const}(= 0), \theta = \pi/2\}$ we studied in Section 14.1.2. For V growing above zero, the geometry returns back symmetrically, i.e. the funnel shrinks to "pinch off" at $V = +1$ and two disconnected regions are left again. The temporary funnel between the two asymptotically flat regions is called the **Schwarzschild throat**, the **Einstein-Rosen bridge** or – for media even better – a **wormhole**. Since (some) people like to cross bridges, the question may arise whether it is possible to cross this one. Clearly no, since one would have to pass through two horizons on the way. The situation is best seen on the Kruskal diagram: it is not possible to travel between the two domains of outer communications ($r > 2M$), the marginal case being that of photon generators of the horizons which can travel from the past to the future horizons through the bifurcate horizon at the origin.

Worth to notice, however, that the absolute futures of the two static Schwarzschildian regions do have non-empty intersection, namely the black-hole interior. The author of [31] writes, as an illustration, that astronauts who fall there from the two universes, "can meet, embrace, and die together". Well, good to first recall the expression $\tau_{\rm fall} = \pi\sqrt{r_{\rm in}^3/(8M)}$ which we found for the proper time of free fall from $r_{\rm in}$ to $r=0$. Writing it in standard units (with $r_{\rm S} \equiv 2GM/c^2$),

$$\tau_{\rm fall} = \frac{\pi}{2}\,\frac{r_{\rm in}}{c}\sqrt{\frac{r_{\rm in}}{r_{\rm S}}} = \frac{\pi}{2}\,\frac{r_{\rm S}}{c}\left(\frac{r_{\rm in}}{r_{\rm S}}\right)^{3/2},$$

we have for $r_{\rm in} = r_{\rm S}$

$$\tau_{\rm fall} = \frac{\pi}{2}\,\frac{r_{\rm S}}{c} \doteq (1.55\cdot 10^{-5}\,\mathtt{s})\,\frac{M}{M_\odot}\,.$$

So do not hope for any cool trip in stellar black holes (with M of maximally $100M_\odot$). Even the black hole in the nucleus of our Galaxy ($M \simeq 4.1\cdot 10^6 M_\odot$) does not offer more than about a minute of free fall. The only reasonable option are still more massive holes. The one in the nucleus of the galaxy M87, famous for its "silhouette" revealed, in 2019, by the Event Horizon Telescope project, has $M \simeq 6.5\cdot 10^9 M_\odot$, and that already provides 28 hours. For the most massive black holes known (which appear to have almost 10 times more mass than the one in M87), the free-fall time comes out about 10 days. This already seems to be enough for a decent vacation, but only in case one has a very good connection to the site.[5]

One may add that supermassive black holes are also (perhaps *mainly*) much more "favourable" from the point of view of tidal forces (curvature). Actually, the Kretschmann scalar $K^2 := R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{48M^2}{r^6}$ yields $K_{\rm H}^2 = \frac{3}{4M^4}$ on the Schwarzschild horizon, which after making square root and in standard units yields

$$K_{\rm H} = \frac{2\sqrt{3}}{r_{\rm S}^2} \doteq 3.35\cdot 10^{15} K_\oplus \left(\frac{M_\odot}{M}\right)^2,$$

where we have denoted by $K_\oplus$ the value of the Kretschmann invariant on the Earth surface,

$$K_\oplus := \frac{4\sqrt{3}\,GM_\oplus}{c^2 R_\oplus^3} \doteq 1.2\cdot 10^{-22}/\mathtt{m}^2\,.$$

A human body is said to be able to withstand, temporarily, as much as $10^{8\div 10}K_\oplus$. Hence, the above formula for $K_{\rm H}$ is a good reason to keep at distance from stellar-mass black holes (and from *all* neutron stars). A "habitable" near-horizon zones start from a black-hole mass of about $1000M_\odot$. For the black hole at our Galactic centre, $K_{\rm H} \doteq 200K_\oplus$, which is totally comfortable, and for the black hole in M87 the horizon field is even much more homogeneous than that on the Earth surface, $K_{\rm H} \doteq K_\oplus/12600$. Anyway, before any voyage to a horizon, we recommend to study section 32.6. of [29].

[5] In passing, one sees in the Kruskal diagram that in order to meet and embrace (otherwise than by pure chance), the astronauts would have to make appointment. However, they can only establish a causal connection (thus get to know of each other) inside the black hole – and only if they entered the hole at sufficiently close values of U and V. Think it over!

14.2.3 Physical sense of the extended Schwarzschild solution?

The new half of the Schwarzschild manifold we discovered in Kruskal-Szekeres coordinates – the one with the "white hole" – can be regarded as exact causal inversion of the original half (containing a black hole). Such an extension might have been expected on the basis of time symmetry of the Einstein equations. Actually, it can be understood as an analogy of the coexistence of retarded and advanced solutions of Maxwell equations (which do not prefer any specific direction of time either). Both kind of solutions are mathematically equally justified, though only the retarded ones are being used to describe physical situations, because the advanced solutions would require very special initial/boundary conditions (incoming waves would have to be arranged in exactly such a manner to be just absorbed by their "sources"). The time asymmetry is – to a *mathematically* symmetrical situation – brought by world-lines of physical observers: their clocks fix a privileged time direction.

The Schwarzschild solution is very robust, it fully follows from spherical symmetry and from Einstein equations with $\Lambda = 0$ and $T_{\mu\nu} = 0$. It surely cannot represent the real Universe, but it can very well approximate the gravitational field in a certain region around a roughly spherical star. However, the maximally extended Schwarzschild manifold certainly does not have (astro)physical sense, because it is vacuum *everywhere*. Even if there was just one "star" in the universe, which would live from i^- and would collapse to a black hole at certain stage, the only piece of the Kruskal diagram which would be present would be a part of one outer region $r > 2M$ and part of the adjacent black-hole region $r < 2M$. Actually, the surface of such a star would first follow some of the $r = \mathrm{const}$ hyperbolas, and then, during the collapse, its world-line would become similar to the world-line of the particle freely falling to the black hole – see Figures 14.2 and 14.3. From the whole Schwarzschild manifold, the only "realized" would be the region to the right of the surface's world-line; this does not contain anything of the new quadrants opened in the Kruskal-Szekeres coordinates, and especially it does not contain the white hole.

Finally, there also exist more subtle reasons why the maximal Schwarzschild manifold is not (astro)physically relevant, for instance, an instability against pair creation in the extremely non-homogeneous field near the past singularity. The generated particles would have to escape from the white hole, either to $\mathscr{I}^+$, to i^+ or to the black hole, so the white hole would gradually radiate away. Needless to say, the particle generation itself would break the assumption of $T_{\mu\nu} = 0$.

Chapter 15

Reissner-Nordström solution of Einstein equations

Soon after the Schwarzschild solution, in 1916 (Reissner) and 1918 (Nordström), its electrically charged extension has been found. The generalization is simple and the metric has the same shape, but still it is worth treating in detail, at least for the following reasons:

- First, due to the charge, the solution also involves EM field. Should the solution be exact, one has to admit that the EM field enters the problem as a source of gravity, through the corresponding energy-momentum tensor. This means that the solution can no longer be vacuum, one has to solve *non-homogeneous* Einstein equations. Still more importantly, besides Einstein equations, one also has to include in the problem the equations governing the "non-gravitational physics" of the given source – in our case, the Maxwell equations. The problem thus becomes more involved, because one has to tackle a *coupled* set of Einstein and Maxwell equations.

- Second reason concerns the global structure of the solution. Similarly as in the Schwarzschild case, the Reissner-Nordström metric describes the gravitational and EM fields outside *any* spherically symmetric massive and charged source. In such a case, the difference from Schwarzschild is typically just tiny, because, as we will see, the effect of charge falls off with distance faster than that of mass. However, if speaking about situation when the *whole* space-time is electro-vacuum ("the black-hole case"), it is no longer true, because the structure of the innermost parts of the Reissner-Nordström solution turns out to be *considerably* different from that of the Schwarzschild black hole. This follows from the fact that Reissner-Nordström in general contains *two* horizons.

15.1 Metric and interpretation of parameters

The Reissner-Nordström solution is an exact solution of Einstein and Maxwell equations describing the gravitational and EM fields outside a spherically symmetric (and static) source.

More accurately, one assumes that cosmological constant is zero, that the only source of gravity is the EM field (note that it should describe the *exterior* of a massive charged body), and that the EM field is source-free ($J^\mu = 0$). In such a case, the system of equations reads[1]

$$R_{\mu\nu}(= G_{\mu\nu}) = 8\pi T_{\mu\nu}\,, \qquad T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}\,g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{15.1}$$

$$F^{\mu\nu}{}_{;\nu} = 0, \qquad F_{\{\mu\nu;\rho\}} = 0\,. \tag{15.2}$$

Regarding that the electric field is a source field, whereas magnetic field is a vortex field, the only EM field compatible with spherical symmetry is the one having just radial electric component. Namely, even if there existed a spherically symmetric distribution of current (current outflow or inflow), the magnetic field would be zero. Starting from the Schwarzschild-type coordinates (t, r, θ, ϕ) again, we have the spherically symmetric metric

$$\mathrm{d}s^2 = g_{tt}(r)\mathrm{d}t^2 + g_{rr}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{15.3}$$

(if already supposing independence of time on the basis of experience with Schwarzschild), and the EM-field tensor together with the corresponding energy-momentum tensor read

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad\Longrightarrow\quad 8\pi T^t{}_t = E^2 g^{tt} g^{rr} = \frac{E^2}{g_{tt}g_{rr}} = 8\pi T^r{}_r\,. \tag{15.4}$$

Two non-trivial field equations have $G^t{}_t$ and $G^r{}_r$ on the left-hand sides as in (12.5), and the above-given $8\pi T^t{}_t = 8\pi T^r{}_r$ on the right-hand sides, so they read

$$\frac{\mathrm{d}g_{rr}}{\mathrm{d}r} = -\frac{g_{rr}}{r}\left(g_{rr} - 1 + \frac{E^2 r^2}{g_{tt}}\right), \qquad \frac{\mathrm{d}g_{tt}}{\mathrm{d}r} = \frac{g_{tt}}{r}\left(g_{rr} - 1 + \frac{E^2 r^2}{g_{tt}}\right). \tag{15.5}$$

Good to comment more on the assumption of stationarity: in the Schwarzschild case, stationarity arose from the third independent equation $G_{tr} = 0$. This equation remains the same here, because $T_{tr} = 0$ for the EM field. So stationarity would follow in the same manner as in Schwarzschild, it is *not* an extra assumption.

By dividing the two field equations, we have

$$\frac{\mathrm{d}g_{tt}}{\mathrm{d}g_{rr}} = -\frac{g_{tt}}{g_{rr}} \quad\Longrightarrow\quad \ln|g_{tt}g_{rr}| = \mathrm{const} \quad\Longrightarrow\quad g_{tt}g_{rr} = \mathrm{const}'\,. \tag{15.6}$$

We will now use this in the first set of Maxwell equations. The latter can be written in terms of partial divergence according to (5.17) (for an anti-symmetric tensor),

$$0 = F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,F^{\mu\nu}\right)_{,\nu} + \cancel{\Gamma^\mu_{\rho\sigma}F^{\rho\sigma}}\,,$$

[1] Remember that $T = 0$ for the EM field, so $R = 0$ as well, and thus $R_{\mu\nu} = G_{\mu\nu}$.

so for $\mu = 0 \equiv t$ we have

$$0 = \left(\sqrt{-g}\, F^{t\nu}\right)_{,\nu} = \left(\sqrt{-g}\, F^{tr}\right)_{,r} = \left(\frac{\sqrt{-g}}{g_{tt}g_{rr}}\, F_{tr}\right)_{,r} = \left(\frac{\sqrt{-g_{tt}g_{rr}}\, r^2 \sin\theta}{-g_{tt}g_{rr}}\, E\right)_{,r} =$$
$$= \left(\frac{r^2 \sin\theta}{\sqrt{-g_{tt}g_{rr}}}\, E\right)_{,r} = \frac{\sin\theta}{\sqrt{-g_{tt}g_{rr}}}\left(r^2 E\right)_{,r} \quad\Longrightarrow\quad Er^2 = \text{const} =: Q\,. \tag{15.7}$$

Indeed, Er^2 of course cannot depend on θ or ϕ, and neither on t since E is "dynamical" (it is coupled with the metric, which however is static); one can also see the latter explicitly by differentiating either of (15.5) by t. Consider that the constant really represents the electric charge, because r is the *area radius*, linked to the area of the $r = \text{const}$ sphere by $4\pi r^2$ – and that is exactly what the Gauss law should involve when computing the flux of $\vec{E}$ across such a sphere. (In particular, the interpretation would *not* work if r represented proper radius.)

Equations (15.5) are solved by the two-parameter (M, Q) family of metrics

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{15.8}$$

or in physical units: $\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2}} + r^2\mathrm{d}\Omega^2$.

15.1.1 Basic features of the Reissner-Nordström metric

- Schwarzschildian limit: for an uncharged source ($Q = 0$), the Reissner-Nordström metric reduces to the Schwarzschild one.

- Asymptotic behaviour: at large radial distances ($r^2 \gg Q^2$), the metric (15.8) first goes over to the Schwarzschild one and then, still farther away, to the flat metric. M thus represents mass of the source again.

- The metric is again static (stationary and diagonal).

- The coordinates (t, r, θ, ϕ) have the same meaning as in the Schwarzschild case.

- Three singularities occur in the metric – one truly physical and two coordinate ones. With experience from Schwarzschild, one guesses $r = 0$ to be the physical one, and really the Kretschmann invariant confirms that,

  $$R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu} = \frac{8}{r^8}\left(6M^2r^2 - 12MQ^2r + 7Q^4\right). \tag{15.9}$$

  Interestingly, with $M = 0$ while $Q \neq 0$, the singularity is even stronger (than in the opposite, Schwarzschild limit).

  Two horizons are present, as given by the coordinate singularities at

  $$\Delta := r^2 - 2Mr + Q^2 = 0 \quad\Longrightarrow\quad r = r_\pm = M \pm \sqrt{M^2 - Q^2}\,. \tag{15.10}$$

They both represent one-way membranes as it will be clear from causal structure. The normal to $r = \mathrm{const}$ hypersurfaces, $\frac{\partial r}{\partial x^\mu}$, turns light-like on the horizons, so the latter are themselves null again. The Killing field $t^\mu = \frac{\partial x^\mu}{\partial t}$ also becomes light-like on the horizons, $g_{\mu\nu}t^\mu t^\nu = g_{tt} \stackrel{r\to r_\pm}{\longrightarrow} 0$, so $r_\pm$ are Killing horizons – and they are static limits as well (four-velocity of a static congruence becomes light-like there, since it is proportional to t^μ). Finally, the horizons again represent infinite-redshift surfaces: between a static observer close to the horizon and another one at $r_{\rm B} > r_+$, the frequency ratio amounts to

$$\frac{\nu(r_{\rm A})}{\nu(r_{\rm B} > r_+)} = \sqrt{\frac{-g_{00}(r_{\rm B} > r_+)}{-g_{00}(r_{\rm A})}} \quad \stackrel{r_{\rm A}\to r_+}{\longrightarrow} \quad \infty. \tag{15.11}$$

[When speaking of singularities, note once more that the source might have bigger radius than r_+, so neither of the singularities need be present. However, we are again mainly interested in the limit case of "point" source when the central region considerably differs from flat space-time.]

- Clearly the meaning of the coordinates t and r switches at the horizons, which implies that the metric is dynamical *between the horizons* (*not* in the whole region below the outer horizon).

- Look at the relation (15.10) once more: if the Schwarzschild centre were gradually being charged from zero, the outer horizon would go down from $r = 2M$, while the inner horizon would rise from $r = 0$. When the charge reaches the value $|Q| = M$, the horizons coalesce on $r = M$; such a case with double-degenerate horizon is called the **extreme black hole**. If the charge rose still more, it would not be possible to satisfy $\Delta = 0$ any more – this function would be everywhere positive and there would be no horizons. In such a case, the centre at $r = 0$ is called the **naked singularity** (it is not "dressed" in horizons).

- Last but not least, light cones generated by purely radial photon world-lines:

$$0 = \mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \quad \Longrightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} \,.$$

The behaviour of radial light cones is clear from Figure 15.1 where the network of radial null world-lines is plotted. (Light cones delimited by non-radial photons are "narrower" as usual.) Compared to the Schwarzschild space-time, the main novelty is the surprising behaviour below the inner horizon: there, in contrast to the dynamical region *between* the horizons, the causal future points in the direction of t back again, so the future-oriented causal world-lines *need not* necessarily approach the central singularity, they may even be radially outgoing there (up to the inner horizon). In particular, the Reissner-Nordström *singularity is time-like* in contrast to the Schwarzschild one! Such a circumstance is always alarming since the time-like singularity can communicate with observers in its vicinity (here with observers in the region $r < r_-$).

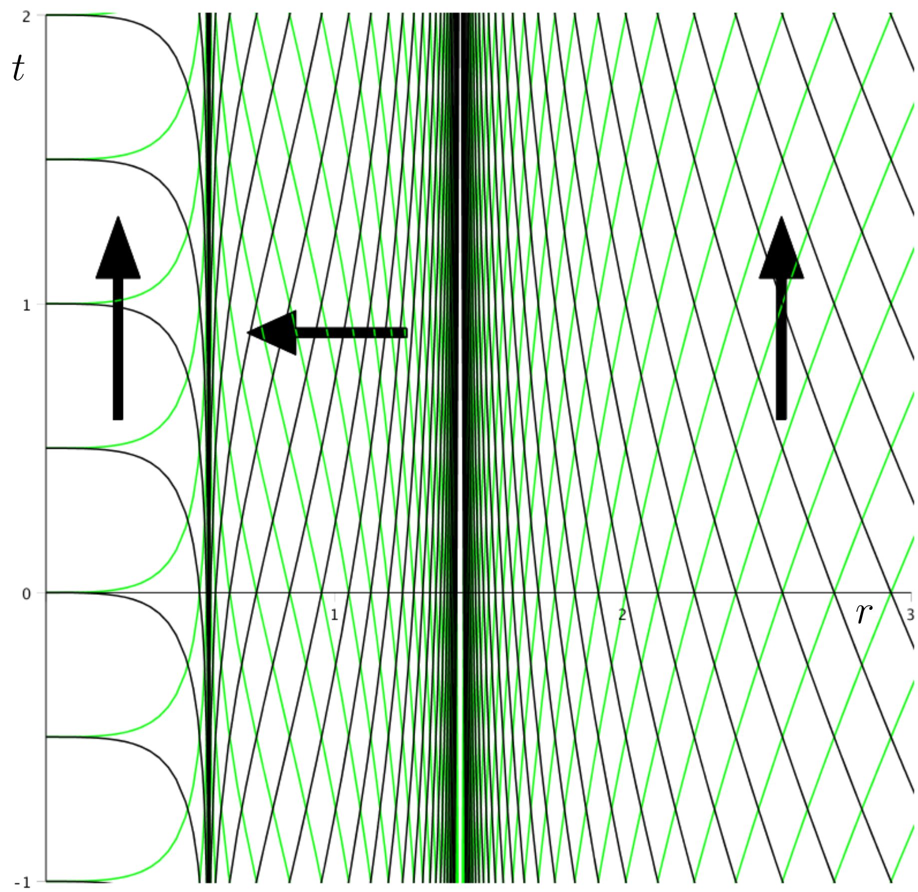


Figure 15.1 *Radial light-like world-lines in the Reissner-Nordström space-time (with charge* $Q = 0.9M$*), drawn in the Schwarzschild coordinates (units are that of* M*). The horizons at* $r_+ \doteq 1.44M$ *and* $r_- \doteq 0.56M$ *are clearly seen thanks to the behaviour of the world-lines. Black are ingoing world-lines, with the slope* $\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{1-\frac{2M}{r}+\frac{Q^2}{r^2}}$ *; they cross the horizons in the inward direction – the outer one via* $t = +\infty$ *and the inner one via* $t = -\infty$*. Green are outgoing world-lines, with the slope* $\frac{\mathrm{d}t}{\mathrm{d}r} = +\frac{1}{1-\frac{2M}{r}+\frac{Q^2}{r^2}}$ *; at the outer horizon they start at* $t = -\infty$ *and at the inner horizon they end at* $t = +\infty$*. Between the horizons, both ingoing and outgoing world-lines point "downwards", and the ingoing lines go there against the direction of* t*. At any point, the local light cone is determined by tangents to the ingoing and outgoing light world-line at that location. Causal future is indicated by arrows in all the three regions. It is seen that (i) far from the centre (*$r \gg M$*) the light cones are* $\pm 45°$ *like in special relativity (*$\mathrm{d}t/\mathrm{d}r = \pm 1$*); (ii) they narrow down towards the outer horizon (*$\mathrm{d}t/\mathrm{d}r \to \pm\infty$*), so that all ingoing causal motions cross* r_+ *at* $t \to +\infty$*; (iii) between the horizons, the ingoing cones are "inward" oriented – just below* r_+ *they have* $\mathrm{d}t/\mathrm{d}r \to \mp\infty$*, so all motions with* $\mathrm{d}r < 0$ *are allowed; on the way to* r_- *they narrow down, but then widen back to* $\mathrm{d}t/\mathrm{d}r \to \mp\infty$*, so the inner horizon is also crossed at* $t = \pm\infty$ *by all motions; (iv) below the inner horizon, the light cones become "vertical" again, similarly as in the domain of outer communications; towards the central singularity* $r = 0$ *they open to* $\mathrm{d}t/\mathrm{d}r \to 0^{\mp}$.

15.2 Analytic extension of the Reissner-Nordström metric

Without repeating the motivation for looking for a coordinate system which would be more suitable for representation of the central part of the RN space-time, and in which the manifold would be geodesically maximal (see the detailed discussion at the Schwarzschild solution), we will just transform to the Kruskal-type coordinates. The only significant difference from Schwarzschild is that now the metric has coordinate singularities on *two* horizons (which are distinct in general), so we will have to repeat the procedure twice – once for removing the singularity at each of the horizons. One will thus obtain two maps – one covering the region $r > r_-$ and the other covering the region $(0, r_+)$ – which will then be matched in the overlapping part (r_-, r_+). The transformation again concerns only the (t, r) part of the metric, the angular part $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 =: \mathrm{d}\Omega^2$ will stay untouched. And, we suppose throughout the "generic black-hole" case, $\underline{0 < Q^2 < M^2}$, with non-degenerate two horizons at $0 < r_- < r_+ < 2M$.

- Tortoise radial coordinate:

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = g_{rr} = \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} = \frac{1}{\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)}$$
$$\Longrightarrow \quad r^* = r + \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r}{r_+} - 1\right| + \frac{r_-^2}{r_- - r_+} \ln\left|\frac{r}{r_-} - 1\right|. \tag{15.12}$$

- Advanced and retarded time (the Eddington-Finkelstein coordinates):

$$t^\pm := t \pm r^* \qquad \Longrightarrow \qquad t^+ + t^- = 2t\,, \quad t^+ - t^- = 2r^*\,, \tag{15.13}$$

$$\mathrm{d}s^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)\mathrm{d}t^+\mathrm{d}t^- + r^2\mathrm{d}\Omega^2 =$$
$$= -\frac{r_+ r_-}{r^2}\left(\frac{r}{r_+} - 1\right)\left(\frac{r}{r_-} - 1\right)\mathrm{d}t^+\mathrm{d}t^- + r^2\mathrm{d}\Omega^2\,. \tag{15.14}$$

15.2.1 Kruskal-Szekeres coordinates above the inner horizon

- Light Kruskal-Szekeres coordinates:

$$u_+ := -\epsilon_+ \exp\left(-\frac{r_+ - r_-}{2r_+^2}\,t^-\right) = -\epsilon_+ \frac{\sqrt{\left|\frac{r}{r_+} - 1\right|}}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{2r_+^2}}} \exp\left[\frac{r_+ - r_-}{2r_+^2}(r - t)\right], \tag{15.15}$$

$$v_+ := \exp\left(\frac{r_+ - r_-}{2r_+^2}\,t^+\right) = \frac{\sqrt{\left|\frac{r}{r_+} - 1\right|}}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{2r_+^2}}} \exp\left[\frac{r_+ - r_-}{2r_+^2}(r + t)\right], \tag{15.16}$$

where $\epsilon_+ := \mathrm{sign}\left(\frac{r}{r_+} - 1\right)$ distinguishes whether being above or below r_+, and we have used

$$\exp\left(\frac{r_+ - r_-}{2r_+^2}\, r^*\right) = \exp\left(\frac{r_+ - r_-}{2r_+^2}\, r\right) \exp\left(\frac{1}{2}\, \ln\left|\frac{r}{r_+} - 1\right|\right) \exp\left(-\frac{r_-^2}{2r_+^2}\, \ln\left|\frac{r}{r_-} - 1\right|\right) =$$
$$= \exp\left(\frac{r_+ - r_-}{2r_+^2}\, r\right) \frac{\sqrt{\left|\frac{r}{r_+} - 1\right|}}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{2r_+^2}}} \, .$$

Like in the Schwarzschild case, we compute the product

$$\begin{aligned} -u_+ v_+ &= \epsilon_+ \, \exp\left[\frac{r_+ - r_-}{2r_+^2}(t_+ - t_-)\right] = \epsilon_+ \, \exp\left(\frac{r_+ - r_-}{r_+^2}\, r^*\right) = \\ &= \frac{\epsilon_+ \left|\frac{r}{r_+} - 1\right|}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{r_+^2}}} \, \exp\left(\frac{r_+ - r_-}{r_+^2}\, r\right) = \frac{\frac{r}{r_+} - 1}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{r_+^2}}} \, \exp\left(\frac{r_+ - r_-}{r_+^2}\, r\right) \end{aligned} \tag{15.17}$$

and employ it in derivation of the relation between $\mathrm{d}t^+\mathrm{d}t^-$ and $\mathrm{d}u_+\mathrm{d}v_+$,

$$\begin{aligned} \mathrm{d}u_+ \, \mathrm{d}v_+ &= -u_+ v_+ \left(\frac{r_+ - r_-}{2r_+^2}\right)^2 \mathrm{d}t^+\mathrm{d}t^- \\ &= \left(\frac{r_+ - r_-}{2r_+^2}\right)^2 \frac{\frac{r}{r_+} - 1}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{r_+^2}}} \, \exp\left(\frac{r_+ - r_-}{r_+^2}\, r\right) \, \mathrm{d}t^+\mathrm{d}t^- \, , \end{aligned} \tag{15.18}$$

thanks to which the metric assumes the form

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{r_+ r_-}{r^2} \left(\frac{2r_+^2}{r_+ - r_-}\right)^2 \left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{r_+^2}+1} \exp\left(-\frac{r_+ - r_-}{r_+^2}\, r\right) \mathrm{d}u_+ \, \mathrm{d}v_+ \\ &+ r^2 \mathrm{d}\Omega^2 \, . \end{aligned} \tag{15.19}$$

- Kruskal-Szekeres coordinates (V_+, U_+):

$$v_+ := V_+ + U_+, \quad u_+ := V_+ - U_+ \quad \Longrightarrow \quad V_+ = \frac{v_+ + u_+}{2}\, , \quad U_+ = \frac{v_+ - u_+}{2}\, . \tag{15.20}$$

We will not be rewriting the metric once more, since one just expresses $-\mathrm{d}u_+ \, \mathrm{d}v_+ = -\mathrm{d}V_+^2 + \mathrm{d}U_+^2$ in it; the metric is valid both above and below the horizon r_+. The final transformation reads, explicitly,

$$V_+ = \frac{\sqrt{\frac{r}{r_+} - 1}}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{2r_+^2}}} \, \exp\left(\frac{r_+ - r_-}{2r_+^2}\, r\right) \, \sinh\left(\frac{r_+ - r_-}{2r_+^2}\, t\right) ,$$

$$U_+ = \frac{\sqrt{\frac{r}{r_+}-1}}{\left(\frac{r}{r_-}-1\right)^{\frac{r_-^2}{2r_+^2}}}\ \exp\!\left(\frac{r_+-r_-}{2r_+^2}\,r\right)\ \cosh\!\left(\frac{r_+-r_-}{2r_+^2}\,t\right) \tag{15.21}$$

in the $r > r_+$ region, while

$$V_+ = \frac{\sqrt{1-\frac{r}{r_+}}}{\left(\frac{r}{r_-}-1\right)^{\frac{r_-^2}{2r_+^2}}}\ \exp\!\left(\frac{r_+-r_-}{2r_+^2}\,r\right)\ \cosh\!\left(\frac{r_+-r_-}{2r_+^2}\,t\right),$$

$$U_+ = \frac{\sqrt{1-\frac{r}{r_+}}}{\left(\frac{r}{r_-}-1\right)^{\frac{r_-^2}{2r_+^2}}}\ \exp\!\left(\frac{r_+-r_-}{2r_+^2}\,r\right)\ \sinh\!\left(\frac{r_+-r_-}{2r_+^2}\,t\right) \tag{15.22}$$

in the $(r_- <)\ r < r_+$ region.

- From the expressions for V_+ and U_+, one easily gets the inverse transformation, i.e. the relations which determine t and r as functions of V_+ and U_+:

$$\frac{V_+}{U_+} = \begin{cases} \tanh\left(\frac{r_+-r_-}{2r_+^2}\,t\right) & \text{for } r > r_+ \\ \coth\left(\frac{r_+-r_-}{2r_+^2}\,t\right) & \text{for } (r_-<)\ r < r_+ \end{cases},$$

$$U_+^2 - V_+^2 = -u_+v_+ = \frac{\frac{r}{r_+}-1}{\left(\frac{r}{r_-}-1\right)^{\frac{r_-^2}{r_+^2}}}\ \exp\!\left(\frac{r_+-r_-}{r_+^2}\,r\right). \tag{15.23}$$

Therefore, in the (V_+, U_+) plane, the $r = \mathrm{const}$ hypersurfaces are represented by hyperbolas with asymptotes $V_+ = \pm U_+$ (which correspond to $r = r_+$), whereas $t = \mathrm{const}$ are straight lines passing through the origin.

- The chapter is titled “analytic extension”, and indeed two new regions open in the new coordinates, similarly as in the Schwarzschild case – the second outer region $r > r_+$ and the second inner region $r_- < r < r_+$. From the inverse transformation it is seen that $r < r_+$ corresponds to $U_+^2 < V_+^2$ and $r > r_+$ corresponds to $U_+^2 > V_+^2$, which means *two* different quadrants in both cases, placed symmetrically with respect to the origin and bounded by the horizons $V_+ = \pm U_+$; in particular, radial infinities are reached, in the Kruskal coordinates, along the directions $U_+ \to \pm\infty$, while the inner horizons $r \to r_-$ are approached in the directions $V_+ \to \pm\infty$.

- The new regions are symmetric by the origin $V_+ = 0$, $U_+ = 0$ with respect to the “old” ones, so the transformations (15.21,15.22) work there as well, just with minuses in front. Hence, in total one needs 4 sets of transformations to cover the four quadrants of the region $r_- < r < \infty$. The inverse transformation (15.23) remains valid *everywhere*.

- Light cones ($\mathrm{d}s^2 = 0$):

$$\mathrm{d}V_+^2 = \mathrm{d}U_+^2 + \frac{\frac{r^4}{r_+ r_-}\left(\frac{r_+ - r_-}{2r_+^2}\right)^2}{\left(\frac{r}{r_-} - 1\right)^{\frac{r_-^2}{r_+^2}+1}} \exp\left(\frac{r_+ - r_-}{r_+^2}\, r\right) \mathrm{d}\Omega^2 \geqslant \mathrm{d}U_+^2 \quad \Rightarrow \quad \left|\frac{\mathrm{d}V_+}{\mathrm{d}U_+}\right| \geqslant 1\,. \quad (15.24)$$

Therefore, the cones are "$45°$" or narrower, with $45°$ applying to purely radial motion.

15.2.2 Kruskal-Szekeres coordinates below the outer horizon

- Light Kruskal-Szekeres coordinates:

$$u_- := -\epsilon_- \exp\left(-\frac{r_- - r_+}{2r_-^2}\, t^-\right) = -\epsilon_- \frac{\sqrt{\left|\frac{r}{r_-} - 1\right|}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \exp\left[\frac{r_- - r_+}{2r_-^2}(r - t)\right], \quad (15.25)$$

$$v_- := -\exp\left(\frac{r_- - r_+}{2r_-^2}\, t^+\right) = -\frac{\sqrt{\left|\frac{r}{r_-} - 1\right|}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \exp\left[\frac{r_- - r_+}{2r_-^2}(r + t)\right], \quad (15.26)$$

where $\epsilon_- := \mathrm{sign}\left(\frac{r}{r_-} - 1\right)$ distinguishes between above and below r_- , and we have used

$$\exp\left(\frac{r_- - r_+}{2r_-^2}\, r^*\right) = \exp\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \exp\left(\frac{1}{2}\, \ln\left|\frac{r}{r_-} - 1\right|\right) \exp\left(-\frac{r_+^2}{2r_-^2}\, \ln\left|\frac{r}{r_+} - 1\right|\right) =$$

$$= \exp\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \frac{\sqrt{\left|\frac{r}{r_-} - 1\right|}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}}\,.$$

We again compute

$$-u_- v_- = -\epsilon_- \exp\left[\frac{r_- - r_+}{2r_-^2}(t_+ - t_-)\right] = -\epsilon_- \exp\left(\frac{r_- - r_+}{r_-^2}\, r^*\right) =$$

$$= -\frac{\epsilon_- \left|\frac{r}{r_-} - 1\right|}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}}} \exp\left(\frac{r_- - r_+}{r_-^2}\, r\right) = -\frac{\frac{r}{r_-} - 1}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}}} \exp\left(\frac{r_- - r_+}{r_-^2}\, r\right) \quad (15.27)$$

and derive the relation

$$\mathrm{d}u_-\, \mathrm{d}v_- = -u_- v_- \left(\frac{r_- - r_+}{2r_-^2}\right)^2 \mathrm{d}t^+ \mathrm{d}t^- =$$

$$= -\left(\frac{r_- - r_+}{2r_-^2}\right)^2 \frac{\frac{r}{r_-} - 1}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}}} \exp\!\left(\frac{r_- - r_+}{r_-^2}\, r\right) \, \mathrm{d}t^+ \mathrm{d}t^- \ ,$$

which makes the metric (15.14) appear

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{r_- r_+}{r^2} \left(\frac{2r_-^2}{r_- - r_+}\right)^2 \left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}+1} \exp\!\left(-\frac{r_- - r_+}{r_-^2}\, r\right) \mathrm{d}u_-\, \mathrm{d}v_- \\ &+ r^2 \mathrm{d}\Omega^2 \,. \end{aligned} \tag{15.28}$$

- Kruskal-Szekeres coordinates (V_-, U_-):

$$v_- := V_- + U_-, \quad u_- := V_- - U_- \quad \Longrightarrow \quad V_- = \frac{v_- + u_-}{2}\,, \quad U_- = \frac{v_- - u_-}{2}\ . \tag{15.29}$$

Therefore, one just substitutes $-\mathrm{d}u_-\, \mathrm{d}v_- = -\mathrm{d}V_-^2 + \mathrm{d}U_-^2$ in the metric; its form is valid everywhere within $0 < r < r_+$. The final transformation reads, explicitly,

$$\begin{aligned} V_- &= -\frac{\sqrt{\frac{r}{r_-} - 1}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \ \exp\!\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \cosh\!\left(\frac{r_- - r_+}{2r_-^2}\, t\right), \\ U_- &= -\frac{\sqrt{\frac{r}{r_-} - 1}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \ \exp\!\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \sinh\!\left(\frac{r_- - r_+}{2r_-^2}\, t\right) \end{aligned} \tag{15.30}$$

in the region $(r_+ >)\, r > r_-$, while

$$\begin{aligned} V_- &= -\frac{\sqrt{1 - \frac{r}{r_-}}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \ \exp\!\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \sinh\!\left(\frac{r_- - r_+}{2r_-^2}\, t\right), \\ U_- &= -\frac{\sqrt{1 - \frac{r}{r_-}}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{2r_-^2}}} \ \exp\!\left(\frac{r_- - r_+}{2r_-^2}\, r\right) \cosh\!\left(\frac{r_- - r_+}{2r_-^2}\, t\right) \end{aligned} \tag{15.31}$$

in the region $r < r_-$.

- Inverse transformation:

$$\frac{V_-}{U_-} = \begin{cases} \coth\left(\frac{r_- - r_+}{2r_-^2}\, t\right) & \text{for} \ \ (r_+ >)\, r > r_- \\ \tanh\left(\frac{r_- - r_+}{2r_-^2}\, t\right) & \text{for} \ \ r < r_- \end{cases} \ ,$$

$$U_-^2 - V_-^2 = -u_- v_- = \frac{1 - \frac{r}{r_-}}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}}} \exp\left(\frac{r_- - r_+}{r_-^2}\, r\right). \tag{15.32}$$

In the (V_-, U_-) plane, the hypersurfaces $r = \text{const}$ thus appear as hyperbolas with asymptotes $V_- = \pm U_-$ (which correspond to $r = r_-$), while $t = \text{const}$ are straight lines passing through the origin.

- Again two quadrants open newly (uncovered by the original t, r) – one more copy for each of the two existing ones. The inverse transformation implies that $r > r_-$ lies where $U_-^2 < V_-^2$ while $r < r_-$ lies where $U_-^2 > V_-^2$, which in both cases corresponds to *two* quadrants situated symmetrically with respect to the origin and bounded by the horizons $V_- = \pm U_-$; in particular, the outer horizon $r = r_+$ is approached in the direction $V_- \to \pm\infty$ while the physical singularity $r = 0$ is localized on hyperbolas

$$U_-^2 - V_-^2 = 1 \quad \Longrightarrow \quad U_- = \pm\sqrt{1 + V_-^2}\,.$$

- In the new regions, transformations (15.30,15.31) are valid as well, just with minuses. In order to cover the whole region $0 < r < r_+$, one again needs 4 sets of transformations, while the inverse transformation (15.32) holds everywhere.

- Light cones:

$$\mathrm{d}V_-^2 = \mathrm{d}U_-^2 + \frac{\frac{r^4}{r_- r_+}\left(\frac{r_- - r_+}{2r_-^2}\right)^2}{\left(1 - \frac{r}{r_+}\right)^{\frac{r_+^2}{r_-^2}+1}} \exp\left(\frac{r_- - r_+}{r_-^2}\, r\right) \mathrm{d}\Omega^2 \geqslant \mathrm{d}U_-^2 \quad \Rightarrow \quad \left|\frac{\mathrm{d}V_-}{\mathrm{d}U_-}\right| \geqslant 1\,, \tag{15.33}$$

so they are 45° or narrower, with 45° valid for purely radial motion.

15.2.3 Kruskal diagram and Penrose-Carter conformal diagram

Kruskal diagram thus consists of two maps, one for the $r_- < r < \infty$ region and the other for the $0 < r < r_+$ region (Figure 15.2). All the discussed features of radial behaviour of the Reissner-Nordström geometry are well seen on it. Most notably, let us once more stress the everywhere-45° shape of light cones and the time-like character of singularities. Thanks to it, there exist more options for where time-like world-lines may go: in Schwarzschild, physical motions ended at time or light infinity, or at the singularity, whereas here it is also possible to travel through the whole diagram, without reaching infinity or hitting the singularity – and even in *finite* proper time. Wait! –That would mean geodesic *non*-maximality! Indeed, a particle which gets below the outer horizon has to continue below the inner horizon; there it can either reach the singularity, or continue towards the inner horizon of "the other universe"; in the second case, it has then no other possibility than to reach the corresponding outer horizon. As we checked in Schwarzschild – and here it is similar – horizons are at *finite*

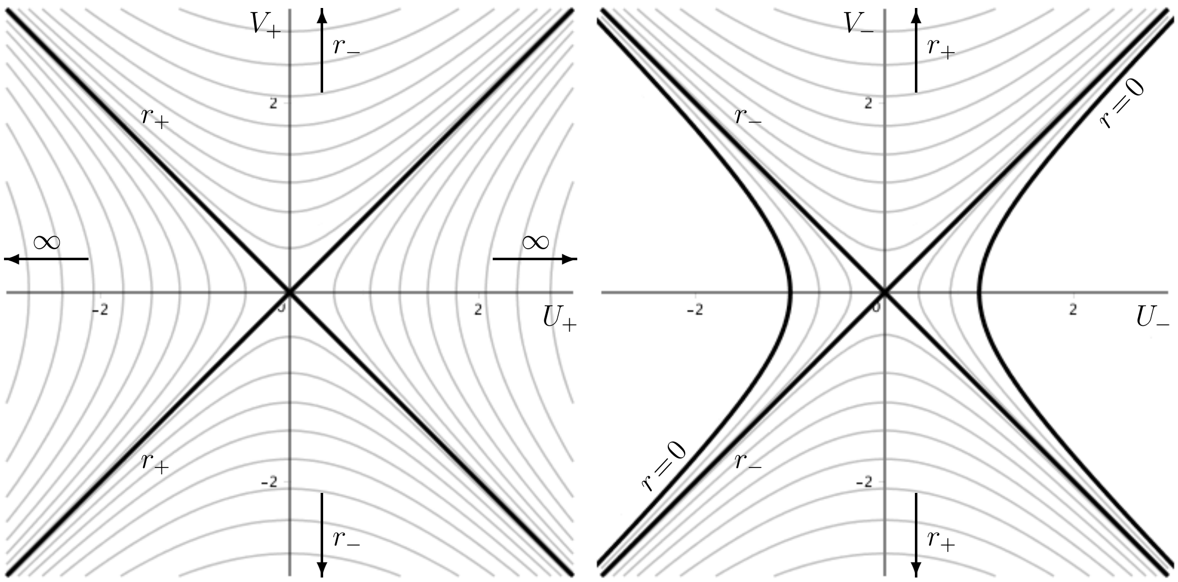


Figure 15.2 *Kruskal diagrams for the "outer" and "inner" regions of the Reissner-Nordström space-time, $r_- < r < \infty$ and $0 < r < r_+$ – on the left and on the right, respectively. Similarly as in the Schwarzschild case, the hypersurfaces $r = \text{const}$ appear as hyperbolas with asymptotes $V_+ = \pm U_+$ (outer horizons r_+) and $V_- = \pm U_-$ (inner horizons r_-), respectively. In the left plot, radial infinity is on the left and on the right, while up and down directions lead to the inner horizons. In the right plot, up and down directions lead (from below) to the outer horizons, while to the left and to the right the radii decrease to singular $r = 0$. The hypersurfaces $t = \text{const}$ appear as straight lines passing through the origin (but we are not showing them). In both parts of the diagram, the light cones for radial motion are $45°$, so the light-like as well as one-way character of the horizons is seen clearly.*

proper distance (and, in accord with that, they are reached in *finite* proper time). Hence, even the two Kruskal maps together cannot represent *maximal* extension of the metric, since certain *non-singular* parts of their boundary are at finite distance. At the same time, we have no other space-time regions at our disposal than those two...

To answer such queries, much more comfortable are conformal diagrams (because in Kruskal diagram, "the other" horizons are at infinity, although they actually lie at finite distance). The conformal diagrams are obtained by the same type of transformation as in the Schwarzschild case, (14.29). The region $r_- < r < \infty$ produces a square rhombus, while the region $0 < r < r_+$ looks like the Schwarzschild conformal diagram turned by $90°$ (but here with outer horizons at places of the Schwarzschild infinities). Now, the non-maximality query can be answered in two ways:

- Rather than just one pair of the regions, take *infinite* number of them, and place them above each other.

- Restrict to only finite number of the diagram pairs (possibly even to the original *one* pair), but identify the $r_- < r_+$ region at the very top with the same region at the very bottom, that is, roll the whole diagram into a cylinder. Crucial here is to recall that *Einstein equations do not restrict topology of space-time*. Actually, by rolling the diagram, its intrinsic geometry (fixed by the Einstein equations) is not altered, since this does not induce any deformations within the "plane of the diagram".

Figure 15.3 (left) shows a part of the Penrose-Carter diagram of the Reissner-Nordström manifold, as obtained by the first (infinite) way of composition. The radial light cones remain 45° everywhere, so it is clear how time-like world-lines can constantly avoid singularities and traverse the whole diagram. Importantly, all the horizons crossed in such a journey are *distinct* – the particle can never return to the *same* space-time region which it left before via the respective future horizon.

The right part of Figure 15.3 adds the Penrose-Carter conformal diagrams

in the extreme case $Q^2\!=\!M^2$ and in the naked-singularity case $Q^2\!>\!M^2$.

It is important to remind that all the above procedure concerned the "general black-hole" case $Q^2\!<\!M^2$. The other cases now being added are in fact simpler, because

- For an extreme black hole, the horizon is double degenerate – it corresponds to double root of the equation $\Delta=0$ (there is no dynamical region), $1-\frac{2M}{r}+\frac{Q^2}{r^2}=\left(1-\frac{M}{r}\right)^2$, so the coordinate singularity at $r\!=\!M$ can be removed by just a single transformation

$$\frac{\mathrm{d}r^*}{\mathrm{d}r}=g_{rr}\quad\Longrightarrow\quad r^*=r\,\frac{r-2M}{r-M}+2M\,\ln\left|\frac{r}{M}-1\right|\,.$$

  In the same way as in the $Q^2<M^2$ case, one introduces $t^\pm:=t\pm r^*$ and thus writes the metric as

$$\mathrm{d}s^2=g_{tt}\,\mathrm{d}t^+\mathrm{d}t^-+r^2\mathrm{d}\Omega^2=-\left(1-\frac{M}{r}\right)^2\mathrm{d}t^+\mathrm{d}t^-+r^2\mathrm{d}\Omega^2\,.$$

  May be worth to add that the extreme horizon is (also) very different from the non-degenerate one in that it is at *infinite* proper radial distance:

$$\int_M^{r>M}\sqrt{g_{rr}}\;\mathrm{d}r=\int_M^{r>M}\frac{\mathrm{d}r}{1-\frac{M}{r}}=\left[r+M\ln\frac{r-M}{M}\right]_M^{r>M}=\infty\,.$$

- For a naked singularity, there are no horizons and the tortoise coordinate is introduced by

$$r^*=r+M\,\ln(r^2-2Mr+Q^2)-\frac{Q^2-2M^2}{\sqrt{Q^2-M^2}}\,\mathrm{arctan}\frac{r-M}{\sqrt{Q^2-M^2}}\,.$$

In both cases, the manifold is already completely covered by the advanced and retarded times $t^\pm\!=\!t\pm r^*$, and the final compactification can be achieved by the same transformation as for flat space-time,

$$\psi=\arctan\frac{t^+}{M}+\arctan\frac{t^-}{M}\,,\qquad\xi=\arctan\frac{t^+}{M}-\arctan\frac{t^-}{M}\,.$$

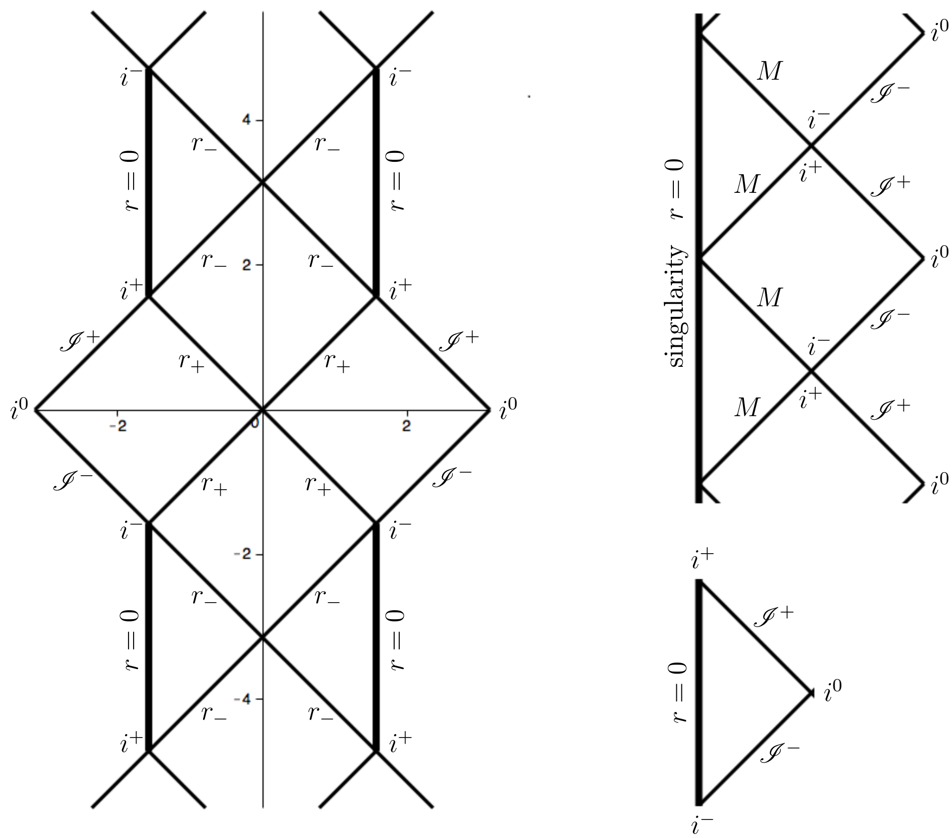


Figure 15.3 *Penrose-Carter conformal diagram of the maximal Reissner-Nordström space-time.* **Left:** *the generic black-hole case $0 < Q^2 < M^2$, as obtained by composing many compactified outer and inner regions from Figure 15.2. The light cones for radial motion are $45°$, so it is possible to traverse any finite part of the diagram in finite proper time, without hitting a singularity. Future inner horizons play the role of Cauchy horizons and the geometry is unstable against perturbations there (see the main text).* **Right:** *the extreme case $Q^2 = M^2$ (top), again obtained by composition of many analytically extended regions (of only one type here), and the naked-singularity case $Q^2 > M^2$ (bottom) where no extension is necessary. Note that the extreme case is clearly different in that the singularities do not asymptotically approach infinities.*

15.2.4 Physical meaning of the extended Reissner-Nordström solution

Interesting to see how different from Schwarzschild the global structure of Reissner-Nordström is, however small electric charge it involves. Yet practical significance of the Reissner-Nordström solution is low. First, due to the differential character of electr(omagnet)ic interaction, the celestial bodies are quite strictly neutral. Second, the "cons" already mentioned in connection with the extended Schwarzschild metric apply here as well. Still, there is an important "third" in addition – an important pathology at the inner horizons:

- For a given pair of domains of outer communications ($r > r_+$), their future inner horizons play the role of the so-called **Cauchy horizons**, namely they are future boundaries of the region where a solution of any Cauchy problem (evolution of any physical system) is determined *uniquely* by initial conditions given on some space-like hypersurface spanning between the two spatial (or possibly also light) infinities. Actually, as nicely seen in Figure 15.3, the inner horizons represent boundaries of a region which already has at least one of the future singularities in its past light cone, i.e. which can be affected by information coming from the singularities (the inner horizons in fact coincide with future light cones of the past ends of the singularities). But a theory cannot provide any prediction about how singularities will behave (just because they are singularities of Einstein equations), so future becomes uncertain above such horizons.[2] In fact a more general conclusion arises: Cauchy horizons necessarily appear in space-times with time-like singularities. Therefore, in such space-times the ideal of classical determinism – that "the future is uniquely determined by the past" – has to be given up or at least weakened.

- At the inner horizons, the RN solution is unstable with respect to perturbations which occurred in some of the corresponding (preceding) domains of outer communications. This is a complicated mathematical result, requiring to make such a perturbation (here of the metric and of the EM field), to write down the perturbed field equations, to isolate from these the equations for the perturbation quantities (namely, to subtract the unperturbed part of the equations), to linearize the resulting equations in the perturbations (or to simply cut them at certain desired order), and to solve them. Let us try to at least illustrate the problem loosely: imagine two colleagues staying at some $r > r_+$. At $t = 0$ ($V_+ = 0$), say, one of them will travel inside the black hole, while the other will stay at rest and will be shining with a torch into the black hole. Imagine that the shining observer (and the battery in the torch) live up to future time infinity i^+. Now look at the conformal diagram, draw the world-lines of both observers as well as world-lines of the photons (diagonals parallel to the corresponding inner horizon). Clearly, before the "black-hole explorer" crosses, in *finite* proper time, the inner horizon, it receives *infinitely* many photons – namely all the photons emitted during the static-observer *infinite* life.

  [This is clearly nothing more than an indication that the space-time has a tendency to amplify every perturbation at the inner horizon. More realistically, the "perturbing" photons can be (and typically are) generated during the formation of the given black hole by a gravitational collapse. In any case, the above instability implies that even if such a collapse produced an isolated and charged spherical black hole, the latter's interior would almost certainly be different from the Reissner-Nordström solution. In particular, the central singularity would likely be space-like rather than time-like.]

[2] Such a conclusion is particularly suggestive if one solved as the Cauchy problem the evolution of the space-time itself, i.e. if one did *not* suppose to have a complete RN space-time right at hand, but rather wished to find it as a result of evolution from some initial data.

Chapter 16

Kerr solution of Einstein equations

Celestial bodies are seldom significantly charged, but what on the contrary is ubiquitous in astrophysics – and the Schwarzschild solution does *not* involve it – is **rotation, spin**. Including rotation is non-trivial in GR since it represents motion, mass current, which itself contributes to gravitation (in addition to the scalar mass itself). No surprise that it took decades before the geometry around a rotating centre was found. It was in 1963 and it was due to R. P. Kerr. Kerr discovered it when tackling Einstein equations for a metric ansatz corresponding to a special structure of space-time curvature (the so-called algebraic type D). In a few years, however, the solution was interpreted as describing a field of a stationary, rotating black hole or naked singularity. S. Chandrasekhar was quoting this result the most important *astrophysical* discovery of the second half of the 20th century.

16.1 Mach, Einstein, Lense & Thirring and *dragging*. – And Kerr…

When E. Mach was deprecating "absolute" elements of thought constructions – those which are not *mutually* interconnected with other elements –, he could not avoid **inertia**. In Newtonian physics, this basic property of matter is an *intrinsic* property, not affected by anything exterior. It manifests itself if one tries to accelerate a body with respect to an absolute space (in fact with respect to any inertial system). In the famous Newton's bucket, rotation makes the water surface paraboloidal, because the bucket spins with respect to inertial systems, and thus centrifugal force acts in it. Mach, instead, viewed inertia as resulting from *interaction* of a given body with other bodies in the Universe. In *his* bucket, water also assumes the parabolical shape, but not because of spinning with respect to any system, but because of spinning *with respect to "distant stars"* – more specifically, with respect to a certain average rest system of the cosmic matter. Mach used to add that the experiment would give the same outcome if the bucket stayed still while "the Universe around" was rotating. More accurately, he was saying something slightly different, though following from the same reasoning: that even in a spinning bucket the water would stay flat if the bucket walls were made very thick and heavy.

When finishing GR, Einstein was very much referring to Mach's views. When supplemented with the equivalence principle, they implied that gravitation should not actually be understood as "generated by this or that body", but rather as following from mutual interaction between all the matter in the Universe. He thus wondered at Schwarzschild's early solution, since it is non-trivial even though generated by one single point. In February 1916, Schwarzschild added an interior solution for a spherical "star", and Reissner soon (in March) generalized the vacuum Schwarzschild metric to the electrically charged case. By that time, Mach had already been deceased. But not so his (and Einstein's) principle.

Yet it should be specified how "all the matter" contributes to the determination of an "inertial field" at a given point. According to every experience, forces fall with distance, so if inertia is linked to gravity, the inertial field should be dominated by heavy bodies which are close to the given location. For example, the Foucault-pendulum plane should of course be kept by "distant stars", but it should also be slightly dragged by the Earth (which is not that massive, but very close-by).[1]

Einstein began to more think about inertia in Prague where he wrote several important papers on gravitation. In 1912 he published the paper *Is there a gravitational effect which is analogous to electrodynamical induction?*[2] He studied in it the behaviour of a free particle inside an isolated massive spherical shell, to conclude – contrary to the Newtonian "no effect" – that the particle does feel when the sphere starts to be pulled: *with respect to the rest system of the sphere*, the particle becomes accelerated, hence inertial force starts to drag it *in the direction in which the sphere is moving*. Einstein sent the paper to Mach and on 25th June 1913 added in a letter: "... your brilliant investigations on the foundations of mechanics will have received a splendid confirmation. For it follows of necessity that inertia has its origin in some kind of interaction of the bodies, exactly in accordance with your argument about Newton's bucket experiment. You will find a first consequence in this sense on the top of page 6 of the paper. Beyond that, the following results have been obtained: 1. If one accelerates an inertial spherical shell S, then, according to the theory, a body enclosed by it experiences an accelerating force. 2. If the shell S rotates about an axis passing through its centre (relative to the fixed stars ('Restsystem'), then a Coriolis field arises inside the shell, i.e., the plane of the Foucault pendulum is being carried along (though with a practically immeasurably small velocity)."

In 1918, Einstein's prediction was confirmed by J. Lense and H. Thirring, already within definitive GR, by analysing the behaviour of a free particle inside as well as outside a slowly rotating massive sphere. The effect thus bears their names, although it was chiefly Einstein himself who explained to them the problem in a correspondence. Anyway, more frequent today may be the term **dragging effects** (dragging of inertial frames by matter), or **gravito(electro)magnetism**, stemming from the analogy with electrodynamics where

[1] The Foucault-pendulum experiment was really proposed for the South Pole, but other plans to detect the dragging effect apparently turned out more promising.

[2] It appeared in the journal *Vierteljahrsschrift für gerichtliche Medizin und öffentliches Sanitätswesen* (English: Quarterly Journal for Forensic Medicine and Public Health Service) as a birthday present to Einstein's friend H. Zangger, a renowned Curych professor of forensic medicine.

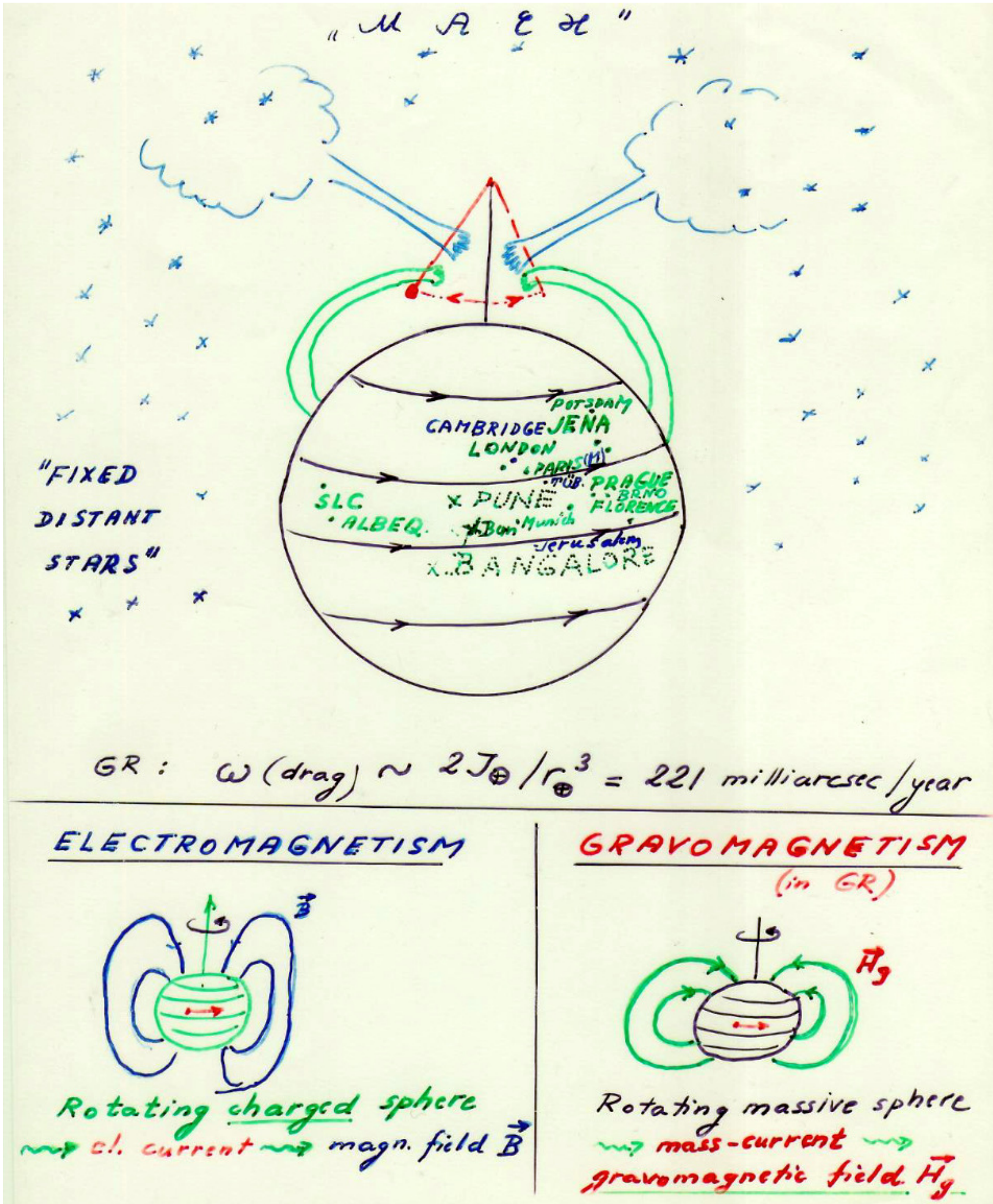


Figure 16.1 All *these lecture notes actually are a tribute to* Jiří Bičák. *However, certain "images" connected with* Jiří*'s teaching emerge in my (O.S.'s) mind more often than others, one of those being the very old and tired "Machian" transparency explaining the rotational dragging on a Foucault pendulum placed at Earth's pole: the pendulum's plane of oscillation is fixed to distant stars basically (blue little hands), yet the Earth* very slightly *drags it along (green little hands) by its rotation.*

currents generate magnetic component of the field (while charges generate electric one).

Kerr metric – the exact vacuum solution of Einstein equations describing the gravitational field of a uniformly rotating black hole or naked singularity – provides a prominent background for the study of rotational dragging, namely of the influence of the centre's spin on the motion of particles, on the precession of gyroscopes and on the structure of external fields.

16.2 Kerr metric

The Kerr solution consists in a family of metrics with two parameters representing mass and rotational angular momentum ("spin") of the centre (it is assumed that $\Lambda = 0$). S. Chandrasekhar begins, in his monograph [6], the chapter on Kerr solution by saying: "It has been stated that 'there is no constructive analytic derivation of the [Kerr] metric that is adequate in its physical ideas, and even a check of this solution of Einstein's equations involves cumbersome calculations' (Landau and Lifshitz). Contrary to this statement, we shall find that, once the basic equations have been properly written and reduced, the derivation of the Kerr metric is really very simple and proceeds with an adequate base of physical and mathematical motivations." Chandrasekhar then proves his reassurance on the following 16 pages where the reader finds 133 numbered equations (of which, moreover, (3) actually contains 13 equations); in addition, the author enters that chapter already knowing canonical form of the stationary and axisymmetric metric, as well as the corresponding components of curvature tensors (all that had been derived in chapter 2 therein).

Indeed, we are not going to derive the Kerr metric. In the cylindrical Kerr-Schild coordinates (T, ρ, z, ψ), it reads

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2 + \frac{2Mr^3}{r^4 + a^2z^2}\left(\mathrm{d}T + \frac{r\rho\,\mathrm{d}\rho - a\rho^2\mathrm{d}\psi}{r^2 + a^2} + \frac{z\mathrm{d}z}{r}\right)^2, \quad (16.1)$$

where r is an oblate radius given by equation

$$r^4 - (\rho^2 - a^2 + z^2)\,r^2 - a^2z^2 = 0$$

and M and a are parameters. The metric reduces to a flat one for $M = 0$, so M likely represents mass of the source (and the Kerr-Schild coordinates generalize standard cylindrical coordinates).

Mostly the metric is being presented in the Boyer-Lindquist coordinates (t, r, θ, ϕ) which are related to the Kerr-Schild ones by

$$\mathrm{d}T = \mathrm{d}t - \frac{2Mr}{\Delta}\,\mathrm{d}r\,, \quad \mathrm{d}\psi = \mathrm{d}\phi - \frac{2Mar}{(r^2+a^2)\Delta}\,\mathrm{d}r\,, \quad \rho = \sqrt{r^2+a^2}\,\sin\theta, \quad z = r\cos\theta\,.$$

The metric assumes in them the form

$$\boxed{\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + g_{\phi\phi}\,(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2}\,, \quad (16.2)$$

where (N below is called the lapse function, it will be important in Chapter 25)

$$N^2 := -g_{tt} - g_{t\phi}\omega = \frac{\Sigma\Delta}{\mathcal{A}}\,, \qquad g_{\phi\phi} := \frac{\mathcal{A}}{\Sigma}\sin^2\theta\,, \qquad \omega := \frac{-g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\mathcal{A}}\,,$$
$$\Sigma := r^2 + a^2\cos^2\theta\,, \qquad \Delta := r^2 - 2Mr + a^2\,,$$
$$\mathcal{A} := (r^2+a^2)^2 - \Delta a^2\sin^2\theta \;= \Sigma(r^2+a^2) + 2Mra^2\sin^2\theta \;= \Sigma\Delta + 2Mr(r^2+a^2)\,.$$

Sometimes a different arrangement of the (t, ϕ) terms is suitable, for example

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left[a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi\right]^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \tag{16.3}$$
$$= -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mr}{\Sigma}\,a\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi + \frac{\mathcal{A}}{\Sigma}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \tag{16.4}$$
$$= -\mathrm{d}t^2 + \frac{2Mr}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2\,. \tag{16.5}$$

16.2.1 Basic features of the Kerr metric

- The Kerr metric is stationary and axially symmetric – there exist two (commuting) Killing vector fields, of which one is time-like (at least at large radii r) with open integral lines while the other is space-like with closed integral lines. If parametrizing the time symmetry by t and the axial symmetry by ϕ, the Killing fields (again) read

  $$t^\mu = \frac{\partial x^\mu}{\partial t}\,, \qquad \phi^\mu = \frac{\partial x^\mu}{\partial \phi}\,.$$

  In other words, the t and ϕ included in the Boyer-Lindquist coordinates are directly related to the space-time symmetries.

- However, as opposed to Schwarzschild, the metric is *not* static: it contains the non-diagonal term $g_{t\phi} = -2Mra\sin^2\theta/\Sigma$ which makes it depending on the direction of time t.[3] Exactly this term brings the new GR effect of dragging which we focus on in the next section.

- The metric is also reflection symmetric with respect to the equatorial plane $\theta = \pi/2$: the trigonometric functions only occur in it in the second power.

- For $a = 0$ the metric reduces to the Schwarzschild form. Hence, M obviously represents mass, and the Boyer-Lidquist coordinates generalize the Schwarzschild ones. The parameter a is clearly connected with rotation.

- The meaning of M is confirmed by the asymptotic behaviour (with respect to a): at radii $r \gg a$, the metric becomes (to linear order in a/r)

  $$\mathrm{d}s^2 = \text{Schwarzschild} - \frac{4Ma}{r}\,\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi\,. \tag{16.6}$$

[3] Geometrically, the non-staticity means that the time Killing field t^μ is not hypersurface-orthogonal, i.e. that it cannot be expressed, globally, as proportional to a gradient of any scalar. (See Section 24.4.1.)

This also elucidates the meaning of a (as the rotational angular momentum per unit mass M), if compared with the generic form of the metric of a stationary quasi-Newtonian source, derived in Section 22.4.3. At radial infinity, the metric finally reduces to a flat metric in spherical coordinates.

- However, the Boyer-Lindquist coordinates are *not* spherical. Actually, if setting $M=0$ in (16.2), one of course obtains flat metric, but not in spherical coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Sigma\,\mathrm{d}r^2}{r^2+a^2} + \Sigma\,\mathrm{d}\theta^2 + (r^2+a^2)\,\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{16.7}$$

Therefore, if $a \neq 0$, the Boyer-Lindquist coordinates are ellipsoidal (spheroidal), more specifically of an *oblate* type. The latter is best seen from relations

$$\frac{\rho^2}{r^2+a^2} + \frac{z^2}{r^2} = 1, \qquad \frac{\rho^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1 \tag{16.8}$$

which are just another reading of the transformation $\rho = \sqrt{r^2+a^2}\,\sin\theta$, $z = r\cos\theta$. Hence, in the Kerr-Schild axes, the surfaces $r=\mathrm{const}$ are oblate rotational ellipsoids and the surfaces $\theta=\mathrm{const}$ are rotational hyperboloids. All have a common focus at $[\rho=a, z=0]$ which corresponds to $[r=0, \theta=\pi/2]$.

Note in particular that r thus no longer stands for an area or circumferential radius, because the $r=\mathrm{const}$ surfaces have areas

$$\int_0^{2\pi}\!\!\int_0^{\pi} \sqrt{g_{\theta\theta}g_{\phi\phi}}\;\mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi}\!\!\int_0^{\pi} \sqrt{\mathcal{A}}\;\sin\theta\;\mathrm{d}\theta\mathrm{d}\phi = 2\pi\int_0^{\pi}\sqrt{\mathcal{A}}\;\sin\theta\;\mathrm{d}\theta \tag{16.9}$$

and the circumferences of the $r=\mathrm{const}$, $\theta=\mathrm{const}$ circles read

$$\int_0^{2\pi}\sqrt{g_{\phi\phi}}\,\mathrm{d}\phi = 2\pi\sqrt{\frac{\mathcal{A}}{\Sigma}}\;\sin\theta\,.$$

These do not even reduce to $4\pi r^2$ and $2\pi r\sin\theta$ for $M=0$. Indeed, for $M=0$, one has $\mathcal{A} = \Sigma(r^2+a^2)$, hence the integrals yield

$$\begin{aligned}\int_0^{2\pi}\!\!\int_0^{\pi} \sqrt{g_{\theta\theta}g_{\phi\phi}}\;\mathrm{d}\theta\mathrm{d}\phi &= 2\pi\sqrt{r^2+a^2}\int_0^{\pi}\sqrt{\Sigma}\;\sin\theta\;\mathrm{d}\theta = \\ &= 2\pi(r^2+a^2) + \frac{2\pi r^2}{a}\sqrt{r^2+a^2}\;\mathrm{arcsinh}\frac{a}{r}\,,\\ \int_0^{2\pi}\sqrt{g_{\phi\phi}}\,\mathrm{d}\phi &= 2\pi\sqrt{r^2+a^2}\;\sin\theta\,.\end{aligned}$$

- The metric is invariant with respect to the transformations

$$(a \to -a,\ t \to -t), \qquad (a \to -a,\ \phi \to -\phi)$$

(similarly as with respect to the inversion $t \to -t,\ \phi \to -\phi$), which confirms that a represents rotation. Its dimension really agrees with the rotational angular momentum (spin) divided by mass, $a = J/M$. Without loss of generality, this parameter is being considered non-negative, $a \geqslant 0$, which simply means that the coordinate ϕ is oriented in the direction of the centre's rotation.

- Similarly as in the Reissner-Nordström case, the metric has three singularities. The **physical singularity** lies where $\Sigma = 0$, as seen from the Kretschmann and the dual (so-called Chern-Pontryagin) invariants[4]

$$K := R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{48M^2}{\Sigma^6}\,(r^2 - a^2\cos^2\theta)(\Sigma^2 - 16r^2a^2\cos^2\theta)\,, \tag{16.10}$$

$$^{*}K := {}^{*}R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{96M^2}{\Sigma^6}\,ra\cos\theta\,(3r^2 - a^2\cos^2\theta)(r^2 - 3a^2\cos^2\theta)\,. \tag{16.11}$$

The scalars reveal a miraculously symmetric curvature structure of the Kerr space-time (see Figure 16.2). In addition, it can be verified that even simpler comes out the modulus of the complex number $K - \mathrm{i}\,{}^{*}K$,

$$|K - \mathrm{i}\,{}^{*}K| \equiv \sqrt{K^2 + {}^{*}K^2} = \frac{48M^2}{\Sigma^3}\,. \tag{16.12}$$

Horizons (coordinate singularities) are given by $\Delta = 0$, similarly as in previous chapter, so they are two again and given by the same expression, just involving a^2 instead of Q^2,

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{16.13}$$

Also similar are thus the three options – a generic black hole ($0 < a < M$, two horizons), an extreme black hole ($a = M$, one double degenerate horizon) and a naked singularity ($a > M$, no horizon). Horizons are again light-like, as it is possible to check by evaluating the norm of the normal to the $r = \mathrm{const}$ hypersurfaces,

$$g^{\mu\nu}\frac{\partial r}{\partial x^\mu}\frac{\partial r}{\partial x^\nu} = g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\Sigma}\,.$$

- Important difference from the Schwarzschild and Reissner-Nordström space-times: Kerr horizons are *neither static limits nor infinite-redshift surfaces*. Actually, as it is clear from $g_{\mu\nu}t^\mu t^\nu = g_{tt}$, the temporal Killing field becomes light-like on surfaces given by $g_{tt} = 0$, so at

$$\Sigma = 2Mr \quad \Longleftrightarrow \quad r = r_{0,1} = M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{16.14}$$

[4] The Chern-Pontryagin scalar is zero in static space-times, such as Schwarzschild or Reissner-Nordström.

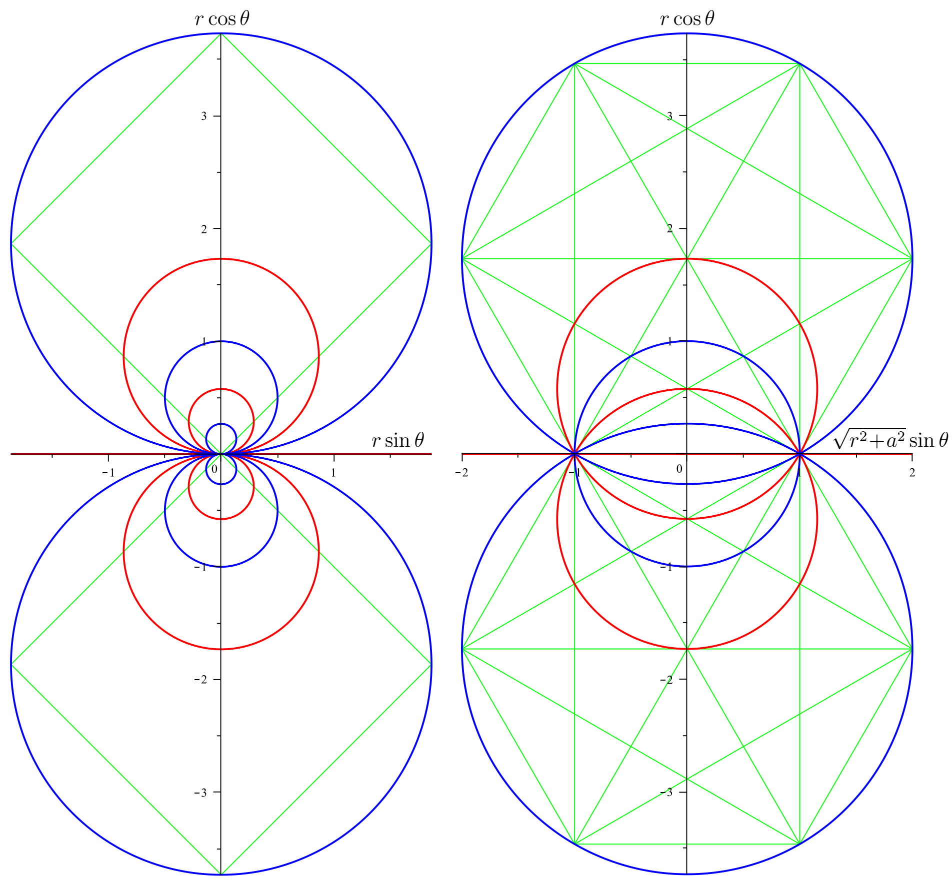


Figure 16.2 *Curvature of the Kerr space-time represented in the Boyer-Lindquist coordinates* $r\sin\theta$, $r\cos\theta$ (**left**) *and in the Kerr-Schild coordinates* $\rho = \sqrt{r^2+a^2}\,\sin\theta$, $z = r\cos\theta$ (**right**). *The six/three blue circles indicate zeros of the Kretschmann invariant (16.10) and the four/two red circles (plus red-coloured horizontal axis) indicate zeros of the Chern-Pontryagin invariant (16.11). The arrangement is quite miraculous in the Kerr-Schild plot: all the circles intersect at the singularity* $(\rho = a, z = 0)$ *and define a* $(\pi/6)$*-segmentation of meridional planes there; a remarkable symmetry of the pattern is revealed on tangents to the circles drawn (in green colour) at the singularity (note, for example, that the tangents only intersect at the circles). In the Boyer-Lindquist picture (left), the pattern based on circles' tangents is of course degenerate and the only other straight lines one can draw are diagonals crossing the circles' at their leftmost/rightmost points.*

These only touch the horizons at the symmetry axis ($\theta = 0, \pi$), whereas elsewhere the surfaces are arranged as $r_1 < r_- \leqslant r_+ < r_0$. The frequency shift between static observers is given by g_{tt} as well, so the infinite-redshift surfaces coincide with the static limits. From the above formula, it is seen that the static limits – in contrast to the horizons – exist for any value of a, even for $a > M$: when $a = M$, the inner and outer static limits r_1 and r_0 join at the axis and for $a > M$ they are represented by a single toroidal surface which surrounds the central singularity.

- Another important difference from static space-times is in topology of the physical singularity. We saw it is given by $\Sigma = 0$ which in Boyer-Lindquist coordinates requires $r = 0$ *and* $\theta = \pi/2$. Strange situation – a point only singular from the equatorial side. Such directional singularities usually indicate a certain degeneracy of the coordinate representation. Actually, in the Kerr-Schild coordinates, the singularity is given by $z = 0$, $\rho = a$, that is, it has a *ring character*. The Kerr-Schild coordinates are also "just coordinates" of course, so it is at place to compute, for example, a proper radius of that circle,

$$\int_0^{\pi/2} \sqrt{g_{\theta\theta}(r=0)}\, \mathrm{d}\theta = \int_0^{\pi/2} \sqrt{\Sigma(r=0)}\, \mathrm{d}\theta = a \int_0^{\pi/2} \cos\theta\, \mathrm{d}\theta = a\,,$$

or the proper area of the whole disc $r = 0$,

$$\int_0^{2\pi}\int_0^{\pi/2} \sqrt{(g_{\theta\theta}g_{\phi\phi})_{r=0}}\, \mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi/2} \sqrt{\mathcal{A}(r=0)}\, \sin\theta\, \mathrm{d}\theta\mathrm{d}\phi = 2\pi a^2 \int_0^{\pi/2} \cos\theta \sin\theta\, \mathrm{d}\theta = \pi a^2\,.$$

We see the quantities are even related in a Euclidean way. In the Kerr-Schild coordinates, the derivation is still simpler, since $g_{\rho\rho}(r = 0) = 1$ and $g_{\psi\psi}(r = 0) = \rho^2$ (the metric (16.1) becomes flat on $r = 0$). Hence, the proper radius and proper area of the disc are obtained like

$$\int_0^a \sqrt{g_{\rho\rho}(r=0)}\, \mathrm{d}\rho = \int_0^a \mathrm{d}\rho = a\,, \qquad \int_0^{2\pi}\int_0^{a} \sqrt{(g_{\rho\rho}g_{\psi\psi})_{r=0}}\, \mathrm{d}\rho\mathrm{d}\psi = \int_0^{2\pi}\int_0^{a} \rho\, \mathrm{d}\rho\mathrm{d}\psi = \pi a^2\,.$$

- Also different has to be discussion of the light cones (for radial motion), because the space-time is not spherically symmetric and so it is not clear what is the *radial direction*. If restricting to the *coordinate* radial direction, thus taking $\mathrm{d}\phi = 0$, one has

$$0 = \mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 \quad \Longrightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{\Sigma}{\sqrt{\Delta(\Sigma - 2Mr)}}\,,$$

so the cones close to the vertical direction at the static limit where $\Sigma = 2Mr$. Well, this only confirms that if something wants to stay at the static limit, it has to be a photon. However, the horizon is still more "downtown", so it is still possible to stay at constant r below the static limit, or even to travel outwards from there, one only needs a suitable angular velocity in the azimuthal direction. We will make this claim precise below, but let us at least add two examples: with the angular velocity $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{a}{r^2+a^2}$ (corresponding to the so-called principal null congruence, see later), one finds from (16.3)

$$0 = \mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\left(1 - \frac{a^2\sin^2\theta}{r^2+a^2}\right)^2\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 = -\frac{\Sigma\Delta}{(r^2+a^2)^2}\,\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2$$
$$\Longrightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r^2+a^2}{\Delta}\,.$$

Similarly, with the angular velocity $\frac{d\phi}{dt} = \omega$ (corresponding to zero axial angular momentum, see below), one finds from (16.2)

$$0 = ds^2 = -\frac{\Sigma\Delta}{\mathcal{A}}\,dt^2 + \frac{\Sigma}{\Delta}\,dr^2 \qquad \Longrightarrow \qquad \frac{dt}{dr} = \pm\frac{\sqrt{\mathcal{A}}}{\Delta}\,.$$

The light cones defined in either of these ways only "close" (and prevent outward travelling) at the horizon.

A generic conclusion is that the light cones behave similarly as in the Reissner-Nordström field, but, due to dragging, they simultaneously get more and more tilted in the positive-ϕ direction as the centre is approached.

16.3 Dragging of inertial frames

In Newton's theory of gravitation, the field of a spherically symmetric source is given by $-GM/r^2$, independently of whether the source rotates or not – there is no difference between static and stationary situation. In electrodynamics, there *is* a difference – rotating charged bodies generate, besides the electric field Q/r^2, also a magnetic one, because moving charges $\equiv$ current. The notions "electric" and "magnetic" are of course observer-dependent, but it is in general not possible to transform out any of these components. In GR it works similarly, only that mass currents play the role instead of charge currents. Two differences (from electrodynamics) may be mentioned:

- Electric charge is invariant, whereas mass increases with relative speed ($m = m_0\gamma$ from special relativity), thus so does the energy $E = mc^2$. By the equivalence principle, this should equally hold for inertial as well as gravitational mass. Hence, the scalar ("gravito-electric") part of the gravitational field is *also* affected by motion – actually, a mutual force between two moving bodies contains the factor γ^2 (given by their relative speed).

- Yet GR does not only differ from Newton's theory in the "(gravito)magnetic" effects known from electrodynamics: it brings curvature of space(-time) in addition.

According to Mach's views, the inertial space should behave kind-of like viscous fluid which is being "mixed" – i.e. dragged along – by matter. Specifically in the case of a *rotating* body, free test particles (inertial frames) around should be carried away along the rotation, the more the closer to the body they are; torque-free test gyroscopes anchored at the axis should precess in the direction of centre's rotation, while those placed in the radial direction to the equatorial plane should precess *against* the direction of rotation (since dragging should be differential, it should drop off with distance from the rotating source). It indeed goes like this in the Kerr space-time (cf. Figure 18.2 later then). However, since the Kerr solution describes an *isolated* source, Mach would certainly ask *relative to what* it actually rotates. It rotates with respect to an asymptotic inertial frame, defined by test particles resting at radial infinity. Mach's "distant celestial masses" are thus represented by boundary conditions at infinity.[5]

[5] Sure, this is a partial retreat from Machian positions. After all, the very notion of inertia of an isolated source is problematic by Mach – there are no masses which the source might be referred to (and the less so which it might interact with). In relativity, even a single test particle in Minkowski does have inertia.

In the following, we show that the Kerr metric really describes *geometry dragged along by rotation of the source*.

16.3.1 Comparison with flat metric in rotating coordinates

We already know from Section 1.4.2 that the metric term $(\mathrm{d}\phi \pm \omega \mathrm{d}t)^2$ is connected with rotation. Actually, by transforming the Minkowski metric

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2$$

from some inertial frame (T, X, Y, Z) to cylindrical coordinates rotating with an angular velocity $-\omega = \mathrm{const}$,

$$T = t, \qquad X = \rho\cos(\phi - \omega t), \qquad Y = \rho\sin(\phi - \omega t), \qquad Z = z\,,$$

one obtains

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \rho^2(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2\,.$$

Now, the Boyer-Lindquist coordinates are "non-rotating", at least asymptotically – they correspond to an asymptotic inertial system since the Kerr metric written in them goes over, at $r \to \infty$, to a flat metric in spherical coordinates. Hence, we can guess that the Kerr metric corresponds to a *rotating geometry in non-rotating coordinates* (anchored to infinity).

Comparison of (16.2) with the above flat metric in rotating coordinates indicates that ω represents angular velocity with which the inertial space / the geometry is being dragged by rotation of the centre, taken with respect to the asymptotic inertial frame. The question might arise naturally: isn't it possible to perform, analogously as above, a transformation to co-rotating coordinates which would erase the dragging term? No, it is not possible because in the Kerr metric ω is *not constant* (it depends on r and θ) – *dragging is differential*. Actually, the transformation $T = t$, $\varphi = \phi - \omega t$, $R = r$, $\vartheta = \theta$ does make the desired job,

$$g_{T\varphi} = \frac{\partial t}{\partial T}\frac{\partial \phi}{\partial \varphi}\, g_{t\phi} + \frac{\partial \phi}{\partial T}\frac{\partial \phi}{\partial \varphi}\, g_{\phi\phi} = g_{t\phi} + \omega g_{\phi\phi} = g_{t\phi} - g_{t\phi} = 0,$$

but, on the other hand, it brings a new non-diagonal component (which is time-dependent on top of that)

$$g_{R\vartheta} = \frac{\partial \phi}{\partial R}\frac{\partial \phi}{\partial \vartheta}\, g_{\phi\phi} = \frac{\partial \omega}{\partial R}\frac{\partial \omega}{\partial \vartheta}\, T^2 g_{\phi\phi} \left(= \frac{\partial \omega}{\partial r}\frac{\partial \omega}{\partial \theta}\, t^2 g_{\phi\phi} \right).$$

Similar time-dependent terms also arise in another components, $g_{RR} = g_{rr} + (\omega_{,r}t)^2 g_{\phi\phi}$ and $g_{\vartheta\vartheta} = g_{\theta\theta} + (\omega_{,\theta}t)^2 g_{\phi\phi}$.

16.3.2 Stationary circular orbits in the Kerr field

Space-time features may very well manifest on families of motions which "very well fit in a given background". In our case, such a privileged family is that of stationary circular motions, i.e. of observers which orbit with uniform angular velocity $\Omega := \mathrm{d}\phi/\mathrm{d}t$ on circles $r = \mathrm{const}$, $\theta = \mathrm{const}$. Namely, such observers exactly follow the space-time symmetries and, consequently, perceive the geometry in their vicinity as stationary.[6] The four-velocity of stationary circular motions is proportional to a combination of the two Killing vector fields,[7]

$$u^\mu = \frac{t^\mu + \Omega\phi^\mu}{|t^\mu + \Omega\phi^\mu|} = \frac{t^\mu + \Omega\phi^\mu}{\sqrt{-g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa)}}\,, \tag{16.15}$$

so, in the BL coordinates where

$$t^\mu = \delta^\mu_t, \quad \phi^\mu = \delta^\mu_\phi, \qquad g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa) = g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2\,,$$

the four-velocity has components

$$u^\mu = u^t(1,0,0,\Omega)\,, \quad u^t = \frac{1}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}} = \frac{1}{\sqrt{N^2 - g_{\phi\phi}(\Omega-\omega)^2}}\,. \tag{16.16}$$

The angular velocity with respect to an asymptotic inertial system, Ω, cannot be arbitrary – too large values would correspond to super-luminal (space-like) motion. The interval of time-like motion has boundaries where u^μ can no longer be normalized by any real u^t, i.e. at the roots of

$$g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = 0\ :$$

$$\begin{aligned}\Omega_{\max,\min} &= \frac{-g_{t\phi} \pm \sqrt{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \\ &= \omega \pm \frac{\sqrt{-g_{t\phi}\omega - g_{tt}}}{\sqrt{g_{\phi\phi}}} \equiv \omega \pm \frac{N}{\sqrt{g_{\phi\phi}}} = \omega \pm \frac{\Sigma\sqrt{\Delta}}{\mathcal{A}\sin\theta}\,.\end{aligned} \tag{16.17}$$

It is illustrated in Figure 16.3. The above simple analysis brings several important observations:[8]

- In the spherically symmetric *Schwarzschild* field (and as well in Reissner-Nordström), $\omega = 0$ and the light cone is of course symmetric about $\Omega = 0$ (the $\pm\phi$ directions are equivalent),

$$\Omega_{\max,\min} = \pm\sqrt{\frac{-g_{tt}}{g_{\phi\phi}}} = \pm\frac{\sqrt{1-\frac{2M}{r}}}{r\sin\theta}\,.$$

[6] These motions are *not* in general geodesic; there do exist circular geodesics, but only as a special subclass of *equatorial* circular orbits. (With one marginal exception, there do not exist circular geodesics off the equatorial plane.)

[7] This combination is even linear, because $\Omega = \mathrm{const}$, but only along each single orbit, so it does not represent a Killing *field*. Stationary circular orbits are thus sometimes called *quasi-Killing trajectories*.

[8] When looking at the formula, it's good to realize that, for $r > 0$, it holds $\Sigma > 0$ and $\mathcal{A} > 0$.

The interval narrows down as $1/r$ asymptotically, while it also shrinks to just $\Omega=0$ at the horizon where $g_{tt}=0$. (This limit value corresponds to a photon standing at the horizon.)

In the Kerr field, the whole time-like interval is shifted in the positive-Ω direction, in accord with how light cones are dragged in the positive-ϕ sense. Apparently it is ω which plays the role of the central value, so it is interpreted as the angular velocity with which the geometry (the inertial space) is dragged along by rotation of the source, as taken with respect to an asymptotic inertial frame.

- Note in particular that it is possible to define the horizon *invariantly* as the surface where the interval of permitted Ωs of circular orbiting shrinks to just a single value. This happens where the lapse function vanishes, $N=0$ (which agrees with $\Delta=0$), and the value reads

$$\Omega_{\rm max} = \Omega_{\rm min} = \omega(N=0) = \frac{2Mr_+a}{\mathcal{A}(r_+)} = \frac{2Mr_+a}{(r_+^2+a^2)^2} = \frac{a}{2Mr_+} =: \omega_{\rm H}\,. \tag{16.18}$$

This value is naturally interpreted as angular velocity of the (outer) horizon, it is positive and *constant all over the horizon*. Actually, it does not depend on θ (and of course not on t, r and ϕ). The horizon thus rotates "as a rigid body"; this result (called the *rigidity theorem*) holds for all stationary horizons and is very important at many places of the black-hole theory.

- With the meaning of ω clarified, one can quantify the inertial dragging: it falls off radially as $2Ma/r^3$, so much faster than deviation from the flat $g_{tt}=-1$ (i.e. $2M/r$). On the other hand, when approaching the black hole, the azimuthal dragging becomes stronger than the radial attraction, since it definitively forces everything to co-rotate at r_0 (while the radial attraction only wins definitively at r_+ which is lower). Actually, the appearance of the static limit at r_0 – of this extreme manifestation of dragging – is very well seen in Figure 16.3, since it lies where $\Omega_{\rm min}$ crosses zero. The formula (16.17) confirms that this happens where $g_{tt}=0$. In the next chapter, we will see that the region between r_+ and r_0 – the so-called ergosphere – has remarkable properties.

- Hitherto, we have supposed the black-hole case ($a \leqslant M$). If there are no horizons, one checks the limit behaviour of $\Omega_{\rm max,min}$ at $r=0$:

$$\omega(\theta\neq\pi/2, r\to 0) = 0\,, \quad \omega(\theta=\pi/2, r\to 0) = \frac{1}{a}\,, \qquad \frac{N}{\sqrt{g_{\phi\phi}}}(r\to 0) = \frac{1}{a\sin\theta}\,.$$

Therefore, there is no dragging over the central disc $r=0$, while its singular rim ($r=0$, $\theta=\pi/2$) rotates with the angular velocity $1/a$.

Invariance of the Killing part of the metric

Above, we stressed that the properties of circular orbits provide an invariant way how to localize the horizon – by $N=0$. This is a suitable moment to also stress that in fact the whole

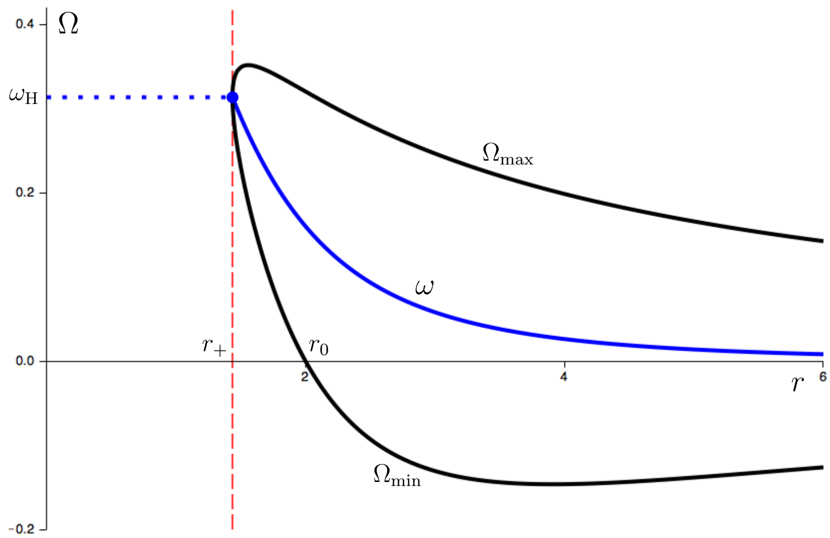


Figure 16.3 *The interval of angular velocities Ω within which the circular motion in the Kerr field (with $a = 0.9M$) is time-like, plotted for equatorial orbits ($\theta = \pi/2$) in dependence on the radius r. The central value ω is in blue. On the horizon r_+, the interval shrinks to a single limit value $\omega_{\rm H}$ (which is constant everywhere on it). Radius r_0 of the outer static limit is clearly visible (in the equatorial plane it is $r_0 = 2M$ independently of a. The radius axis is in the units of M, the Ω axis is in the units of $1/M$. Correspondingly to how light cones are tilted by dragging in the positive-ϕ direction, the interval of permitted Ωs is shifted towards $\Omega > 0$ with respect to Schwarzschild where it is symmetric about $\Omega = 0$.*

"Killing" part of the metric has an invariant meaning, because it is completely determined by scalar products of the Killing vector fields,

$$g_{tt} = g_{\iota\kappa}t^\iota t^\kappa\,, \quad g_{t\phi} = g_{\iota\kappa}t^\iota \phi^\kappa\,, \quad g_{\phi\phi} = g_{\iota\kappa}\phi^\iota\phi^\kappa\,. \tag{16.19}$$

Consequently, any quantity solely given by these metric components is invariant as well – to ω and N it applies in particular.

Finally, let us also derive four-acceleration of the four-velocity (16.16),

$$\begin{aligned} a_\mu &= \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - \Gamma^\iota{}_{\mu\kappa}u_\iota u^\kappa = -\Gamma_{\iota\mu\kappa}u^\iota u^\kappa = -\frac{1}{2}(g_{\iota\mu,\kappa}+g_{\kappa\iota,\mu}-g_{\mu\kappa,\iota})u^\iota u^\kappa = -\frac{1}{2}\,g_{\kappa\iota,\mu}u^\iota u^\kappa = \\ &= -\frac{1}{2}(u^t)^2(g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2) = \frac{1}{2}\,\frac{g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2}{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2}\;; \end{aligned} \tag{16.20}$$

we have used the stationarity of the motion, thus constancy of u_μ along the orbit, and symmetry of $u^\iota u^\kappa$ due to which the term $(g_{\iota\mu,\kappa} - g_{\mu\kappa,\iota})$ antisymmetric in (ι,κ) drops out in the multiplication. The result holds for any stationary and axisymmetric space-time (at least if it is also *orthogonally transitive*, see later), and its main aspect is that the components a_t and a_ϕ are always zero.

Equatorial circular geodesics

In reflection symmetric space-times where the equatorial plane exists, it is natural to specifically study the motions in that plane. First, the dragging effects are typically the strongest in the equatorial plane (ω really increases from $\theta = 0$ to $\theta = \pi/2$). Second, such motions are simpler since their "vertical" acceleration (a_θ in the BL coordinates) identically vanishes. Focusing back on the stationary circular motions, one can now look for equatorial *geodesics*. They are given by vanishing of the remaining, radial component of acceleration (16.20),

$$a_r(\theta\!=\!\pi/2) = 0 \qquad \Longleftrightarrow \qquad \Omega = \Omega_\pm = \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}} = \frac{1}{a \pm \sqrt{r^3/M}}\,, \tag{16.21}$$

with the upper/lower signs representing the prograde and retrograde senses of orbiting (with respect to ω). The effect of the centre's rotation is again revealed by comparing the parameters of the prograde and retrograde circular geodesics: generally, the prograde geodesics lie closer to the horizon, so the centre attracts them less then the retrograde ones.[9] One may specifically illustrate this on **photon circular geodesics** whose radii are determined by equations $\Omega_\pm = \Omega_{\rm max,min}$ and which, for $a \leqslant M$, come out as

$$r_{\rm ph\pm} = 2M\left\{1 + \cos\left[\frac{2}{3}\arccos\left(\mp\frac{a}{M}\right)\right]\right\}. \tag{16.22}$$

The prograde photon orbit always lies below the retrograde one, $r_{\rm ph+} < r_{\rm ph-}$, specifically in the extreme limit $a\!=\!M$ one finds $r_{\rm ph+}\!=\!M$, $r_{\rm ph-}\!=\!4M$.

16.3.3 To orbit or not to orbit: ZAMOs

Are you sure about how long the month is? Astronomers offer 5 answers at least, depending on with respect to what the Moon orbit is taken as "completed" – with respect to "fixed stars", with respect to the Sun-Earth connecting line, with respect to the ecliptic plane, with respect to a fixed ecliptic longitude, or as taken from perigee to perigee...

We see now that in GR there is still another possibility – to orbit (or not to orbit) *with respect to the geometry*. For Schwarzschild it is not an issue since the field is spherically symmetric (no dragging), but for a rotating centre this might undoubtedly be a better option than to refer to infinity, because "standing with respect to infinity" (i.e. having $\Omega = 0$) is not at all time-like below the static limit. However, well justified options (useful down to the very horizon) are several, either tied to physical parameters or geometric characteristics of the orbit itself, or, for instance, to the behaviour of gyroscopes carried along it. Without going into details of these reasonable alternatives, let us only mention the subclass of stationary circular motions characterized by $\Omega\!=\!\omega$.

---

[9] This corresponds to the above-mentioned dependence of the attraction between masses on the relative γ-factor squared. One might also mention the analogy with electromagnetism: parallel electric currents attract each other magnetically, while anti-parallel ones repel. In gravitation, the signs are opposite, similarly as in "electric" component of the fields (electric charges of the same sign repel each other, whereas masses of the same sign attract each other).

- The angular velocity ω lies right in the middle of the time-like interval $(\Omega_{\rm min}, \Omega_{\rm max})$, so it represents "central line of the light cone".

- The axial angular momentum (per unit mass)

$$\tilde{L} \equiv u_\phi = g_{\phi\iota} u^\iota = g_{\phi\phi} u^\phi + g_{t\phi} u^t = g_{\phi\phi} u^t (\Omega - \omega) \tag{16.23}$$

  is clearly zero for $\Omega = \omega$, which is why the observers in such circular orbits are called **zero-angular-momentum observers** (ZAMOs). For more or less obvious reasons, they are also sometimes called **locally non-rotating**.

- Consider the following exercise. Imagine that an observer in stationary circular motion launches two photons in the opposite directions and forces them to fly along the same circular orbit (for example by using a waveguide or a cylindrical mirror). The photons go around the centre and return to the observer. It turns out that they return *at the same moment* if and only if the observer's angular velocity is $\Omega = \omega$.

- In special relativity, an observer at rest with respect to a given inertial system has $u^i = 0$; this can be translated in a geometrical language by saying that the four-velocity u^μ is orthogonal to the hypersurfaces $t = \text{const}$. Now consider the ZAMOs: they have $u_\phi = 0$, thus $u_\mu = (u_t, 0, 0, 0)$, so the latter's scalar product with any vector $s^\mu = (0, s^i)$ tangent to $t = \text{const}$ (t being the Killing time now) is clearly zero, $u_\mu s^\mu = u_t s^t + u_i s^i = 0$. Hence, the ZAMOs are **orthogonal to hypersurfaces** (of constant t, specifically), which in itself is a privilege – it means, for example, that the congruence of such motions has zero vorticity (see Frobenius theorem in Section 24.4).

- That the ZAMO congruence very well fits into the geometry is also seen from its four-acceleration: taking the generic circular-orbit formula (16.20) and checking that, for $\Omega = \omega \equiv -g_{t\phi}/g_{\phi\phi}$,

$$g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega = -N^2 \,,$$
$$g_{tt,\mu} + 2g_{t\phi,\mu}\omega + g_{\phi\phi,\mu}\omega^2 = (g_{tt} + g_{t\phi}\omega)_{,\mu} = (-N^2)_{,\mu} = -2NN_{,\mu} \,,$$

  one obtains

$$a_\mu = \frac{1}{2}\frac{-2NN_{,\mu}}{-N^2} = \frac{N_{,\mu}}{N} \,. \tag{16.24}$$

  The lapse is often being expressed in terms of the gravitational potential Φ, as $N = e^\Phi$; then the ZAMO's acceleration is just $a_\mu = \Phi_{,\mu}$.

- A very intuitive picture, finally: imagine a set of test particles released from rest from radial infinity, with zero angular momentum L. They freely fall towards the centre, just following what the field does with them. Since they are geodesic, L is conserved, so it has to stay zero all along the fall. However, we have seen in (16.23) that zero L necessarily means $\Omega = \omega$ (the formula is general, not restricted to just stationary circular motions), so rather than falling radially, they spiral towards the centre.[10] This is a sound illustration of dragging, because in the Newtonian gravity such particles would fall perfectly radially, irrespectively of whether the centre rotates or not. (The picture is really simple since it turns out, in addition, that such particles exactly follow $\theta = \mathrm{const}$ – see Section 17.3.10.)

16.4 More on spatial structure of Kerr

16.4.1 Through the hoop: the second sheet of the metric

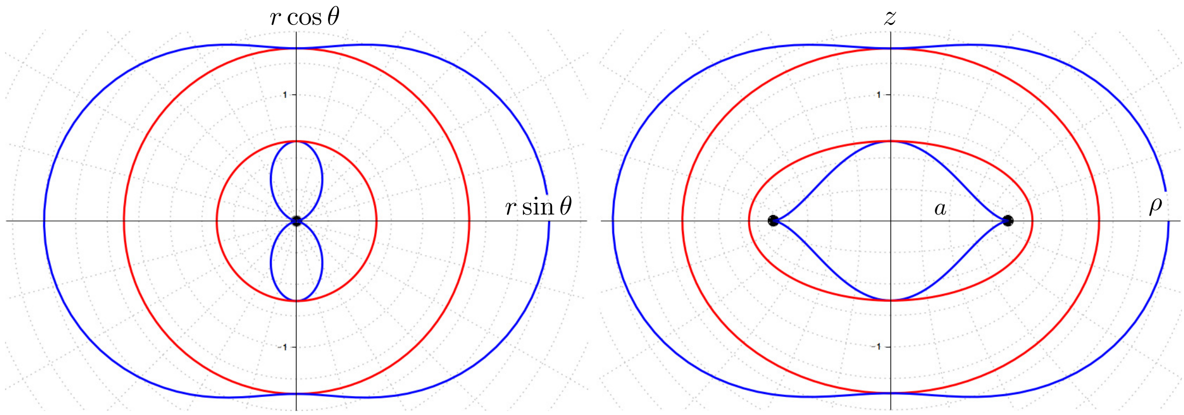


Figure 16.4 *Meridional section through the central part of the Kerr space-time with $a = 0.93M$, in the Boyer-Lindquist (**left**) and Kerr-Schild (**right**) coordinates. Blue are static limits and red are horizons. The inner static limit goes to the singularity which is indicated as a black bullet. Dotted grey is the BL coordinate mesh, in particular, the ellipsoids $r/M = 0.25,\ 0.5,\ 0.75,\ 1,\ \ldots$ and the hyperboloids $\theta = 15°,\ 30°,\ 45°,\ \ldots$ are shown. Axes denote the equatorial plane and the symmetry axis, in the units of M. The right plot is obtained from the left one by stretching the central region so that the point $r=0$ becomes the whole disc $\rho \leqslant a$ spanned by the singularity.*

Figure 16.4 shows the meridional section through the central part of the Kerr space-time with $a = 0.93M$, in the Boyer-Lindquist (**left**) as well as Kerr-Schild (**right**) coordinates. The surfaces $r_0 \geqslant r_+ \geqslant r_- \geqslant r_1 \geqslant 0$ we already know, as well as the ring singularity at $z=0$, $\rho=a$ (the fat point in the section; in the BL coordinates it degenerates to the origin). Focus now on the central disc spanned by the singularity, i.e. $r=0, \theta<\pi/2$ in the BL and $z=0, \rho<a$ in the KS coordinates. Let us stress once again that this disc is *not* singular. Actually, it is easy to find that $N=1$, $g_{\phi\phi}=a^2\sin^2\theta$, $\omega=0$, so the metric reduces there to

$$\mathrm{d}s^2(r=0, \theta\neq\pi/2) = -\mathrm{d}t^2 + a^2\left(\sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\theta^2\right),$$

[10] In fact they wind about the horizon indefinitely before plunging into the black hole. Indeed, we know the angular velocity remains finite at the horizon, $\Omega \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega(r=r_+) \equiv \omega_{\mathrm{H}} = \frac{a}{2Mr_+}$, whereas the coordinate radial velocity $\mathrm{d}r/\mathrm{d}t$ vanishes there (similarly as in the Schwarzschild case, this is due to an infinite dilation between t and any proper time at the horizon). Hence, $\mathrm{d}r/\mathrm{d}\phi$ must vanish there as well, which corresponds to infinite winding.

the Kretschmann invariant (16.10) amounts to $K = -\frac{48M^2}{(a\cos\theta)^6}$ and the Chern-Pontryagin invariant (16.11) vanishes.

What happens if one crosses that disc? At $r \to 0^+$, the radial derivative $g_{\theta\theta,r} = 2r$ goes to zero, but the normal gradient of other metric components does *not* vanish,

$$g_{tt,r} \to \frac{2M}{a^2\cos^2\theta}\,, \quad g_{t\phi,r} \to -\frac{2M\sin^2\theta}{a\cos^2\theta}\,, \quad g_{rr,r} \to \frac{2M\cos^2\theta}{a^2}\,, \quad g_{\phi\phi,r} \to \frac{2M\sin^4\theta}{\cos^2\theta}\,.$$

Since crossing the $r = 0$ means switching the radial-gradient sign, one experiences jump in the normal gradient of the metric. Metric gradient means field, so there occurs jump in the normal field – which implies there has to be a matter layer over the disc! From Einstein equations, it is possible to compute the corresponding energy-momentum tensor, and if interpreting the latter as a dust layer, one obtains a negative surface density $\sigma = -\frac{M}{2\pi a^2\cos^2\theta} = -\frac{Ma}{2\pi(a^2-\rho^2)^{3/2}}$. The overall mass in space-time has to come out positive (M) of course, and this is ensured by *positively* infinite density at the very singularity, but the negative-density layer is anyway not very satisfactory. Is there any solution? The only one is to allow for negative radii r: if r continued, across the disc $r = 0$, to negative values, there would be no field jump across that disc – the metric would continue smoothly to the new region. This new "sheet" of the metric just differs by the sign of r, which however only matters in the terms $2Mr$, so if preferring to keep the radius r non-negative everywhere, we see the new sheet physically differs in that the source mass M appears there negative (which also means reverse of the sign of dragging since $J = Ma$, $\omega = 2Mra/\mathcal{A}$). Anyway, with $2Mr$ negative, neither Δ nor g_{tt} can be made vanish, so the second sheet contains neither horizons nor static limits.

16.4.2 The third horizon, or what?

Still the $r < 0$ region is far from boring: notice that it contains a time machine. *Wow!!!* Consider the function $\mathcal{A}$, best in the form $\mathcal{A} = \Sigma(r^2 + a^2) + 2Mra^2\sin^2\theta$. At $r > 0$ it is everywhere positive, including the $r = 0$ disc where it reduces to $a^4\cos^2\theta$; at the singularity, it vanishes (similarly as Σ). However, the second term, linear in $2Mr$, makes $\mathcal{A}$ *negative* in a certain toroidal region which surrounds the singularity in the second metric sheet. Negative $\mathcal{A}$ (and r) implies $N^2 \equiv \Sigma\Delta/\mathcal{A} < 0$, $\omega \equiv 2Mra/\mathcal{A} > 0$ and, mainly, $g_{\phi\phi} \equiv (\mathcal{A}/\Sigma)\sin^2\theta < 0$.

Imagine now someone on a circular orbit (the one we studied above). Along such an orbit,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{\phi\phi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2\,.$$

If $\mathcal{A} > 0$, the second term contributes positively, so one needs to also add a sufficiently large (negative) first term in order that the motion be time-like. But if $\mathcal{A} < 0$, the second term is negative and thus itself ensures time-like character of the motion! It is *not* necessary to add the first term, so time t may stay constant. Moreover, if t *should* tick, it is better to have $\mathrm{d}t < 0$, because $\omega > 0$ and so such an option makes the $\mathrm{d}s^2$ more negative than the opposite one.[11] To be on the safe side, the observer's proper time ticks normally of course, but in

[11] Notice that the first term of the interval is positive, because $N^2 \equiv \Sigma\Delta/\mathcal{A}$ is *negative* in the $\mathcal{A} < 0$ region – so the negativity of the second term is really necessary for the motion to be time-like.

terms of time of an observer at $|r| \gg M$ such a motion directs towards the past. Now, it is possible to arrange for the following show: two colleagues stay far from the centre; at a certain moment, one of them flies "down", gets into the $\mathcal{A}<0$ region and orbits against the sense of the far-observer time for a sufficiently long period; then this traveller returns to the distant colleague. During flying down and back, t was ticking onwards of course, but if the traveller was orbiting in the "time-machine" region for long enough, they can return *before* starting the trip. In the naked-singularity case ($a > M$) when there are no horizons, such a mission can even be accomplished from the $r>0$ side. The occurrence of time machine plays a very disturbing role in physics, so in most theorems it is supposed there do not exist any **closed time-like curves** (CTCs) in the space-time, at least not outside black holes.

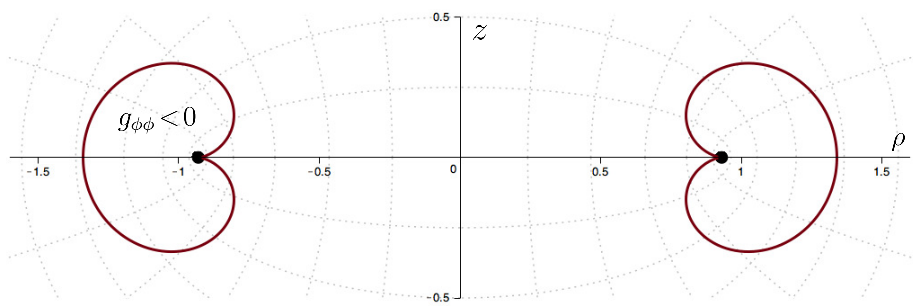


Figure 16.5 *The chronology-violating region where $\mathcal{A} < 0$ (hence $g_{\phi\phi} < 0$), surrounding the singularity (black bullets) in the second sheet ($r<0$) of the Kerr space-time (here specifically with $a=0.93M$). The plot is drawn in the Kerr-Schild coordinates, with the mesh of Boyer-Lindquist coordinates indicated in dotted grey. Values along the axes (equatorial plane and symmetry axis) are in the units of M. The region is* dynamical, *yet mainly very weird (see the main text).*

In order to more grasp the nature of the $\mathcal{A}<0$ region, recall the behaviour of the metric coefficients across the static limits and across the horizons:

- At the static limits, the time Killing field is null, $g_{\mu\nu}t^\mu t^\nu = g_{tt} = 0$, turning from time-like (outside) to space-like (between them). The surfaces are given by $\Sigma = 2Mr$ which does not mean anything so special for the rest of the metric.

- At the horizons, the vector field $t^\mu + \omega\phi^\mu$ is null,

$$g_{\mu\nu}(t^\mu + \omega\phi^\mu)(t^\nu + \omega\phi^\nu) = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega = -N^2 = 0\,,$$

  turning from time-like (outside) to space-like (between them). When entering between the horizons, both N^2 and g_{rr} turn from positive to negative (the "dynamical region"). And recall that, on the horizons, $t^\mu+\omega\phi^\mu$ is a Killing field, because *there* ω is constant. Besides that, horizons are null hyper-surfaces, because $g^{\mu\nu}r_{,\mu}r_{,\nu}=g^{rr}=0$ on them. Hence, they are the *Killing horizons*.

- Now compare the $\mathcal{A}=0$ surface: there, the axial Killing field is null, $g_{\mu\nu}\phi^\mu\phi^\nu = g_{\phi\phi} = 0$, turning from positive (outside) to negative (inside). Similarly as at the horizons, both N^2

and $g_{\phi\phi}$ turn from positive to negative (with N^2 jumping between $\pm\infty$ rather than crossing zero, however); the $\mathcal{A}<0$ region is thus *dynamical*, similarly as that between the horizons. However, its surface is *not* a null hypersurface, as verified by computing the norm of the $\mathcal{A}=\text{const}$ surfaces,

$$g^{\mu\nu}\mathcal{A}_{,\mu}\mathcal{A}_{,\nu} = g^{rr}(\mathcal{A}_{,r})^2 + g^{\theta\theta}(\mathcal{A}_{,\theta})^2 = \frac{\Delta}{\Sigma}\left[(\mathcal{A}_{,r})^2 + 4a^4\cos^2\theta\sin^2\theta\right].$$

This only vanishes in the equatorial plane (we do not consider $\theta=0$ since $\mathcal{A}=0$ never extends to the axis), if $\mathcal{A}_{,r}$ vanishes there, so $\mathcal{A}=0$ is certainly not a null hypersurface.

Finally, let us repeat once more that in the $\mathcal{A}<0$ region, dragging points in the opposite direction than in the rest of the $r<0$ sheet, namely ω is *positive* there (strangely enough, jumping between $\pm\infty$ across its surface $\mathcal{A}=0$, similarly as N^2).

- Finally, just to recall: none of the above surfaces is a physical singularity, because the curvature invariants (16.10) and (16.11) only blow up at $\Sigma=0$.

To summarize, the $\mathcal{A}<0$ region is a *dynamical* region, similarly as the one between the horizons, yet still its boundary is *not* null, so it is not a horizon. It rather has similar character as the static limit, only that it is the Killing field ϕ^μ (rather than t^μ) which becomes null there.

16.4.3 Repulsion at the bottom

Last interesting point. Recall the recipe (16.20) for the four-acceleration of stationary circular orbits. By computing the metric derivatives, one finds that its non-zero components read

$$a_r = \frac{(u^t)^2}{\Sigma^2}\left[M(r^2-a^2\cos^2\theta)(1-a\Omega\sin^2\theta)^2 - r(\Sigma\Omega\sin\theta)^2\right], \tag{16.25}$$

$$a_\theta = -\frac{(u^t)^2}{2\Sigma^2}\left\{2Mr[a-(r^2+a^2)\Omega]^2 + \Delta\Sigma^2\Omega^2\right\}\sin 2\theta\,. \tag{16.26}$$

For $r>0$ and $\Delta>0$ (i.e., outside the dynamical region where circular orbits are impossible), we clearly see that $a_\theta\leqslant 0$, which means that in the latitudinal direction all test motions on stationary circular orbits are being pulled towards the equatorial plane. What about radial pull? Anything static with respect to infinity ($\Omega=0$) has

$$a_r = M\,\frac{(u^t)^2}{\Sigma^2}(r^2-a^2\cos^2\theta)\,,$$

which means that in the region $r^2>a^2\cos^2\theta$ it is being pulled in the negative radial sense (a_r has to be positive in order to guarantee staying at rest), but inside that region the opposite holds. One would thus describe the effect of the centre as "attraction" at $r^2>a^2\cos^2\theta$, whereas as "repulsion" at $r^2<a^2\cos^2\theta$. Note that in the Kerr-Schild coordinates the border $r^2=a^2\cos^2\theta$ is given by

$$\rho^2+z^2 \equiv (r^2+a^2)\sin^2\theta + r^2\cos^2\theta \equiv r^2+a^2\sin^2\theta = a^2\cos^2\theta + a^2\sin^2\theta \equiv a^2\,,$$

so it is a sphere spanned by the singularity (at $\rho = a$, $z = 0$). Admittedly, due to dragging, we know $\Omega = 0$ is not the best option for "staying at rest". However, look at (16.25) once more: if $r^2 < a^2 \cos^2\theta$, it is in fact impossible to reach $a_r > 0$ by *any* choice of Ω. Note also that specifically at $r = 0$ (where $\omega = 0$, hence where $\Omega = 0$ *is* a very reasonable angular velocity) the particles at rest have

$$a_r = -\frac{M}{a^2\cos^2\theta}\,, \qquad a_\theta = 0\,; \qquad\qquad g^{\mu\nu}a_\mu a_\nu = g^{rr}(a_r)^2 = \frac{M^2}{a^4\cos^6\theta}\,.$$

The "repulsive" effect is clearly connected with what we encountered when trying to interpret the Kerr space as solely the $r \geqslant 0$ region – the negative-mass layer was induced over the circle $r = 0$.

16.5 Maximal extension: Kruskal and Penrose-Carter diagrams

The space-time structure is almost the same as for the Reissner-Nordström, the only difference being that now the space-time is not spherically symmetric, so it is different at different θ. More specifically, the diagrams are exactly the same along $\theta = \pi/2$, whereas along different θs there is not a singularity at $r = 0$ – instead, there opens the second sheet of the metric behind the (now non-singular) line $r = 0$; that sheet, however, is simple (like the naked-singularity one) since it contains no horizons. The conformal diagrams are given in Figure 16.6.

16.6 Kerr-Newman solution of Einstein equations

The Kerr solution might have been called Newman solution if E. T. Newman did not make a sign mistake in calculations.[12] Nevertheless, in 1965 he generalized, together with collaborators, the Kerr solution to a charged case. The resulting metric is the same as (16.2), or also (16.3), the only difference appearing in the Δ function. Effectively, the term $2Mr$ has to be changed for $2Mr - Q^2$ everywhere (Q denoting the electric charge as in Reissner-Nordström),

$$2Mr \to 2Mr - Q^2 \qquad \Longrightarrow \qquad \Delta = r^2 - 2Mr + Q^2 + a^2\,, \quad \omega = \frac{a}{\mathcal{A}}(2Mr - Q^2)\,.$$

Due to this change, also some of the expressions for $\mathcal{A}$ are modified (the first remains the same)

$$\mathcal{A} := (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta = \Sigma(r^2 + a^2) + (2Mr - Q^2)a^2\sin^2\theta = \Sigma\Delta + (2Mr - Q^2)(r^2 + a^2)\,.$$

And one must not forget about the electromagnetic part. Actually, the solution is not vacuum, there is the EM field as well. The EM four-potential reads, in the BL coordinates (t, r, θ, ϕ),

$$A_\mu = \frac{Qr}{\Sigma}\,(-1, 0, 0, a\sin^2\theta)\,, \tag{16.27}$$

[12] Any lesson?

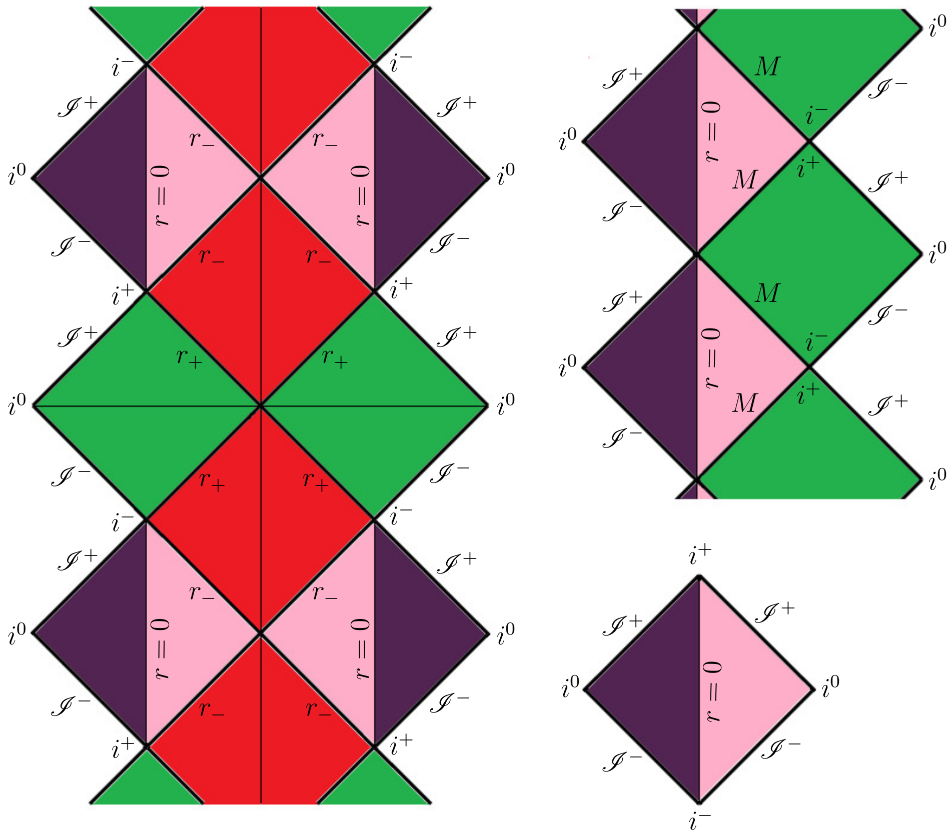


Figure 16.6 *Penrose-Carter conformal diagram of the analytical extension of some of the surfaces $(t, r, \theta \neq \pi/2)$ of the Kerr space-time. The black-hole case $(0<a<M)$ is on the left, the extreme case $(a=M)$ is on the top right and the naked-singularity case $(a>M)$ is on the bottom right. Light cones for radial motion are 45° as usual. Green are domains of outer communications (outside the horizons), red are dynamical regions between the horizons, rose are regions where one can communicate with the singularity (either the $r_->0$ regions in black-hole cases or the $r>0$ region in the naked-singularity case), and dark violet are the second sheets $(r<0)$ of the manifolds. Infinities of the second sheets have similar meaning as those of the first sheet. If we instead plotted the extended Kerr equatorial plane $(\theta=\pi/2)$, the diagrams would be the same as for Reissner-Nordström (Figure 15.3), i.e., they would end at $r=0$ which from $\theta=\pi/2$ is singular, and would not involve the $r<0$ sheets. [Super enjoyed placing all those labels in LaTeX...]*

and the corresponding EM-field tensor has non-zero components

$$F_{tr} = -\frac{Q}{\Sigma^2}\,(2r^2-\Sigma)\,, \qquad F_{\phi r} = \frac{Q}{\Sigma^2}\,(2r^2-\Sigma)\,a\,\sin^2\theta\,,$$
$$F_{t\theta} = \frac{Qr}{\Sigma^2}\,a^2\sin 2\theta\,, \qquad F_{\phi\theta} = -\frac{Qr}{\Sigma^2}\,(r^2+a^2)\,a\,\sin 2\theta\,. \tag{16.28}$$

The corresponding invariants come out

$$F^{\mu\nu}F_{\mu\nu} = -\frac{2Q^2}{\Sigma^4}\left[(r^2-a^2\cos^2\theta)^2 - 4r^2a^2\cos^2\theta\right],$$

$$
\begin{aligned}
{}^*F^{\mu\nu}F_{\mu\nu} &= \frac{8Q^2 ra\cos\theta}{\Sigma^4}\,(r^2-a^2\cos^2\theta)\,,\\
\left|F^{\mu\nu}F_{\mu\nu}-\mathrm{i}\,{}^*F^{\mu\nu}F_{\mu\nu}\right| &= \sqrt{(F^{\mu\nu}F_{\mu\nu})^2+({}^*F^{\mu\nu}F_{\mu\nu})^2} = \frac{2Q^2}{\Sigma^2}\;.
\end{aligned}
$$

Note that the EM field does not at all depend on M. For $a=0$ it reduces to the Reissner-Nordström $F_{tr}=-\frac{Q}{r^2}$ and for $Q=0$ it vanishes of course.

There again exist two horizons given by $\Delta=0$,

$$r_\pm = M \pm \sqrt{M^2-Q^2-a^2}\;, \tag{16.29}$$

two static limits given by $\Sigma-2Mr+Q^2\,(=\Delta-a^2\sin^2\theta)=0$,

$$r_{0,1} = M \pm \sqrt{M^2-Q^2-a^2\cos^2\theta}\;, \tag{16.30}$$

and the ring sigularity at $\Sigma=0$. The Kretschmann and Chern-Pontryagin invariants are somewhat longer than in Kerr. Like in the Reissner-Nordström case, they even remain non-zero for $M=0$ (but $Q\neq 0$). However unphysical the latter circumstance may be, the metric does not have problem with it – it corresponds to a naked singularity ($\Delta>0$ everywhere) which induces dragging with angular velocity $\omega=-aQ^2/\mathcal{A}$. In a generic naked-singularity situation ($Q^2+a^2>M^2$), the chronology-violating region $\mathcal{A}<0$ also partly reaches to the $r>0$ sheet of the space-time.

Yet a remark to the EM field: looking at the above invariants while recalling from Section 7.3.1 that $F^{\rho\sigma}F_{\rho\sigma}=2\hat{B}^2-2\hat{E}^2$ with $\hat{E}_\mu=F_{\mu\nu}\hat{u}^\nu$ and $\hat{B}_\mu=-{}^*F_{\mu\nu}\hat{u}^\nu$ standing for fields measured by some (arbitrary) observer $\hat{u}^\nu$, we see that the simplest option how to design the invariants would be

$$\hat{B}^2=\frac{4Q^2}{\Sigma^4}\,r^2a^2\cos^2\theta\;,\qquad \hat{E}^2=\frac{Q^2}{\Sigma^4}(r^2-a^2\cos^2\theta)^2\;.$$

Could be of interest to learn *which* particular observer family measures such fields (and whether it is sensible, at least timelike). Independence of t and ϕ indicates it has to be a subfamily of stationary circular motions, with $\hat{u}^\mu=u^t(1,0,0,\Omega)$, for which we compute

$$
\begin{aligned}
\hat{E}^2 &\equiv g^{\mu\nu}F_{\mu\alpha}\hat{u}^\alpha F_{\nu\beta}\hat{u}^\beta = g^{\mu\nu}(u^t)^2(F_{\mu t}+F_{\mu\phi}\Omega)(F_{\nu t}+F_{\nu\phi}\Omega) =\\
&= g^{rr}(u^t)^2(F_{rt}+F_{r\phi}\Omega)^2+g^{\theta\theta}(u^t)^2(F_{\theta t}+F_{\theta\phi}\Omega)^2 =\\
&= \frac{Q^2}{\Sigma^5}\,(u^t)^2\left\{\Delta(\Sigma-2r^2)^2(1-\Omega a\sin^2\theta)^2+\left[a-(r^2+a^2)\Omega\right]^2 a^2r^2\sin^2 2\theta\right\}.
\end{aligned}
$$

Simple cases – when one of the brackets vanish – are $\Omega=\frac{1}{a\sin^2\theta}$ and $\Omega=\frac{a}{r^2+a^2}$. The former behaves badly on the axis, so let us try the latter. It corresponds to

$$
\begin{aligned}
(u^t)^2 &= \frac{1}{-g_{tt}-2g_{t\phi}\Omega-g_{\phi\phi}\Omega^2} =\\
&= \frac{\Sigma(r^2+a^2)^2}{(\Sigma-2Mr+Q^2)(r^2+a^2)^2+2(2Mr-Q^2)(r^2+a^2)a^2\sin^2\theta-\mathcal{A}\,a^2\sin^2\theta} =
\end{aligned}
$$

$$= \frac{\Sigma(r^2+a^2)^2}{\mathcal{A}\Sigma - (2Mr - Q^2)(r^2+a^2)\Sigma} = \frac{\Sigma(r^2+a^2)^2}{\Delta\Sigma^2} = \frac{(r^2+a^2)^2}{\Delta\Sigma} ,$$

hence the electric-field square reads

$$\hat{E}^2 = \frac{Q^2}{\Sigma^5}\frac{(r^2+a^2)^2}{\Delta\Sigma}\,\Delta\,\frac{(\Sigma-2r^2)^2\Sigma^2}{(r^2+a^2)^2} = \frac{Q^2}{\Sigma^4}(\Sigma-2r^2)^2 = \frac{Q^2}{\Sigma^4}(r^2-a^2\cos^2\theta)^2 .$$

Good. We thus naturally "discovered" another important family of stationary observers in the Kerr-Newman space-time, known as the **Carter (canonical)** family. It has four-velocity $\hat{u}^\mu = \frac{1}{\sqrt{\Delta\Sigma}}(r^2+a^2,0,0,a)$ and its main privilege is explained in Section 17.3.7. Experience is that most of the calculations in Kerr-Newman appear especially simple in the BL-coordinate-adapted frame attached to these observers.

16.7 True shape of the horizon

In Boyer-Lindquist coordinates, horizons are spherical (they lie on constant r), but that does not necessarily tell anything about its true, intrinsic geometry. Let us evaluate important circumferences of the outer horizon: the equatorial circumference is

$$\int_0^{2\pi}\sqrt{g_{\phi\phi}(r=r_+)}\,\mathrm{d}\phi = \int_0^{2\pi}\sqrt{\frac{\mathcal{A}_+}{\Sigma_+}}\,\mathrm{d}\phi = 2\pi\frac{r_+^2+a^2}{r_+} = 2\pi\frac{2Mr_+ - Q^2}{r_+} = 4\pi M\left(1-\frac{Q^2}{2Mr_+}\right)$$

(which for $Q=0$ yields the Schwarzschildian value $4\pi M$ *irrespectively* of the rotational parameter a), while the meridional (polar) circumference is

$$2\int_0^{\pi}\sqrt{g_{\theta\theta}(r=r_+)}\,\mathrm{d}\theta = 2\int_0^{\pi}\sqrt{\Sigma_+}\,\mathrm{d}\theta = 4\sqrt{r_+^2+a^2}\;E\left(\frac{a}{\sqrt{r_+^2+a^2}}\right),$$

where $E(k) := \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\;\mathrm{d}\theta$ denotes the complete elliptic integral of the 2nd kind. When gradually spinning up a *Kerr* horizon ($Q=0$), its polar circumference decreases from $4\pi M$ to $4M\sqrt{2}\;E(1/\sqrt{2}) \doteq 0.608\cdot(4\pi M)$, so it apparently gets more and more oblate.

An interesting footnote: when $a>\sqrt{3}M/2 \doteq 0.87M$ (this is still a moderately spinning black hole), the Gauss curvature of the Kerr horizon (as a 2D surface $t=\mathrm{const}$, $r=r_+$) becomes negative at the axis; the region of negative curvature then spreads with increasing a. We may try this exercise. If not knowing any "shortcut", Gauss curvature of a 2D surface is the only independent component of the (twice mixed) Riemann tensor, in other words, half of the corresponding Ricci scalar. The outer horizon $\{t=\mathrm{const}, r=r_+\}$ has – for any Kerr-Newman black hole – the metric

$$\mathrm{d}s_+^2 = \frac{\mathcal{A}_+}{\Sigma_+}\sin^2\theta\,\mathrm{d}\phi^2 + \Sigma_+\,\mathrm{d}\theta^2 = \frac{(r_+^2+a^2)^2}{r_+^2+a^2\cos^2\theta}\sin^2\theta\,\mathrm{d}\phi^2 + (r_+^2+a^2\cos^2\theta)\,\mathrm{d}\theta^2 ,$$

for which non-zero Christoffel symbols read

$$\Gamma_{\theta\theta\theta} = -a^2 \sin\theta\cos\theta\,,$$
$$\Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{(r_+^2+a^2)^3}{\Sigma_+^2}\,\sin\theta\cos\theta\,.$$

The Ricci scalar is given by (A$=\theta,\phi$)

$$R = R^\theta_\theta + R^\phi_\phi = R^{A\theta}{}_{A\theta} + R^{A\phi}{}_{A\phi} = 2R^{\theta\phi}{}_{\theta\phi} = 2g^{\phi\phi}R^\theta{}_{\phi\theta\phi}\,,$$

and this is twice the Gauss curvature. A bit of computation,

$$\begin{aligned} R^\theta{}_{\phi\theta\phi} &= \Gamma^\theta{}_{\phi\phi,\theta} - \cancel{\Gamma^\theta{}_{\phi\theta,\phi}} + \Gamma^\theta{}_{\theta A}\Gamma^A{}_{\phi\phi} - \Gamma^\theta{}_{\phi A}\Gamma^A{}_{\phi\theta} = \\ &= \Gamma^\theta{}_{\phi\phi,\theta} + \Gamma^\theta{}_{\theta\theta}\Gamma^\theta{}_{\phi\phi} - \Gamma^\theta{}_{\phi\phi}\Gamma^\phi{}_{\phi\theta} = \\ &= \frac{(r_+^2+a^2)^3(r_+^2-3a^2\cos^2\theta)}{\Sigma_+^4}\,\sin^2\theta\,. \end{aligned}$$

Hence,

$$\text{horizon Gauss curvature} = g^{\phi\phi}R^\theta{}_{\phi\theta\phi} = \frac{(r_+^2+a^2)(r_+^2-3a^2\cos^2\theta)}{\Sigma_+^3}\,. \tag{16.31}$$

This most easily becomes negative at $\cos^2\theta = 1$, and that happens if $r_+^2 < 3a^2$. Substituting here, for Kerr specifically ($Q=0$), $r_+ = M+\sqrt{M^2-a^2}$, one finds that it indeed corresponds to $a>\sqrt{3}M/2$. In the extreme limit of $a=M$, $r_+=M$, the negative-curvature region reaches from the axis to $\cos^2\theta = 1/3$, i.e. to $\theta \doteq 55°$. From there down to the equatorial plane, the curvature is always positive.

- Footnote to footnote: Kind of a quick check is provided thanks to the Gauss-Bonnet theorem. For closed surfaces, the theorem says that the integration of the Gauss curvature over the surface is connected with genus of the surface ("number of handles", p) by relation

$$\int\limits_A (\text{Gauss})\,\mathrm{d}A = 4\pi(1-p)\,.$$

Our horizon is spheroidal, so $p=0$, and by integration of the above result one obtains

$$\int\limits_0^{2\pi}\int\limits_0^{\pi}\sqrt{(g_{\theta\theta}g_{\phi\phi})_+}\,(\text{Gauss})\,\mathrm{d}\theta\mathrm{d}\phi = 2\pi(r_+^2+a^2)^2\int\limits_0^\pi \frac{r_+^2-3a^2\cos^2\theta}{(r_+^2+a^2\cos^2\theta)^3}\,\sin\theta\,\mathrm{d}\theta = 4\pi\,.$$

16.8 True shape of the static limit

It is also educative to consider the true geometry of the static limit. Let us again analyse it for Kerr ($Q=0$). Recall that the static-limit surface is given by $g_{tt}=0$, i.e. $\Sigma = 2Mr$, which yields $r = r_0 \equiv M \pm \sqrt{M^2 - a^2\cos^2\theta}$. In still another words, $r^2 - 2Mr = -a^2\cos^2\theta$, the latter implying

$$(r-M)\mathrm{d}r = a^2\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$\Longrightarrow \quad \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 \stackrel{r=r_0}{=} \frac{2Mr_0}{a^2\sin^2\theta}\,\mathrm{d}r^2 = \frac{2Mr_0}{a^2\sin^2\theta}\,\frac{a^4\sin^2\theta\cos^2\theta}{(r_0-M)^2}\,\mathrm{d}\theta^2 = \frac{2Mr_0\,a^2\cos^2\theta}{M^2-a^2\cos^2\theta}\,\mathrm{d}\theta^2\,.$$

Adding the latitudinal metric term $g_{\theta\theta}\mathrm{d}\theta^2 = \Sigma\mathrm{d}\theta^2$, now equaling $2Mr_0\mathrm{d}\theta^2$, we have

$$\frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \stackrel{r=r_0}{=} \frac{2Mr_0\,\mathrm{d}\theta^2}{M^2-a^2\cos^2\theta}(\cancel{a^2\cos^2\theta} + M^2 - \cancel{a^2\cos^2\theta}) = \frac{2M^3r_0\,\mathrm{d}\theta^2}{M^2-a^2\cos^2\theta}\,,$$

and, finally, adding the last non-zero metric term

$$g_{\phi\phi} = \frac{\mathcal{A}}{\Sigma}\sin^2\theta = \left(r^2 + a^2 + \frac{2Mr}{\Sigma}\,a^2\sin^2\theta\right)\sin^2\theta =$$
$$\stackrel{r=r_0}{=} (r_0^2 + a^2 + a^2\sin^2\theta)\,\sin^2\theta = 2(Mr_0 + a^2\sin^2\theta)\,\sin^2\theta\,,$$

we arrive at

$$\mathrm{d}s^2(t=\mathrm{const}, r=r_0) = 2(Mr_0 + a^2\sin^2\theta)\,\sin^2\theta\,\mathrm{d}\phi^2 + \frac{2M^3r_0\,\mathrm{d}\theta^2}{M^2-a^2\cos^2\theta}\,. \tag{16.32}$$

An arbitrary circle $\theta = \mathrm{const}$ on the static-limit surface has proper circumference

$$\int_0^{2\pi}\sqrt{g_{\phi\phi}(r=r_0)}\,\mathrm{d}\phi = 2\pi\sqrt{g_{\phi\phi}(r=r_0)} = 2\pi\sqrt{2Mr_0 + 2a^2\sin^2\theta}\,\sin\theta\,.$$

This amounts to $2\pi\sqrt{4M^2+2a^2}$ in the equatorial plane (where $r_0 = 2M$), while it vanishes (as $2\pi\sqrt{2Mr_0}\,\sin\theta$) when approaching the axis. On the other hand, proper distance in the latitudinal direction behaves according to

$$\int\sqrt{\frac{2M^3r_0}{M^2-a^2\cos^2\theta}}\,\mathrm{d}\theta = \sqrt{2}\,M\int\sqrt{\frac{1+\sqrt{1-\frac{a^2}{M^2}\cos^2\theta}}{1-\frac{a^2}{M^2}\cos^2\theta}}\,\mathrm{d}\theta\,.$$

In the small-θ limit, the ratio of these lengths approaches

$$\lim_{\theta\to 0^+}\frac{2\pi\sqrt{2Mr_0}\,\mathrm{d}\theta}{\sqrt{\frac{2M^3r_0}{M^2-a^2}}\,\mathrm{d}\theta} = 2\pi\sqrt{1-\frac{a^2}{M^2}}\,.$$

Hence, whenever the black hole is rotating, the ratio comes out *smaller than* 2π, so the static limit has a conical cusp at the axis (although it is a locally flat surface there in the Boyer-Lindquist image – see Figure 16.4). In the extreme limit ($a \to M$), the cusp even becomes "infinitely long" (the polar-circumference element diverges). Namely, imagining that one gradually increases a, the extreme case corresponds to the moment when the outer and the inner static limits just touch smoothly at the axis, at the horizon radius $r = M$. For a increased still more, the horizon disappears and the static-limit surface (now united already) detaches from the axis and begins shrinking towards the equatorial plane.

Let us stress that the above result has been obtained for the $t = \mathrm{const}$ slices ("snapshots") of the static limit and need not (and really does not) hold if considering different "constant-time" slices.

CHAPTER 17

More on motion in black-hole fields

Hitherto, we have only studied geodesic motion in the Schwarzschild field. Two addenda are worth discussing in some detail:

- Two applications of the geodesic motion in Schwarzschild were particularly important in the early years of GR, for Einstein himself and for the acceptance of the then new (and then quite mysterious) theory – prediction (actually explanation) of an anomalous effect in the apsidal precession of bound orbits and of light bending in the gravitational field. Both these effects can be derived easily from the relativistic Binet formula which in itself is useful to know of.

- It turns out, rather surprisingly, that the geodesic motion (even the electro-geodesic one, including Lorentz force) remains completely integrable in the Kerr-Newman space-time. This was found elegantly using the Hamilton-Jacobi formalism. Bad tongues say this is the only example where the HJ formalism has brought something previously unknown, which in itself is a sufficient reason to go through the derivation.

17.1 Apsidal advance and light bending in Schwarzschild

Pericentre advance and light bending provided historically the first tests of GR, and we also mentioned them first in Section 10.6. Here we derive the respective GR predictions. The key ingredient will be the Binet formula which describes the (necessarily plane) motion in the spherically symmetric field in terms of the $r(\phi)$ behaviour, i.e., it yields the orbit shape (rather than its time evolution).

17.1.1 Binet formula

Equation for $r(\phi)$ is obtained by dividing $p^r \equiv m\frac{\mathrm{d}r}{\mathrm{d}\tau}$ by $p^\phi \equiv m\frac{\mathrm{d}\phi}{\mathrm{d}\tau}$, so let us divide equation (12.23) by the second of equations (12.21), to get

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4}{L^2}\left[E^2 - \left(1-\frac{2M}{r}\right)\left(m^2+\frac{L^2}{r^2}\right)\right],$$

where $E \equiv -p_t$ and $L \equiv p_\phi$ are constants of geodesic motion. Introducing the reciprocal radius $u := 1/r$, one rewrites

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \left(-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{L^2}\left[E^2 - (1-2Mu)(m^2 + L^2u^2)\right] .$$

Differentiating by ϕ yields

$$2\,\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} = \frac{1}{L^2}\left(2m^2M + 6ML^2u^2 - 2L^2u\right)\frac{\mathrm{d}u}{\mathrm{d}\phi}\,,$$

from where, assuming that the orbit is not circular, $\frac{\mathrm{d}u}{\mathrm{d}\phi} \neq 0$, follows the **relativistic Binet formula**

$$\boxed{\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{m^2M}{L^2} + 3Mu^2}\,. \tag{17.1}$$

Compared to its classical form, extra is the last term. In the Newtonian limit of the $m \neq 0$ case, this last term is negligible with respect to the term before,

$$\frac{3Mu^2}{\frac{m^2M}{L^2}} = \frac{3L^2}{m^2r^2} = \frac{3(g_{\phi\phi}p^\phi)^2}{m^2r^2} = 3(ru^\phi)^2 \doteq 3\left(\hat{\gamma}\hat{v}^{\hat{\phi}}\right)^2 \ll 1\,;$$

the last expression follows from the second equation in (12.31) which yields

$$\frac{L^2}{m^2r^2} = \frac{E^2}{m^2}\frac{(\hat{v}^{\hat{\phi}})^2}{1-\frac{2M}{r}} \xrightarrow{r\gg M} (\hat{\gamma}\hat{v}^{\hat{\phi}})^2\,,$$

where $\hat{v}^{\hat{\phi}}$ is the linear tangential velocity of the particle measured by a far static observer and $\hat{\gamma}$ is the corresponding Lorentz factor (energy per unit rest mass m). In short, the term is small if linear azimuthal velocity is small with respect to the speed of light.

If interested in motion in a weak field such as that in the Solar system, where the Newtonian limit of GR suffices, one may solve the Binet equation iteratively – first *without* the relativistic term, and only then "switch on" that term as a small perturbation.

17.1.2 Perihelium advance

Precession of bound orbits concerns massive test bodies, so let us denote $\tilde{L} := L/m$ as in the chapter on Schwarzschild solution. Well, so we first tackle the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{M}{\tilde{L}^2}\,.$$

The homogeneous equation is solved by $\mathrm{const}\cdot\cos(\phi-\phi_0)$, where ϕ_0 can be chosen zero by adjusting the origin of ϕ. By adding the particular solution $\frac{M}{\tilde{L}^2}$ we obtain, as zero approximation, the well known Newtonian solution of the Kepler problem – the conic

$$u_{(0)} = \frac{M}{\tilde{L}^2}(1 + A\cos\phi)\,. \tag{17.2}$$

Being interested in bound orbits, we will only focus on the ellipse case ($|A| < 1$) in the following. Comparing the result with the form $\frac{1}{r} = \frac{1+e\cos\phi}{p}$ usual in classical mechanics [in which $p = b^2/a = a(1-e^2)$, with a, b and e standing respectively for the semi-major and semi-minor axes and the numerical eccentricity], we see that $\tilde{L}^2/M = p$, $A = e$.

Now we switch on the relativistic term in the Binet formula and try to solve it in the form $u = u_{(0)} + u_{(1)}$, where $u_{(1)}$ is a small correction of $u_{(0)}$ (we leave it in the equation up to linear order only). Substituting the splitting to (17.1) and subtracting the already satisfied part for $u_{(0)}$, we are left with

$$\frac{\mathrm{d}^2 u_{(1)}}{\mathrm{d}\phi^2} + u_{(1)} = 3Mu_{(0)}^2 + \cancel{3Mu_{(1)}^2} + \cancel{6Mu_{(0)}u_{(1)}} = \frac{3M^3}{\tilde{L}^4}\left(1 + 2e\cos\phi + e^2\cos^2\phi\right), \quad (17.3)$$

where the cross term $6Mu_{(0)}u_{(1)}$ has been neglected *as well*, because in the Newtonian ($\Rightarrow$ weak-field) limit the linearly small guy $u_{(1)}$ is multiplied there by $6Mu_{(0)} = 6M/r_{(0)}$ which is itself small. Actually, it is three times the ratio of the central-object Schwarzschild radius $2M$ to the radius of the given orbit; specifically for the orbit of Mercury, $6Mu_{(0)} \simeq 1.5 \cdot 10^{-7}$.

Planning to apply the exercise to Solar-system planetary orbits (for which $e \ll 1$), one can neglect the e^2 term. The homogeneous solution we already know from the zeroth approximation; the particular solution is easily seen to be $\frac{3M^3}{\tilde{L}^4}(1 + e\phi\,\sin\phi)$, so the complete solution reads

$$u = \frac{M}{\tilde{L}^2}\left[1 + \frac{3M^2}{\tilde{L}^2} + e\left(\cos\phi + \frac{3M^2}{\tilde{L}^2}\phi\,\sin\phi\right)\right].$$

Let us use the small-angle trigonometric trick to put the result in a more elegant form. Since, in the Solar system,

$$\frac{3M^2}{\tilde{L}^2} = \frac{3M}{p} \simeq \frac{3M}{a} \ll 1\,,$$

we may write

$$\frac{3M^2}{\tilde{L}^2}\phi \simeq \sin\left(\frac{3M^2}{\tilde{L}^2}\phi\right), \qquad 1 \simeq \cos\left(\frac{3M^2}{\tilde{L}^2}\phi\right),$$

hence

$$\begin{aligned} u &= \frac{M}{\tilde{L}^2}\left\{1 + \frac{3M^2}{\tilde{L}^2} + e\left[\cos\left(\frac{3M^2}{\tilde{L}^2}\phi\right)\cos\phi + \sin\left(\frac{3M^2}{\tilde{L}^2}\phi\right)\sin\phi\right]\right\} = \\ &= \frac{M}{\tilde{L}^2}\left[1 + \frac{3M^2}{\tilde{L}^2} + e\,\cos\left(\phi - \frac{3M^2}{\tilde{L}^2}\phi\right)\right]. \end{aligned} \quad (17.4)$$

Two relativistic corrections can be noticed: The first is the term $\frac{3M^2}{\tilde{L}^2}$ after the unity which, however, represents only a very tiny modification of the parameter p. More important is the difference in the argument of cosine. Since the latter is not given merely by ϕ, the cosine now does not assume the same value after 2π, but only after $2\pi + 2\pi\,\frac{3M^2}{\tilde{L}^2}$. The whole orbit thus turns, within one revolution, by the angle

$$\delta\phi = \frac{6\pi M^2}{\tilde{L}^2} = \frac{6\pi M}{a(1-e^2)}\,, \qquad \text{in standard units}: \quad \delta\phi = \frac{6\pi GM}{c^2 a(1-e^2)}\,, \tag{17.5}$$

in the "prograde" direction (the direction of orbiting) – hence "pericentre *advance*".[1] This value is *also* tiny, but it cumulates in time, so after many periods it does reveal. The shift comes out the largest for inner planets (with a small semi-major axis a) with a large eccentricity e (though still small enough relative to unity, in order that the derivation be valid). The best candidate obviously is the Mercury; the expression (17.5) yields 43 arc seconds per century for it.

The above value was actually known from 19th century already (the first to point out the precession excess likely was Le Verrier in 1845; he estimated the value at 35"). This was quite an achievement since the Mercury orbit seems to precess by 5025"/century due to the precession of the Earth rotational axis, and it really precesses by some 532"/century due to the influence of other planets (since they modify the Solar potential $-M/r$), so these major effects have to be subtracted first. One might guess that celestial mechanics of the 19th century could not have been concerned with such a minute anomaly of 43"/century, but the truth is contrary: the 10th edition of Encyclopaedia Britannica from 1902 states, for instance, that "either Mercury must be acted upon by some unknown body or the theory of gravitation needs modification".

17.1.3 Apsidal-advance prominenti: double pulsar and the S2 star

Systems have been observed in which the pericentre shift – and relativistic effects at all – are much larger. Prominent "relativistic laboratories" are double pulsars. To understand pulsars as such, imagine that a moderately rotating star with a moderate magnetic field collapses to a neutron star, i.e. to an object having circa 10km and between 1 and 2 solar masses, supported against gravity by pressure of a degenerate neutron gas. Since nature favours conservation of angular momentum and conservation of magnetic flux, the neutron stars tend to rotate extremely fast (today the fastest one spins 716 times per second) and many of them have extremely strong magnetic fields (from 10^4 to 10^{11} tesla). The magnetic field is of roughly dipole shape and guides the charged particles approaching the star to almost exclusively hit the surface at the magnetic poles. Besides that, within the magnetic funnels at the poles, charged particles are being strongly accelerated. The poles thus act as a powerful source of radiation. In some neutron stars, the rotational axis is misaligned with the magnetic axis, which makes the "hot spots" gyrate and the object works as a lighthouse – on the cone defined by the magnetic-axis precession, an observer perceives pulses with the rotational period. The period is very stable (though modern atomic clocks are still several orders of magnitude more "precise").

[1] The orbiter is thus pulled towards pericentre more than in the Newtonian case, which intuitively confirms that the relativistic centre attracts stronger than the Newtonian one. Were the gravitational centre *weaker* than according to the Newtonian $-M/r$ formula, the orbit would precess in the retrograde sense.

The first pulsar was discovered in 1967 by J. Bell & A. Hewish (Nobel Prize in 1974). In 1974, R. Hulse & J. Taylor discovered the first double pulsar called PSR B1913+16 (Nobel Prize in 1993). In it, a pulsar orbits another neutron star. A truly remarkable system was discovered in 2003 under the name PSR J0737-3039. It consists of two neutron stars (of 1.34 and 1.25 $M_\odot$) which have *both* been identified as pulsars (22.7 $\mathtt{ms}$ and 2.77 $\mathtt{s}$). Their orbital separation is 800 000 $\mathtt{km}$, orbital period is 2.45 hours and orbital speed 300 $\mathtt{km/s}$. The orbit shrinks by 7 millimetres per day, in perfect accordance with GR quadrupole formula for energy loss in gravitational radiation. The elliptical-type orbit about the common centre of mass turns by as much as 17° per year (which means during some 3600 orbits) due to the relativistic periapsis advance.

Several more highlights are connected with PSR J0737-3039. The orbital plane of the system is almost edge-on to us (inclination about 89°), which makes the faster pulsar subject to partial eclipses by its weaker companion (lasting for about half minute). Also, due to the relativistic spin precession, the slower pulsar ceased to be visible in 2008 and has been predicted to be back with us in 2035.

Another relativistic laboratory has become the orbital dynamics of the innermost stars orbiting the black hole supposed to exist in our Galaxy nucleus. A particular fame has built the star called S2 (also S02) which has already been tracked for 27 years in 2020 (Nobel Prize of 2020 for R. Genzel and A. Ghez). Its orbit indicates the presence of a central mass (the black hole) of some $4.25{\cdot}10^6\,M_\odot$. S2 passed pericentre in May 2002 and again in May 2018, which was an occasion to measure the periapsis effect. The pericentre was at 120 $\mathtt{AU}$ (about 1400 Schwarzschild radii of the central hole) and S2 passed it at 7700 $\mathtt{km/s}$. The Doppler effect as well as gravitational redshift have been confirmed and, very recently, the periapsis advance has been announced of 12' per one orbital period. Therefore, if taken per one orbit, the S2 star presently seems to be a clear record holder.

17.1.4 Light bending

Photons have $m = 0$, so only the second, relativistic term appears on the right-hand side of the Binet formula (17.1). Restricting to the Solar system as above, it is again adequate to solve the motion iteratively. The zeroth (Newtonian) approximation is given by solution of the homogeneous equation,

$$u_{(0)} = \frac{\cos\phi}{r_{\min}}\,. \tag{17.6}$$

This is a straight line with $r_{\min}$ denoting its minimal distance from the centre. Therefore, there is no effect in the Newton theory. Should be emphasized that the Newtonian result exactly means here the one *with infinite speed of light*. If, instead, the Newton gravity is combined with the corpuscular theory of light and the correct speed c, there does arise a non-zero effect (specifically, it amounts to *half* of the relativistic result).

The inhomogeneous equation we again solve in the form $u = u_{(0)} + u_{(1)}$. We thus start from

$$\frac{\mathrm{d}^2(u_{(0)}+u_{(1)})}{\mathrm{d}\phi^2} + u_{(0)} + u_{(1)} = 3Mu_{(0)}^2 + 6Mu_{(0)}u_{(1)} + 3Mu_{(1)}^2\,,$$

which after subtracting the zeroth order $\frac{d^2 u_{(0)}}{d\phi^2} + u_{(0)} = 0$ yields

$$\frac{d^2 u_{(1)}}{d\phi^2} + u_{(1)} = 3Mu_{(0)}^2 + 6Mu_{(0)}u_{(1)} + 3Mu_{(1)}^2 \,.$$

Since $u_{(0)} \gg u_{(1)}$, it holds $3Mu_{(0)}^2 \gg 6Mu_{(0)}u_{(1)} \gg 3Mu_{(1)}^2$. The $3Mu_{(1)}^2$ term we clearly omit, but $6Mu_{(0)}u_{(1)}$ is to be omitted as well. Indeed, the latter is not only much smaller than $3Mu_{(0)}^2$, but also than the $u_{(1)}$ order of the left-hand side, because $u_{(1)}$ is multiplied in it by

$$6Mu_{(0)} \equiv \frac{6M}{r_{\min}} \,, \quad \text{i.e., in standard units,} \quad \frac{6GM}{c^2 r_{\min}} \ll 1$$

(the minimal radius $r_{\min}$ is assumed to be much larger than $6GM/c^2$, i.e. than twice the radius of the photon circular orbit). Consequently, we have the equation

$$\frac{d^2 u_{(1)}}{d\phi^2} + u_{(1)} = 3Mu_{(0)}^2 = \frac{3M\cos^2\phi}{(r_{\min})^2} = \frac{3M}{2(r_{\min})^2}(1+\cos 2\phi) \,. \tag{17.7}$$

This can be satisfied in the form $u_{(1)} = A + B\cos 2\phi + C\sin 2\phi$, specifically, it follows that

$$u_{(1)} = \frac{M}{2(r_{\min})^2}(3-\cos 2\phi) \qquad \text{– namely,} \quad A = \frac{3M}{2(r_{\min})^2} \,, \quad B = -\frac{M}{2(r_{\min})^2} \,, \quad C = 0 \,.$$

Hence the result

$$u = \frac{\cos\phi}{r_{\min}} + \frac{M}{2(r_{\min})^2}(3-\cos 2\phi) = \frac{\cos\phi}{r_{\min}} + \frac{M}{(r_{\min})^2}(2-\cos^2\phi) \,. \tag{17.8}$$

The bending consists in the difference between directions of the ray far before the centre and far behind it, where "far" means $u \to 0$. On the right-hand side, we again use the fact that in a weak field the effect is very small, so the asymptotic values are $\phi \doteq \pm\pi/2$ and thus

$$\cos\phi = \sin\left(\frac{\pi}{2} - \phi\right) \doteq \frac{\pi}{2} - \phi \,, \qquad \cos^2\phi \doteq 0 \,.$$

After multiplication by $r_{\min}$, we thus have from (17.8), at infinities,

$$0 \doteq \frac{\pi}{2} - \phi + \frac{2M}{r_{\min}} \quad \Longrightarrow \quad \phi \doteq \frac{\pi}{2} + \frac{2M}{r_{\min}} \,.$$

The term $\frac{2M}{r_{\min}}$ stands for the deflection of the asymptotic ray direction from the direction the ray has at pericentre $r_{\min}$. Owing to the mirror symmetry with respect to that central moment, the total deflection between the asymptotic directions is twice that value,

$$\delta\phi = 4M/r_{\min} \,, \qquad \text{in standard units} \quad \delta\phi = \frac{4GM}{c^2 r_{\min}} \,. \tag{17.9}$$

For a ray passing by the Sun and having $r_{\min}$ just above the solar surface, this amounts to 1.75 arc second.

Well yes, bending can reach any amount in a sufficiently strong field, such as that around black holes. We saw at the end of Section 12.3.4 that if a photon approaches a Schwarzschild black hole with impact parameter $|\ell| = 3\sqrt{3}\,M$, it "winds up" to the circular orbit on $r = 3M$ (suffering indefinite bending in a sense); with $|\ell|$ less than that, it just ends under the horizon.

17.1.5 Gravitational lensing

After confirmation of the GR light-bending prediction in 1919, The New York Times wrote "Lights all askew in the heavens"... But that was not so much new, right? Things *are* different from how they seem to be, and not only due to GR. Already in 1676, O. Rømer provided the first decent confirmation that light propagates with a finite speed. Hence, this composes – already by Galilei transformation of course – non-trivially with the relative velocity between the source and the observer. Several consequences immediately follow:

- **aberration:** a source is seen in a slightly different direction than where it "really" was at the moment of light emission (i.e., than where it could have been seen "instantaneously at distance", by means of an infinitely fast signal). In the observer system, the radiation pattern of the source is concentrated to the direction of their relative motion, so an approaching/receding source is perceived as brighter/dimmer (the co-called *beaming* effect)
- **deformation:** in the longitudinal direction (along the relative motion), a source appears longer/shorter when it is approaching/receding. Besides that, it is seen as slightly rotated: if, for example, a cube would be flying along a straight line, tangentially with respect to us and so that one of its walls would be exactly orthogonal to the direction of motion and the other would just face us, we would still also see the "rear" wall a bit, because the photons which finally reach us could start from that wall slightly askew "backwards" (with respect to the cube)
- **Doppler effect:** a source appear to emit higher/lower-frequency radiation when approaching/receding.

Further modification of the appearance of objects brings special relativity:

- the velocities (of light and of the relative motion) compose in a different, Lorentz way, and the effects of time dilation and length contraction come into the play (implying the new, transverse Doppler effect, in particular).

In general relativity, furthermore, the above optical illusions are supplemented by the gravitational shift of frequency (Chapter 4) and by the space-time curvature (most of other chapters). The **gravitational lensing** effect is the most intriguing consequence of the latter, directly stemming from the gravitational light bending. A historical breakthrough date is April 17, 1936. That Friday, Einstein was paid a visit in Princeton by Rudi W. Mandl, a Czech born in Vsetín who received electrical-engineering education in Wien and who came to US after quite an eventful life (also including prisoning in Siberia during the 1st World War). He was washing dishes for restaurants in Washington, D.C., and also had some income from drawing of eggshells. Besides that, he was interested in GR, and one spring day of 1936 came to Science News Letter offices to show his calculations and ideas concerning gravitational lensing: "You see," he said, "the light from a distant star will be bent as it passes the nearer star and the effect will be a great brightening that anyone can see with a small telescope." Journalists offered Mandl to pay his trip to Princeton to visit Einstein, and that happened on April 17.

Einstein accepted Mandl cordially, even made some calculations with him, but mainly to show that the effect is too tiny to be observable. However, Mandl was arguing that if the source, the lens and the detector were arranged in a line and at suitable distances, the detector could see the source considerably brighter, because then there exist more trajectories along which light can reach it (in an ideal case, the trajectories even form a continuous "spindle" which should be seen as a ring). In addition, the cosmic bodies are slightly moving relative to each other, so a passage of three bodies through the roughly aligned configuration should be observable as a "real time" brightening. Mandl then added several letters in which he tried to persuade Einstein to publish something on the subject. He was warning that otherwise astrologists and other charlatans would take up and misuse the topic. He also enclosed several drafts in which he was explaining, as a result of gravitational-lensing flashes, the evolutionary breaks in the history of terrestrial life, especially for large extinctions he was suggesting a "burn down" by a particularly strong lensing effect with Earth "in the focus". Einstein advised against trying to publish this "outreach" of the ideas, but finally (in December) he agreed to publish a short note in the Science journal. The note starts with kind-of apology: "Some time ago, R. W. Mandl paid me a visit and asked me to publish the results of a little calculation, which I had made at his request. This note complies with his wish." And to the editor of Science Einstein wrote: "Let me also thank you for your cooperation with the little publication, which Mr. Mandl squeezed out of me. It is of little value, but it makes the poor guy happy."

The ideas on tangential effect of gravity on light are centuries older and go to J. Michell (the same Michell who contemplated about radial effect and "predicted black holes") and his friend H. Cavendish who derived a correct Newtonian bending formula in 1802 in an unpublished note. Similarly, using the corpuscular picture, the bending was derived by J. G. von Soldner in 1804. Already within GR, several authors discussed the effect after the Eddington-Dyson 1919 measurements – besides Eddington himself, it was most notably O. D. Chwolson who seems to have been the first to mention a possibility of multiple images and of a "halo effect" when the source, lens and observer get nearly aligned. The first (and thorough) account on gravitational *lensing* (not just light bending) is due to F. Link and was published on 16th March 1936. Link wrote two papers on the subject which basically include all the optics involved (including finite-size effects).

Still Einstein's late 1936' Mandl-inspired note is much more taken as "classics" in the field. Anyway, justice *was* done actually, only that Einstein did not stress it: he himself was thinking about the lensing effect in 1911-12 when, in Prague, he was working out his 1907' conclusion that the principle of equivalence (together with the Newton theory) implies time dilation, this in turn implies the dependence of the speed of light on potential (Einstein's early conjecture), and from that it follows, finally, that the rays which are not parallel to the direction of weight are being bent. Einstein did not publish anything because he believed the effect was negligible, but the typical figure of the ray course and an estimate of the intensity enhancement survived in his notebook (lensing notes are presumably from his journey to Berlin in April 1912; they still involve just half value of the correct, GR bending angle, since Einstein did not yet consider curvature at that time).

In 1979, the first double image was observed, of the quasar Q0957+561, and in 1987 the first "Einstein(-Chwolson) ring" (the ideal-alignment consequence) – the quasar image MG1131+0456 projected by an intermediate galaxy. Having started as a GR curiosity, gravitational lenses quickly became a very valuable tool for the study of distant sources and, in the

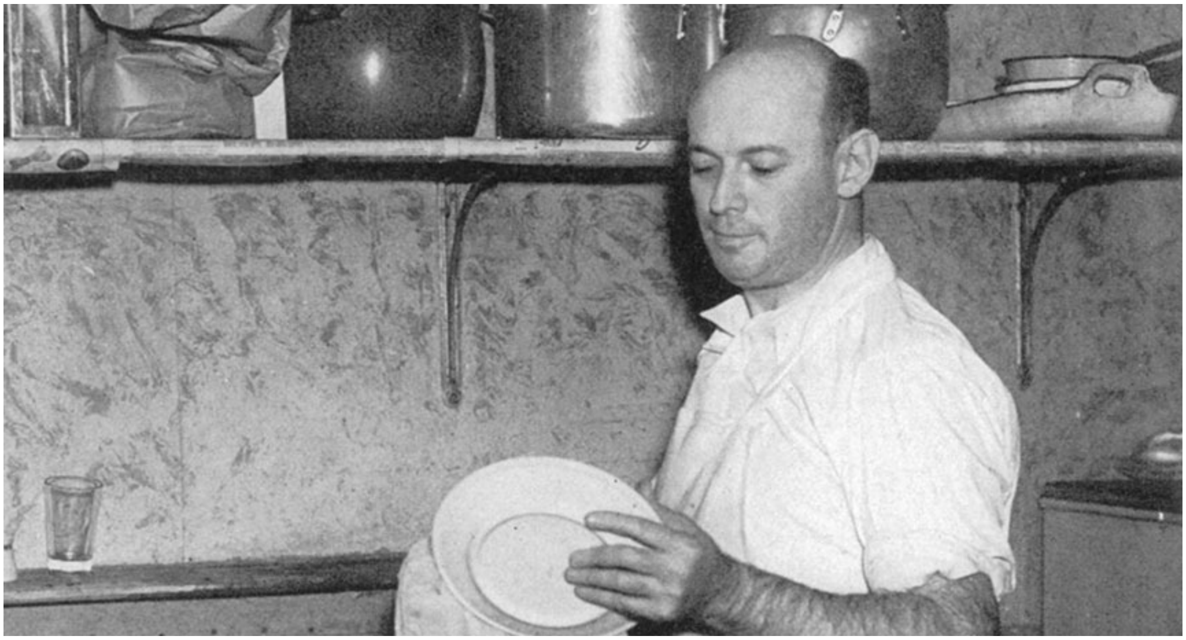


Figure 17.1: Rudi Mandl – electrical engineer, dishwasher, inventor, relativist...

opposite sense, an indicator of distribution of the intermediate (lensing) matter. Analysis of the images of quasars and far galaxies is crucial for cosmology, e.g. for estimating the amount and configuration of dark matter. Extensive monitoring programs also focus on *microlensing* in which the sources as well as lenses are individual stars.

Gravitational lensing provides an almost exclusive tool in cases when an object is not visible and only manifests itself through the gravitational influence. Such as *dark matter*. It was also in 1930s when F. Zwicky first pointed out, on the basis of studying the Coma cluster of galaxies, that "dark matter is present in much greater amount than luminous matter". No surprise that Zwicky realized that gravitational lensing could reveal the amount of such matter in various cosmic structures. He also realized that the large-scale objects (other galaxies and their clusters) "offer a much better chance than stars for observation of gravitational lens effects". In 1937 he published two papers on lensing by "extragalactic nebulae". The first of them (titled Nebulae as gravitational lenses) begins as follows: "Einstein recently published some calculations concerning a suggestion made by R. W. Mandl, namely, that a star B may act as a 'gravitational lens' for light coming from another star A which lies closely enough on the line of sight behind B. As Einstein remarks the chance to observe this effect for stars is extremely small. Last summer Dr. V. K. Zworykin (to whom the same idea had been suggested by Mr. Mandl) mentioned to me the possibility of an image formation through the action of gravitational fields." :-)

- Impossible to end without the following postscript. Look at *Long Beach Independent* of May 2, 1948 (it is easy at the Historical Newspaper Archives website). In a sport page (p. 34), there is a column "Gent will prevent games from being rained out" informing about a guy who is offering to control the weather over sport stadiums. His letter had been sent

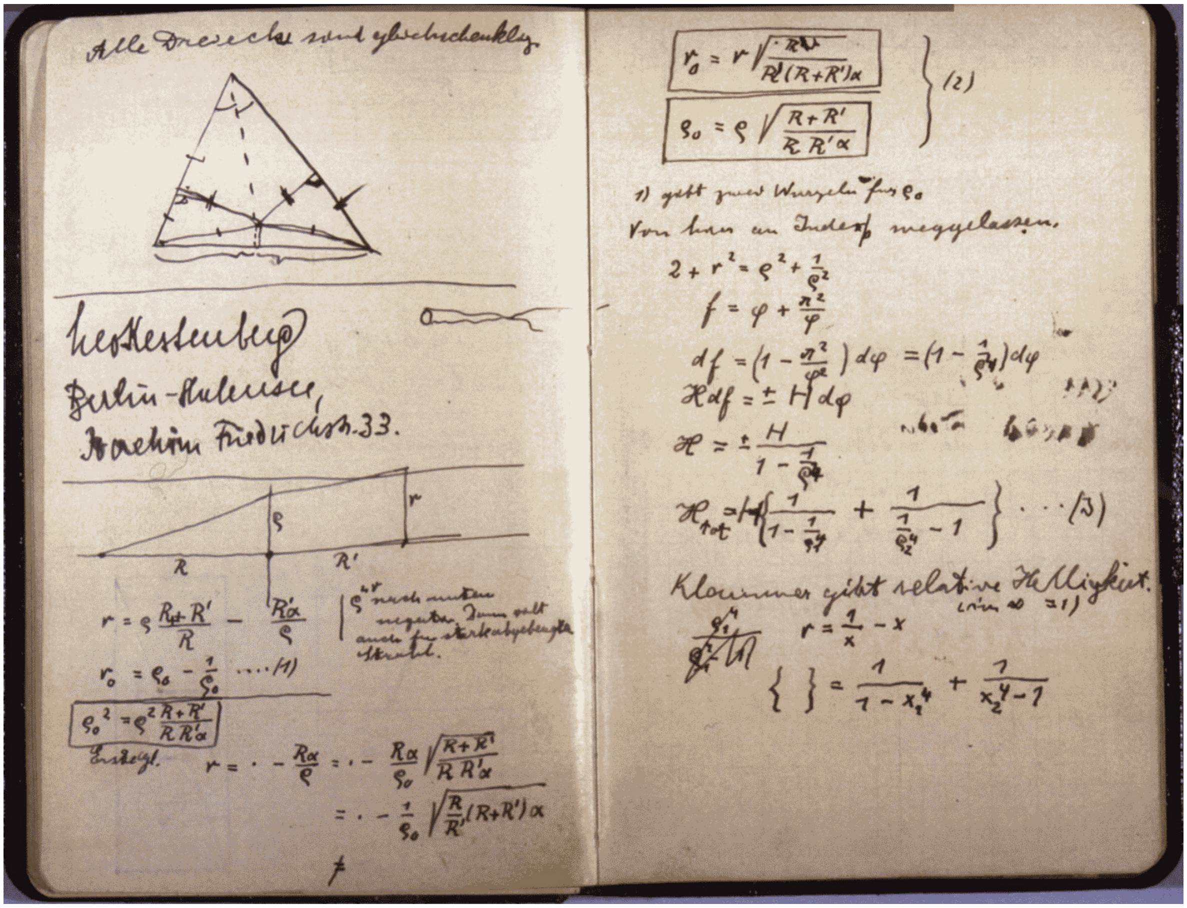


Figure 17.2 *Einstein's notebook with calculations from April 1912 estimating the gravitational-lensing effect. From the formulas shown it follows that the angular radius of the Einstein(-Chwolson) ring is* $\vartheta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}}$, *with* M *mass of the lensing object,* $D_{\rm L}$ *distance from observer to the lensing object,* $D_{\rm S}$ *distance from observer to the source, and* $D_{\rm LS}$ *distance between the lens and the source. Note that in very large scales the distances have to be understood carefully (as the so-called angular-diameter distances).*

to most baseball clubs in the area: “I am an inventor of a machine which will prevent rain from falling inside your stadium and, therefore, will eliminate rain checks. For information about me, I refer you to the Science News Letter of December 19, 1936. You will see that I collaborated with Prof. Einstein then and a few times since that time ...” Signed Rudi W. Mandl of Los Angeles.

17.2 Shapiro time delay

Consider once more, like in the light-bending section, a photon passing by a gravitating object while travelling between two points. Take the object to be the Schwarzschild centre, denote by r_1 and r_2 the radii of the emission and detection locations, and denote by r_0 the radius of the photon's closest approach to the centre. Were the photon's path straight, its length is given by the Pythagoras theorem, $\sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}$, and hence also follows the time of flight (considering the standard speed of light). In order to compute the GR result, take first the equation (12.38) at the moment of the closest approach ($r = r_0$, $(\hat{v}^{\hat{\phi}})^2 = 1$), hence obtain $\ell^2 = r_0^2/(1 - 2M/r_0)$, and use this, together with $E \equiv -p_t = -g_{tt}p^t$, in equation (12.35) for radial motion of photons:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(\frac{p^r}{p^t}\right)^2 = \frac{E^2\left(1 - \frac{1-\frac{2M}{r}}{r^2}\,\ell^2\right)}{\frac{E^2}{\left(1-\frac{2M}{r}\right)^2}} = \left(1 - \frac{2M}{r}\right)^2 \left(1 - \frac{r_0^2}{r^2}\,\frac{1 - \frac{2M}{r}}{1 - \frac{2M}{r_0}}\right).$$

The corresponding integral for time t leads to a very long expression involving elliptic integrals, so we better solve it with integrand only expanded to linear order in M (thus assuming $r \geqslant r_0 \gg M$),

$$t(r_0, r) = \int_{r_0}^{r} \frac{\mathrm{d}r}{\left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{r_0^2}{r^2}\frac{1-\frac{2M}{r}}{1-\frac{2M}{r_0}}}} \simeq \int_{r_0}^{r} \frac{r + 2M + \frac{Mr_0}{r+r_0}}{\sqrt{r^2 - r_0^2}}\,\mathrm{d}r =$$

$$= \sqrt{r^2 - r_0^2} + 2M \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} + M\sqrt{\frac{r - r_0}{r + r_0}}\,. \tag{17.10}$$

Adding results from the "ingoing" and "outgoing" phases ($r = r_1$ and $r = r_2$), we see that the first part is the classical term, and that the remaining two terms are *positive*,

$$\Delta t(r_1 \!\to r_0 \!\to r_2) =$$

$$= 2M \ln \frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2} + M\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + M\sqrt{\frac{r_2 - r_0}{r_2 + r_0}} > 0\,, \tag{17.11}$$

so in GR the photon travels *longer* – hence **Shapiro's delay** (result from 1964).

For a round trip of a radar signal from the Earth (r_1), *by Sun* ($M = M_\odot$), to some satellite or celestial body (r_2), and back, one obtains twice the above difference. In terms of the terrestrial-observer proper time, the result still has to be multiplied by the "redshift" factor $\sqrt{-g_{tt}} \simeq \sqrt{1 - 2M_\odot/r_1}$. –Yes, the mass of the *Sun* is really correct here, because – maybe surprisingly – the potential due to the Sun at Earth-orbit radius ($GM_\odot/r_1$) is $8.85 \cdot 10^8$ J/kg, while the own Earth's potential on its surface ($GM_\oplus/R_\oplus$) is only $6.25 \cdot 10^7$ J/kg. Within Solar system, the effect has a typical order of hundreds of microseconds (for the Earth-Sun-Mars-Sun-Earth trip, it amounts to $240\mu s$). Together with other effects, the Shapiro delay is clearly important in precise arrival times of signals from space, in particular of those from pulsars.

17.3 Motion of test particles in the Kerr-Newman field – Carter equations

The Kerr(-Newman) metric does not look entirely simple, but it has turned out that many problems lead in it – though possibly after extensive calculations – to surprisingly elegant results. Apparently it is not only connected with Killing nature of the t and ϕ coordinates included in the Boyer-Lindquist system. People have learned, among others, that several important physical equations can be solved by separation – e.g. Maxwell equations, wave equation for scalar field or for gravitational perturbations, or equation of motion for (possibly charged) test particles. This integrability properties, first seemed miraculous, are connected with very special multipolar structure of the space-time. There is a deep geometric structure under these properties, stemming from the existence of the so called non-degenerate closed conformal Killing-Yano 2-form; we will mention this connection at the end of this section.

Our task will be to demonstrate that similarly as in the Schwarzschild field the (electro-)geodesic motion in the Kerr-Newman field is completely integrable, i.e. that four independent integrals of motion exist. Realize right away that this is not trivial since in Schwarzschild one of the integrals actually was $u^\theta = 0$ thanks to the planar character of the motion, valid for any spherically symmetric field. Here the space is just axially symmetric and the motion is *not* planar.

Therefore, our problem is the equation of motion

$$\frac{\mathrm{D}p_\mu}{\mathrm{d}\tau}\,(= m a_\mu) = qF_{\mu\nu}u^\nu \qquad \Longleftrightarrow \qquad \frac{\mathrm{D}\Pi_\mu}{\mathrm{d}\tau} = qA_{\nu;\mu}u^\nu\,, \tag{17.12}$$

where $\Pi_\mu := p_\mu + qA_\mu$ is a generalized momentum, $a_\mu := \frac{\mathrm{D}u_\mu}{\mathrm{d}\tau}$ is the four-acceleration, m is the rest mass of the test particle and q is its electric charge. If the particle or the centre are uncharged, i.e. if $qQ = 0$, the Lorentz force on the right-hand side vanishes and the equation reduces to the geodesic equation. The complete set of first integrals of this equations was found by B. Carter in 1968 using the separated solution of the Hamilton-Jacobi equation. However, two integrals follow immediately from space-time symmetries:

$\boxed{\text{Lemma}}$ If there exists in space-time a Killing vector field ξ^μ, the projection $\Pi_\mu\xi^\mu$ is conserved along the world-lines of charged particles. (This is an extension of the property from Section 11.4.1 which in general only holds for geodesics.)
<u>Proof</u>:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(\Pi_\mu\xi^\mu) &= \frac{\mathrm{D}}{\mathrm{d}\tau}(\Pi_\mu\xi^\mu) = \frac{\mathrm{D}\Pi_\mu}{\mathrm{d}\tau}\,\xi^\mu + \Pi_\mu\frac{\mathrm{D}\xi^\mu}{\mathrm{d}\tau} = qA_{\nu;\mu}u^\nu\xi^\mu + \Pi_\mu\xi^\mu{}_{;\nu}u^\nu = \\ &= qA_{\nu;\mu}\xi^\mu u^\nu + m\,\cancel{\xi_{\mu;\nu}u^\mu u^\nu} + q\,\xi^\mu{}_{;\nu}A_\mu u^\nu = q\left(A_{\nu;\mu}\xi^\mu + \xi^\mu{}_{;\nu}A_\mu\right)u^\nu = \\ &= q\left(\pounds_\xi A_\nu\right)u^\nu = 0\,.\end{aligned}$$

Besides the equation of motion, we also used the Killing property $\xi_{\mu;\nu}u^\mu u^\nu = \xi_{(\mu;\nu)}u^\mu u^\nu = 0$ and the fact that the EM field has the same symmetry as the gravitational one, i.e. $\pounds_\xi A_\nu = 0$.[2] If one wanted to check the latter in (BL) coordinates, consider that the Kerr-Newman EM four-potential has the form $A_\mu = (A_t, 0, 0, A_\phi)$ while the Killing fields read $(1,0,0,0)$ and $(0,0,0,1)$, so

$$\pounds_\xi A_\nu = A_{\nu;\mu}\xi^\mu + \xi^\mu{}_{;\nu}A_\mu = A_{\nu,\mu}\xi^\mu + \xi^\mu{}_{,\nu}A_\mu = 0\,. \qquad \square$$

[2] If the EM field was a *test* one, it certainly would not need to follow the space-time symmetries (consider, for example, that of a point electric charge in generic motion), but here the EM field is *dynamical*, it means "gravitating", coupled to the gravitational field.

The stationarity and axial symmetry of space-time means the existence of the Killing fields $t^\mu = \partial x^\mu/\partial t$ and $\phi^\mu = \partial x^\mu/\partial\phi$. Regarding again their BL expression $t^\mu = \delta^\mu_t$, $\phi^\mu = \delta^\mu_\phi$, the conserved projections read

$$E = -\Pi_\mu t^\mu = -(p_\mu + qA_\mu)t^\mu = -mu_t - qA_t \qquad \text{(energy at infinity)}\,,$$
$$L = \Pi_\mu \phi^\mu = (p_\mu + qA_\mu)\phi^\mu = mu_\phi + qA_\phi \qquad \text{(axial angular momentum at } \infty)\,.$$

17.3.1 Equations for time and azimuthal components of four-velocity

The "Killing" components of four-velocity thus follow directly by solution of the set

$$E = -mu_t - qA_t = -mg_{tt}u^t - mg_{t\phi}u^\phi - qA_t\,,$$
$$L = mu_\phi + qA_\phi = mg_{t\phi}u^t + mg_{\phi\phi}u^\phi + qA_\phi\,.$$

Finding u^t: multiply the first relation by $g_{\phi\phi}$ while the second by $g_{t\phi}$, and add the results. Then compute the factor at u^t and the terms involving A_μ,

$$(g_{t\phi})^2 - g_{tt}g_{\phi\phi} = \Delta\sin^2\theta\,, \qquad g_{\phi\phi}A_t - g_{t\phi}A_\phi = -\frac{Qr}{\Sigma}(r^2+a^2)\sin^2\theta\,,$$

and multiply the resulting equation by $\frac{\Sigma}{\sin^2\theta}$:

$$m\Delta\Sigma u^t = \mathcal{A}E - (2Mr - Q^2)aL - qQr(r^2+a^2).$$

Finding u^ϕ: multiply the first relation by $g_{t\phi}$ while the second by g_{tt}, and add the results. In what comes out, the factor at u^ϕ we already know from above, and the terms with A_μ yield

$$g_{tt}A_\phi - g_{t\phi}A_t = -\frac{Qr}{\Sigma}a\sin^2\theta\,,$$

so, by multiplying the result by $\frac{\Sigma}{\sin^2\theta}$ we obtain

$$m\Delta\Sigma u^\phi = (2Mr - Q^2)aE + (\Delta - a^2\sin^2\theta)\frac{L}{\sin^2\theta} - qQra\,.$$

There exists a special case of motion – the one fixed to the equatorial plane, i.e. with $u^\theta = 0$, the latter indeed being solution since the equatorial plane is the symmetry plane (such a motion is even stable in the normal direction). Were we only interested in such a motion, we would be done, since that is completely fixed by the above equations for u^t and u^ϕ and by normalization $g_{\mu\nu}u^\mu u^\nu = -1$ (the latter yields the third component u^r).

For generic motion (with all components of u^μ non-zero), the two constants (E, L) plus normalization are not enough – one would need some fourth constant to determine u^μ completely. B. Carter really found it in looking for a separated solution of the Hamilton-Jacobi equation. He thus completed the set of first integrals of the equation of motion (17.12). In order to show how to derive the result, we have to compose the Hamilton-Jacobi equation, so we need Lagrangian and Hamiltonian of a charged particle in the Kerr-Newman field. Let us first recall how the exercise looked in special relativity.

17.3.2 Hamilton variational principle in special relativistic mechanics

Two basic approaches exist how to understand the variation of the action $S = \int \mathcal{L}(x^\mu, u^\mu)\, \mathrm{d}\tau$.

- Either one strictly respects that proper time is specific for each world-line, so a varied world-line is parameterized by a *varied* parameter whose increment $\mathrm{d}\tau^{*2} = -\eta_{\mu\nu}\mathrm{d}x^{*\mu}\mathrm{d}x^{*\nu}$ is different from $\mathrm{d}\tau^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ valid along the actual world-line. This approach has the advantage that τ and τ^* really represent proper times along the respective world-lines, which in turn ensures that the tangent vectors $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ and $u^{*\mu} = \mathrm{d}x^{*\mu}/\mathrm{d}\tau^*$ really represent the corresponding four-velocities (they are properly normalized along all the world-lines). Thanks to that, $\eta_{\mu\nu}p^\mu p^\nu = -m_0^2$ is the same as well, with m_0 thus having the meaning of rest mass along all the world-lines.

  The variational principle reads in this case

  $$0 = \delta S = \int (\delta\mathcal{L}\, \mathrm{d}\tau + \mathcal{L}\, \delta \mathrm{d}\tau)$$

  and its Euler-Lagrange equations provide Lagrange equations of the 2nd kind

  $$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial \mathcal{L}}{\partial u^\mu} + \left(\frac{\partial \mathcal{L}}{\partial u^\nu}\, u^\nu - \mathcal{L}\right) u_\mu\right] = 0\,. \tag{17.13}$$

  The Lagrangian of charged particle in the EM field is given by

  $$\mathcal{L} = -m_0 + qA_\mu u^\mu\,,$$

  where the rest mass m_0 – similarly as the charge q – is not only invariant (and same on all world-lines), but also constant (thanks to the orthogonality to u^μ of the Lorentz force, m_0 does not change in time).

- Or one parameterizes the whole bunch of virtual world-lines by proper time of the actual world-line, τ. Since τ is *not* proper time on the varied world-lines, the tangent vector $u^{*\mu} = \mathrm{d}x^{*\mu}/\mathrm{d}\tau$ is *not* four-velocity on varied world-lines in this case, which in turn implies that $-\eta_{\mu\nu}p^\mu p^\nu$ only determines rest mass along the actual world-line. This approach thus requires to be careful when performing gradients in general direction (not necessarily along the actual world-line), in particular, the usual normalization of four-velocity cannot be taken as everywhere valid in it.

  On the other hand, the corresponding Hamilton principle is of course simpler: the stationary-action condition

  $$0 = \delta S = \int \delta\mathcal{L}\, \mathrm{d}\tau$$

  leads to the Lagrange equations of the 2nd kind in the "classical" form[3]

  $$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) = 0\,. \tag{17.14}$$

[3] Note that the extra term in the Lagrange equations of the first formulation (17.13), $\left(\frac{\partial \mathcal{L}}{\partial u^\nu}\, u^\nu - \mathcal{L}\right)$, is just the Hamiltonian; the latter is surely constant of the motion, since the Lagrangian does not depend explicitely on τ.

The Lagrangian has to somehow reflect that here the variation "spoils", in the linear order already, the four-velocity normalization. One settles this by writing out the factor ± 1 standing at m_0 as some power of $\eta_{\mu\nu}u^\mu u^\nu$ with a suitable coefficient, while remembering that $\eta_{\mu\nu}u^\mu u^\nu = -1$ is only allowed to be used *after substitution in the Lagrange equations* (17.14). (This is allowed since those equations already hold along the actual world-line.[4]) Most frequent options for the Lagrangian are

$$\mathcal{L} = \frac{1}{2}\, m_0 \eta_{\mu\nu}u^\mu u^\nu + qA_\mu u^\mu \qquad \text{or} \qquad \mathcal{L} = -m_0\sqrt{-\eta_{\mu\nu}u^\mu u^\nu} + qA_\mu u^\mu\,.$$

17.3.3 Transition to general relativity

Comma-goes-to-semicolon rule... We have $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$, and the rest mass is now denoted by m. We will employ the second variational method and the Lagrangian

$$\mathcal{L} = \frac{1}{2}\, m g_{\mu\nu}u^\mu u^\nu + qA_\mu u^\mu\,. \tag{17.15}$$

The corresponding canonical momentum reads

$$\Pi_\alpha \equiv \frac{\partial\mathcal{L}}{\partial u^\alpha} = m g_{\alpha\nu}u^\nu + qA_\alpha \;\; (= p_\alpha + qA_\alpha) \quad \left[\Leftrightarrow\; u^\mu = \frac{1}{m}\, g^{\mu\alpha}(\Pi_\alpha - qA_\alpha)\; \left(= \frac{p^\mu}{m}\right)\right]$$

and hence the Hamiltonian

$$\begin{aligned}\mathcal{H} = \mathcal{H}(x^\mu, \Pi_\mu) \equiv \Pi_\mu u^\mu - \mathcal{L} = \Pi_\mu u^\mu - \frac{m}{2}\, g_{\mu\nu}u^\mu u^\nu - qA_\mu u^\mu &= \\ = p_\mu u^\mu - \frac{m}{2}\, g_{\mu\nu}u^\mu u^\nu = \frac{1}{2m}\, g^{\mu\nu}p_\mu p_\nu &= \\ = \frac{1}{2m}\, g^{\mu\nu}(\Pi_\mu - qA_\mu)(\Pi_\nu - qA_\nu)\,.& \end{aligned} \tag{17.16}$$

The Hamiltonian does not depend on τ and is thus constant of the motion – sure, the four-velocity normalization does hold along the actual world-line, so $\mathcal{H} = -m/2$.

Let us check that the Hamilton equations yield the correct equation of motion:

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\partial\mathcal{H}}{\partial\Pi_\alpha} \qquad \Leftrightarrow \qquad \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{1}{m}(\Pi^\alpha - qA^\alpha) = u^\alpha\,, \tag{17.17}$$

$$\begin{aligned}\frac{\mathrm{d}\Pi_\alpha}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial x^\alpha} \qquad &\Leftrightarrow \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} + qA_{\alpha,\beta}u^\beta = -\frac{1}{2m}\, g^{\mu\nu}{}_{,\alpha}p_\mu p_\nu + \frac{1}{m}\, g^{\mu\nu}qA_{\mu,\alpha}p_\nu \\ &\Leftrightarrow \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} - \frac{1}{2}\, g_{\mu\nu,\alpha}p^\mu u^\nu = q(A_{\mu,\alpha} - A_{\alpha,\mu})u^\mu \\ &\Leftrightarrow \qquad \frac{\mathrm{D}p_\alpha}{\mathrm{d}\tau} = qF_{\alpha\mu}u^\mu\,. \end{aligned} \tag{17.18}$$

[4] More accurately, it is already allowed after substitution for $\frac{\partial\mathcal{L}}{\partial x^\mu}$, $\frac{\partial\mathcal{L}}{\partial u^\mu}$, because the derivative by τ already acts purely along the actual world-line.

In the second equation, we have used the relation

$$g^{\mu\kappa}g_{\kappa\lambda} = \delta^\mu_\lambda \quad \Rightarrow \quad g^{\mu\kappa}{}_{,\alpha}g_{\kappa\lambda} = -g^{\mu\kappa}g_{\kappa\lambda,\alpha}\,\Big|\cdot g^{\nu\lambda} \quad \Rightarrow \quad g^{\mu\nu}{}_{,\alpha} = -g^{\mu\kappa}g^{\nu\lambda}g_{\kappa\lambda,\alpha}$$

to rewrite – later also employing (2.9) –

$$\frac{1}{2m}\,g^{\mu\nu}{}_{,\alpha}p_\mu p_\nu = -\frac{1}{2m}\,g_{\kappa\lambda,\alpha}p^\kappa p^\lambda = -\frac{1}{2m}\left(\Gamma_{\lambda\alpha\kappa}+\Gamma_{\kappa\alpha\lambda}\right)p^\kappa p^\lambda = -\Gamma_{\kappa\alpha\lambda}p^\kappa u^\lambda$$

and thus to obtain the absolute derivative.

17.3.4 The Hamilton-Jacobi equation

The Hamilton-Jacobi equation $-\frac{\partial S}{\partial\tau} = \mathcal{H}\left(x^\mu, \frac{\partial S}{\partial x^\mu}\right)$ reads, with the Hamiltonian (17.16),

$$-\frac{\partial S}{\partial\tau} = \frac{1}{2m}\,g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - qA_\nu\right). \tag{17.19}$$

In the Boyer-Lindquist coordinates, we substitute the four-potential

$$A_\mu = \frac{Qr}{\Sigma}(-1, 0, 0, a\sin^2\theta) \tag{17.20}$$

and the non-zero components of the inverse Kerr-Newman metric

$$g^{tt} = -\frac{\mathcal{A}}{\Sigma\Delta}\,,\quad g^{t\phi} = -\frac{(2Mr-Q^2)a}{\Sigma\Delta}\,,\quad g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\,,\quad g^{rr} = \frac{\Delta}{\Sigma}\,,\quad g^{\theta\theta} = \frac{1}{\Sigma} \tag{17.21}$$

as following from the equations $g^{\mu\sigma}g_{\sigma\alpha} = \delta^\mu_\alpha$. In such a way, we obtain the explicit form of the HJ equation. By multiplying it by $2m\Sigma$ and some shuffling on the right-hand side, one has

$$\begin{aligned}-2m\Sigma\,\frac{\partial S}{\partial\tau} = &-\frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial\phi} + qQr\right]^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial t}\,a\sin^2\theta + \frac{\partial S}{\partial\phi}\right)^2 + \\ &+\Delta\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{\partial S}{\partial\theta}\right)^2.\end{aligned} \tag{17.22}$$

Thanks to independence of the Hamiltonian of τ, t and ϕ, Carter found that the equation has the separated solution

$$S = \frac{1}{2}\,m\tau - Et + L\phi + S_r(r) + S_\theta(\theta)\,.$$

Actually, by plugging in this ansatz, one has

$$\frac{1}{\Delta}\big[(r^2+a^2)E - aL - qQr\big]^2 - m^2r^2 - \Delta\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 = \left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 + (ma\cos\theta)^2 + \left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2,$$

where the left-hand side only depends on r while the right-hand side only on θ, so both have to be constant – let us denote this constant by $\mathcal{K}$:

$$\mathcal{K} = \frac{1}{\Delta}\left[(r^2+a^2)E - aL - qQr\right]^2 - m^2r^2 - \Delta p_r^2 \tag{17.23}$$

$$= \left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2 + (ma\cos\theta)^2 + p_\theta^2\,. \tag{17.24}$$

The existence of this last, fourth independent constant of motion leads to the complete set of separated first integrals of the equation of motion. We may right notice, from the second expression, that $\mathcal{K}$ cannot be negative, and that it only vanishes if all of the following holds: i) the particle moves along $\theta = \text{const}$; ii) it is tied to the equatorial plane ($\cos\theta = 0$) or $m = 0$; iii) $L = aE\sin^2\theta$.

17.3.5 Equations for radial and latitudinal components of four-velocity

The remaining two integrals of the equation of motion are simply found by expressing, from the above forms (17.23,17.24),

$$\Delta p_r^2 = \Delta(g_{rr}p^r)^2 = \frac{(m\Sigma u^r)^2}{\Delta} \qquad \text{and} \qquad p_\theta^2 = (g_{\theta\theta}p^\theta)^2 = (m\Sigma u^\theta)^2\,.$$

Finally, we list the full set of Carter equations:

$$\boxed{\begin{aligned}
m\Delta\Sigma u^t &= (r^2+a^2)R - \Delta\Theta a\sin^2\theta \\
&= \mathcal{A}E - (2Mr - Q^2)aL - (r^2+a^2)qQr\,, && (17.25\text{a})\\
m\Delta\Sigma u^\phi &= aR - \Delta\Theta \\
&= (2Mr - Q^2)aE + (\Delta - a^2\sin^2\theta)\frac{L}{\sin^2\theta} - qQar\,, && (17.25\text{b})\\
(m\Sigma u^r)^2 &= R^2 - \Delta(m^2r^2 + \mathcal{K})\,, && (17.25\text{c})\\
(m\Sigma u^\theta)^2 &= \mathcal{K} - (ma\cos\theta)^2 - \Theta^2\sin^2\theta\,, && (17.25\text{d})
\end{aligned}}$$

where

$$\begin{aligned}
R = R(r) &:= (r^2+a^2)E - aL - qQr\,,\\
\Theta = \Theta(\theta) &:= aE - \frac{L}{\sin^2\theta}\,.
\end{aligned}$$

Note that the equations are separated (solved with respect to individual components of u^μ), but they remain coupled. In particular, the right-hand sides of u^r and u^θ only depend on r and θ, respectively, but the Σ on their left-hand sides depends on both the coordinates. Fortunately, this factor is the same for both the equations, which makes it possible to rescale the time accordingly, $\mathrm{d}\tau \to \mathrm{d}\lambda := \mathrm{d}\tau/\Sigma$ (the so-called Mino time), and thus decouple the two meridional components of motion.

17.3.6 The case of massless particles

We have assumed massive particles so far, but it should be clear how to obtain equations for photons: one just writes p^μ instead of mu^μ on the left-hand sides, and puts $m = 0$ and $q = 0$

on the right-hand sides, i.e.

$$\boxed{\begin{aligned} \Delta\Sigma p^t &= (r^2+a^2)R - \Delta\Theta a\sin^2\theta = \mathcal{A}E - (2Mr-Q^2)aL\,, && (17.26a)\\ \Delta\Sigma p^\phi &= aR - \Delta\Theta = (2Mr-Q^2)aE + (\Delta - a^2\sin^2\theta)\frac{L}{\sin^2\theta}\,, && (17.26b)\\ (\Sigma p^r)^2 &= R^2 - \Delta\mathcal{K} = \big[(r^2+a^2)E - aL\big]^2 - \Delta\mathcal{K}\,, && (17.26c)\\ (\Sigma p^\theta)^2 &= \mathcal{K} - \Theta^2\sin^2\theta = \mathcal{K} - \left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2\,, && (17.26d) \end{aligned}}$$

where

$$R = R(r) := (r^2+a^2)E - aL\,, \qquad \Theta = \Theta(\theta) := aE - \frac{L}{\sin^2\theta}\,.$$

17.3.7 Principal null congruences

When illustrating the dragging phenomenon, we mentioned (uncharged massive) particles which freely fall from radial infinity with $L=0$. They have $E=m$ and $\mathcal{K}=a^2m^2$, and thus they move exactly along $\theta=\mathrm{const}$, as given by (17.25d). In the azimuthal direction, they have angular velocity $\Omega=\omega$ as it is necessary for vanishing of their conserved L.

In the case of massless particles when the term $(ma\cos\theta)^2$ is no longer present in the latitudinal-motion equation (17.25d), it is also possible to permanently annul the right-hand side of the latter by making vanish both the remaining terms *individually* – by choosing, in (17.26d), $\mathcal{K}=0$ and $L=aE\sin^2\theta$ (where θ is *the* particular latitude along which the motion proceeds). These are the photons of the **principal null congruences** (PNC). Substituting to the remaining Carter equations, one finds

$$\begin{aligned} \Delta\Sigma p^t &= E\big[\mathcal{A} - (2Mr-Q^2)a^2\sin^2\theta\big] = E\Sigma(r^2+a^2) && \Longrightarrow \quad p^t = E\,\frac{r^2+a^2}{\Delta}\,,\\ \Delta\Sigma p^\phi &= E\big[(2Mr-Q^2)a + (\Delta - a^2\sin^2\theta)a\big] = E\Sigma a && \Longrightarrow \quad p^\phi = E\,\frac{a}{\Delta}\,,\\ (\Sigma p^r)^2 &= E^2(r^2+a^2-a^2\sin^2\theta)^2 = E^2\Sigma^2\,, && \Longrightarrow \quad p^r = \pm E\,. \end{aligned}$$

The two solutions, only differing in the sign of the radial component, are mostly denoted by k^μ (outgoing) and l^μ (ingoing). The normalization factor (the energy E) can be chosen arbitrarily (but constant), so it can actually be absorbed in the parameter of the respective congruence. One is thus simply left with

$$k^\mu = \frac{1}{\Delta}\,(r^2+a^2, \Delta, 0, a) \quad \leftrightarrow \quad k_\mu = \left(-1, \frac{\Sigma}{\Delta}, 0, a\sin^2\theta\right), \tag{17.27}$$

$$l^\mu = \frac{1}{\Delta}\,(r^2+a^2, -\Delta, 0, a) \quad \leftrightarrow \quad l_\mu = \left(-1, -\frac{\Sigma}{\Delta}, 0, a\sin^2\theta\right). \tag{17.28}$$

The motion along such world-lines has azimuthal angular velocity $\Omega = \frac{k^\phi}{k^t} \equiv \frac{l^\phi}{l^t} = \frac{a}{r^2+a^2}$; it is slightly faster than the angular velocity of dragging ω,

$$\frac{\omega}{\frac{a}{r^2+a^2}} = \frac{\frac{a}{\mathcal{A}}(2Mr-Q^2)}{\frac{a}{r^2+a^2}} = \frac{(2Mr-Q^2)(r^2+a^2)}{\mathcal{A}} = 1 - \frac{\Sigma\Delta}{\mathcal{A}} = 1 - N^2\,.$$

At the horizon ($\Delta = 0$), the angular velocity thus becomes $\omega_{\rm H}$, so, if adjusting the normalization suitably, the outgoing PNC photons coincide there with the generators of the horizon, $k^\mu \sim (1, 0, 0, \omega_{\rm H})$.

Observers which follow stationary circular orbits with the angular velocity Ω of the PNC photons (i.e. those who see the principal directions to be purely radial) are the Carter (canonical) observers we have already met in Section 16.6. They have four-velocity

$$u^\mu = \frac{1}{\sqrt{\Sigma\Delta}}\,(r^2+a^2, 0, 0, a)\,, \qquad u_\mu = \sqrt{\frac{\Delta}{\Sigma}}\,(-1, 0, 0, a\sin^2\theta) \tag{17.29}$$

which – in passing – is just parallel to the EM four-potential $A_\mu = \frac{Qr}{\Sigma}(-1, 0, 0, a\sin^2\theta)$. Similarly as ZAMOs, they are time-like everywhere outside the horizon.

The principal null directions are very important in the curvature structure of space-time – they are eigen-directions of the Riemann (or Weyl) tensor and thus are crucial in its algebraic classification (see Section 30.5). Curvature of the outer Kerr-Newman space-time is *algebraically special* (type D, or II-II), which according to the Goldberg-Sachs theorem is equivalent to the existence of a *shear-free and geodesic null vector field* (Chapters 24 and 30). Here in Kerr-Newman, more accurately, *two* such fields exist, k^μ and l^μ, each representing a double eigen-direction of the Riemann (or Weyl) tensor.[5] At the same time, they also fit nicely into a simple coordinate picture of the Kerr-Newman space: in the 3D Kerr-Schild–type coordinates (ρ, ϕ, z), they are straight-line generators of the hyperboloids $\theta = \mathrm{const}$ (within the projection to the coordinate *plane* (ρ, z), they are of course hyperbolas, but remember that they also have a certain component in the ϕ direction).

17.3.8 Carter constant and Killing tensor of the Kerr-Newman space-time

From Appendix B we know that the existence of a Killing tensor field $\xi_{\mu\ldots\nu}$ ensures that the scalar $\xi_{\mu\ldots\nu}u^\mu\ldots u^\nu$ is conserved along any geodesic. And there does exist such a tensor in Kerr-Newman. Actually, the above projection may even be conserved along certain accelerated world-lines, provided of course that the acting force is "symmetric" as well – typically in electro-vacuum space-times where the EM field follows the same symmetry as the geometry. In the case of Kerr-Newman space-time, specifically, there exists the 2nd-rank Killing tensor $\xi_{\mu\nu}$, and the corresponding scalar indeed remains constant along the world-lines of (Lorentz-force affected) charged test particles; it is exactly the "fourth" constant due to Carter, $\mathcal{K} = \xi_{\mu\nu}p^\mu p^\nu$. Let us demonstrate it.

$\boxed{\text{Lemma}}$ The bivector $Y_{\mu\nu}$ of non-zero BL components ($\omega = t, \phi$)

$$Y_{\omega r} = a\cos\theta\left(1, 0, 0, -a\sin^2\theta\right), \quad Y_{\omega\theta} = r\sin\theta\left(-a, 0, 0, r^2+a^2\right) \tag{17.30}$$

---

[5] If given by (17.27) and (17.28), both the fields are really geodesic (and affinely parameterized) and shear-free. In the literature, the principal null fields are being normalized in many ways which are not always so clear in this respect, namely so that at least one of the vectors ceases to be geodesic (or at least affinely parameterized) and/or shear-free. In particular, as basis vectors of a suitable null tetrad, k^μ and l^μ are often normalized so that $k^\mu l_\mu = -1$, which is natural for the tetrad-work purposes, but since $g_{\mu\nu}k^\mu l^\nu = -2\Sigma/\Delta$, it requires to adjust the fields using the factor Σ/Δ which however is *not* constant along the Kerr-Newman geodesics, so such a choice does not correspond to a "permitted" choice of the energy E in the solution of Carter equations.

is the Killing-Yano tensor of the Kerr-Newman space-time. (The covariant components are totally independent of M and Q!)

Proof: To be proved is the equality $Y_{\mu(\nu;\alpha)} = 0$. It's straightforward ... and tedious, so we beg to leave it to the reader (or to a computer algebra).

Corollary: As shown in Appendix B, the existence of the KY tensor implies the existence of the 2nd-rank Killing tensor $\xi_{\mu\nu} = Y_{\mu\alpha}Y_\nu{}^\alpha$. Here in Kerr-Newman its BL components read

$$\xi_{tt} = \frac{a^2}{\Sigma}\left(\Delta\cos^2\theta + r^2\sin^2\theta\right), \qquad \xi_{t\phi} = -\frac{a\sin^2\theta}{\Sigma}\left[\Sigma\Delta + (2Mr - Q^2)r^2\right],$$
$$\xi_{rr} = -\frac{\Sigma}{\Delta}\,a^2\cos^2\theta\,, \qquad \xi_{\theta\theta} = r^2\Sigma\,, \qquad \xi_{\phi\phi} = \frac{\sin^2\theta}{\Sigma}\left(r^2\mathcal{A} + \Sigma\Delta a^2\sin^2\theta\right).$$

Another option how to write it is

$$\xi_{\mu\nu} = \Delta k_{(\mu}l_{\nu)} + r^2 g_{\mu\nu}\,, \tag{17.31}$$

where k_μ, l_μ are the principal null congruences introduced in the previous subsection. Such a form makes it very easy to compute basic invariants connected with the tensor:

$$g^{\mu\nu}\xi_{\mu\nu} = -2\Sigma + 4r^2 = 2(r^2 - a^2\cos^2\theta)\,, \qquad \xi^{\mu\nu}\xi_{\mu\nu} = 2r^4 + 2a^4\cos^4\theta\,.$$

In passing, the invariants directly given by the Killing-*Yano* tensor are strikingly simple as well,

$$Y^{\mu\nu}Y_{\mu\nu}\ (= g^{\mu\nu}\xi_{\mu\nu}) = 2(r^2 - a^2\cos^2\theta)\,, \qquad {}^*Y^{\mu\nu}Y_{\mu\nu} = 4ra\cos\theta\,,$$
$$\left|Y^{\mu\nu}Y_{\mu\nu} - \mathrm{i}\,{}^*Y^{\mu\nu}Y_{\mu\nu}\right| = \sqrt{(Y^{\mu\nu}Y_{\mu\nu})^2 + ({}^*Y^{\mu\nu}Y_{\mu\nu})^2} = 2\Sigma\,.$$

$\boxed{\text{Lemma}}$ In the Kerr-Newman field, the quantity $\mathcal{K} = \xi_{\mu\nu}p^\mu p^\nu$ is conserved along the world-lines of charged test particles. (It is an extension of the knowledge from Appendix B – there it only refers to geodesics.)

Proof: Using the definition property $\xi_{(\mu\nu;\alpha)} = 0$ of the Killing tensor and substituting from the equation of motion (17.12), we find

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\tau} = \frac{\mathrm{D}\mathcal{K}}{\mathrm{d}\tau} = m^2\xi_{\mu\nu;\alpha}u^\mu u^\nu u^\alpha + 2m^2\xi_{\mu\nu}u^\mu{}_{;\alpha}u^\alpha u^\nu =$$
$$= m^2\xi_{(\mu\nu;\alpha)}u^\mu u^\nu u^\alpha + 2m\xi_{\mu\nu}F^\mu{}_\lambda u^\lambda u^\nu = 0 + 0\,.$$

The second term is really zero as well, because the expression $\xi_{\mu\nu}F^\mu{}_\lambda$ is skew-symmetric in indices $[\nu, \lambda]$:

$$2\xi_{\mu(\nu}F^\mu{}_{\lambda)} = \Delta\left(k_{(\mu}l_{\nu)}F^\mu{}_\lambda + k_{(\mu}l_{\lambda)}F^\mu{}_\nu\right) + 2r^2F_{(\nu\lambda)} = 0\,.$$

Above, vanishing of $F_{(\nu\lambda)}$ is automatic and vanishing of the expression in parenthesis follows from the relations

$$F_{\mu\nu}k^\mu = -\frac{Q}{\Sigma^2}(2r^2 - \Sigma)\,k_\nu\,, \qquad F_{\mu\nu}l^\mu = \frac{Q}{\Sigma^2}(2r^2 - \Sigma)\,l_\nu \tag{17.32}$$

which can be checked easily by plugging there (16.28) and (17.27,17.28). □

Finally, we should show that the scalar thus obtained is indeed the Carter constant:

$$\begin{aligned}\xi_{\mu\nu}p^\mu p^\nu &= \Delta k^t l^t (p_t)^2 + 2\Delta k^t l^\phi p_t p_\phi + \Delta k^\phi l^\phi (p_\phi)^2 + \Delta k^r l^r (p_r)^2 - m^2 r^2 = \\ &= \frac{1}{\Delta}\left[(r^2+a^2)(E+qA_t) - a(L-qA_\phi)\right]^2 - \Delta (p_r)^2 - m^2 r^2 = \\ &= \frac{1}{\Delta}\left[(r^2+a^2)E - aL - qQr\right]^2 - \Delta (p_r)^2 - m^2 r^2 \,.\end{aligned}$$

17.3.9 Spherical photon orbits

Let us add one more application of the Carter equations which provides another illustration how the Kerr-Newman geometry works, and also supports r as a privileged radial coordinate. Consider the question whether there exist photon world-lines that keep $r = \mathrm{const}$ (hence "spherical") while orbiting in the ϕ as well as θ coordinate (in the latitudinal direction, the orbit is only complete if $L = 0$, otherwise it is not possible to approach the rotational axis arbitrarily). For such orbits, $p^r = 0$ must hold *constantly*. By differentiation of (17.26c) with respect to the affine parameter (assumed to be normalized so that $p^\mu = \dot{x^\mu}$), we have

$$2\Sigma p^r (\dot{\Sigma} p^r + \Sigma \dot{p}^r) = 4\left[(r^2+a^2)E - aL\right] r\dot{r}E - 2(r-M)\,\dot{r}\mathcal{K}\,,$$

which, after dividing by $2p^r \equiv 2\dot{r}$ and (then) substituting $p^r = 0$, becomes

$$\Sigma^2 \dot{p}^r = 2rE\left[(r^2+a^2)E - aL\right] - (r-M)\,\mathcal{K}\,.$$

Hence, we reach the conditions

$$\begin{aligned} p^r = 0 \qquad &\Longleftrightarrow \qquad \mathcal{K} = \frac{1}{\Delta}\left[(r^2+a^2)E - aL\right]^2 \,, \\ \dot{p}^r(p^r = 0) = 0 \qquad &\Longleftrightarrow \qquad \mathcal{K} = \frac{2rE}{r-M}\left[(r^2+a^2)E - aL\right] \,. \end{aligned}$$

The option $\mathcal{K} = 0 = (r^2 + a^2)E - aL$ is excluded, because (17.26d) would in such a case reduce to

$$(p^\theta)^2 = -\frac{E^2}{a^2 \sin^2\theta}$$

which is impossible. The non-zero option is solved by

$$\ell := \frac{L}{E} = -\frac{2r\Delta - (r-M)(r^2+a^2)}{a\,(r-M)}\,, \qquad \frac{\mathcal{K}}{E^2} = \frac{4r^2\Delta}{(r-M)^2}\,. \tag{17.33}$$

The spherical photon orbits only exist at radii for which the equation (17.26d) offers some latitudinal interval, i.e. at such radii for which the right-hand side of (17.26d) is *somewhere* non-negative. By solving this condition, one finds that the limits of latitudinal motion, symmetric with respect to the equatorial plane, are only non-empty within certain interval of radii whose minimal and maximal radii are given by prograde and retrograde equatorial circular photon geodesics. Actually, for Kerr space-time ($Q=0$), the limit condition $p^\theta=0$, i.e. $\frac{\mathcal{K}}{E^2}=\left(a\sin\theta-\frac{\ell}{\sin\theta}\right)^2$ yields, after inserting the above values of ℓ and $\mathcal{K}/E^2$,

$$(\cos^2\theta)_{\max}=r\,\frac{-r^3+3M^2r-2Ma^2+2\sqrt{M\Delta(2r^3-3Mr^2+Ma^2)}}{a^2(r-M)^2}$$

for the latitudinal boundaries. As expected, they go as far as $(\cos^2\theta)_{\max}=1$ (i.e. up to the axis) when $\ell=0$, while they shrink to just $(\cos^2\theta)_{\max}=0$ at radii satisfying the equation

$$r(r-3M)^2=4Ma^2\,.$$

This is exactly the equation for equatorial circular photon orbits, with the roots given by (16.22). Outside them, there is no room for spherical photon motion.

Therefore, there exists the whole continuous class of spherical photon orbits, travelling at all possible inclinations with respect to the equatorial plane and having the prograde and retrograde equatorial photon orbits as their limits. Since "spherical" means here that these orbits stay on constant r, the exercise shows that r is a very plausible radial coordinate.

17.3.10 Radial free fall from infinity

In a sense, the following also confirms that r and θ are preferred coordinates: particles freely falling from rest from radial infinity with zero angular momentum L (thus having $\Omega=\omega$) travel along $\theta=\mathrm{const.}$ Really, to fall from rest from infinity means to have $E=m$. Then, since $L=0$, and at infinity also $p_\theta=0$, we can evaluate (17.24) *there* and find $\mathcal{K}=m^2a^2$. To summarize, our particles have constants

$$E=m,\quad L=0,\quad \mathcal{K}=m^2a^2.$$

Substitution to the Carter equation (17.25d) reveals that $u^\theta=0$. If $q=0$ (uncharged particle), the remaining Carter equations become

$$\Delta\Sigma u^t=\mathcal{A}\,,\qquad \Delta\Sigma u^\phi=(2Mr-Q^2)a\,,\qquad (\Sigma u^r)^2=(2Mr-Q^2)(r^2+a^2)\,.$$

One may check, for example, that $\Omega=\frac{u^\phi}{u^t}=\frac{2Mr-Q^2}{\mathcal{A}}\,a=\omega$, as we already knew before from the conservation of $L=mu_\phi=mu^tg_{\phi\phi}(\Omega-\omega)$ (along geodesics). Note that if the particle is charged, the conserved quantity is $L=mu_\phi+qA_\phi$, which modifies the consequent Ω.

CHAPTER 18

Observer frames and Fermi-Walker transport

In the theory of relativity, one constantly concerns with what depends on coordinates and what is invariant. Despite their absolute validity, even invariants may not provide an insight, since they do not necessarily correspond to anything what some observer could actually experience. For a simple example, hardly anybody *directly* experiences the four-velocity invariant $(-c^2)$.

A direct answer to the question of measurements starts from establishing a family of "observers" – a congruence of time-like world-lines – and local frames tied to them. In order to yield plausible data and help intuition, both the observers and their frames should be "physically natural" and have reasonable (simple) mathematical description. The observer frames $\{e^\mu_{\hat{\alpha}}\}_{\hat{\alpha}=0..3}$ (with $e^\mu_{\hat{0}} \equiv \hat{u}^\mu$) are usually chosen orthonormal,

$$g_{\mu\nu} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \eta_{\alpha\beta} \quad \Leftrightarrow \quad \eta^{\alpha\beta} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = g^{\mu\nu} \, .$$

If the measurement is not just quasi-local ("at one point"), the tetrad needs to be defined along a finite segment of the observer world-line (or of the whole congruence). Such a definition should mathematically be provided by a certain transport operating along the congruence. Since the observer four-velocity u^μ is automatically taken as the basic (zeroth) tetrad vector, and since it is typically chosen a priori, the transport has to be adjusted to it – the four-velocity has to automatically transport "correctly". The second requirement is that the transport conserves scalar product of vectors, so that the frame remains orthonormal. (This is a rather obvious requirement: were it not true, the transport would change the vectors' norm.)

Please notice that *parallel transport does not satisfy the first requirement*, so it is not useful for setting up evolution of observer tetrads. Indeed, parallel transport only keeps the four-velocity tangent to a world-line if the world-line is a *geodesic*. (This in fact is the defining property of geodesics: they are such world-lines along which their tangent vector transports parallelly.) For *generic*, accelerated world-lines, their tangent vector functions are *not* parallel, so – the other way round – if using the parallel transport, what was a tangent vector at a certain point is generally not tangent further then along the world-line. (See Figure 18.1.)

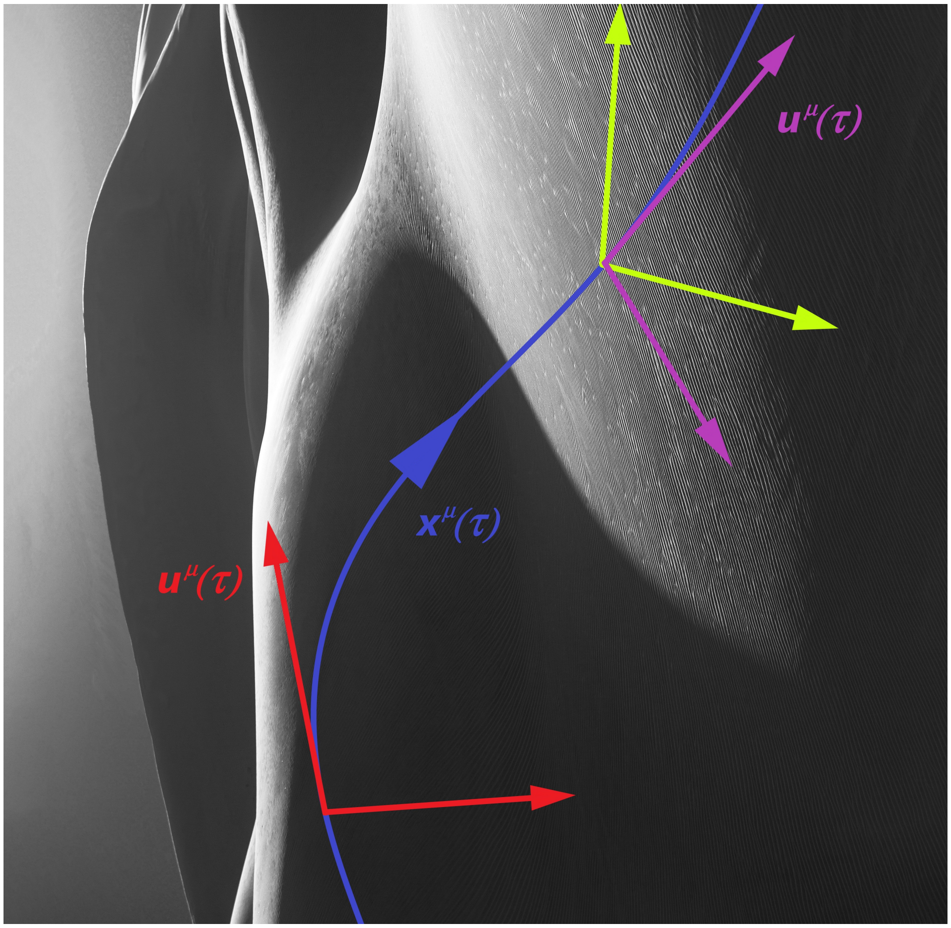


Figure 18.1 *A 2D illustration of the difference between the parallel and the Fermi-Walker transport. Along the blue world-line $x^\mu(\tau)$, the initial, red-depicted orthonormal basis having four-velocity u^μ as the time vector evolves into the same-type basis (the violet one) according to the Fermi-Walker transport, whereas the basis obtained by parallel transport (the light-green one) is "generic" (its time vector is no longer tangent to x^μ).*

Remark: Students somehow often say that the flaw of the parallel transport is that it can only be used along a geodesic. This is not true! Consider that it can actually be used along any *smooth curve, even a space-like one.*

Should not the transport be tied to any special geometric structure or physical element, the evolution of the four-velocity along its world-line ought to have the form

$$\frac{\mathrm{D}\hat{u}^\mu}{\mathrm{d}\hat{\tau}} \;(=:\hat{a}^\mu) = f(\hat{u}^\alpha, \hat{a}^\alpha; g_{\kappa\lambda})\,,$$

where $\hat{a}^\alpha$ is the corresponding acceleration. Since, due to normalization, $\hat{u}^\alpha$ can only change in the (normal) direction of $\hat{a}^\alpha$, it must perform a rotation in the $(\hat{u}^\alpha, \hat{a}^\alpha)$ plane. Such a rotation is described by a matrix given by antisymmetrized product of the two vectors, so one suggests the transport law[1]

$$\boxed{\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} = \left(u^\mu a_\nu - a^\mu u_\nu\right) V^\nu}\,. \tag{18.1}$$

This formula, called formula for the **Fermi-Walker transport**, really embodies that the tangent vector automatically satisfies it, since for $V^\mu \equiv u^\mu$ it reduces to $a^\mu = a^\mu$. Let us show that it has all the other desirable properties as well:

- Fermi-Walker transport conserves scalar product. Indeed, for arbitrary two vectors V^μ and W^μ,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}V^\mu W^\nu\right) = \frac{\mathrm{D}}{\mathrm{d}\tau}\left(g_{\mu\nu}V^\mu W^\nu\right) &= g_{\mu\nu}\,\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau}\,W^\nu + g_{\mu\nu}V^\mu\,\frac{\mathrm{D}W^\nu}{\mathrm{d}\tau} = \\ &= g_{\mu\nu}(u^\mu a^\sigma - u^\sigma a^\mu)V_\sigma W^\nu + g_{\mu\nu}V^\mu(u^\nu a^\sigma - u^\sigma a^\nu)W_\sigma = \\ &= (\cancel{u^\mu a^\sigma} - \cancel{u^\sigma a^\mu})V_\sigma W_\mu + (\cancel{u^\nu a^\sigma} - \cancel{u^\sigma a^\nu})V_\nu W_\sigma = 0\,.\end{aligned}$$

- In the case of a geodesic ($a^\mu = 0$), the Fermi-Walker transport coincides with the parallel transport,

$$\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} = 0\,.$$

- Vectors orthogonal to u^μ ("purely spatial") transport according to a shorter formula

$$\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} = u^\mu a^\nu V_\nu \tag{18.2}$$

  called **Fermi transport**. This tells that the projection of an absolute derivative of V^μ to the three-space orthogonal to u^μ vanishes,

$$\frac{\mathrm{D}V^\nu}{\mathrm{d}\tau}(\delta^\mu_\nu + u^\mu u_\nu) = \frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} + u^\mu\,\frac{\mathrm{D}V^\nu}{\mathrm{d}\tau}\,u_\nu = \frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} - u^\mu V^\nu\,\frac{\mathrm{D}u_\nu}{\mathrm{d}\tau} \equiv \frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} - u^\mu V^\nu a_\nu \;(=0)\;.$$

Therefore, the parallel transport keeps the "space-time direction" of vectors, irrespectively of the transport path, whereas the Fermi-Walker transport keeps the direction of vectors *with respect to the tangent vector of the transport world-line*.

18.1 Transport of spin and gyroscopes

The tangent vector to any world-line is unique, but the spatial triad may in general be chosen in many ways and it may also evolve along its host world-line in many ways (of course while continually satisfying the requirement that its vectors are orthogonal to the tangent

[1] Let us omit the hats from now on – we are simply speaking about *some* world-line, without needing to stress that we consider an observer on it.

and mutually orthonormal). Hence, the Fermi-Walker formula must be prescribing a certain specific behaviour for the spatial triad. Let us show how to understand it.

Consider a *gyroscope*. Technically, a light, symmetric, torque-free gyroscope (if forces act on it, they only act at its centre of mass). Anyway, we will basically imagine it naively, as something which keeps its direction in space, as given by its spin vector s^μ. In special relativity, such a property means that the spin does not change with respect to inertial systems (in spite of possible acceleration of its centre of mass). Hence, according to the equivalence principle,[2] in a certain local inertial system with respect to which the gyroscope is (at least momentarily) at rest – i.e. where the four-velocity u^μ of its centre of mass has but the time component – the spatial components of its spin do not change. Bearing also in mind that the spin is purely spatial with respect to u^μ, we may summarize

$$\exists \text{ system (“hatted”)}: \qquad u^{\hat{\imath}} = 0, \quad s^{\hat{0}} = 0, \quad \frac{\mathrm{d}s^{\hat{\imath}}}{\mathrm{d}\tau} = 0\,.$$

The covariant content of the above assumptions is that the gyroscope moves along a time-like world-line, $u_\mu u^\mu = -1$, that its spin is orthogonal to the four-velocity, $u_\mu s^\mu = 0$, and that the change of s^μ along u^μ is purely temporal with respect to u^μ, i.e. $\frac{\mathrm{D}s^\mu}{\mathrm{d}\tau} = \lambda u^\mu$, with λ some scalar. Differentiating the orthogonality condition and using the last relation, we have

$$0 = \frac{\mathrm{D}u_\mu}{\mathrm{d}\tau}\, s^\mu + u_\mu \frac{\mathrm{D}s^\mu}{\mathrm{d}\tau} = a_\mu s^\mu - \lambda \qquad \Longrightarrow \qquad \lambda = a_\mu s^\mu\,.$$

Therefore, the spin vector is subject to equation

$$\frac{\mathrm{D}s^\mu}{\mathrm{d}\tau} = u^\mu a_\nu s^\nu\,, \tag{18.3}$$

which is exactly the equation for Fermi transport.

Hence also the interpretation of the frame imagined at the beginning of this chapter: the spatial triad whose orthonormal vectors are Fermi-transported physically corresponds to a basis made of three orthogonal gyroscopes. Among all possible spatial bases which may be attached to a given world-line, the Fermi-transported ones thus represent those which are *non-rotating* in the usual 3D sense.

18.1.1 A *rotating* basis

Is it possible to write an evolution equation for a *generic* (“rotating”) spatial basis and see how exactly it rotates with respect to the Fermi-Walker transported one? Yes, one just has to add, to the FW formula (18.1), a term representing rotation in a plane orthogonal to u^μ (which may or may not contain a^μ). So consider, instead of the FW formula (18.1), a more general one

$$\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau} = (u^\mu a^\nu - a^\mu u^\nu)\, V_\nu + \epsilon^{\mu\nu\kappa\lambda} V_\nu \Omega_\kappa u_\lambda\,, \tag{18.4}$$

[2] The gyroscope is considered to be *negligibly small with respect to the curvature length-scale*, otherwise it would be affected by non-homogeneity of the field, i.e. by curvature, so the equivalence principle would not apply – see Section 18.1.2 below.

where the angular-velocity vector Ω^μ is orthogonal to u^μ and specifies the plane of rotation. The formula remains trivially satisfied for $V^\mu = u^\mu$, since the added last term is orthogonal to u^μ, so the tangent vector remains transported as it should be. The new term also does not harm conservation of the scalar product,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\,(g_{\mu\nu}V^\mu W^\nu) = \frac{\mathrm{D}}{\mathrm{d}\tau}\,(g_{\mu\nu}V^\mu W^\nu) &= g_{\mu\nu}\,\frac{\mathrm{D}V^\mu}{\mathrm{d}\tau}\,W^\nu + g_{\mu\nu}V^\mu\,\frac{\mathrm{D}W^\nu}{\mathrm{d}\tau} = \\ &= g_{\mu\nu}(... + \epsilon^{\mu\sigma\kappa\lambda}\Omega_\kappa u_\lambda)V_\sigma W^\nu + g_{\mu\nu}V^\mu(... + \epsilon^{\nu\sigma\kappa\lambda}\Omega_\kappa u_\lambda)W_\sigma = \\ &= ... + ... + \epsilon^{\mu\sigma\kappa\lambda}\Omega_\kappa u_\lambda(V_\sigma W_\mu + W_\sigma V_\mu) = 0\end{aligned}$$

(we have denoted by dots the “old” terms we already know to cancel out). The new term apparently describes rotation of V^μ in the plane perpendicular to u^μ and Ω^μ. The meaning of this rotation is revealed by considering, besides the above vector V^μ, also another one, say U^μ, which transports along the same world-line in the Fermi-Walker way (thus without the last term in the equation), and which at a certain moment coincides with V^μ. Evaluating the derivative of their difference *just at that moment*, the Fermi-Walker terms cancel out mutually and one is left with

$$\frac{\mathrm{D}(V^\mu - U^\mu)}{\mathrm{d}\tau} = \epsilon^{\mu\nu\kappa\lambda}V_\nu\Omega_\kappa u_\lambda\,.$$

In the special case when V^μ and U^μ are spatial-basis vectors (V^μ belonging to a general basis and U^μ to the Fermi-Walker transported one), so when they are both perpendicular to u^μ, the whole equation “lives” in the three-space orthogonal to u^μ and one can rewrite the result in a three-vector notation as

$$\frac{\mathrm{D}(\vec{V} - \vec{U})}{\mathrm{d}\tau} = \vec{\Omega} \times \vec{V}\,,$$

with $\times$ standing for the vector product in that three-space.

The additional term $\epsilon^{\mu\nu\kappa\lambda}V_\nu\Omega_\kappa u_\lambda \equiv (\vec{\Omega} \times \vec{V})^\mu$ is thus called *spatial rotation*; specifically, it represents rotation which is *not* restricted to the plane spanned by u^μ and a^μ, and so which is not solely forced by the fact that the vectors u^μ and a^μ themselves rotate in space-time (namely that they do not transport parallelly).

18.1.2 Applicability of the Fermi-(Walker) formula

The Fermi-Walker transport describes the evolution of a *test* gyroscope with a certain initial value of spin (proper rotational angular momentum) along a *prescribed* world-line – the corresponding acceleration is taken as *independent of the evolution of the gyroscope*. Physically, this is only adequate for small spin. Namely, the spin magnitude bounds, from below, the size of the gyroscope – roughly to $\gtrsim \frac{s}{m}$, where m is its mass, in order that (at least some) gyro elements would not have to rotate with superluminal speed. (A small body simply cannot have very large spin, because $\boldsymbol{s} = m\boldsymbol{r} \times \boldsymbol{v}$.) If the spin is large, the size of the gyro cannot be negligible, so tidal forces enter the play, caused by non-homogeneity of the gravitational field and described by the Riemann tensor. In such a case, the spin behaviour does affect the motion, so one has to solve a coupled problem for momentum (equation of motion) and spin (equation for precession). In the case when the body does not bear significant higher multipoles, such a problem leads to the equations of the Mathisson-Papapetrou-Dixon type, see Section 6.4.1.

Below, we focus on three specifically relativistic contributions to the gyroscope precession, while restricting to the case when the FW description is appropriate, i.e. when the gyroscope itself does not contribute to the field, and when there is no feedback between the gyro's spin evolution and its orbital motion.

18.2 Thomas precession

Consider a gyroscope carried along an accelerated world-line in the *Minkowski background*. Consider a continuous sequence of inertial frames with respect to which the gyroscope is *momentarily* at rest; components with respect to these will be denoted by a hat. Every such frame is related to the "laboratory" inertial frame (no special notation) by the Lorentz boost ($c \equiv 1$)

$$\hat{t} = \gamma\,(t - \boldsymbol{v}\cdot\boldsymbol{r})\,, \qquad \hat{\boldsymbol{r}} = \boldsymbol{r} + \frac{\gamma-1}{v^2}\,(\boldsymbol{v}\cdot\boldsymbol{r})\,\boldsymbol{v} - \gamma\,t\,\boldsymbol{v} \qquad \ldots \quad \text{direct boost}\,, \tag{18.5}$$

$$t = \gamma\left(\hat{t} + \boldsymbol{v}\cdot\hat{\boldsymbol{r}}\right)\,, \qquad \boldsymbol{r} = \hat{\boldsymbol{r}} + \frac{\gamma-1}{v^2}\,(\boldsymbol{v}\cdot\hat{\boldsymbol{r}})\,\boldsymbol{v} + \gamma\,\hat{t}\,\boldsymbol{v} \qquad \ldots \quad \text{inverse boost}\,, \tag{18.6}$$

where $\boldsymbol{v}$ is the instantaneous spatial velocity of the gyro with respect to the laboratory system, and γ is the corresponding Lorentz factor. Since in every instantaneously comoving frame the gyro's spin is purely spatial, $s^{\hat{\alpha}} = (0, \hat{\boldsymbol{s}})$, its laboratory components $s^\mu = (s^0, \boldsymbol{s})$ read

$$s^0 = \gamma\,\boldsymbol{v}\cdot\hat{\boldsymbol{s}}\,, \qquad \boldsymbol{s} = \hat{\boldsymbol{s}} + \frac{\gamma-1}{v^2}\,(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\,\boldsymbol{v}\;.$$

Take now the Fermi-transport equation (18.3) in Minkowski, $\frac{\mathrm{d}s^\mu}{\mathrm{d}t}\gamma = u^\mu a_\nu s^\nu$, and substitute there

$$u^\mu = \gamma(1,\boldsymbol{v}) \quad \Longrightarrow \quad a^\mu := \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \gamma^2\left[\gamma^2(\boldsymbol{v}\cdot\boldsymbol{a}),\,\boldsymbol{a}+\gamma^2(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\right], \quad \text{with } \boldsymbol{a} := \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\,,$$

$$\begin{aligned}
a_\nu s^\nu &= -\gamma^5(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}}) + \gamma^2\left[\boldsymbol{a}+\gamma^2(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\right]\cdot\left[\hat{\boldsymbol{s}} + \frac{\gamma-1}{v^2}(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{v}\right] = \\
&= \gamma^2\left[(\boldsymbol{a}\cdot\hat{\boldsymbol{s}}) + \frac{\gamma-1}{v^2}(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\right], \\
\frac{\mathrm{d}s^0}{\mathrm{d}t} &= \gamma^3(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}}) + \gamma(\boldsymbol{a}\cdot\hat{\boldsymbol{s}}) + \gamma\left(\boldsymbol{v}\cdot\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}\right), \\
\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t} + \frac{(\gamma-1)^2(\gamma+2)}{v^4}(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{v} + \frac{\gamma-1}{v^2}\left[(\boldsymbol{a}\cdot\hat{\boldsymbol{s}})\boldsymbol{v} + \left(\boldsymbol{v}\cdot\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}\right)\boldsymbol{v} + (\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{a}\right].
\end{aligned}$$

Next, in order to get rid of the products $\left(\boldsymbol{v}\cdot\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}\right)$, combine from above

$$\frac{v^2}{\gamma-1}\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} - \frac{1}{\gamma}\frac{\mathrm{d}s^0}{\mathrm{d}t}\boldsymbol{v} = \frac{v^2}{\gamma-1}\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t} + \frac{\gamma-1}{v^2}(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{v} + (\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{a}$$

and compare it to the equivalent combination of the right-hand sides of the Fermi-transport equation,

$$\left(\frac{v^2}{\gamma-1}-\frac{1}{\gamma}\right)(a_\nu s^\nu)\,\boldsymbol{v}=\frac{1}{\gamma^2}(a_\nu s^\nu)\,\boldsymbol{v}=\left[(\boldsymbol{a}\cdot\hat{\boldsymbol{s}})+\frac{\gamma-1}{v^2}(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\right]\boldsymbol{v}\,.$$

One thus arrives at equation

$$\frac{v^2}{\gamma-1}\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}=(\boldsymbol{a}\cdot\hat{\boldsymbol{s}})\boldsymbol{v}-(\boldsymbol{v}\cdot\hat{\boldsymbol{s}})\boldsymbol{a}\;\equiv\;\hat{\boldsymbol{s}}\times(\boldsymbol{v}\times\boldsymbol{a})\,. \tag{18.7}$$

Hence, the spin precesses according to the equation

$$\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}=-\boldsymbol{\Omega}_{\mathbf{T}}\times\hat{\boldsymbol{s}}\,,\qquad\text{where}\quad\boldsymbol{\Omega}_{\mathbf{T}}:=\frac{\gamma-1}{v^2}(\boldsymbol{v}\times\boldsymbol{a})\,. \tag{18.8}$$

Clearly the Thomas precession is only present if the gyroscope moves with respect to the laboratory ($\boldsymbol{v}\neq 0$) along an accelerated world-line ($\boldsymbol{a}\neq 0$), if the two vectors are not parallel ($\boldsymbol{v}\times\boldsymbol{a}\neq 0$), and if the spin $\hat{\boldsymbol{s}}$ is not perpendicular to the plane they span (i.e. it is not parallel to $\boldsymbol{\Omega}_{\mathbf{T}}$). It remains to take your right hand now and apply the cross-product rule twice, or to look at the original form (18.7), in order to see that the spin change i) is orthogonal to $\hat{\boldsymbol{s}}$, ii) lies in the $(\boldsymbol{v},\boldsymbol{a})$ plane (thus in the plane of the 3D trajectory), and that its angular velocity iii) is opposite (retrograde) with respect to an instantaneous *orbital* angular velocity (defined using an osculating circle to the spatial trajectory).

18.2.1 With respect to what the gyro precesses?

In derivation of the Thomas precession, this is often "lost in translation" (or rather in rotation). Actually, the defining assumption that it keeps direction in its proper, comoving frame, may seem to be in contradiction with that the above equation describes rotation of the *locally measured three-vector*, not of the spatial components of the four-spin in the laboratory frame. The point is that the quantities written in boldface are not measured with respect to the frame which the gyro carries along (this proper frame has nowhere been used in this section), but – once more – with respect to a continuous sequence of inertial frames with respect to which the gyroscope is *momentarily* at rest. Such frames only *instantaneously* coincide with the gyro's proper frame, of course, because the gyro is accelerated.

Why such a complicated picture? Every "instantaneous inertial frame" of the gyro is obtained by pure boost (Lorentz transformation without rotation of the spatial coordinate axes) from the laboratory inertial frame, so it represents, for a given relative three-velocity $\boldsymbol{v}$, a benchmark of *what is not turned with respect to the laboratory axes*. So the sequence of inertial frames set along the gyro's world-line realizes, along that world-line, the best possible counter-part of laboratory axes. Last query: why such a construction, why not to simply refer the spin behaviour to the laboratory axes trivially transferred to the momentary position of the gyro? Because the gyro is moving in a general direction (not necessarily along some of the laboratory-system axes), so its instantaneous-frame axes are *contracted* relative to the laboratory ones in the direction of the relative velocity, which makes the frames *non-orthogonal* with respect to each other. To summarize very shortly: the precession is to be understood as happening *with respect to the laboratory frame*. (Thus, clearly, it depends on how the laboratory frame is precisely defined.)

18.3 Geodetic precession

General relativity adds two more important components to the gyroscope precession. In a generic situation (generic motion in a generic background), the behaviour of the gyro may be quite a complicated mixture of all the effects, so we will only consider simple settings in which a given contribution reveals clearly.

Let a gyroscope (with spin s^μ) move on a geodesic (with four-velocity u^μ) in the Schwarzschild field. For zero acceleration, the Fermi-Walker transport goes over to parallel transport, so we have equation

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = -\Gamma^\mu{}_{\kappa\lambda}u^\kappa s^\lambda \,.$$

Imagine the simple case of a circular orbit, which in the Schwarzschild coordinates means

$$u^\mu = u^t(1,0,0,\Omega), \quad u^t = \frac{1}{\sqrt{-g_{tt} - g_{\phi\phi}\Omega^2}}\,, \quad \Omega := \frac{\mathrm{d}\phi}{\mathrm{d}t}$$
$$\Longrightarrow \quad \frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = -u^t\left(\Gamma^\mu{}_{t\lambda} + \Gamma^\mu{}_{\phi\lambda}\Omega\right)s^\lambda \,.$$

Remember that s^μ has to be orthogonal to u^μ, $g_{\kappa\lambda}u^\kappa s^\lambda = g_{tt}u^t s^t + g_{\phi\phi}u^\phi s^\phi = 0$, so the time component of spin has to read $s^t = \frac{g_{\phi\phi}\Omega s^\phi}{-g_{tt}}$. Remember also that it is sufficient to fix this at one point of the world-line, because parallel transport conserves scalar product. And here are the non-zero Christoffel symbols for Schwarzschild:

$$\Gamma^r{}_{tt} = \frac{M(r-2M)}{r^3}\,, \qquad \Gamma^t{}_{tr} = -\Gamma^r{}_{rr} = \frac{M}{r(r-2M)}\,, \qquad \Gamma^r{}_{\theta\theta} = \frac{\Gamma^r{}_{\phi\phi}}{\sin^2\theta} = -(r-2M)\,,$$
$$\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}\,, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta\,, \qquad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}\,.$$

Geodesic motion is planar in the spherically symmetric field, so we choose this plane to be the equatorial one ($\theta = \pi/2$) as usual. Components of the transport equation thus read

$$\frac{\mathrm{d}s^t}{\mathrm{d}\tau} = -u^t s^r \Gamma^t{}_{tr} = -\frac{M\,u^t s^r}{r(r-2M)}\,, \qquad \frac{\mathrm{d}s^\phi}{\mathrm{d}\tau} = -u^t s^r \Gamma^\phi{}_{\phi r}\Omega = -\frac{u^t s^r}{r}\,\Omega\,, \qquad \frac{\mathrm{d}s^\theta}{\mathrm{d}\tau} = 0\,,$$
$$\frac{\mathrm{d}s^r}{\mathrm{d}\tau} = -u^t s^t \Gamma^r{}_{tt} - u^t s^\phi \Gamma^r{}_{\phi\phi}\Omega = -u^t s^\phi \Omega\left(\Gamma^r{}_{\phi\phi} - \frac{g_{\phi\phi}}{g_{tt}}\,\Gamma^r{}_{tt}\right) = u^t s^\phi \Omega\,(r-3M)\,.$$

- $\frac{\mathrm{d}s^t}{\mathrm{d}\tau}$ is not very important, because s^t is anyway given by $s^t = \frac{g_{\phi\phi}\Omega s^\phi}{-g_{tt}}$. However, by comparing it with the derivative of this last relation, while substituting for $\frac{\mathrm{d}s^\phi}{\mathrm{d}\tau}$, one easily obtains the value of the orbital angular velocity: $\Omega^2 = \frac{M}{r^3}$.

- If erecting the spin exactly perpendicular to the equatorial plane, $s^\lambda = (0,0,s^\theta,0)$, it clearly remains such, without any change. Otherwise, in a generic situation, the behaviour of s^θ is "uninteresting", so let us focus on the remaining components.

- Since $u^t \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau}$, it is shorter to write the remaining equations as

  $$\frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -\frac{s^r}{r}\,\Omega\,, \qquad \frac{\mathrm{d}s^r}{\mathrm{d}t} = s^\phi\Omega\,(r-3M)\,.$$

  The spin certainly rotates with respect to the coordinates with some constant angular velocity, call it $\Omega_{\rm gyro}$, so we may write

  $$s^r(t) = s^r(0)\cos(\Omega_{\rm gyro}t)\,.$$

  Now it is easy to check that the equations are satisfied if

  $$s^\phi(t) = -\frac{1}{r}\frac{\Omega}{\Omega_{\rm gyro}}\,\sin(\Omega_{\rm gyro}t)\,, \qquad \frac{\Omega^2_{\rm gyro}}{\Omega^2} = \frac{r-3M}{r} \;\;(<1)$$

  (zero initial s^ϕ has been chosen). So the gyro's rotation is *slower* than the orbital revolution: the gyro does not point in the same direction after one full orbit, it somewhat lags behind. Yes, the sign changes at $r=3M$, but that is the photon orbit – the last free circular orbit, where a physical gyroscope can no longer orbit freely. The difference arises due to curvature and it is called the **geodetic precession** or the **de Sitter effect**. (Sometimes even the word "geodesic" is being used, but that is not the best choice, because the effect does not only act along geodesics, of course.)

18.4 Lense-Thirring (gravitomagnetic) precession

In Chapter 16 on the Kerr solution, we discussed thoroughly how a rotating massive body drags the surrounding space-time into co-rotation. We saw that the dragging is strongly differential, falling off with distance as $1/r^3$. This must bring another precession effect to our collection. It actually brings *two* effects – one "local", influencing ordinary gyroscopes along their orbits (or even possibly staying at rest, as we will see), and the other "global", making the whole orbits precess with respect to the asymptotic frame. We will focus on the first, local effect here, the global one being best studied using the Carter equations (Section 17.3).[3]

In order to reveal the effect of dragging in a pure form (not coexisting with the Thomas and/or geodetic precession, if possible), we consider two special cases of motion in the Kerr space-time – a free radial fall along the rotation axis, and a gyroscope on stationary circular orbit in the equatorial plane.

18.4.1 Gyroscope freely falling along the Kerr symmetry axis

Consider the Kerr space-time and a gyroscope which freely falls from rest at radial infinity, with zero orbital angular momentum, $u_\phi = 0$ ($\Rightarrow \Omega = \omega$), first along a *general* θ. For such a motion, the energy with respect to infinity is just the rest mass,

$$1 = \frac{E}{m} \equiv -u_t = -g_{tt}u^t - g_{t\phi}u^\phi = u^t(-g_{tt} - g_{t\phi}\omega) \equiv u^t N^2 \qquad \Longrightarrow \quad u^t = \frac{1}{N^2}\,.$$

[3] Both effects are often studied in the *gravito-electromagnetic* analogy, see Section 22.7.

We also saw in Section 17.3.10 that this motion follows $\theta = \mathrm{const}$. From normalization $g_{\mu\nu}u^\mu u^\nu = -1$, we thus obtain

$$-1 = (u^t)^2(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2) + g_{rr}(u^r)^2 = -(u^t)^2N^2 + g_{rr}(u^r)^2 = -\frac{1}{N^2} + g_{rr}(u^r)^2$$

$$\Longrightarrow \quad u^\mu = \left(\frac{1}{N^2}\,,\ -\sqrt{\frac{1-N^2}{N^2 g_{rr}}}\,,\ 0\,,\ \frac{\omega}{N^2}\right) = \left(\frac{\mathcal{A}}{\Sigma\Delta}\,,\ -\frac{\sqrt{2Mr(r^2+a^2)}}{\Sigma}\,,\ 0\,,\ \frac{2Mra}{\Sigma\Delta}\right).$$

The orthogonality of s^μ to u^μ is satisfied if

$$0 = g_{\mu\nu}u^\mu s^\nu = u_t s^t + \cancel{u_\phi} s^\phi + u_r s^r = -s^t + g_{rr}u^r s^r$$

$$\Longrightarrow \quad s^t = g_{rr}u^r s^r = -\frac{\sqrt{2Mr(r^2+a^2)}}{\Delta}\, s^r\,.$$

The Fermi transport reduces, for free fall, to the parallel transport, so

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = -\Gamma^\mu{}_{\kappa\lambda}u^\kappa s^\lambda = -u^t\big(\Gamma^\mu{}_{t\lambda} + \Gamma^\mu{}_{\phi\lambda}\omega\big)s^\lambda - \Gamma^\mu{}_{r\lambda}u^r s^\lambda\,.$$

Focus now on the gyroscope just falling along the symmetry axis ($\theta = 0$). For such a special case, the four-velocity slightly simplifies,

$$u^\mu = \left(\frac{r^2+a^2}{\Delta}\,,\ -\sqrt{\frac{2Mr}{r^2+a^2}}\,,\ 0\,,\ \frac{2Mra}{\Delta(r^2+a^2)}\right).$$

It is a quick computer task (the more if your laptop has already reached Quantum Supremacy) to find that of the Christoffel symbols $\Gamma^\mu{}_{t\lambda}$, $\Gamma^\mu{}_{\phi\lambda}$ and $\Gamma^\mu{}_{r\lambda}$, the following remain non-zero on the axis:

$$\Gamma^r{}_{tt} = \frac{M\Delta(r^2-a^2)}{(r^2+a^2)^3}\,, \quad \Gamma^r{}_{rr} = -\frac{M(r^2-a^2)}{\Delta(r^2+a^2)}\,, \quad \Gamma^t{}_{tr} = \frac{M(r^2-a^2)}{\Delta(r^2+a^2)}\,, \quad \Gamma^\phi{}_{tr} = \frac{Ma(r^2-a^2)}{\Delta(r^2+a^2)^2}\,,$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{r}{r^2+a^2}\,, \quad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}\left(1 + \frac{2Mra^2\sin^2\theta}{\Sigma^2}\right), \quad \Gamma^\phi{}_{t\theta} = -\frac{\cos\theta}{\sin\theta}\,\frac{2Mar}{\Sigma^2}$$

(the last two being divergent there, we have better given them in an exact generic form). The evolution equation thus yields

$$\frac{\mathrm{d}s^t}{\mathrm{d}\tau} = -\Gamma^t{}_{tr}(u^t s^r + u^r s^t)\,, \qquad \frac{\mathrm{d}s^r}{\mathrm{d}\tau} = -\Gamma^r{}_{tt}u^t s^t - \Gamma^r{}_{rr}u^r s^r = 0\,, \qquad \frac{\mathrm{d}s^\theta}{\mathrm{d}\tau} = -\Gamma^\theta{}_{r\theta}u^r s^\theta$$

(equation for s^ϕ is considerably longer). So the evolution of s^θ is completely decoupled from the evolution of s^t and s^r. And, of the latter, s^r does not change at all during the fall, while s^t evolves so as to be constantly related to (the constant) s^r by $s^t = g_{rr}u^r s^r$. However, s^r is certainly not *the interesting* component in this situation, so if it stays constant, why not to choose it zero? Doing so, one has $s^t = 0$ as well, so the equations reduce to

$$\frac{\mathrm{d}s^\theta}{\mathrm{d}\tau} = -\Gamma^\theta{}_{r\theta}u^r s^\theta\,, \qquad \frac{\mathrm{d}s^\phi}{\mathrm{d}\tau} = -\Gamma^\phi{}_{r\phi}u^r s^\phi - u^t(\Gamma^\phi{}_{t\theta} + \Gamma^\phi{}_{\phi\theta}\omega)s^\theta\,,$$

where, however, a very nice thing happens with the uncomfortable two Gammas:

$$\Gamma^{\phi}{}_{t\theta}+\Gamma^{\phi}{}_{\phi\theta}\omega=-\frac{2Mra^3\sin\theta\cos\theta}{\Sigma\mathcal{A}}\quad\xrightarrow{\theta\to 0}0\,.$$

Consequently, finally we are only left with

$$\frac{\mathrm{d}s^{\theta}}{\mathrm{d}\tau}=-\Gamma^{\theta}{}_{r\theta}u^r s^{\theta}\,,\qquad \frac{\mathrm{d}s^{\phi}}{\mathrm{d}\tau}=-\Gamma^{\phi}{}_{r\phi}u^r s^{\phi}\qquad\Longrightarrow\quad \frac{\mathrm{d}s^{\phi}}{\mathrm{d}s^{\theta}}=\frac{s^{\phi}}{s^{\theta}}\quad\Rightarrow\quad\frac{s^{\phi}}{s^{\theta}}=\mathrm{const}\,.$$

Admittedly, "azimuthal component" is not the best idea on the axis, so let us add how nicely the result appears in the (Kerr-Schild–type) Cartesian-like coordinates

$$x=\sqrt{r^2+a^2}\,\sin\theta\cos\phi\,,\qquad y=\sqrt{r^2+a^2}\,\sin\theta\sin\phi\,,\qquad z=\sqrt{r^2+a^2}\,\cos\theta\,.$$

Transforming the spin vector in a standard way,

$$\begin{aligned}
s^x&=\frac{\partial x}{\partial r}\cancel{s^r}+\frac{\partial x}{\partial\theta}s^{\theta}+\frac{\partial x}{\partial\phi}s^{\phi}\quad\xrightarrow{\theta\to 0}\quad\frac{\partial x}{\partial\theta}s^{\theta}=\sqrt{r^2+a^2}\,\cos\phi\,s^{\theta}\,,\\
s^y&=\frac{\partial y}{\partial r}\cancel{s^r}+\frac{\partial y}{\partial\theta}s^{\theta}+\frac{\partial y}{\partial\phi}s^{\phi}\quad\xrightarrow{\theta\to 0}\quad\frac{\partial y}{\partial\theta}s^{\theta}=\sqrt{r^2+a^2}\,\sin\phi\,s^{\theta}\,,\\
s^z&=\frac{\partial z}{\partial r}\cancel{s^r}+\frac{\partial z}{\partial\theta}s^{\theta}+\cancel{\frac{\partial y}{\partial\phi}}s^{\phi}\quad\xrightarrow{\theta\to 0}\quad 0\,,
\end{aligned}$$

we obtain, for the evolution of s^x and s^y,

$$\begin{aligned}
\frac{\mathrm{d}s^x}{\mathrm{d}\tau}&=\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{r^2+a^2}\,\cos\phi\right)s^{\theta}+\sqrt{r^2+a^2}\,\cos\phi\,\frac{\mathrm{d}s^{\theta}}{\mathrm{d}\tau}=\\
&=\left(\cancel{\frac{ru^r}{\sqrt{r^2+a^2}}\cos\phi}-\sqrt{r^2+a^2}\,\sin\phi\,u^{\phi}\right)s^{\theta}-\cancel{\sqrt{r^2+a^2}\,\cos\phi\,\frac{ru^r}{r^2+a^2}}\,s^{\theta}=\\
&=-\sqrt{r^2+a^2}\,\sin\phi\,u^t\omega\,s^{\theta}=-u^t\omega\,s^y\,,\\
\frac{\mathrm{d}s^y}{\mathrm{d}\tau}&=\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{r^2+a^2}\,\sin\phi\right)s^{\theta}+\sqrt{r^2+a^2}\,\sin\phi\,\frac{\mathrm{d}s^{\theta}}{\mathrm{d}\tau}=\\
&=\left(\cancel{\frac{ru^r}{\sqrt{r^2+a^2}}\sin\phi}+\sqrt{r^2+a^2}\,\cos\phi\,u^{\phi}\right)s^{\theta}-\cancel{\sqrt{r^2+a^2}\,\sin\phi\,\frac{ru^r}{r^2+a^2}}\,s^{\theta}=\\
&=\sqrt{r^2+a^2}\,\cos\phi\,u^t\omega\,s^{\theta}=u^t\omega\,s^x\,,
\end{aligned}$$

which means that the gyro rotates about the axis exactly with the dragging angular velocity $\Omega_{\rm gyro}=\omega$:

$$\frac{\mathrm{d}s^x}{\mathrm{d}t}=-\omega\,s^y\,,\qquad\frac{\mathrm{d}s^y}{\mathrm{d}t}=\omega\,s^x\,.\tag{18.9}$$

Note that since $\omega=\frac{2Mra}{(r^2+a^2)^2-\Delta a^2\sin^2\theta}$, it increases from the axis towards the equatorial plane, hence the gyro pointing in the θ direction precesses "along" the centre's rotation (the picture is at least clear when close to the axis).

18.4.2 Gyroscope on circular orbit in the Kerr equatorial plane

Second, we wish to consider a gyroscope "at rest" in the equatorial plane, $\theta = \pi/2$. To stay at rest certainly means to keep constant r and θ, but with ϕ it is rather unclear. Indeed, when treating the rotational dragging in the Kerr space-time, we argued that it is not clear, in the field of a rotating source, what it means to "non-orbit" (in the direction of the source rotation, i.e. ϕ). We also suggested stationary circular motion with zero angular momentum (ZAMO congruence) as one of sensible options – see Section 16.3.3. However, let us start from the stationary motion ($\Omega = \mathrm{const}$) along a *generic* circular orbit, as studied in Section 16.3.2. For such,

$$u^\mu = u^t(1,0,0,\Omega), \quad u^t = \frac{1}{\sqrt{N^2 - g_{\phi\phi}(\Omega-\omega)^2}}\,, \qquad a_\nu = \cancel{u_{\nu,\lambda}u^\lambda} - \Gamma^\kappa{}_{\lambda\nu}u_\kappa u^\lambda\,,$$

so the Fermi-transport equation $\frac{\mathrm{D}s^\mu}{\mathrm{d}\tau} = u^\mu a_\nu s^\nu$ assumes the form

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = -(\delta^\mu_\kappa + u^\mu u_\kappa)\,\Gamma^\kappa{}_{\lambda\nu}u^\lambda s^\nu \qquad \ldots\ \times\ \frac{1}{u^t}$$

$$\Longrightarrow \quad \frac{\mathrm{d}s^\mu}{\mathrm{d}t} = -(\Gamma^\mu{}_{t\nu} + \Gamma^\mu{}_{\phi\nu}\Omega)s^\nu - u^\mu\left(\Gamma^t{}_{t\nu}u_t + \Gamma^\phi{}_{t\nu}u_\phi + \Gamma^t{}_{\phi\nu}u_t\Omega + \Gamma^\phi{}_{\phi\nu}u_\phi\Omega\right)s^\nu\,.$$

The orthogonality of s^μ to u^μ implies that s^t and s^ϕ have to keep a certain relation,

$$0 = u_t s^t + u_\phi s^\phi \qquad \text{(while } s^r \text{ and } s^\theta \text{ are not constrained)}\,.$$

In the equatorial plane, the Fermi-transport formula has special properties:

- The θ-component of the equation vanishes, and s^θ disappears from it altogether. Hence, in particular, if s^i is chosen perpendicular to the equatorial plane (with only s^θ non-zero) at some point, it stays so forever – it is a permanent solution of the equation, similarly as it held in Schwarzschild.

- So the equation reduces to the r and ϕ components (with s^t determined by $u_t s^t = -u_\phi s^\phi$),

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} = -\Gamma^r{}_{tt}s^t - \Gamma^r{}_{t\phi}s^\phi - \Gamma^r{}_{t\phi}\Omega s^t - \Gamma^r{}_{\phi\phi}\Omega s^\phi\,,$$

$$\frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -(\Gamma^\phi{}_{tr} + \Gamma^\phi{}_{\phi r}\Omega)s^r - u^t\Omega\left(\Gamma^t{}_{tr}u_t + \Gamma^\phi{}_{tr}u_\phi + \Gamma^t{}_{\phi r}u_t\Omega + \Gamma^\phi{}_{\phi r}u_\phi\Omega\right)s^r\,.$$

  Clearly it describes rotation within the equatorial plane. Actually, abbreviating the equations as

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} = \alpha(r)\,s^\phi, \qquad \frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -\beta(r)\,s^r, \qquad\qquad r = \mathrm{const}\,, \tag{18.10}$$

  we may, for example, choose the initial spin as purely radial, $s^r(0) \neq 0$, $s^\phi(0) = 0$, and solve them in the form

$$s^r(t) = s^r(0)\cos(\Omega_{\mathrm{gyro}}t) \qquad \Longrightarrow \qquad \frac{\mathrm{d}s^r}{\mathrm{d}t} = -s^r(0)\,\Omega_{\mathrm{gyro}}\sin(\Omega_{\mathrm{gyro}}t) \quad \ldots = \alpha(r)\,s^\phi(t)$$

$$\Longrightarrow \quad \frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -s^r(0)\,\frac{\Omega^2_{\rm gyro}}{\alpha(r)}\,\cos(\Omega_{\rm gyro}t) \quad ... = -\beta(r)\,s^r(t) = -\beta(r)s^r(0)\cos(\Omega_{\rm gyro}t)$$
$$\Longrightarrow \quad \Omega^2_{\rm gyro} = \alpha\beta \;\;(\text{must be} \geqslant 0)\,. \tag{18.11}$$

Let us analyse in more detail two specific cases, $\Omega=\omega$ (gyroscope tied to the ZAMO) and $\Omega=0$ (gyroscope tied to the static observer):

- For the ZAMO, the *orbital* dragging effects (and so the geodetic-precession contribution as well) should be eliminated as much as possible. $\Omega=\omega$ implies $u_\phi=0$ (this is the angular momentum), so $s^t=0$ (sure: ZAMO's u^μ is orthogonal to $t=\mathrm{const}$, so s^t has to vanish in order that s^μ be orthogonal to u^μ), and $u^t u_t=-1$. Therefore, the equations reduce to

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} = -(\Gamma^r{}_{t\phi}+\Gamma^r{}_{\phi\phi}\omega)s^\phi = \frac{Ma\Delta}{\mathcal{A}r^2}(3r^2+a^2)\,s^\phi\,,$$
$$\frac{\mathrm{d}s^\phi}{\mathrm{d}t} = \left(\Gamma^t{}_{tr}\omega+\Gamma^t{}_{\phi r}\omega^2-\Gamma^\phi{}_{tr}-\Gamma^\phi{}_{\phi r}\omega\right)s^r = -\frac{Mar^2}{\mathcal{A}^2}(3r^2+a^2)\,s^r\,.$$

  Comparison with (18.10) and (18.11) yields the precession angular velocity

$$\Omega^2_{\rm gyro} = \frac{M^2a^2\Delta}{\mathcal{A}^3}(3r^2+a^2)^2\,. \tag{18.12}$$

- For the static gyroscope ($\Omega=0$), the precession equations reduce to only

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} = -\Gamma^r{}_{tt}s^t-\Gamma^r{}_{t\phi}s^\phi\,, \qquad \frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -\Gamma^\phi{}_{tr}s^r\,.$$

  Since $u_\kappa=g_{\kappa\lambda}u^\lambda=g_{\kappa t}u^t$, it holds $s^t=-\frac{g_{t\phi}}{g_{tt}}\,s^\phi$, so we have

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} = \left(\Gamma^r{}_{tt}\frac{g_{t\phi}}{g_{tt}}-\Gamma^r{}_{t\phi}\right)s^\phi = \frac{Ma\Delta}{r^3(r-2M)}\,s^\phi\,, \qquad \frac{\mathrm{d}s^\phi}{\mathrm{d}t} = -\Gamma^\phi{}_{tr}s^r = -\frac{Ma}{\Delta r^2}\,s^r\,,$$

  which implies that the precession angular velocity is given by

$$\Omega^2_{\rm gyro} = \frac{M^2a^2}{r^5(r-2M)}\,. \tag{18.13}$$

  The static observer may be considered as "effectively counter-rotating", because its angular velocity is smaller than the dragging one, $0=\Omega<\omega$, and, consequently, it has negative angular momentum,

$$u_\phi = g_{\phi\iota}u^\iota = g_{\phi t}u^t = \frac{g_{t\phi}}{\sqrt{-g_{tt}}} = -\frac{2Ma}{\sqrt{r(r-2M)}}\,.$$

  In spite of that, its (initially purely radial) gyroscope precesses *against* the centre's rotation, which exactly reveals how the differentially rotating geometry drags the origin of the gyro faster than its end (see Figure 18.2).

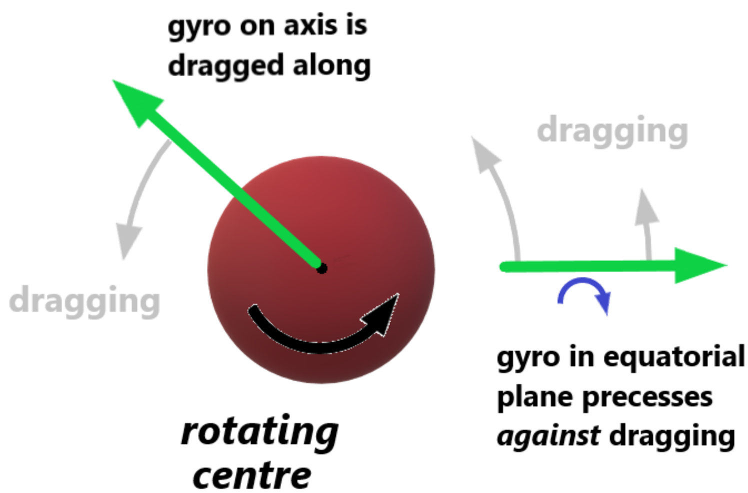


Figure 18.2 *Two gyroscopes (green) in the field of a rotating centre, as viewed along the axis of rotation. The gyroscope fixed to the axis (left one) follows dragging, i.e. it precesses in the same azimuthal sense in which the centre is rotating. On the contrary, the gyroscope fixed to the equatorial plane precesses in the* opposite *sense because dragging falls off with radius.*

CHAPTER 19

Gravitational collapse and black holes

Black hole is a very robust concept, because it is given by mere **causal structure**, it is independent of anything else. In brief, if sufficient mass concentrates in a sufficiently small volume, a black hole is formed just because nothing can move arbitrarily fast. However, what does depend on "microphysics" of matter are conditions under which sufficient concentration of mass ("gravitational collapse") can happen. If such a concentration were impossible in reality, the black holes would still be a robust concept, but just occurring as a mathematical curiosity in certain solutions of Einstein equations.

Yet physics does know situations in which the formation of a black hole is possible, in some cases it even does not know of anything which could prevent the gravitational collapse. Such an assertion might seem surprising, because gravitation is the weakest interaction, so one might expect that the necessary compression is in all circumstances easily prevented by structure of the matter – by its internal pressure, in microscopic terms, by the EM interaction, in some limit cases possibly also by strong interaction, by uncertainty relations and by the Pauli exclusion principle governing the behaviour of fermions. Indeed, the horizon formation requires truly extreme compression: should the mass M be concentrated within its Schwarzschild radius $r_{\rm S}=\frac{2GM}{c^2}$, its mean density would have to reach

$$M=\frac{4}{3}\pi r_{\rm g}^3\bar{\rho} \quad\Longrightarrow\quad \bar{\rho}=\frac{3M}{4\pi r_{\rm S}^3}=\frac{3c^6}{32\pi G^3M^2}=\frac{3c^6}{32\pi G^3M_\odot^2}\frac{M_\odot^2}{M^2}\doteq 1.85{\cdot}10^{19}\,\mathtt{kg/m^3}\left(\frac{M_\odot}{M}\right)^2$$

($M_\odot$ being mass of the Sun). For moderate-mass stars, the density required is greater than the nuclear density ($r_{\rm S}$ being about $3\mathrm{km}$ for Sun), while for Earth (it would have to be compressed under the radius of $9\mathrm{mm}$!), it comes out about $2\cdot 10^{30}\,\mathtt{kg/m^3}$; for "things of common usage", their Schwarzschild radius is deep below the scale of elementary particles. In spite of this, it turned out, notably in the 1930s, that if the contracting matter has more that some $2M_\odot$, gravitation *is* able to compress it below its Schwarzschild radius. Again, this is due to its long-range character and due to its *universally* attractive character. These two properties make it – contrary to other interactions – cumulate within the body. Still it looks like matter has to suffer very extreme conditions which are totally beyond our experience.

Not necessarily. The above mean density depends on $1/M^2$, and there also exist supermassive black holes in galactic nuclei. For $M \sim 10^{9\div10}M_\odot$, the necessary density equals that of the air on the Earth surface. The volume enclosed by the horizon of $10^9 M_\odot$ is about 100 times larger than the volume of 10^9 suns (we mean *our* Sun), so you may imagine to put together such a number of suns – better having switched off gravity first – while leaving 100 times more empty volume in between them: after switching on gravity, a horizon would enclose the cluster. (Sure that the suns could not stay at their positions then, they would all collapse to the central singularity irrespectively of any other physics – the picture is only to illustrate that black holes need not result from extreme conditions only.) Besides that, as already stressed in Section 14.2.2, the local physics at such a horizon is very far from extreme – the gravitational acceleration is $56\times$ bigger than on the Sun surface and $1500\times$ bigger than on the Earth surface (however the acceleration itself being *locally* irrelevant thanks to the equivalence principle, i.e. thanks to the possibility to transform it out by going freely falling), and the field non-homogeneity (curvature) is $75\times$ lower than on the Sun and $300\times$ lower than on the Earth.

Yes, volume should actually be computed differently in the curved space, but we will see in Section 20.3.1 (equation for mass) that in GR the mass contained in a spherical ball of radius r is really given by $M = \frac{4}{3}\pi r^3 \bar{\rho}$ as in the Newtonian case. Also, the "gravitational acceleration" we mentioned refers to the rigorous (and invariant) relativistic quantity of *surface gravity* we will introduce later in this chapter, as well as the non-homogeneity of the field which follows as the square root of the Kretschmann invariant.

Let us focus on the **origin of real black holes** now. First, when asked about origin of anything, one may answer in two basic ways: either it is here since the beginning of times, or it has arisen later. Ad the first option, it is really possible that the Universe started with some regions already "a priori" containing super-critical amount of matter, thus being born right enclosed in horizons, but it is very hard to say something more specific in this direction, simply because this option goes to the very initial conditions of the Universe. One can only put, on the basis of "anthropic" arguments following from current observations, some upper bound on how numerous such **primordial black holes** may be. We will thus focus on the second option and will ask which processes are able to concentrate enough mass to a small volume. Current physics is quite certain about one situation in which nothing should be able to balance gravity: sufficiently heavy nuclei of stars in which all the available nuclear energy has been exhausted. The region where the thermonuclear chain has terminated either slowly contracts to a white-dwarf state until the pressure of degenerate electrons counter-balances gravity, or it rapidly shrinks to a neutron star until the pressure of degenerate neutrons stops further collapse, or it collapses completely below its horizon. Generally, heavier bodies tend to be more prone to complete collapse, but in astrophysical reality the picture is very complicated and may in fact also end by explosion and complete disruption of the body. Anyway, there appears to be a certain mass value called the **Chandrasekhar limit** (1930, see Section 21.5) which is a fundamental threshold for whether the degenerate fermion gas (the Pauli principle) is able to counterbalance gravity or not.

Without trying to explain anything more concerning the extremely complicated final stages of stellar evolution (besides the contraction of the star nucleus also accompanied by more or less spectacular explosion of the outer star layers), let us at least show that GR can describe the complete gravitational collapse to a black hole consistently. We will follow the famous paper by J. R. Oppenheimer and H. Snyder which was published in Physical Review on 1st September 1939. Regarding the tragedy of that period, no surprise that much more interest incited then the paper by N. Bohr and J. A. Wheeler "Mechanisms of nuclear fission" which appeared just next but one to Oppenheimer & Snyder's. Also rather interestingly, only a month later, in Annals of Mathematics, Einstein himself published a paper, with the following conclusion: "The essential result of this investigation is a clear understanding as to why the 'Schwarzschild singularities' do not exist in physical reality." (At that time, 'Schwarzschild singularity' meant what we now call the black-hole horizon.) Einstein considered a "star" made of particles orbiting, symmetrically, on circular paths in their own gravitational field. He found that if the star radius had been below a certain value, the particles would have had to move faster than light. This result was correct, only the conclusion is different today...

19.1 Collapse of a spherical ball of incoherent dust

In every problem involving an extended body, one has to solve three issues – behaviour of the interior, behaviour of the exterior, and matching the two properly on the surface. Oppenheimer and Snyder considered the following very simple situation:

- A spherically symmetric "star". This immediately implies that the exterior has already been solved – it is Schwarzschild.
- A star made of incoherent dust, i.e. a pressure-free ideal fluid. According to the Euler equations (7.33), such kind of matter moves on geodesics.
- A star with homogeneous and isotropic interior. Such a body is locally identical to some homogeneous and isotropic cosmological model.

Hence, we will select a suitable cosmological model and describe the interior of the star accordingly. In order that the interior and the exterior solution can match on the stellar surface, we will compare the exterior and interior description of particles freely falling on that surface and relate the parameters accordingly. Finally, it is to be shown that the matching is conserved during the collapse. We will at least demonstrate that this holds for the two metrics.

19.1.1 Interior of the ball as a part of the closed FLRW universe

First, we will naturally set $\Lambda = 0$ in order that there is no other effective source than the dust itself (Λ is not important on stellar scale anyway). Further, if planning to describe a collapse from an initially static state, it is necessary to employ such a cosmological dust model which *is* static at some moment. This only fulfils the closed, spherical model – the remaining two (the flat and the hyperboloidal one) evolve in a monotonous way, without any turning point. The history of the spherical model is described by the cycloid (13.48), according to which the universe expands, from a big bang, to a certain maximal size and then collapses back symmetrically. Obviously, the collapse will be described as the contraction phase of a spherical ball cut out of the closed FLRW universe. The ball will be defined by $\chi \leqslant \chi_0 \, (< \pi/2)$. It is important to stress that the angular radius χ_0 will remain *constant* during the collapse – the ball will collapse as a part (a spherical "cap") of the whole cosmological model (technically, due to cycloidal decrease of the latter's expansion factor a).

The only modification we will perform is to shift the conformal time η by π (back), in order that $\eta = 0$ now correspond to the moment of maximal expansion (start of the collapse) rather than to big bang. In equations (13.48), this just reverses signs of the goniometric functions. As known from Chapter 13, in the FLRW models the cosmic fluid moves on geodesics, so we can safely use the equations given there to describe the collapse of our star. Let us write down the equation for the free fall of its surface in terms of the area radius. The latter is related to χ by $r = a \sin\chi$, so, from (13.48) with shifted η, the surface moves according to

$$R_{\rm F} = \frac{a_{\rm max} \sin\chi_0}{2} \left(1 + \cos\eta\right), \qquad \tau_{\rm F} = \frac{a_{\rm max}}{2} \left(\eta + \sin\eta\right), \tag{19.1}$$

where we have used τ instead of t to remind that the cosmic time actually has the meaning of proper time of the fluid.

19.1.2 Exterior of the ball as a part of Schwarzschild

The "exterior treatment" is simple – the particles on the ball's surface move according to the equations we derived in Section 14.1.4, in the paragraph on radial free fall from rest. From equations (14.12)–(14.14), we obtain for their area radius and proper time

$$R_{\rm S} = \frac{R_{\rm in}}{2} \left(1 + \cos\eta\right), \qquad \tau_{\rm S} = \sqrt{\frac{R_{\rm in}^3}{8M}} \left(\eta + \sin\eta\right), \tag{19.2}$$

where $R_{\rm in}$ naturally stands for the initial value of the radius.

19.1.3 Matching the two solutions

The above Friedmannian and Schwarzschildian equations for R and τ match on the surface if the parameters of the two solutions are related by

$$R_{\rm S} = R_{\rm F} \quad \Longrightarrow \quad R_{\rm in} = a_{\rm max} \sin\chi_0 \,, \tag{19.3}$$

$$\tau_{\rm S} = \tau_{\rm F} \quad \Longrightarrow \quad M = \frac{R_{\rm in}^3}{2a_{\rm max}^2} = \frac{a_{\rm max}}{2} \sin^3\chi_0 \,. \tag{19.4}$$

Remark: Does it really has good sense to compare the Friedmann-like interior solution and the Schwarzschild exterior solution? Actually, one may doubt whether the meaning of what we call "the conformal time" (η) is the same in Schwarzschild space-time as well as in cosmology (which are very different settings). However, the conformal time is only a parameter, nobody measures it, so one takes is pragmatically and argues the other way round: the consistency of the interior and the exterior solutions is just verified by the fact that equating the Friedmannian and the Schwarzschildian formulas for R and τ really yields time-independent relations between constants of the two solutions.

The recipe for gravitational collapse may now be summarized as follows:

- Take the closed $\Lambda = 0$ dust FLRW universe having (some chosen) $a = a_{\max}$ at the instant of maximal expansion ($\eta = 0$). Cut out of it the spherical region $\chi \leqslant \chi_0 \, (< 90°)$ – that will represent the interior of the star – and throw out the rest.

- Take Schwarzschild with mass $M = \frac{a_{\max}}{2} \sin^3 \chi_0$ at the instant $t = 0$ ($\Leftrightarrow \eta = 0$). Cut out of it and throw out the spherical region $r < R_{\rm in} = a_{\max} \sin \chi_0$; the rest will represent the exterior of the star.

- Join the two regions on the surface. The result is a momentarily static star with radius $R_{\rm in} = a_{\max} \sin \chi_0$ and mass $M = \frac{a_{\max}}{2} \sin^3 \chi_0$. Recalling from cosmology that $a_{\max} = \frac{8\pi}{3} \rho_0 a_0^3$ and that $\rho_0 a_0^3$ remains constant during the cosmic evolution, we may evaluate it just at the maximal-expansion instant, $a_{\max} = \frac{8\pi}{3} \rho_{\rm in} a_{\max}^3$, and from there obtain the initial density of the star $\rho_{\rm in} = 3/(8\pi a_{\max}^2)$. Let us also remember that we actually well know the above picture from Section 14.1.2: there, we were embedding the Schwarzschild equatorial plane in $\mathbb{E}^3$, also considering – besides the pure vacuum Schwarzschild – the case with a central star of constant density. The exterior Schwarzschild was represented by a rotational paraboloid, while the interior by a spherical cap just matching the paraboloid on the star surface. This is exactly what we have now assembled.

- Let it collapse freely, according to the Einstein equations...

Oppenheimer & Snyder showed that the smoothness of surface matching is conserved during the collapse. Let us at least check the basic requirement that the interior and exterior metrics match at the surface. On the surface, the FLRW metric reduces to

$$\mathrm{d}s_{\rm F}^2(\chi = \chi_0) = -\mathrm{d}\tau_{\rm F}^2 + a^2 \sin^2 \chi_0 \, \mathrm{d}\Omega^2 \,, \tag{19.5}$$

while the Schwarzschild metric reads

$$\mathrm{d}s_{\rm S}^2(r = R_{\rm S}) = -\left(1 - \frac{2M}{R_{\rm S}}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{R_{\rm S}}} + R_{\rm S}^2 \, \mathrm{d}\Omega^2 \,. \tag{19.6}$$

Since the area radii are clearly the same, $a^2 \sin^2 \chi_0 = \frac{R_{\rm in}}{2}\,(1 + \cos \eta) = R_{\rm S}$, it also applies to the whole angular terms. It thus remains to analyse the Schwarzschild (t, r) part. Recall the equations (14.10) and (14.11) describing the radial free fall from rest from $r = r_{\rm in}$ in Schwarzschild, i.e.

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \frac{1 - \frac{2M}{r_{\rm in}}}{\left(1 - \frac{2M}{r}\right)^2} \,, \qquad \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \frac{2M}{r} - \frac{2M}{r_{\rm in}} \,.$$

Applying them to our case $r = R_{\rm S}$, $r_{\rm in} = R_{\rm in}$ and $\mathrm{d}\tau = \mathrm{d}\tau_{\rm S}$, we can express

$$\mathrm{d}t^2 = \frac{1 - \frac{2M}{R_{\rm in}}}{\left(1 - \frac{2M}{R_{\rm S}}\right)^2} \, \mathrm{d}\tau_{\rm S}^2 \,, \qquad \mathrm{d}r^2 = \left(\frac{2M}{R_{\rm S}} - \frac{2M}{R_{\rm in}}\right) \mathrm{d}\tau_{\rm S}^2 \,,$$

which, when substituted to the above Schwarzschild-metric part, yields

$$-\left(1-\frac{2M}{R_{\rm S}}\right){\rm d}t^2+\frac{{\rm d}r^2}{1-\frac{2M}{R_{\rm S}}}=\left(-\frac{1-\cancel{\frac{2M}{R_{\rm in}}}}{1-\frac{2M}{R_{\rm S}}}+\frac{\frac{2M}{R_{\rm S}}-\cancel{\frac{2M}{R_{\rm in}}}}{1-\frac{2M}{R_{\rm S}}}\right){\rm d}\tau_{\rm S}^2=-{\rm d}\tau_{\rm S}^2\,.$$

However, at the proper-time element ${\rm d}\tau$, the indices "F" and "S" do not make any difference, because proper time is unique (the indices only indicate "from which side" we have obtained it), so the metrics (19.5) and (19.6) really match exactly on $\chi=\chi_0 \Leftrightarrow r=R_{\rm S}$.

19.1.4 Realistic collapse: with pressure and without symmetries

Physicists mostly deemed Oppenheimer & Snyder's setting to be too simplistic to accept their result as decisive. However, the need for a more realistic description gave way then to the study of just the opposite processes – the nuclear-bomb races began. In the 1950-60s, experts say, the programs simulating the explosions were basically used to model the collapse, only necessary was to substitute in them more mass (like $10M_\odot$) and to switch on its gravity... They confirmed the occurrence of a horizon.

With non-zero pressure (and non-zero pressure gradient in general), the elements of the star are not free, so they do not move on geodesics. It is no longer possible to assume homogeneity and isotropy, even if pressure was constant: at the star surface, it would anyway jump to zero, so the gradient would be infinite there. Consequently, after the collapse would have started, the outer layers would be tossed away and a rarefication wave would propagate inwards, destroying the homogeneity and isotropy. However, with a suitable pressure profile, the surface matching is possible and the results are not significantly different from the pressure-free case.

More complications may arise if the assumption of spherical symmetry is released. In the case of large asymmetry – especially with fast rotation – the problem becomes very complicated. Generally, asymmetries rather act against contraction, in particular, rotation typically leads to disintegration of the body at a certain stage. The query of whether it is possible to say, already before horizon is formed, that it *will* inevitably be formed, turned out to be quite hard. Several **"hoop conjectures"** have been stated which answer that question in terms of how short a hoop has to be which is able to encircle the collapsing body in every direction. And similar statements have also been suggested which confine the object by closed surfaces.)

Thanks to the **singularity theorems**, it is sure that inside a black hole there appears a singularity. However, in a generic (not spherically symmetric) situation, much less can be said about how large a fraction of the collapsing matter ends at the singularity.

19.1.5 Cosmic censorship hypothesis

In 1969, in a seminal paper on gravitational collapse and black holes, R. Penrose formulated a conjecture that a collapse should never produce a naked singularity. A bit more accurately, in an originally regular space-time, there should not occur any singularity visible from infinity. This opinion still remains plausible, although it is not proved as a theorem, and although several counter-examples have been provided. Namely, these counter-examples are numerical evolutions from certain very special (yet not unphysical) initial configurations, like an "imploding" asymmetric cloud of photons, when no horizon was found while the density had already reached arbitrarily large values at certain locations. The delicate point is to analyse generic validity and *stability* of such scenarios. Cosmic censorship thus stands as one of yet open questions of general relativity.

Naked singularities anyway pose problems for various types of reasoning, mainly for those based on causal relations, so in the theorems they are usually assumed to be absent. Actually, we have seen they prevent a unique solution of the Cauchy problem, and sometimes also induce the occurrence of a region with chronology or even causality violation. However, it is of course hard to strictly exclude **primordial naked singularities**, although these too are prone to various instabilities (e.g. to being "dressed" due to accretion of matter, or to quantum polarization). After all, singularities of the classical GR are supposed to be "resolved" by its quantum counter-part, which should eliminate their unphysical features. Therefore, it is not unreasonable to even study *undressed* alternatives in various situations.

Yet $a > M$ is *not* bizarre

It's worth to add a remark. The naked singularities are suggested as "over-extreme", bizarre options, so it might seem that a decent object should not have $a > M$ (or $|Q| > M$). The truth is contrary, it is a rule rather than exception that celestial as well as microscopic bodies largely exceed the "extreme" limits. Actually, you may check that for Earth $a/M \sim 880$, similarly as for Jupiter, for example. For stars this ratio tends to be of the order of unity, but very young stars or protostars may likely be rotating as fast as $a/M \sim 10^4$. For neutron stars (even the fastest, millisecond pulsars), on the contrary, the ratio is below unity. Hence, before shrinking to a small object, the large bodies must get rid of most of their angular momentum. This is very easily seen from the classical formula for a spherical rigid rotator: its spin angular momentum being

$$J = \frac{2}{5}MR^2\omega = \frac{2}{5}MRv\,,$$

it is seen that if this were to be conserved during contraction of the object, the equatorial linear speed $v = R\omega$ would have to increase reciprocally with decrease of the radius R. For Earth, for instance, the Schwarzschild radius is about $9\,\mathrm{mm}$, so to make a black hole from Earth would require to decrease the Earth radius 710 million times. The Earth rotation is slow, but still it corresponds to the linear speed $v \doteq 460\,\mathrm{m/s}$ of the equatorial regions. Shrinking the radius 710 million times and conserving J would thus require v to increase to $1100c$.

However, even more surprising may be to check these values for elementary particles. For electron, for example – let us calculate it carefully in SI units –, the ratios a/M and Q/M have extremely large values,

$$\frac{\frac{a}{c}}{\frac{GM}{c^2}} = \frac{\frac{J}{Mc}}{\frac{GM}{c^2}} = \frac{\frac{\hbar/2}{m_\mathrm{e}c}}{\frac{Gm_\mathrm{e}}{c^2}} = \frac{\hbar c}{2Gm_\mathrm{e}^2} = 2.854\cdot 10^{44}\,,$$

$$\frac{\frac{\sqrt{G}|Q|}{\sqrt{4\pi\epsilon_0}\,c^2}}{\frac{GM}{c^2}} = \frac{|Q|}{\sqrt{4\pi\epsilon_0 G}\,M} = \frac{e}{\sqrt{4\pi\epsilon_0 G}\,m_\mathrm{e}} = 2.041\cdot 10^{21}\,.$$

19.2 Black-hole uniqueness theorems

The stationary black-hole solutions we studied in preceding chapters may seem to represent just very special, simple examples selected so that they can be managed in an undergraduate course. It is not entirely so. The assumptions of isolation, high symmetries and asymptotic flatness certainly *make* the solutions special, but it is remarkable that if they are satisfied, the field equations uniquely lead to the metrics of the Kerr-Newman type:

> Theorem: Every isolated stationary black hole in an asymptotically flat space-time, which contains no singularities and no closed time-like curves elsewhere than possibly under the horizon, is necessarily of the Kerr-Newman type, i.e. – among others – it is axially symmetric, it has a horizon of spherical topology, and it is completely characterized by at most 3 parameters (M, Q and a). Specially, if it is even static, then it is of the Reissner-Nordström type, i.e. spherically symmetric and described by just 2 parameters (M, Q).

Factually this is a "synthetic" form of several theorems which had been derived under slightly different assumptions, most notably by W. Israel (Schwarzschild and Reissner-Nordström), B. Carter, D. C. Robinson (Kerr), P. O. Mazur and G. Bunting (Kerr-Newman).
Note: the parameters describing the isolated stationary black hole can actually be *four*, but the fourth one would correspond to a not-ever-observed magnetic monopole, so we do not include it.

Hard to guess the narrative of the years one did not directly experience, but the first theorem of this type – W. Israel's proof of uniqueness of the Schwarzschild black hole (1967) – rather made the community surprised and sceptical, because it seemed like if the complete gravitational collapse was restricted to the "zero-measure", exactly spherically symmetric situation. It was notably R. Penrose who, in the already-mentioned paper from 1969, stressed that in dynamical situation all the higher multipoles should be radiated away and, consequently, "if an absolute event horizon develops in an asymptotically flat space-time, then the solution exterior to this horizon approaches a Kerr-Newman solution asymptotically with time" (Penrose called this the "generalized Israel conjecture"). Within several years, his suggestion was confirmed by detailed calculations. The most important claim here is not the above theorem itself, but rather the belief that an isolated object – even though suffering such a violent process as the gravitational collapse – before long settles to a stationary state.

19.2.1 Why and how the collapsing object loses its hair

In official slang, the uniqueness theorems are referred to as "no-hair" theorems; "hair" means independent parameters which characterize the object. These are certainly being lost in the collapse, since the progenitor star was undoubtedly characterized by density, pressure, temperature, entropy and luminosity profiles, and by many other quantities describing nuclear reactions, radiation transfer, convection, turbulences, eruptions, etc. etc. Why and how this happens? Very loosely, it is simply because "hairy" object is energetically higher than the "bald" one. In particular, any deviation from axial symmetry tends to be radiated away since an object with a changing mass quadrupole emits gravitational waves. Similarly, any object with changing electric dipole emits EM waves. Both these fundamental interactions propagate with the speed of light, and an object on the verge of becoming a black hole is extremely small, so the characteristic time of "balding" and "calming down" is not many orders greater than $r_{\rm S}/c$, which for stellar-mass objects means fractions of a second. Let us jump to recent times and refer to the *gravitational-wave transient catalog* where signals of some 10 clear events are shown: they were typically generated in collisions of $10\div 50M_{\odot}$ black holes, and their characteristic time scale is seen to be 0.1 second.

Yet still, why and how almost all the hair is lost? You know that EM radiation is a dipole radiation. This means that if you perform a multipole decomposition of an EM field, then radiative (i.e. transporting energy) can be the components corresponding to dipole or higher multipoles. Indeed, monopole = charge and that is exactly conserved by the continuity equation $J^{\mu}{}_{;\mu}=0$, so charge is the only multipole that can*not* be radiated away. With gravitational radiation it is similar, only that it is a quadrupole one, so gravitational waves can carry away all the multipoles *beyond* dipole. Also different in gravity is that mass(-energy) does not equal monopole, because it also involves contributions from rotation (i.e. dipole) etc. Similarly, spin (rotational angular momentum) need not only include the "dipole" part. (Connected with this is also the fact that the centre of mass is *not* an invariant notion.) So gravitational radiation can carry away all moments except monopole and dipole parts of mass and spin.

In the classical theory of radiation, there in fact exists a theorem of that kind which applies to all "massless fields with integer spin". This is a particle-like slang for long-range fields (falling off according to the Coulomb law) with integer helicity. Actually, already on classical level it is possible to recognize what will be the spin (s) of particles obtained by quantization of a given field: a plane wave of that field is symmetric under rotation by $2\pi/s$ about the direction of its propagation, and it can be decomposed in two linearly polarized components whose polarization vectors make the angle $\pi/(2s)$. The classical counter-part of spin, of equal value, is called **helicity**. Now, the theorem says that, for such fields, radiative (able to transport energy) are the components corresponding to multipoles $l\geqslant s$. For EM field $s=1$ while for gravitational field $s=2$.

The last "stone in a mosaic" is called the **Price theorem** and it simply says that what *can* be radiated away *is* indeed radiated away, i.e. that really just the $l<s$ multipoles are left. Technically, the collapse scenario goes as follows:

- Have an isolated source of gravitation and electromagnetism. Let it only slightly deviate from the Reissner-Nordström–type centre. (This assumption is necessary for the multipole expansion to have good sense.)

- Decompose both the fields into spherical harmonics which represent components generated by the respective multipoles of the source – the EM field decomposes into vector harmonics and the gravitational field into tensor harmonics. (If scalar field was considered as well, it would decompose into scalar harmonics.)

- Let the object begin to collapse. During this, the deviations from spherical symmetry will tend to grow considerably.
- The growth of higher multipoles will lead to emission of waves of both fields. However, only the $l \geqslant s$ multipoles will be being radiated away, where $s=1$ for the EM and $s=2$ for the G field. (Possible scalar waves, $s=0$, could involve all the modes.)
- The horizon is formed as an extremely dynamical, non-symmetric object, but in a dynamical time-scale of the order of (r_{S}/c) it "rings down" to a stationary state. All the $l \geqslant s$ multipoles are completely radiated away, which means that only left are the monopole component of the EM field (electric charge is *exactly* conserved during the collapse) and the monopole plus dipole components of the gravitational field (the object keeps the monopole and dipole parts of its mass and the dipole part of its rotational angular momentum). If the source originally generated a scalar field as well, that would be radiated away completely, so neither the scalar charge would remain.
- Except for some matter possibly shed during the collapse and now orbiting around, the object ends as a steadily rotating, axisymmetric black hole.

Also important is to add *where the emitted waves go*. When pronouncing "radiated away", relativists necessarily move their hands in a *characteristic* way – i.e. along the outgoing light-like directions in which the radiation aims at future null infinity. However, as the object approaches the black-hole state, its vicinity is more and more curved, which makes larger and larger part of the outgoing radiation **scatter on curvature**, and this in turn makes more and more of the radiation return back and eventually end below the horizon finally formed. It was pointed out that in the closest vicinity of the horizon (below the circular photon orbit, approximately), the outgoing waves partially interfere *destructively* with the ones just scattered back, so the external observer cannot learn anything (or at least *not everything*) about temporal bumps excited on the horizon.

A remark: consider that the conservation of Q and (of certain parts of) M and $J = Ma$ well agrees with the fact that exactly these quantities are linked to the Gaussian integrals of fluxes of the corresponding fields over surfaces surrounding the source. (In fact such integrals provide a reasonable possibility how to define those quantities, otherwise than from asymptotic behaviour of the field components.)[1] The conservation of these imprints of Q, M and J in the external fields thus plays the role of boundary conditions of collapse, including its final stages when the sources of the fields have already disappeared behind the horizon or have even been destroyed in the singularity (or emerged in other universe). It is true, after all, that from the point of view of any external observer the sources *never* cross the horizon...

19.2.2 External-observer experience with collapse

Gravitational collapse is very fast for a comoving clock, but, at the same time, there is dilation between it and a remote clock which in the final stages diverges (the dilation even approaches

[1] A total momentum of the isolated system should certainly be conserved as well. However, one assumes the whole picture is depicted in the centre-of-mass system where the momentum is zero.

"double infinity" – due to gravitation as well as due to the Doppler effect). So how the collapsing star really *looks* for a remote observer? Let us illustrate it on situation when

- the star surface shines on some frequency which is constant according to a clock comoving with the surface
- the observer stays at fixed r, θ, ϕ "at infinity" ($r \gg M$)
- the star surface collapses by free fall.

Then the observer receives frequency

$$\nu_\infty \propto \exp\left(-\frac{t}{4M}\right), \tag{19.7}$$

and flux corresponding to a total luminosity

$$L_\infty \propto \exp\left(-\frac{2}{3\sqrt{3}}\,\frac{t}{2M}\right). \tag{19.8}$$

This result is very intuitive concerning the doubts whether it is at all reasonable to talk (in astrophysics) about black holes if the matter never crosses the horizon as taken with respect to the distant clock: the latter is right, but still – even for the distant observer – the object approaches the black-hole state *exponentially quickly*. If you evaluate the formulas on computer, you can *see* that, basically, "the star is shining and then suddenly disappears". As already estimated, the characteristic time of "turning off" is

$$t_{\rm char} \sim 2M \sim (10^{-5}\mathtt{s})\,\frac{M}{M_\odot}\,. \tag{19.9}$$

19.2.3 Loss of information in collapse

"No-hair" behaviour means almost absolute loss of features – an extreme rise of entropy, extreme simplification of the object. Actually, Chandrasekhar was stressing that black holes are the simplest objects in the Universe (and we stressed already how nicely he demonstrated it in [6]). If a complex material reality undergoes gravitational collapse, after the dust settles (which takes just a moment), you are not able to infer what was the progenitor of the black hole just formed – it might have been a star, a merger of two black holes, a strong concentration of photons, interaction of gravitational waves, or a dense cloud of anti-matter.[2] More generally speaking, if a black hole has been formed, it is not possible to completely reconstruct the past, even if knowing the present state arbitrarily precisely. The appearance of black holes thus implies another restriction of classical determinism (the first was the limitation of prediction past the Cauchy horizons).

It should be stressed that the above **loss of information** remains under an intensive study from the quantum point of view, since it breaks the unitarity of evolution (conservation

[2] Another plausible progenitor is a super-critical accumulation of books and reprints (on GR and collapse) in Jiří Bičák's office.

of the Hilbert-space scalar product) which is one of the basic principles of quantum theory. Practically, some information *is* of course lost, but this is a *fundamental* theoretical question whether *in principle* it is still not possible to reconstruct the past, precisely, on the basis of information carried away by the waves and of information released (?) by a black hole later through quantum processes ("quantum evaporation").

19.3 Horizons as boundaries of black holes

Horizon is the central concept of the black-hole theory. We have seen that the Kerr-Newman horizon is a null hypersurface which plays the role of a one-way causal membrane, that in the static case it coincides with the static limit and the infinite-redshift surface, and we have also learnt that in some cases it may represent a Cauchy horizon. However, there are actually several different notions of a horizon in GR:

- We first met, in Schwarzschild, the "future horizon" as the boundary between the region of outer communications and the black-hole region, i.e. between the regions from where it is/isn't possible, at least by one time-like or light-like world-line (or by a combination of such world-lines), to reach future null infinity. In short, the **future horizon is a boundary of the causal past of $\mathscr{I}^+$**. Symmetrically, the **past horizon is the boundary of the causal future of $\mathscr{I}^-$**. A horizon defined as a causal boundary (thus as a one-way membrane for physical motions) is called **the event horizon**. It is a null hypersurface whose generators are null geodesics. In a generic, dynamical situation, the null generators may enter the horizon at some point, but once having entered it, they never leave it and neither ever intersect any other of the generators [Penrose].

  Event horizon actually is a more general concept, it does not only represent history of the black-hole "surface" – more generally, it stands for a *boundary of the causal past of a certain observer*. For example, such horizons naturally appear in cosmology (as a boundary of the region from where signals can ever reach a given observer), and also within the causal relations of an accelerated observer: in the case of hyperbolic motion (accelerated by constant force in Minkowski), the event horizon is clearly represented by the light cone whose generator the observer asymptotically approaches. In these situations, however, the event horizons bound the causal past of specific observers, whereas as the boundaries of black (or white) holes they have "absolute" meaning (actually, the black-hole event horizons have also been called *absolute horizons*).

  Important remark: if you think of a trip towards a black hole, and if you plan to return back, the event horizon is not suitable for the safety control (although, at the same time, exactly *it* is crucial for your future). Namely, it is defined with the reference to future light infinity, hence, in order to determine its location at any time, one has to know *all the future* of the host space-time. Imagine an observer at rest near the horizon of a Schwarzschild black hole (it may be a huge black hole, so there need not be any problem with tidal forces). Imagine further that a massive spherical shell falls to that black hole. The region from where it is impossible to escape to infinity will gradually grow, until the shell crosses its boundary (i.e. the event horizon). Therefore, the observer may easily find oneself *below* the horizon without any warning – actually, they may even get there *before* the passage of the shell, thus without noticing any change (a spherical shell does not generate a field in its interior). Due to this global, "teleological" nature, the event horizon is not a suitable concept in self-consistent problems where the future of space-time is typically not known in advance (in numerical relativity in particular).

- In 1964, R. Penrose shed new light on the black-hole research. K. Thorne writes in chapter 13 of [48]: "Nineteen sixty-four was a watershed year. It was the year that Roger Penrose revolutionized the mathematical tools that we use to analyze the properties of spacetime." New was the usage of the methods of differential topology, new was the quasi-local way of evaluation of the gravitational-field "strength", and new and not so much expected was the implication: inside every black hole, there necessarily occurs some kind of space-time singularity. To grasp the second point, have a closed space-like two-surface, and imagine the two congruences of light-like geodesics normal to it ("outgoing" and "ingoing" one). Calculate the *expansion* scalar $\Theta = k^{\mu}{}_{;\mu}$ for both and evaluate it on the surface. Where the surface is convex, the outgoing congruence typically is expanding ($\Theta > 0$) while the ingoing one is contracting ($\Theta < 0$). For a concave surface it is vice versa. Close to a very strong source of gravity, however, even the *outgoing* photons perpendicular to a "convex" surface may *converge*, simply because even *they* may be pulled towards the source rather than escaping outwards. The surfaces for which such a circumstance ($\Theta < 0$) holds everywhere for both the normal congruences are called **trapped surfaces**. The region filled with trapped surfaces is called the trapped region and its boundary is called the **apparent horizon**; the history of the latter is called the **trapping horizon**. The apparent horizon is thus a marginally trapped surface – the outgoing null congruence has zero expansion on it (while the ingoing-one expansion is negative). The Penrose' implication was that inside every trapped surface there must be a singularity.

  Apparent horizon only depends on how the space-time behaves on a given space-like surface, it does not "depend on future" as the event horizon. On the other hand, it depends on how one slices the history of the trapped region, so on observer. This is the main reason why the "absolute" event horizon is more useful in most black-hole theorems. If the space-time only hosts attractive matter (if weak energy conditions holds), the apparent (or trapping) horizon generally lies inside the event horizon. In stationary situations, the horizons usually coincide.

  The vanishing of expansion of one of the normal null congruences is also crucial in the much more recent definition of **dynamical horizons** and their "locally stationary" versions – **isolated horizons**. Dynamical horizons are space-like hypersurfaces, while isolated horizons are light-like. These concepts try to require as little as possible about the bulk space-time (asymptotic flatness or symmetries).

- If a Killing vector field exists which is time-like in a certain space-time region, it is possible to define the **Killing horizon** as the hypersurface on which that field becomes light-like (though this may nowhere happen of course). Physically, the existence of time symmetry means stationarity of space-time, and the Killing horizon encloses a region where a certain class of motions (those with four-velocity proportional to the time Killing field) ceases to be physical. If there exist more than one Killing fields, the concept of the Killing horizon may not be unique, since one may obtain other Killing fields (at least somewhere time-like) by linear combinations of the original ones, and these new fields may become null elsewhere than the original ones.

In the black-hole fields we have been studying in recent chapters, all the above definitions of horizons yields the same outcome. In particular, their horizon is really a Killing horizon, since the field $t^\mu + \omega_{\rm H}\phi^\mu$ becomes null on it:

$$g_{\mu\nu}(t^\mu+\omega_{\rm H}\phi^\mu)(t^\nu+\omega_{\rm H}\phi^\nu) = g_{tt}+2g_{t\phi}\omega_{\rm H}+g_{\phi\phi}(\omega_{\rm H})^2 = -N^2-(g_{t\phi}+g_{\phi\phi}\omega_{\rm H})(\omega-\omega_{\rm H})\,,$$

which at $r=r_+$ (thus $N\equiv -g_{tt}-g_{t\phi}\omega=0$ and $\omega=\omega_{\rm H}$) reduces to zero. And $t^\mu+\omega_{\rm H}\phi^\mu$ is indeed a Killing field since the horizon value of the dragging angular velocity $\omega_{\rm H}$ is a constant. Note that it is *not* the "original" time field t^μ itself – that one already becomes light-like on the static-limit surface ($g_{tt}=0$).

19.3.1 Stationary horizons and circular space-times

In the uniqueness theorems we saw that the isolated stationary black hole is necessarily axially symmetric. This even holds for *any* black hole, not necessarily an isolated one: in a regular asymptotically flat space-time containing physically reasonable matter, every stationary black hole is either static or axially symmetric. This result, due to S. W. Hawking, is known as the *strong rigidity theorem*. Namely, its crucial point is that every stationary horizon rotates with constant angular velocity, thus being a Killing horizon. This is clearly in contradiction with the possibility to arrange, outside the horizon, some matter in a non-axisymmetric yet stationary way. Yes, this is indeed impossible: due to the rotational dragging by the black hole, either the external structure would start rotating and thus – as being non-axisymmetric – emitting gravitational waves, or, if the external structure were held firmly at its place (from infinity, say), it would gradually brake the hole's rotation, on the contrary. Hence, steadily rotating black holes have to be axisymmetric. It was also shown that in an asymptotically flat case the stationarity and axial symmetry always commute [B. Carter].

However, one important supplement has to be added. The metric of a stationary and axially symmetric space-time can only be written in the "Kerr-like" form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{\phi\phi}(\mathrm{d}\phi-\omega\mathrm{d}t)^2 + g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2$$

if the space-time is **orthogonally transitive** in addition. Quite intuitively, this means that *there have to exist meridional planes* – there have to exist 2D integral submanifolds which are everywhere orthogonal to both the existing Killing vector fields t^μ and ϕ^μ. These submanifolds $\{t=\mathrm{const}, \phi=\mathrm{const}\}$ can be covered by the remaining two coordinates x^p, $p\neq t,\phi$, which implies that the metric does not involve the corresponding cross terms, $g_{tp}=0$, $g_{\phi p}=0$.

Imagine a 3D version of such a requirement. First, local planes orthogonal to some vector field are integrable if that vector field has zero rotation; in such a case, it is proportional to a gradient of a scalar function – *the* scalar function which has the desired integral surfaces as its level surfaces. Here we have *two* vector fields and want that their complementary direction (orthogonal to both) be integrable. This does not necessarily entails that the rotations of the fields vanish, only that the rotation of each of them has to be orthogonal to the other field. In 4D, the rotation (also vorticity or twist) form is defined by (Section 24) $\omega_\mu[V]=\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}V^\nu V^{\kappa;\lambda}$, so in our case of t^μ and ϕ^μ the requirements read

$$\omega_\mu[t]\,\phi^\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\phi^\mu t^\nu t^{\kappa;\lambda}=0\,, \qquad \omega_\mu[\phi]\,t^\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}t^\mu \phi^\nu \phi^{\kappa;\lambda}=0\,,$$

which is mostly being written as

$$\phi_{[\mu}t_\nu t_{\kappa;\lambda]}=0\,, \qquad t_{[\mu}\phi_\nu \phi_{\kappa;\lambda]}=0\,.$$

The conditions can also be expressed in terms of curvature (as the so-called Ricci circularity). Let us show it properly.

$\boxed{\text{Theorem}}$ [due to A. Papapetrou] The circularity conditions $\phi_{[\mu}t_\nu t_{\kappa;\lambda]}=0$, $t_{[\mu}\phi_\nu\phi_{\kappa;\lambda]}=0$ are equivalent to the conditions

$$\phi_{[\mu}t_\lambda R_{\kappa]\nu}t^\nu=0\,, \qquad t_{[\mu}\phi_\lambda R_{\kappa]\nu}\phi^\nu=0\,.$$

Proof: The derivation is the same for both the conditions, and since it starts from properties which hold for *any* Killing vector field, we will at this stage denote the latter generically as ξ^μ. Multiplying the definition $\omega_\mu[\xi]=\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\xi^{\nu;\kappa}\xi^\lambda$ by $\epsilon^{\mu\beta\gamma\delta}$ and using the formula (A.5), i.e., explicitly,

$$\epsilon_{\mu\nu\kappa\lambda}\epsilon^{\mu\beta\gamma\delta}=-\delta^\beta_\nu\delta^\gamma_\kappa\delta^\delta_\lambda-\delta^\delta_\nu\delta^\beta_\kappa\delta^\gamma_\lambda-\delta^\gamma_\nu\delta^\delta_\kappa\delta^\beta_\lambda+\delta^\beta_\nu\delta^\delta_\kappa\delta^\gamma_\lambda+\delta^\gamma_\nu\delta^\beta_\kappa\delta^\delta_\lambda+\delta^\delta_\nu\delta^\gamma_\kappa\delta^\beta_\lambda\,,$$

we easily obtain

$$\omega_\mu\epsilon^{\mu\beta\gamma\delta}=-\xi^\beta\xi^{[\gamma;\delta]}-\xi^\delta\xi^{[\beta;\gamma]}-\xi^\gamma\xi^{[\delta;\beta]}=-\xi^\beta\xi^{\gamma;\delta}-\xi^\delta\xi^{\beta;\gamma}-\xi^\gamma\xi^{\delta;\beta}=-\xi^{\{\beta}\xi^{\gamma;\delta\}}\,.$$

Differentiation of the latter by x^β yields

$$\begin{aligned}\omega_{\mu;\beta}\epsilon^{\mu\beta\gamma\delta} &= -(\xi^\beta\xi^{\gamma;\delta})_{;\beta}-(\xi^\delta\xi^{\beta;\gamma})_{;\beta}-(\xi^\gamma\xi^{\delta;\beta})_{;\beta}= \\ &= -\cancel{\xi^\beta{}_{;\beta}}\xi^{\gamma;\delta}-\xi^\beta\xi^{\gamma;\delta}{}_\beta-\cancel{\xi^\delta{}_{;\beta}\xi^{\beta;\gamma}}-\xi^\delta\xi^{\beta;\gamma}{}_\beta-\cancel{\xi^\gamma{}_{;\beta}\xi^{\delta;\beta}}-\xi^\gamma\xi^{\delta;\beta}{}_\beta= \\ &= -\xi_\beta\xi^{\gamma;\delta\beta}+\xi^\delta\,\Box\,\xi^\gamma-\xi^\gamma\,\Box\,\xi^\delta=-\cancel{\xi_\beta R^{\gamma\delta\beta\sigma}\xi_\sigma}+\xi^\gamma R^\delta_\beta\,\xi^\beta-\xi^\delta R^\gamma_\beta\,\xi^\beta\,, \end{aligned} \quad (19.10)$$

where the Killing property and the formulas (11.26), (11.27) have been employed. Multiplication of this relation by $\epsilon_{\alpha\nu\gamma\delta}$ leads to twice the same term on the right-hand side, while on the left one has

$$\omega_{\mu;\beta}\epsilon^{\mu\beta\gamma\delta}\epsilon_{\alpha\nu\gamma\delta}=2\,\omega_{\mu;\beta}(\delta^\mu_\nu\delta^\beta_\alpha-\delta^\mu_\alpha\delta^\beta_\nu)=2(\omega_{\nu;\alpha}-\omega_{\alpha;\nu})\equiv 4\,\omega_{[\nu;\alpha]}\,,$$

so we arrive at the formula for gradient of (any) Killing-vector twist,

$$\omega_{[\nu;\alpha]}=\frac{1}{2}\,\epsilon_{\alpha\nu\gamma\delta}\,\xi^\gamma R^\delta_\beta\,\xi^\beta\,. \quad (19.11)$$

Now, let us specify to our $\xi^\mu \equiv t^\mu$, $\omega_\mu \equiv \omega_\mu[t]$ case (with ϕ^μ the second existing Killing field) and consider the derivative

$$\begin{aligned}(\phi^\nu\omega_\nu)_{,\alpha} &= \phi^\nu{}_{;\alpha}\omega_\nu + \phi^\nu\omega_{\nu;\alpha} = \phi^\nu{}_{;\alpha}\omega_\nu + \phi^\nu\omega_{\alpha;\nu} + 2\phi^\nu\omega_{[\nu;\alpha]} = \\ &= \cancel{(\pounds_\phi \omega_\alpha)} + \epsilon_{\alpha\nu\gamma\delta}\,\phi^\nu t^\gamma R^\delta_\beta\, t^\beta\,. \end{aligned} \tag{19.12}$$

This result confirms that

$$\phi_{[\mu}t_\nu t_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad (\phi^\nu\omega_\nu[t])_{,\alpha} = 0 \quad \Longleftrightarrow \quad \phi_{[\mu}t_\lambda R_{\kappa]\nu}t^\nu = 0\,.$$

Similarly one would verify that

$$t_{[\mu}\phi_\nu\phi_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad (t^\nu\omega_\nu[\phi])_{,\alpha} = 0 \quad \Longleftrightarrow \quad t_{[\mu}\phi_\lambda R_{\kappa]\nu}\phi^\nu = 0\,.$$

The opposite implications are also based on the relation (19.12). Since $\phi_\mu = g_{\mu\phi}$ vanishes on the symmetry axis,[3] also trivial there is $\omega^\mu[\phi] = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\phi_\nu\phi_{\kappa;\lambda}$. Consequently, both the invariants $\phi_\nu\omega^\nu[t]$ and $t_\nu\omega^\nu[\phi]$ vanish on the axis as well. Now, if the space-time is Ricci-circular, i.e. $\phi_{[\mu}t_\lambda R_{\kappa]\nu}t^\nu = 0$ and $t_{[\mu}\phi_\lambda R_{\kappa]\nu}\phi^\nu = 0$, implying that the gradients of both the invariants are everywhere zero, $(\phi_\nu\omega^\nu[t])_{,\alpha} = 0$ and $(t_\nu\omega^\nu[\phi])_{,\alpha} = 0$, then the invariants are themselves zero everywhere, which is the circularity condition. □

Since the circularity properties trivially hold for the metric tensor,

$$\phi_{[\mu}t_\lambda g_{\kappa]\nu}t^\nu = \phi_{[\mu}t_\lambda t_{\kappa]} = 0\,, \qquad t_{[\mu}\phi_\lambda g_{\kappa]\nu}\phi^\nu = t_{[\mu}\phi_\lambda\phi_{\kappa]} = 0\,,$$

one may use Einstein equations and translate the Ricci-circularity conditions to

$$\phi_{[\mu}t_\lambda T_{\kappa]\nu}t^\nu = 0\,, \qquad t_{[\mu}\phi_\lambda T_{\kappa]\nu}\phi^\nu = 0\,. \tag{19.13}$$

Immediately clear is that *vacuum* stationary and axisymmetric space-times are necessarily circular. Actually, circular is every space-time in which sources move purely along stationary circular trajectories (along the Killing directions, i.e. with four-velocity satisfying $u^{[\nu}t^\kappa\phi^{\lambda]} = 0$). This is illustrated on an ideal fluid, $T_{\kappa\nu} = (\rho + P)u_\kappa u_\lambda + Pg_{\kappa\lambda}$: the second part is circular automatically and the first one has to satisfy $\phi_{[\mu}t_\lambda u_{\kappa]} = 0$, resp. $t_{[\mu}\phi_\lambda u_{\kappa]} = 0$ (which is the same).

Note that if an EM field is also present and acting gravitationally, the circularity conditions has to also hold for *its* sources, i.e. the electric current has to only flow, steadily, along circular orbits, $J^{[\nu}t^\kappa\phi^{\lambda]} = 0$. In other words, such space-times where the electric or magnetic fields have non-zero azimuthal component are *not* circular. Admittedly, often "stationary axisymmetric" space-times are *automatically* supposed to be circular, while much less is known about those which are not. However, solenoidal motions are *not* unlikely around astrophysical black holes, and electric currents flowing in meridional directions are even crucial for the putative extraction of their rotational energy (see below the Blandford-Znajek process).

[3] On a regular axis, $g_{\phi\phi} \equiv \phi_\mu\phi^\mu$ has to vanish since it determines proper circumference about the axis (along a circular orbit of ϕ^μ at some given radius). This is *not* due to ϕ^μ becoming null (light-like) there, but because $\phi_\mu = g_{\mu\phi}$ shrinks there to *zero* (while $\phi^\mu = \partial x^\mu/\partial\phi$ everywhere).

19.4 Laws of black-hole (thermo)dynamics

In 1972, J. M. Bardeen, B. Carter and S. W. Hawking found that if black holes participate in physical processes, their suitably defined parameters behave in a way that resembles the behaviour of thermodynamic quantities. Since, however, the black-hole quantities follow from pure geometry, the authors were stressing that however lovely the resemblance may be, it is [in Carter's words:] "only an analogy whose significance should not be exaggerated". It seemed clear in particular that a black hole cannot have *temperature*: in thermodynamics, a body with temperature necessarily emits thermal radiation, whereas a black hole cannot emit anything.

The laws were originally formulated and proved using the notion of the event horizon, so they required the corresponding global behaviour of space-time (asymptotic flatness, no singularities or closed time-like curves), plus some of the energy conditions for matter and fields also possibly involved. Later, the laws have been confirmed using the more local notions of horizon, thus requiring much less from space-time around.

- Zeroth law of black-hole dynamics:

  There exists an invariant quantity (called surface gravity and denoted by $\kappa_{\rm H}$) given by the first metric derivatives which on a *stationary horizon* is everywhere the same. On the extreme horizon, in particular, the surface gravity is zero.
  Thermodynamical counter-part: the zeroth law of thermodynamics – in thermal equilibrium, temperature is constant in the whole volume of the system.

- First law of black-hole dynamics:

  If a black hole transfers between two close stationary states, its parameters change according to

  $$\delta M = \frac{\kappa_{\rm H}}{8\pi}\,\delta A + \omega_{\rm H}\delta J + \varphi_{\rm H}\delta Q\,, \tag{19.14}$$

  where M, J and Q denote mass, rotational angular momentum and electric charge, A is the proper area of the horizon, $\omega_{\rm H}$ is its angular speed with respect to infinity, and $\varphi_{\rm H}$ is its electric potential. (If other sources were present, M has the meaning of *total* mass, and on the right-hand side it is necessary to add contributions from the changes of these external sources.)
  Thermodynamical counter-part: the first law of thermodynamics.

- Second law of black-hole dynamics:

  The horizon area A can never decrease. (If more black holes participate in a process and their merger can happen, it applies to the sum of the horizon areas.)
  Thermodynamical counter-part: the second law of thermodynamics — the entropy of an isolated system cannot decrease.

- Third law of black-hole dynamics:

  A generic black hole (the one with $\kappa_{\rm H} \neq 0$) cannot be changed to an extreme one (with $\kappa_{\rm H} = 0$).

Thermodynamical counter-part: the third law of thermodynamics – a body cannot be cooled to absolute temperature zero.

The most suggestive role in the laws is being played by the area A and by the surface gravity $\kappa_{\rm H}$. If defining the entropy and (thus) temperature of the horizon by

$$S := \frac{kc^3 A}{4G\hbar} = \frac{kA}{4(l_{\rm Planck})^2}\,, \qquad T := \frac{\hbar\kappa_{\rm H}}{2\pi kc}\,,$$

with k the Boltzmann constant, the term $\frac{\kappa_{\rm H}c^2}{8\pi G}\,\delta A$ on the right-hand side of the first law just equals $T\delta S$.

Below, we illustrate the above laws on the situation with a single, isolated black hole of the Kerr-Newman type.

19.4.1 Zeroth law in Kerr-Newman

Intuitively, the strength of the field can be characterized by magnitude of four-acceleration which some suitable observers need in order to "keep themselves at a given orbit". Such a concept is of course ambiguous, but remember that the zeroth law applies to *stationary* horizons. We realized above that stationarity of a rotating horizon implies (in fact *requires*) axial symmetry, which in turn means that there do exist certain "invariant" orbits – the stationary circular ones we studied in the Kerr(-Newman) field. The time-like range of their angular velocity shrinks, in a limit sense, to a single value $\omega_{\rm H} = \omega(r = r_+)$ at the horizon, so it is natural to define the surface gravity from the magnitude of four-acceleration of such motions, specifically, from the magnitude of four-acceleration of some particular subclass of these motions whose angular velocity has $\omega_{\rm H}$ as its horizon limit. One of default options is to take the ZAMO congruence having $\Omega = \omega$. [Whereas, for instance, it is not possible to consider *static* ($\Omega = 0$) congruence for this purpose, since that is only time-like outside the static limit.]

However, the above limit represents the null generator of the horizon – the photon which just stays on the horizon, keeping constant r and θ while orbiting with $\Omega = \omega_{\rm H}$ in the azimuthal direction. No other time-like or light-like world-line can lie on the horizon. This means that in the horizon limit ($N \to 0$), the magnitude of the circular-orbit acceleration undoubtedly diverges; on the ZAMO acceleration (16.24) which we plan to use it is seen at first sight. Yet there is a natural way how to regularize such a divergence: multiply the acceleration by N. This has a clear meaning since N represents the dilation factor between the proper time of ZAMO and the Killing time t. Since by (16.24) the ZAMO acceleration reads $a_\mu = N_{,\mu}/N$, we have for this congruence

$$u^t \equiv \frac{{\rm d}t}{{\rm d}\hat\tau} = \frac{1}{N} \quad\Longrightarrow\quad N_{,\mu} = N a_\mu = \frac{a_\mu}{u^t} \equiv \frac{{\rm D}u_\mu}{{\rm d}\hat\tau}\,\frac{{\rm d}\hat\tau}{{\rm d}t}\,.$$

Therefore, one obtains the ZAMO acceleration taken "with respect to the asymptotic inertial time".

So let us define the **surface gravity** $\kappa_{\rm H}$ by

$$\kappa_{\rm H}^2 := \lim_{N\to 0}(N^2 g^{\mu\nu} a_\mu a_\nu) = \lim_{N\to 0}(g^{\mu\nu} N_{,\mu} N_{,\nu})\,. \tag{19.15}$$

Finally, the derivation of κ_{H} for Kerr-Newman. In the Boyer-Lindquist coordinates, the lapse reads $N=\sqrt{\frac{\Delta\Sigma}{\mathcal{A}}}$. Since the horizon ($N=0$) corresponds to $\Delta(r=r_+)=0$ and since $\Delta_{,\mu}$ only has the radial component $\Delta_{,r}=2(r-M)$, we obtain, at the horizon,

$$N_{,\mu}=\frac{\frac{\Delta_{,\mu}\Sigma+\Delta\Sigma_{,\mu}}{\mathcal{A}}-\frac{\Delta\Sigma\mathcal{A}_{,\mu}}{\mathcal{A}^2}}{2\sqrt{\frac{\Delta\Sigma}{\mathcal{A}}}}\ \xrightarrow{\Delta\to 0}\ \left[\sqrt{\frac{\mathcal{A}}{\Delta\Sigma}}\,\frac{\Delta_{,\mu}\Sigma}{2\mathcal{A}}\right]_{r\to r_+}=\left[\sqrt{\frac{\Sigma}{\Delta\mathcal{A}}}\,(r-M)\right]_{r\to r_+}\delta^r_\mu\,.$$

Hence,

$$\kappa_{\mathrm{H}}^2=\lim_{r\to r_+}(g^{rr}N_{,r}N_{,r})=\left[\frac{\Delta}{\Sigma}\,\frac{\Sigma}{\Delta\mathcal{A}}\,(r-M)^2\right]_{r=r_+}=\left(\frac{r_+-M}{r_+^2+a^2}\right)^2=\frac{M^2-Q^2-a^2}{(r_+^2+a^2)^2}\,, \tag{19.16}$$

where we have substituted $\mathcal{A}_+=(r_+^2+a^2)^2$. The result is independent of θ, so it is indeed the same everywhere on the horizon. Note also that it monotonously decreases from the Schwarzschild value $\frac{1}{4M}$ to zero for the extreme horizon at $r_+=M$. The Schwarzschild value may also be written $\frac{M}{r_{\mathrm{S}}^2}$, which perhaps is the most cogent argument for calling κ_{H} the "surface gravity".

19.4.2 First law in Kerr-Newman

The outer Kerr-Newman horizon has proper area

$$A=\int_0^{2\pi}\int_0^{\pi}\sqrt{(g_{\theta\theta}g_{\phi\phi})_{r=r_+}}\,\mathrm{d}\theta\,\mathrm{d}\phi=2\pi\int_0^{\pi}\sqrt{\mathcal{A}_+\sin^2\theta}\,\mathrm{d}\theta=2\pi\int_0^{\pi}(r_+^2+a^2)\sin\theta\,\mathrm{d}\theta=$$

$$=2\pi(r_+^2+a^2)\int_0^{\pi}\sin\theta\,\mathrm{d}\theta=4\pi(r_+^2+a^2)\,. \tag{19.17}$$

Recalling that $r_+=M+\sqrt{M^2-Q^2-a^2}$, we obtain by variation

$$\frac{\delta A}{8\pi}=r_+\delta r_++a\delta a=r_+\left(\delta M+\frac{M\delta M-Q\delta Q-a\delta a}{\sqrt{M^2-Q^2-a^2}}\right)+a\delta a\,,$$

hence, after multiplication by $\sqrt{M^2-Q^2-a^2}=r_+-M$,

$$(r_+-M)\frac{\delta A}{8\pi}=r_+^2\delta M-r_+Q\delta Q-Ma\delta a\,. \tag{19.18}$$

Rewriting the last term as $Ma\delta a=a\delta J-a^2\delta M$ and solving the equation for δM, we have

$$\delta M=\frac{r_+-M}{r_+^2+a^2}\,\frac{\delta A}{8\pi}+\frac{a}{r_+^2+a^2}\,\delta J+\frac{r_+Q}{r_+^2+a^2}\,\delta Q\,. \tag{19.19}$$

You should recognize the surface gravity $\frac{r_+-M}{r_+^2+a^2}=\kappa_{\rm H}$ in the first term and the angular velocity of the horizon $\frac{a}{r_+^2+a^2}=\omega_{\rm H}$ in the second term. In the third term, what stands in front of δQ is the electric potential of the horizon. Actually, scalar potential represents the "time component" of the four-potential A^μ, i.e. the "minus time component" of A_μ; covariantly, it is the projection of minus A_μ on some time-like vector field. There should be little doubts which vector to use in circular space-times: $n^\mu:=t^\mu+\omega\phi^\mu$ is time-like down to the horizon (light-like in a limit), it is solely given by the symmetries, and down there it becomes the Killing null generator of the horizon. (We denote it by n^μ since we know it is orthogonal to the $t=\mathrm{const}$ hypersurfaces.) So, substituting the Boyer-Lindquist components

$$A_\mu=\frac{Qr}{\Sigma}(-1,0,0,a\sin^2\theta)\,,\qquad n^\mu=(1,0,0,\omega)\quad\text{with}\quad\omega=\frac{2Mr-Q^2}{\mathcal{A}}a\,,$$

we get

$$\varphi:=-A_\mu n^\mu=\frac{Qr}{\Sigma}(1-a\omega\sin^2\theta)=\frac{Qr}{\Sigma\mathcal{A}}\left[\mathcal{A}-(2Mr-Q^2)a^2\sin^2\theta\right]=\frac{Qr}{\mathcal{A}}(r^2+a^2)$$
$$\Longrightarrow\quad\varphi_{\rm H}=-(A_\mu n^\mu)_{\rm H}=\left[\frac{Qr}{\mathcal{A}}(r^2+a^2)\right]_{r=r_+}=\frac{Qr_+}{r_+^2+a^2}\,.\tag{19.20}$$

The first law thus indeed reads as given in (19.14),

$$\delta M=\frac{\kappa_{\rm H}}{8\pi}\,\delta A+\omega_{\rm H}\delta J+\varphi_{\rm H}\delta Q\,.$$

19.4.3 Second law in Kerr-Newman

Using the first law (19.19), one can rewrite the second law $\delta A\geqslant 0$ in the form

$$\delta M\geqslant\frac{a\delta J+r_+Q\delta Q}{r_+^2+a^2}\,.\tag{19.21}$$

Interestingly, it is possible to extract energy from a Kerr-Newman black hole without violating the second law. Actually, if writing out the area of the horizon (19.17) as

$$A=4\pi(r_+^2+a^2)=4\pi(2Mr_+-Q^2)=4\pi\left(2M^2+2M\sqrt{M^2-Q^2-a^2}-Q^2\right),$$

it is seen that the energy of the black hole (i.e. its mass M) *can* be diminished, but only if simultaneously Q and/or a decrease sufficiently. Imagine a sequence of reversible steps in which the parameters M, Q and a would be decreasing in such a manner that the area A would stay constant, $\delta A=0$. Such a process can only proceed until both Q and a fall to zero. At that moment, the mass M reaches the lowest possible value capable to generate a horizon with the given area A – it is the mass of the *Schwarzschild* black hole with that area A. This limit value of M, determined by the relation $A=16\pi M_{\rm irr}^2$, is called the **irreducible mass** of a given black hole (specified by its surface area A). Comparing the two expressions for A,

$$4\pi\left(2M^2-Q^2+2M\sqrt{M^2-Q^2-a^2}\right)=16\pi M_{\rm irr}^2\,,$$

yields the relation

$$4M^2(M^2-Q^2-a^2) = (4M_{\rm irr}^2-2M^2+Q^2)^2 \quad\Longrightarrow\quad 16M_{\rm irr}^2M^2 = (4M_{\rm irr}^2+Q^2)^2+4M^2a^2\,,$$

and hence the formula showing how the "pure mass" $M_{\rm irr}$, EM energy proportional to Q^2 and rotational energy proportional to $J = Ma$ contribute to the total mass-energy M of the black hole,

$$M^2 = \left(M_{\rm irr} + \frac{Q^2}{4M_{\rm irr}}\right)^2 + \frac{J^2}{4M_{\rm irr}^2}\,. \tag{19.22}$$

Clearly the composition is non-linear.

Second law and test-particle motion

One is naturally more interested in breaking the (second) law than in following it. The first eligible process is to shoot into a black hole a particle with low energy (→ small increase of black-hole mass) but as large as possible charge and/or angular momentum (→ large increase of black-hole charge and/or spin). A generic electro-geodesic motion being described by the Carter equations (Section 17.3), let us express the particle energy with respect to infinity E from the meridional-plane equations (17.25c), (17.25d), or, in other words, from equality of the two expressions (17.23) and (17.24) for $\mathcal{K}$:

$$\mathcal{A}E^2 - 2\mathcal{B}E + \mathcal{C} = 0 \quad\Longrightarrow\quad E_\pm = \frac{\mathcal{B} \pm \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}}\,, \tag{19.23}$$

where

$$\begin{aligned}
\mathcal{A} &= (r^2+a^2)^2 - \Delta a^2 \sin^2\theta \qquad \text{(as above)}\,,\\
\mathcal{B} &:= (aL + qQr)(r^2+a^2) - \Delta a L = aL(2Mr - Q^2) + qQr(r^2+a^2)\,,\\
\mathcal{C} &:= (aL+qQr)^2 - \frac{\Delta L^2}{\sin^2\theta} - m^2\Delta\Sigma - m^2\Sigma^2\left[(u^r)^2 + \Delta(u^\theta)^2\right].
\end{aligned}$$

The relation determines the energy as the function of M, Q, a, m, q, r, θ, L, u^r and u^θ. The sign is fixed e.g. by considering an uncharged particle ($q=0$) static ($u^i=0$, $L=0$) at radial infinity ($r\to\infty$). For it, $\mathcal{A}\sim r^4$, $\mathcal{B}=0$, $\mathcal{C}\sim -m^2r^4$, which yields $E_\pm = \pm\sqrt{-\mathcal{C}/\mathcal{A}} \to \pm m$, so physically relevant is the upper sign (the lower one corresponds to a particle aiming to the past). Figure 19.1 shows an example of how the plane of possibilities typically looks. Of the above listed parameters, M and m serve as scale factors, $Q = 0.1M$, $a = 0.8M$, $q = 0.2m$, $r = 1.8M$, $\theta = \pi/2$, L is the horizontal-axis variable and u^r, u^θ determine the "vertical" position (value of E). The latter is just a one-dimensional dependence, since the four-velocity components are constrained by normalization (and u^t, u^ϕ are uniquely related to E and L).

Now, we need the energy to be as small as possible. As already mentioned, for a particle with given m, q and L, at a given location r, θ in the field of a black hole with given M, Q and a, the value of E is fixed by the values of u^r and u^θ. Regarding that $\mathcal{A}>0$ (everywhere

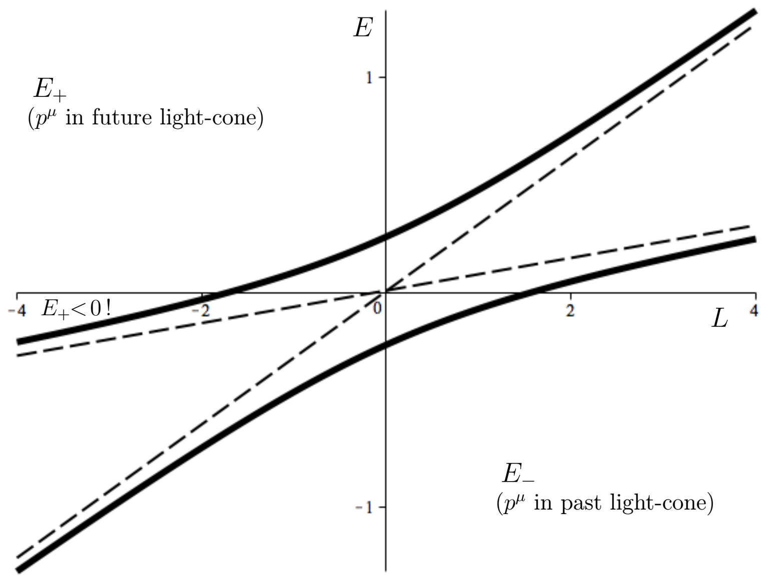


Figure 19.1 *Possible values of energy with respect to infinity of a test particle with charge $q = 0.2m$ at the location $r = 1.8M$, $\theta = 90^\circ$ in the field of a black hole with parameters $Q = 0.1M$, $a = 0.8M$. The energy is plotted, in the units of test-particle rest mass m, in dependence on the angular momentum L; the values corresponding to a particle aiming at future/past lie above the curve $E_+^{\min} = E_+(u^r = 0, u^\theta = 0)$ / below the curve $E_-^{\max} = E_-(u^r = 0, u^\theta = 0)$, respectively (these curves are drawn thick). Between the curves, the value of E cannot lie. Dashed are the lines $E_+^{\min}$ and $E_-^{\max}$ for photons ($m = 0$) – they represent asymptotes of the massive-particle curves. For a given L, the value of E is given by u^r, u^θ (of which just one is independent, because E, L, u^r and u^θ are bound by the u^μ normalization). Note that there exist "plus-states" with negative E (and, vice versa, "minus-states" with positive energy). These do not exist for a non-rotating centre ($a = 0$) when the diagram is symmetric with respect to $L = 0$.*

---

at $r > 0$), to minimalize the energy (19.23) requires to minimalize the square root, which in turn means to maximalize $\mathcal{C}$ – and that is done by setting $u^r = 0$, $u^\theta = 0$.[4] The explicit formula is lengthy, so let us further focus on the moment when the particle enters the horizon, since at that moment the parameters of the black hole are supposed to be changed. Evaluating the energy at the horizon ($\Delta = 0$), we see that for E to be minimal, it is sufficient to set $u^r = 0$: then, however,

$$\mathcal{A}_+ = (r_+^2 + a^2)^2, \quad \mathcal{B}_+ = (aL + qQr_+)(r_+^2 + a^2)\,, \quad \mathcal{C}_+(u^r = 0) = (aL + qQr_+)^2$$
$$\Longrightarrow \quad \mathcal{B}_+^2 - \mathcal{A}_+\mathcal{C}_+ = 0\,, \qquad E_+^{\min}(r = r_+) = E_+(r = r_+; u^r = 0) = \frac{aL + qQr_+}{r_+^2 + a^2}\,.$$

---

[4] Note that u^r and u^θ are bound, together with u^t and u^ϕ (fixed by E/m, L/m), by normalization of u^μ, so only one of them can be chosen in general. However, that is just enough for our purposes, as it will be clear immediately (namely, u^r alone will be relevant).

Since the parameters of the black hole are altered according to

$$\delta M = E, \qquad \delta Q = q, \qquad \delta J = L, \tag{19.24}$$

the above result $E_+(r=r_+) \geqslant E_+^{\min}(r=r_+)$ can be rewritten

$$\delta M \geqslant \delta M\left[E=E_+^{\min}(r=r_+)\right] = \frac{a\delta J + r_+ Q\delta Q}{r_+^2 + a^2}\,. \tag{19.25}$$

But this is exactly the second law, (19.21).

In the above picture, the equality clearly is just an idealized limit, because it would correspond to a particle injected into the black hole with $u^r = 0$ at the horizon – but this is only possible for a null horizon generator (i.e. a photon emitted from there "outwards" and with $L = 0$). In fact the whole consideration has only a certain limit validity, since it is in fact inconsistent: the Carter equations apply to *test* particles, those having zero effect on the geometry, yet we said that once they enter the horizon, they change the black-hole parameters (which *are* of course felt by the metric).

Remarks on area and entropy

- Intuitively, a black hole cannot be split, and it would indeed be against the second law: imagine a Schwarzschild black hole of mass M and two $M/2$ (Schwarzschild) black holes – the area before splitting would be $16\pi M^2$ and after $2\cdot(4\pi M^2)$. The opposite process is "OK", just that the second law restricts the amount of energy which may in principle be radiated away during the merger.

- The most flagrant violation of the area law would be to destroy a black hole completely, namely to change an *extreme* black hole to a naked singularity. Substituting the extreme-hole properties $M^2 = Q^2 + a^2$, $r_+ = M$ to (19.21), we have

$$\delta M \geqslant \frac{Ma\delta a + a^2\delta M + MQ\delta Q}{M^2 + a^2} \quad\Leftrightarrow\quad M\delta M \geqslant a\delta a + Q\delta Q \quad\Leftrightarrow\quad \delta(M^2 - Q^2 - a^2) \geqslant 0\,,$$

  so such a process is impossible.

- One may write the "intensive quantities" (angular velocity, surface gravity and electrostatic potential of the horizon) using the horizon area. For Kerr-Newman, specifically,

$$\omega_{\mathrm{H}} = \frac{4\pi a}{A}\,, \qquad \kappa_{\mathrm{H}} = \frac{4\pi}{A}(r_+ - M)\,, \qquad \varphi_{\mathrm{H}} = \frac{4\pi Q r_+}{A}\,. \tag{19.26}$$

  These relations are good for verifying the **Smarr formula** – a very useful relation generally valid for stationary black holes,

$$M = \frac{\kappa_{\mathrm{H}} A}{4\pi} + 2\omega_{\mathrm{H}} J + \varphi_{\mathrm{H}} Q\,. \tag{19.27}$$

- Entropy is popularly known as the measure of disorder in the system. Highly disordered is such a system which is in a very generic state (which can result from a large number of evolutionary paths) and from which thus only little information can be extracted. Hence, one would expect that black holes have extremely high entropy. Indeed, for a Schwarzschild black hole,

$$S = \frac{kc^3 A}{4G\hbar} = \frac{4\pi k G M_\odot^2}{\hbar c}\left(\frac{M}{M_\odot}\right)^2 \doteq 1.05\cdot 10^{77} k \left(\frac{M}{M_\odot}\right)^2 ,$$

which for $M \sim M_\odot$ yields about 10^{19}-times (!) greater value then is the estimated entropy of the Sun.

"Quantum evaporation" of black holes

The black-hole area law was discovered by S. W. Hawking in 1971. In 1972, J. Bekenstein realized that in order that black holes do not transcend or violate the second law of *standard* thermodynamics, they must have an entropy. He conjectured that this entropy is proportional to the horizon area and showed that such a picture can save the controversy. It would then follow that black holes should, thermodynamically, behave similarly as any other body. In particular, they should have temperature (proportional to surface gravity) and thus emit thermal radiation. Otherwise, if only absorbing, their equilibrium with the surroundings could never establish. S. Hawking was particularly dissatisfied with such a flawed image. He set out to prove that Bekenstein was wrong and that the black-hole temperature is invariably zero. Using quantum theory on Schwarzschild and later also Kerr(-Newman) background, he instead proved, in 1974, that black holes indeed emit particles in accord with their temperature. This discovery confirmed that the laws of black-hole dynamics are *not* only analogies to thermodynamics.[5]

Black hole is an ideal black body, so in order to at least estimate the quantum-evaporation budget, let us employ the Stefan-Boltzmann black-body formula for the total radiated power ($\sigma A T^4$). Plugging there the Stefan-Boltzmann constant σ and the Schwarzschild values for area A and temperature (surface gravity), i.e. (in physical units)

$$\sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2}\,, \qquad A = 16\pi\frac{G^2M^2}{c^4}\,, \qquad \kappa_{\rm H} = \frac{c^4}{4GM} \qquad \Longrightarrow \qquad T = \frac{\hbar\kappa_{\rm H}}{2\pi k c} = \frac{\hbar c^3}{8\pi k G M}\,,$$

we obtain

$$-c^2\frac{{\rm d}M}{{\rm d}t} = \sigma A T^4 = \frac{\hbar c^6}{15360\,\pi G^2 M^2} \doteq (9\cdot 10^{-22}\,\texttt{erg/s})\left(\frac{M_\odot}{M}\right)^2 . \tag{19.28}$$

For black holes resulting from gravitational collapse (whose masses M are at least about $3M_\odot$), the Hawking effect is extremely weak. Still, theoretically interesting is the final stage

[5] Bekenstein's 1972 paper ends by the words: "One sees from (1) that the natural unit of area of a black hole is the Planck length squared. One cannot help wondering about the possible connection between this feature and the expected quantum structure of space-time at a scale of the order of the Planck length."

of evaporation – the radiated power increases as $1/M^2$ and, simultaneously, heavier and heavier particles are being emitted. Hawking himself titled his seminal 1974 paper "Black hole explosions?". It is not yet clear what should be left after complete evaporation: a "Planck-scale" black hole or naked singularity?, some "Planckian particle"?, nothing? However, if no black holes exist with masses *considerably* smaller than $M_\odot$ (primordial?), the Hawking effect is of zero astrophysical significance, because the total time necessary for complete evaporation comes out

$$-\int_{M_{\rm in}}^{0} M^2\,{\rm d}M = \int_{0}^{t_{\rm evap}} \frac{\hbar c^4\,{\rm d}t}{15360\,\pi G^2}$$

$$\Longrightarrow\quad t_{\rm evap} = \frac{5120\,\pi G^2 M_{\rm in}^3}{\hbar c^4} \doteq (2.1\cdot 10^{67}\,\texttt{years})\left(\frac{M_{\rm in}}{M_\odot}\right)^3. \tag{19.29}$$

This is only less than the age of the Universe (14 milliard years) for the initial mass $M_{\rm in}$ smaller than approximately $10^{-19}M_\odot = 2\cdot 10^8\,\texttt{tons}$ (5 times less than has a cubic kilometre of water).

19.4.4 Extraction of energy from a black hole

We saw the area law does not forbid extraction of (rotational and EM) energy from a black hole. How to practically make it? Concerning the rotational energy, it is fairly clear: one can construct a suitable rigid frame and place it around a rotating black hole; dragging will pull it to co-rotate with the hole, so at the other end it can make some work (it may e.g. drive a dynamo, Figure 19.2).

Let us discuss somewhat more the process which R. Penrose suggested in his famous paper of 1969. Its core is in the fact that in the Kerr ergosphere (the area between the static limit and the horizon) there exist orbits with *negative* energy with respect to infinity. If a particle with such an energy falls into a black hole, the black-hole energy is *diminished*. Imagine a body falling freely from outside to the ergosphere with some energy with respect to infinity E, and imagine that in the ergosphere the body splits in such a manner that one part gets on a geodesic with negative energy, $E_1 < 0$, and is absorbed by the hole, while the second escapes back with some energy E_2. Supposing that four-momentum (and thus energy, in particular) is conserved in the break up, the escaping part of the body must have greater energy than the original body, $E_2 > E$. And, keeping the energy of the whole system (hole + bodies), it is clear that the energy gain goes to the expense of the black hole. Why ergosphere is important: energy with respect to infinity is given by $E = -p_\mu t^\mu$, where the momentum p^μ is necessarily (everywhere) time-like, so the scalar product $p_\mu t^\mu$ may only yield positive result (thus making E negative) in a region where the time Killing vector field t^μ is space-like. The latter holds below the static limit.[6]

[6] Note that the possibility of negative energy was already seen in Figure 19.1. Negative energy may sound strange, but keep in mind that it is an energy with respect to a distant observer while the orbits with $E < 0$ must *entirely* lie below the static limit. Every physical measurement, performed *at the point where the particle momentarily is*, would find *positive* local energy of course.

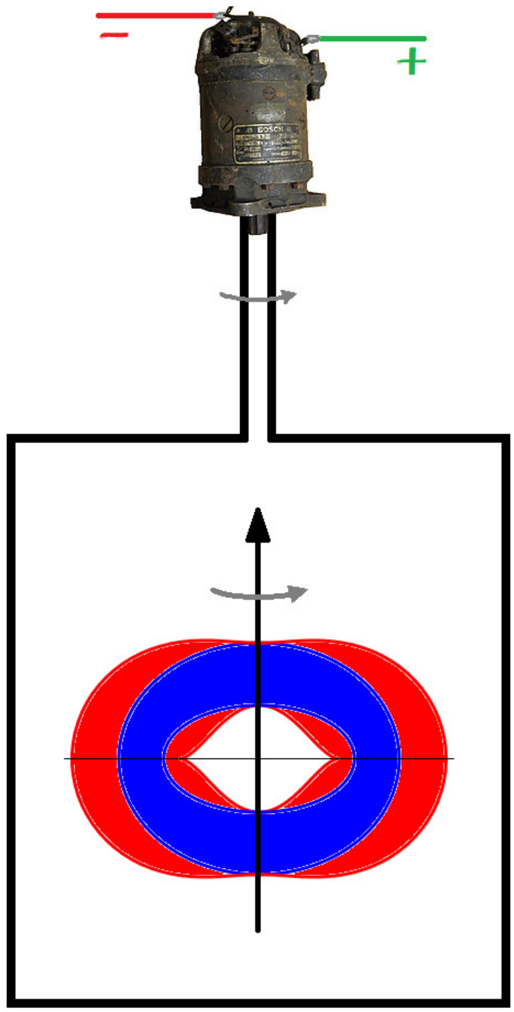


Figure 19.2 *A scheme of how dragging can do its job by driving a dynamo rotor. (The dynamo is of course firmly fixed to infinity.)*

It can be verified that the particles with $E<0$ have to be counter-rotating $(L<0)$. Consider general relations (valid for any motion in any circular space-time) directly given by definition of the angular momentum and energy, and substitute from the former to the latter,

$$L = p_\phi = g_{\phi\sigma}p^\sigma = g_{t\phi}p^t + g_{\phi\phi}p^\phi \qquad \Rightarrow \qquad p^\phi = \frac{L}{g_{\phi\phi}} + \omega p^t\,,$$
$$E = -p_t = -g_{t\sigma}p^\sigma = -g_{tt}p^t - g_{t\phi}p^\phi \overset{\downarrow}{=} (-g_{tt} - g_{t\phi}\omega)p^t + \omega L = N^2 p^t + \omega L.$$

The quantities $N^2 \equiv -g_{tt} - g_{t\phi}\omega$, p^t and $\omega = \frac{-g_{t\phi}}{g_{\phi\phi}}$ are positive outside the horizon, so E may only be made negative by $L<0$.

Exactly due to its key role in the energy extraction the ergosphere got its name (from

Greek $\epsilon\rho\gamma o\nu$ – work). Note that for a *charged* particle in the Kerr-*Newman* field,

$$E = (-p_\mu - qA_\mu)t^\mu = -p_t - qA_t = -p_t + \frac{qQr}{\Sigma}\,,$$

so if qQ is negative/positive, the region where E may be negative is larger/smaller than the ergosphere.

The **Penrose process** also has its wave counter-part – the so-called **superradiance** effect. Loosely speaking, if illuminating a rotating black hole, the scattered radiation may carry more energy than the original, incident radiation. The gain again goes to the expense of the black hole. Both the mechanical and the EM effect might even work practically, but since they require rather special initial conditions, it is unlikely that they have an astrophysical significance. On the contrary, a serious astrophysical discussion is already going on for decades concerning the relevance of the **Blandford-Znajek mechanism** (1977) which is based on interaction of a rotating black hole with an external EM field in a sufficiently dense "magnetosphere" (ionized environment). The key ingredient is the so-called *unipolar induction* – the effect of separation of charges of different signs on a conducting sphere which rotates in a magnetic field [M. Faraday, 1851]. If the magnetic field is parallel to the spin of the sphere, negative charges concentrate around the poles while positive charges in the equatorial region of the sphere.[7] If attaching electrodes to the pole and the equator, one obtains voltage which might be used somehow if a circuit closes. It was shown (notably by [49]) that a black-hole horizon can be regarded as a conducting sphere (its impedance is the same as that of free space, i.e. around 377 Ω) and that a similar induction effect should work there provided there is some external magnetic field (this really is supposed to exist around accreting black holes, being generated by current loops in an accretion disc). If there is a sufficiently dense charged-particle environment around in order that the circuit could be established, the rotational energy of the hole might be, by means of the described "battery", invested in matter through which the current would flow like through a machine. The Blandford-Znajek mechanism is most often being considered in connection with the acceleration of cosmic jets emanating from some active galactic nuclei and also from several binaries involving an ultracompact body. It indeed seems that jets rather occur in systems whose central objects rotate very rapidly.

19.5 Black holes in astrophysics

6th November, 1919. London, Piccadilly, Burlington House, plenary meeting of the Royal Society and, in particular, of the Royal Astronomical Society. A. S. Eddington and F. W. Dyson report on the results of their two expeditions which measured positions of stars around Sun disc during solar eclipse of 29th May. The purpose was to decide whether and how much the light rays bend in passing by the Sun. Eddington interpreted the results as confirming Einstein's prediction and the society accepted that. (It was not that automatic, the less at *those* times, concerning that Newton was English while Einstein was German.) An observation was however also resonating in the hall that GR is hardly comprehensible. After the meeting, L.

[7] The surface density of induced charge is $\sigma = \frac{BR\omega}{8\pi}(5\sin^2\theta - 2)$ in obvious notation.

Silberstein – considered to be one of the experts in GR – was reported to approach Eddington in putting on cloaks, provoking: "Eddington, people have been murmuring, however, there only exist 3 persons in the world who really understand Einstein's theory." [silence] "Well, Arthur, don't be so modest..." Then Eddington replied: "Oh no, I was just wondering who the third one might be."

We mentioned already, in Section 10, how *less expectedly* Eddington – such an early proponent of GR – refused the picture of the gravitational collapse. He was not alone. L. D. Landau – such a proponent of quantum theory – checked Chandrasekhar's ideas on limited possibilities of degenerate-gas pressure in supporting stars against their gravity, confirmed the argument, yet concluded: "All stars more massive than $1.5M_\odot$ contain regions where the laws of quantum mechanics are violated." He was rather willing to restrict the validity of quantum theory than to accept the gravitational collapse.[8] Actually, still at the beginning of 1970s when one of us (J.B.) started teaching this course, the black holes were not the best conversation topic for an astronomical conference. How different the situation is today. It is hard to follow even a narrow part of relativistic astrophysics. Black holes have become yet another celestial bodies, rather boring in comparison with wormholes, loops, strings, branes, time machines...

Above, we quoted Thorne on how 1964 was a watershed in the history of black holes thanks to the new mathematical approach suggested by Penrose. Just at the same period, several astronomical discoveries (X-ray sources 1962, quasars 1963, pulsars 1967) made an even stronger kick. Besides, in 1963, Kerr found his solution later included in models of many of these high-energy sources. Already in 1964 people (Zel'dovich, Salpeter) suggested accretion onto black holes as a possible strong and long-term source of energy for quasars and X-ray sources. In 1969, Lynden-Bell described the picture of active galactic nuclei driven by supermassive black holes. In 1973, Shakura & Sunyaev provided a Newtonian model of an accretion disc, and Novikov & Thorne added its relativistic version living in the Kerr space-time. This is a very short sketch of the beginning of a new research area – **relativistic astrophysics**.

For an astrophysical study of black holes, it is crucial whether they have anything to

[8] Also symptomatic is to follow A. Silberstein's world-line somewhat further. He actually started studying relativity years before Eddington – he already wrote "The Theory of Relativity" textbook in 1914, later expanding it by the GR part. On Wikipedia, Silberstein is thus acknowledged as a "Polish-American physicist who helped make special relativity and general relativity staples of university coursework". Still later, facing predictions of the theory, he became doubtful, and finally wrote to Einstein, in 1935, that he had found a solution the theory admits but which apparently was wrong. It was the static and axially symmetric configuration of two particles fixed at some distance from each other. Such a configuration was not supposed to exist (as a *static* solution), because the particles would certainly fall towards each other due to their mutual attraction. Einstein tried to explain the point, but Silberstein, unconvinced, did not resist publishing the paper "Fatal blow to relativity issued here" in Toronto's newspaper *The Evening Telegram* (!) of March 7, 1936. The solution in fact highlights the *cleverness* of the field equations: they do recognize that such a system cannot stay in equilibrium and add a supporting "strut" between the particles (a singularity along their connecting line); one actually can adjust the metric so that the axis between the particles remains regular, but then, on the contrary, the parts of the axis lying "behind" the particles turn singular – physically, the particles "hang from infinity" (instead of being supported by the strut lying in-between). However, L. Silberstein also contributed positively in many respects, especially to clear mathematical formulation of physical topics. For example, J. L. Synge about whom we mainly wrote in section on geodesics, was apparently much influenced by Silberstein's lectures in Toronto.

interact with. Namely, a black hole is the deepest possible potential well, so everything incoming there has a huge potential – and then kinetic – energy. Being driven by the hole's strong gravity and by its own orbital angular momentum (thus by centrifugal force), such a matter typically forms an accretion disc. Since, at least around stellar-mass black holes, the field is very non-homogeneous, the orbital velocity of matter quickly falls off with radius, which leads to an efficient viscous release of binding energy due to a significant friction between neighbouring orbits.[9] The matter thus heats up to high temperatures; for stellar-mass black hole, inner parts of the disc have about $10^7\mathtt{K}$ and shine strongly in the X-ray band; for supermassive black holes, the field is much more homogeneous and the temperature only reaches about $10^5\mathtt{K}$. Another effect of viscous torques is that the angular momentum of matter is transported out and the matter gradually spirals towards the centre. From about the innermost stable circular orbit (which in Schwarzschild is on $r=6M$, for example), it rather quickly falls to the hole (if it is not expelled away due to some local release of energy).

Therefore, although the black holes are mainly known by their tendency to capture everything, in modern astrophysics they operate as engines of the most energetic sources. X-ray sources are widely interpreted as black holes or neutron stars accreting matter from their "ordinary" stellar companions due to a stellar wind or a Roche-lobe overflow. Nuclei of galaxies showing extraordinary activity (their luminosity may be as high as $10^{4\div5}$ normal galaxies although chiefly generated in a region smaller than the Solar system) are very likely driven by supermassive ($10^{6\div10}M_{\odot}$) black holes accreting gas from the environment of the nucleus and (from) stars of the central cluster. Several subclasses of active galactic nuclei (AGNs) have been recognized, probably differing mainly in angle under which the active nucleus is observed – blazars (lacertids), quasars, Syfert galaxies and radiogalaxies. Some of the hole-disc systems produce **jets** of matter and radiation leaving the core – often in relativistic speeds – along its rotational axis, and typically reaching far into the intergalactic space without losing their collimation. Besides hydrodynamics, jets (and accretion discs themselves) are probably governed by magnetic fields. Actually, though without considerable global charge, discs appear to be able to maintain very strong azimuthal electric currents as a result of magneto-rotational instability. Thus generated magnetic field is supposed to play a role in the Blandford-Znajek mechanism (rather "electrodynamism") already outlined above. Advanced MHD simulations support the belief that the above accretion mechanism can be efficient on stellar as well as galactic scales.[10]

The accreting- black-hole (or neutron-star) interpretation, first just based on energy-budget considerations and the phenomenon of jets, has since been supported by a number of detailed observations and arguments: short variability of the active sources evidences their small size, broader velocity distribution evidences the presence of a compact object, characteristic shape of the spectrum with lines deformed by Doppler effect, gravitational redshift, aberration and lensing evidences the presence of a very strong field, or quasi-periodic oscillations whose frequencies reasonably agree with characteristic times of a system dominated

[9] The efficiency may indeed reach several tens percent, much more than that of nuclear burning.

[10] However, jets as such rather seem to be tied to rotating structures than *necessarily* to a magnetic field. Actually, directional outflows following the rotational axis are also known from protostars or from pulsars (a suggestive time-lapse video can be found of an outflow from the Crab-nebula pulsar).

by a compact body. Polarimetric measurements map the flow of plasma and magnetic fields in the systems. Different aspects of behaviour of the studied sources are typically time-correlated, which indicates that the sources switch between several accretion regimes, mainly in dependence on supply of the material to the black-hole vicinity.

However, some black holes are not interacting strongly and some factually live completely isolated. Most of "normal" (inactive) galaxies turned out to also likely have supermassive black holes in their centres. Actually, it is the fairly calm nucleus of our Galaxy (the Sagittarius A* source) where the presence of a black hole (of mass $4.25 \cdot 10^6\, M_\odot$) has been most clearly evidenced observationally – on the basis of direct tracking of star orbits in its vicinity. In recent years, a new family of "dark" black holes has been discovered thanks to the detections of gravitational waves: most of the GW events yet recorded have been interpreted as emitted during the last phases of inspiral (merger) of two rather massive black holes (tens to almost hundred $M_\odot$). The events were not observed in EM radiation, so it is clear that the black holes involved were not surrounded by an environment, so the gravitational signal was the only chance to discover them. Still there *is* one possibility how to come across even a completely isolated black hole – via gravitational lensing. Not speaking about how highly improbable such an event would be, it is possible in principle that some lensing events will be detected without a visible lensing object.

The above accretion scenario with energetic jet outflow is also mostly accepted as a model of **gamma-ray bursts** (GRBs), first detected in 1967 but only properly interpreted in the 1990s. These flashes of gamma radiation are registered to happen isotropically over the sky, obviously (luckily) coming from cosmological distances and (thus) belonging to the most energetic events in the Universe (in which up to about a solar-mass equivalent of energy can be released). Interestingly, two families of these phenomena clearly exist, one typically lasting just fractions of a second while the other being longer (sometimes even of the order of weeks to months, though in softer-than-gamma bands in later times), with the dividing line lying around 2 seconds. The longer GRBs are being explained as produced in a **gravitational collapse of a massive-star nucleus**: due to gravitation and centrifugal force, a central region where thermonuclear reactions have halted collapses to a dense disc of some 100 km with a black hole (or maybe a neutron star) occurring at its centre; the disc partially falls to the hole just formed but partially is ejected in jets along the rotational axis; these jets break through the outer stellar envelope, which leads to an extreme heating detectable in gamma radiation (later gradually softening) if observed just in counter-direction. The short GRBs are being interpreted as final phases of a **merger of two neutron stars** (or of a neutron star and a black hole): in such an event, the neutron star deforms/disintegrates in an ultra-dense neutron "accretion disc" which in a fraction of a second collapses to a black hole, with a certain part again ejected – with extreme energy – along the rotational axis of the system; from counter-directions, the collimated explosions appear as gamma-ray flashes. The scenario just described is chiefly celebrated as a strong source of gravitational waves (the waves are however mainly emitted along the plane orthogonal to the rotational axis of the system).

Thanks to current computer facilities, the accretion-ejection models already involve many MHD, radiation-transfer and chemical details. What still remains to be provided is a more accurate answer to where and how jets are formed, in particular, how important is the Blandford-Znajek and/or other electrodynamic mechanisms, as e.g. those of magnetic reconnection. It is also to be clarified how exactly pure hydrodynamics, electromagnetism and radiation effects co-operate in the collimation of jets over such an enormous range of scales. In general, the questions of stability of accretion flow (and actually of the accreting objects as well) against various possible perturbations are neither answered completely. The second group are the evolution problems. Had the first black holes been generated solely by collapse of nuclei of the first generation of stars? Or were there (also) any "primordial" black holes? How such a huge black holes could already exist less than a milliard years after the big bang? How frequent are/were such black holes and how often they were interacting and merging? Did these black holes act as seeds for galaxies, or did most of the supermassive holes rather form together with their galaxies (or even later)? How exactly the nuclear black holes interact with their galactic surroundings over the cosmic ages? How so many young stars may have occurred in the vicinity of the black hole in our Galactic centre (if it was thought that in galactic nuclei conditions are not favourable for star formation)? Has our Galactic centre always been "quiet", or has is ever gone through an active period? How (un)common are intermediate-mass black holes (with $100 \div 10000 M_\odot$)?

New input into the above queries is expected from gravitational-wave observations. Actually, the waves have already brought a surprise in how massive black holes apparently float, otherwise hardly detectable, out there in empty space. Of several dozens of existing gravitational-wave events, almost all have been interpreted as mergers of stellar black holes of larger masses than those yet identified through EM (mostly X-ray) observations, with an almost $150 M_\odot$ hole announced to result from one of the events.

And hard not to mention the last *big* observational result – the 2019' publication of the first **silhouette of the black hole** in the nucleus of the M87 galaxy. In fact nothing new was discovered, the extreme-resolution radio measurements just confirmed what was expected, but this observation had a great symbolic meaning – in a sense, it culminated one whole period of black-hole research. And, undoubtedly – similarly as the detection of gravitational waves (plus many others) – it further confirmed Einstein's incredible legacy.

CHAPTER 20

Relativistic stellar models

Preceding chapters were devoted to extreme implications of GR – the black holes. However, relativistic effects can also play a role in the physics of stars, especially at the end of their thermonuclear evolution when their central part typically contracts to a very dense object. These compact remnants – white dwarfs and neutron stars (if not black holes) – will be discussed in the next chapter. Here we'll go through the main items of the relativistic theory of stellar structure. This part is widely based on Kip Thorne's exposition [47], and the radial-oscillation section 20.4 follows section 26 of [29].

20.1 Separation of short-range and long-range forces

Speaking of stellar structure means speaking of stellar interior, thus of *non-vacuum* region. This in itself implies that the problem will be more complicated (than the vacuum problem), because the field equations will be non-homogeneous, with some energy-momentum tensor on the right-hand side. However, more important is that the "interior solution" is not only governed by gravity – it brings other branches of physics on scene: thermodynamics, hydrodynamics (or kinetic theory), electrodynamics and radiation, nuclear and particle physics. Sure that one need not "support" all these properly, but crucial is that the "microphysics" might be coupled to gravitation in a complicated way. Imagine, for example, that the equation of state, thermonuclear reactions or Compton-scattering cross section depend on curvature. In such a case, Einstein equations would already be involved in the "microphysics" and the problem would be hopelessly entangled.

You should remark now that at the beginning of GR we assumed, in the equivalence principle, that this is not the case – that curvature does not enter physical laws (otherwise it would not in general be possible to transform out gravity completely by going over to the local inertial frame). Exactly. Now we can check what such an assumption physically means. Clearly it is the question of scales: curvature *would be* coupled to microphysics if its characteristic length scale were *not* large in comparison with the micro-scale(s). It is thus important to compare the length scale of curvature with those of physical processes going on in the matter.

- **Nuclear forces:**
  Characteristic length of nuclear forces is given by classical size of nucleons, i.e. $\sim 10^{-13}\,\mathtt{cm}$.

- **Compton wavelength:**
  For a particle of rest mass m, the (reduced) Compton wavelength reads $\lambdabar_{\rm C} = \frac{\hbar}{mc}$. This scale is a natural unit (of reciprocal mass) in the quantum theory, but in the theory of relativity it plays an important role as well. Namely, from the Heisenberg uncertainty relations $\Delta x \Delta p \gtrsim \hbar$ one knows that if something is more and more restricted in space, its momentum uncertainty has to grow, and thus the mean value of momentum has to grow either. If a particle is confined down to the size of its reduced Compton wavelength, one finds

$$\Delta x \sim \frac{\hbar}{mc} \quad \Rightarrow \quad \Delta p \gtrsim \frac{\hbar}{\Delta x} \sim mc \quad \Rightarrow \quad p \gtrsim mc\,,$$

  so the particle becomes relativistic.

- **Electromagnetic interaction:**
  For a globally neutral matter (in which atomic nuclei and electrons are distributed roughly uniformly, without accumulation of charges of the same sign), the characteristic length of EM interaction is approximated by the distance between nuclei. Denoting by $m_{\rm N}$ the mass of nuclei and by ρ the rest-mass density, then $(\rho/m_{\rm N})$ represents the number of nuclei in a unit volume, so $(m_{\rm N}/\rho)$ is the volume per one nucleus and $(m_{\rm N}/\rho)^{1/3}$ is the characteristic distance between nuclei.

- **"Macroscopic element" of matter:**
  "Infinitesimal elements" of the macroscopic theory should correspond to sufficiently large real elements of matter, in order that the quantities come out properly smooth after averaging over them and in order that they do not suffer significant fluctuations (the fluctuations are known to be proportional to $1/\sqrt{N}$, where N is the number of particles in a system). Hard to decide precisely what already is a sufficiently "macroscopic" piece, but let us take an element containing 10^{21} atoms (for hydrogen this represents some $1/600$ of gram). From previous estimate, we obtain the characteristic size $(10^{21} m_{\rm N}/\rho)^{1/3} = 10^7\,(m_{\rm N}/\rho)^{1/3}$.

- **Gravitation (curvature):**
  The characteristic scale of curvature is the radius of curvature, given by $1/\sqrt{R}$, with R the curvature. For $\Lambda = 0$, the Einstein equations are traced to $-R = \frac{8\pi G}{c^4} T$, which for $T \sim -\rho c^2$ (exactly valid for incoherent dust) yields $R \sim \frac{8\pi G}{c^2}\rho$, so thus the curvature radius $\frac{1}{\sqrt{R}} \sim \frac{c}{\sqrt{8\pi G\rho}}$.

With growing density, the curvature radius decreases the fastest, yet still only reaches the micro-scales at extremely high densities. The "quantum" length scales (nuclear-force scale and reduced Compton wavelength) are density-independent. Of the micro-scales, for $\rho \gtrsim 10^{28}\,\mathtt{g/cm^3}$ the largest is the reduced Compton wavelength of an electron $\lambdabar_{\rm C}({\rm e}) \sim 10^{-11}\,\mathtt{cm}$, while for $\rho \lesssim 10^{28}\,\mathtt{g/cm^3}$ the largest is the "macroscopic element" $10^7\,(m_{\rm N}/\rho)^{1/3}$, so these are to be compared to the curvature radius within the respective density ranges. The comparison

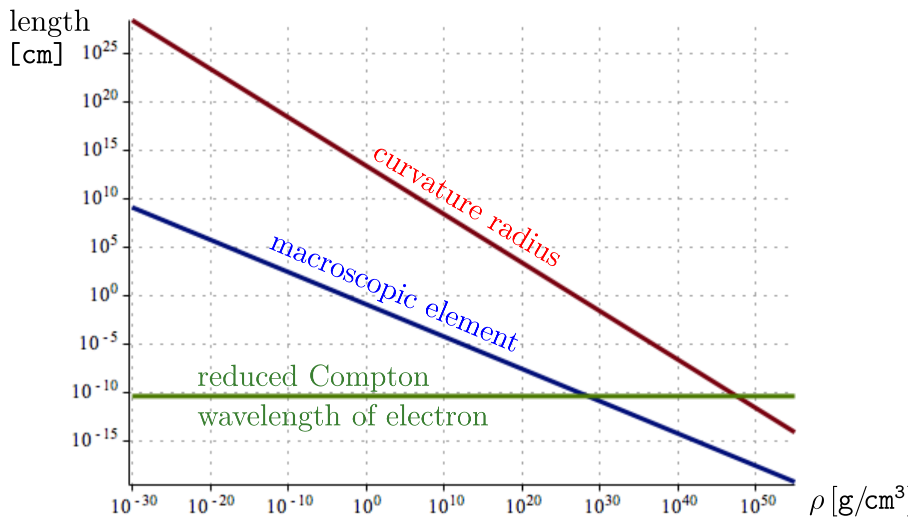


Figure 20.1 *Comparison of the length scales in a logarithmic graph ranging from cosmological values of density ($\sim 10^{-29}\,\mathrm{g/cm^3}$) to extremely high values. Curvature radius is seen to be safely greater then the micro-scales up to some $10^{35\div 40}\,\mathrm{g/cm^3}$.*

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is summarized in Figure 20.1, with the nuclear mass represented by that of a proton ($m_{\mathrm{N}} = m_{\mathrm{p}}$).

$\boxed{\text{Conclusion}}$ For $\rho \ll 10^{47}\,\mathrm{g/cm^3}$, the length-scales of "short-range" forces are (really) much shorter than the curvature length-scale, so in local reasoning it is absolutely fine to use ordinary results of non-gravitational physics, and to only take into account GR in a global scale. The opposite only holds at very initial moments after the big bang and at the very final moments of a complete gravitational collapse.

Remarx: When speaking of the characteristic length of an EM interaction, the **Debye length** may have occurred to you, which quantifies a characteristic scale of the exponential (Yukawa-type) cutoff describing shielding of the electrostatic potential of a charged particle placed in an overall neutral plasma. The formula is

$$\lambda_{\mathrm{D}} = \sqrt{\frac{kT}{4\pi\bar{n}e^2}} \simeq \sqrt{\frac{\bar{m}_{\mathrm{B}}kT}{4\pi\rho e^2}}\,,$$

with $\bar{n}$ the mean number density of electrons and protons. (For a degenerate charged gas, one uses its Fermi energy instead of kT.) One can thus estimate that λ_{D} is of the order of $10^{-12}\,\mathrm{cm}$ in a neutron star, $10^{-10}\,\mathrm{cm}$ in a white dwarf, $10^{-9}\,\mathrm{cm}$ in the solar core, $10^{-4}\,\mathrm{cm}$ in the solar atmosphere, $10\,\mathrm{m}$ in an interstellar space, and $10^5\,\mathrm{m}$ in an intergalactic space. Comparing with Figure 20.1, one sees that λ_{D} is generally orders of magnitude shorter than the macroscopic element.

Another length-scale, important as an indicator of "how much quantum" is the gas, is the **thermal de Broglie wavelength**

$$\lambda_{\mathrm{dBth}} = \frac{h}{\sqrt{2\pi m E}}\,,$$

E standing for a typical kinetic energy of a given kind of particles. If it is much smaller than the typical inter-particle separation, the gas behaves as a classical, Maxwell-Boltzmann gas. In the opposite case, the wave functions of the particles overlap and quantum effects are important; such a gas follows the Fermi-Dirac or the Bose-Einstein description. The formula generally yields very small values: $\lesssim 10^{-12}\,\mathtt{cm}$ for neutrons in a neutron star, about $10^{-10}\,\mathtt{cm}$ for electrons in a white dwarf (again, the pertinent Fermi energy is important instead of thermal energy kT for these degenerate objects), about $10^{-9}\,\mathtt{cm}$ for electrons in the Sun, and $5\cdot 10^{-8}\,\mathtt{cm}$ for an interstellar medium. (For an intergalactic space, it varies a lot, since the temperature of plasma there ranges from 1 to some 10^8 K.) Anyway, the de Broglie wavelength *does not depend on density* (or only indirectly, like in objects supported by degenerate gas).

20.2 Description of a static spherically symmetric star

As often admitted, there is little hope that an *analytical* solution of Einstein equations will ever be found which would describe a *realistic, rotating extended body*. Even a *stationary* and *uniform* (rigid-body) rotation has yet been exactly solved in a thin-disc limit only. Hence, no surprise that this introductory chapter on relativistic stars will restrict to *non-rotating, spherically symmetric* case. The star will be allowed to really act like a star, i.e. to generate, transport and release energy in non-stationary processes, but its gravitational field (metric) will be assumed to be *static*. Remembering the result (12.4), we can state right away that the metric will have the form

$$\mathrm{d}s^2 = g_{tt}(r)\mathrm{d}t^2 + g_{rr}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{20.1}$$

where we will parametrize

$$g_{tt} = -e^{2\Phi(r)}, \qquad g_{rr} = \frac{1}{1-\frac{2m(r)}{r}}\,, \tag{20.2}$$

with Φ representing Newtonian-like gravitational potential (in the Newtonian limit, i.e. for $\Phi \ll 1$, one has $g_{tt} = -1 - 2\Phi$ as it should be) and $m(r)$ representing mass in the sphere with radius r (this follows from the experience with Schwarzschild). The gravitational field will thus be described by two functions of radius, $\Phi(r)$ and $m(r)$, with the latter having the value M for $r \geqslant R$ (radius of the star).

In total, the structure of the static, spherically symmetric star containing B types of baryons will be described by $16+3B$ quantities depending on the radial coordinate (area radius) r and on proper time τ of clocks staying at rest at a given r. Note that the "different types of baryons" does not mean protons and neutrons, but nucleons *in different nuclei*. Therefore, $B=2$ for a star containing just hydrogen and helium, while $B>2$ for a star also containing higher elements.

20.2.1 Basic numbers

- $A\,(=\mathrm{const})$ is the total number of baryons in a star,
  A_k is the total number of "type-k" baryons ($k = 1,\ldots,B$) and

$Z_k := A_k/A$ is the corresponding fractional abundance.
Naturally, $\sum_k A_k = A$ and $\sum_k Z_k = 1$.

- The number of baryons in a sphere of radius r will be denoted by $a(r)$ and the proper number density of baryons by n, so

$$a(r) = \int_V n\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} n\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \int_0^r \frac{4\pi r^2 n\,\mathrm{d}r}{\sqrt{1-\frac{2m}{r}}}$$

$$\iff \quad \frac{\mathrm{d}a}{\mathrm{d}r} = \frac{4\pi r^2 n}{\sqrt{1-\frac{2m}{r}}}\,. \tag{20.3}$$

The assumed conservation of the baryon number A is locally ensured by the continuity equation

$$(nu^\mu)_{;\mu} = \frac{\mathrm{d}n}{\mathrm{d}\tau} + nu^\mu{}_{;\mu} = 0\,. \tag{20.4}$$

20.2.2 Thermodynamic quantities

- $\bar{m}_\mathrm{k}$ ... mean rest mass of type-k baryons, equal to the rest mass of the type-k nucleus (in a ground state) divided by number of nucleons in that nucleus

- $\bar{m}_\mathrm{b}$ ... mean rest mass of baryons,

$$\bar{m}_\mathrm{b}A = \sum_k \bar{m}_\mathrm{k}A_k \quad \Longrightarrow \quad \sum_k \bar{m}_\mathrm{k}Z_k = \bar{m}_\mathrm{b}\,.$$

In a thermonuclear synthesis, the nuclei get over their mutual Coulomb repulsion and are bound by strong nuclear force. Synthesis provides energy if the nucleons get in a deeper potential well in it, which is the case from hydrogen up to iron and nickel. This means that the mass defect per nucleon grows during the thermonuclear evolution, so $\bar{m}_\mathrm{b}$ decreases.

- ϵ ... proper density of internal energy

- ρ ... proper density of total energy, $\rho = \bar{m}_\mathrm{b}n + \epsilon$ (rest + internal)

- P ... pressure ($P = 0$ identifies the stellar surface, $r \equiv R$)

- T ... temperature (also supposed to vanish at the surface)

- s ... entropy per baryon ... for instance, according to statistical definition, $s = k\ln W$, where k is the Boltzmann constant and W is the number of all possible (quantum) states of a baryon (of given type) divided by the number of those baryons; hence, for type-k baryons,

$$W = \frac{1}{A_k}\,\frac{\text{phase-space volume}}{\text{volume per one fermionic quantum state}(=h^3)}\,(2s+1)\,,$$

where $(2s+1)$ is the number of fermions (with all possible different projections of spin s) which can occupy the elementary cell h^3.

20.2.3 Nuclear & radiation characteristics

- nuclear chemical potentials ... $\mu_k(\rho, n, s, Z_1, ..., Z_B)$
- rates of nuclear-abundance change ... $\alpha_k(\rho, T, Z_1, ..., Z_B) := \frac{\mathrm{d}Z_k}{\mathrm{d}\tau}$
  (clearly $\sum_k Z_k = 1 \implies \sum_k \alpha_k = 0$)
- rate of thermonuclear-energy generation ... $q := -\frac{\mathrm{d}\bar{m}_\mathrm{b}}{\mathrm{d}\tau} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_k \bar{m}_\mathrm{k} Z_k = -\sum_k \bar{m}_k \alpha_k$
  (remember that $\bar{m}_k$ do *not* change – they denote baryons in *different* nuclei)
- q generated in the form of neutrinos ... $q_{(\nu)}$
- total luminosity at r (power which passes the sphere of radius r) ... L_r
  (total energy which crosses the sphere of radius r, measured by static observers at that radius, in a unit of their proper time τ; it includes all kinds of energy transport – radiation, conduction, convection, neutrinos)
- neutrino luminosity (neutrino part of L_r) ... $L_\mathrm{r}^{(\nu)}$
- radiative absorption coefficient (per unit mass) ... $\kappa_\mathrm{rad}(\rho, T, Z_1, \ldots, Z_B)$.
  When multiplied by ρ, it represents the fractional attenuation per unit proper distance of the intensity of a beam of light, i.e., as expressed by the equation of radiative transfer,

  $$\frac{\mathrm{d}I_\mathrm{rad}}{\mathrm{d}l} = -\kappa_\mathrm{rad}\rho I_\mathrm{rad}\,,$$

  where the radiation intensity I_rad represents radiative L_r per unit proper area and $\mathrm{d}l$ is the proper-distance element.
- thermal conductivity coefficient ... $\lambda_\mathrm{cond}(\rho, T, Z_1, \ldots, Z_B)$
  It relates the heat-conduction energy-flux density (intensity of conductive luminosity) to the temperature gradient,

  $$\vec{I}_\mathrm{cond} = -\lambda_\mathrm{cond}\vec{\nabla}T\,.$$

  (Note: in the introduction of the above coefficients, no gravitation is taken into account.)

20.2.4 Description summary

The structure of a static and spherically symmetric star containing B types of baryons has been described by $16 + 3B$ quantities:

$$\boxed{\Phi,\, m;\, n,\, a,\, \bar{m}_\mathrm{b},\, \epsilon,\, \rho,\, P,\, T,\, s,\, q,\, q_{(\nu)},\, L_\mathrm{r},\, L_\mathrm{r}^{(\nu)},\, \kappa_\mathrm{rad},\, \lambda_\mathrm{cond};\, Z_k,\, \mu_k,\, \alpha_k\ (k = 1, \ldots, B)}\,.$$

The equilibrium behaviour of these quantities is governed by $16+3B$ equations. Some of the equations follow directly from definition of the respective quantities (like that for the number of baryons $a(r)$, or $\sum_k Z_k = 1$ and $\sum_k \alpha_k = 0$), some are algebraical (e.g. $\rho = \bar{m}_\mathrm{b} n + \epsilon$) and some (seven, exactly) are differential. The latter are treated in the following section.

20.3 Equations of stellar equilibrium

20.3.1 Equation for mass and equation for gravitational potential

From the GR point of view, we are mainly interested in the gravitational field. It is described by the metric functions m, Φ and these are determined by the Einstein equations. We already know how their left-hand side looks in the spherically symmetric case from the chapter about Schwarzschild solution – non-zero are the components (12.5) of the Einstein tensor; here, assuming staticity, we can omit time derivatives in addition, so G_{tr} is zero trivially and only two relevant equations are left. We will describe the stellar matter as ideal fluid, so $T^{\mu}{}_{\nu} = (\rho + P)u^{\mu}u_{\nu} + P\delta^{\mu}_{\nu}$, where the fluid's four-velocity has spherical-coordinate components

$$u^{\mu} = \left(\frac{1}{\sqrt{-g_{00}}}\,, 0, 0, 0\right) = \left(e^{-\Phi}, 0, 0, 0\right), \qquad u_{\mu} = (-\sqrt{-g_{00}}\,, 0, 0, 0) = \left(-e^{\Phi}, 0, 0, 0\right).$$

The "tt" and "rr" field equations thus read

$$G^{t}{}_{t} = 8\pi T^{t}{}_{t}: \qquad -\frac{r\frac{\mathrm{d}g_{rr}}{\mathrm{d}r} + g_{rr}(g_{rr}-1)}{r^2(g_{rr})^2} = -8\pi\rho\,,$$

$$G^{r}{}_{r} = 8\pi T^{r}{}_{r}: \qquad \frac{r\frac{\mathrm{d}g_{tt}}{\mathrm{d}r} - g_{tt}(g_{rr}-1)}{r^2 g_{tt} g_{rr}} = 8\pi P\,.$$

Substituting here (20.2), i.e. $\frac{\mathrm{d}g_{rr}}{\mathrm{d}r} = 2(g_{rr})^2\left(\frac{1}{r}\frac{\mathrm{d}m}{\mathrm{d}r} - \frac{m}{r^2}\right)$, $\frac{\mathrm{d}g_{tt}}{\mathrm{d}r} = 2g_{tt}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$, $g_{rr} - 1 = 2g_{rr}\frac{m}{r}$, one arrives at equations

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho \qquad \text{... \textbf{equation for mass}}\,, \tag{20.5}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{g_{rr}}{r^2}(m + 4\pi r^3 P) = \frac{m + 4\pi r^3 P}{r(r-2m)} \qquad \text{... \textbf{equation for potential}}\,. \tag{20.6}$$

Equation for mass is clear at first sight – it is just the same as in the Newtonian case –, but exactly because of this it is *unclear* at second sight: in GR, one would expect in it the element of *proper* radial distance $\sqrt{g_{rr}}\,\mathrm{d}r$ rather than just the coordinate element $\mathrm{d}r$! Namely, one would expect the mass $m(r)$ to be related to its proper density ρ in a similar manner as the number of baryons $a(r)$ is related to its proper density n through equation (20.3). Let us rewrite the mass equation in an integral form and try to understand it in terms of integration over *proper* volume V:

$$m = \int_0^r 4\pi r^2\rho\,\mathrm{d}r = \int_0^r \frac{4\pi r^2\rho}{\sqrt{1-\frac{2m}{r}}}\sqrt{1-\frac{2m}{r}}\,\mathrm{d}r = \int_V \rho\sqrt{1-\frac{2m}{r}}\,\mathrm{d}V =$$

$$= \int_V \rho\left(1 - 1 + \sqrt{1-\frac{2m}{r}}\right)\mathrm{d}V = \int_V \rho\,\mathrm{d}V - \int_V \rho\left(1 - \sqrt{1-\frac{2m}{r}}\right)\mathrm{d}V\,.$$

The first term represents the rest + internal energy of *gravitationally non-interacting* baryons. The second term becomes understandable in the weak-field limit ($r \gg m$) when one can approximate $\sqrt{1-\frac{2m}{r}} \doteq 1-\frac{m}{r}$ and rewrite

$$-\int\limits_V \rho\left(1-\sqrt{1-\frac{2m}{r}}\right)\mathrm{d}V \doteq -\int\limits_V \frac{\rho\, m}{r}\,\mathrm{d}V\,.$$

This is clearly the gravitational potential energy of the baryons integrated over the given sphere. Hence, the density ρ *is* "in fact" integrated over proper volume, which naturally yields *greater* result than in the Newtonian case (because $g_{rr} > 1$), but this excess is exactly compensated by the (negative) contribution of the gravitational binding energy.

20.3.2 Tolman-Oppenheimer-Volkoff equation

This chapter is about stellar *equilibria*, so central should be the equilibrium equation. Since we describe the stellar matter as ideal fluid, we know how such an equation should look – it is the **equation for hydrostatic equilibrium** (7.38), expressing balance between gravitational and pressure-gradient forces. In the spherically symmetric case, the equilibrium equation reads $-\frac{\mathrm{d}P}{\mathrm{d}r} = (\rho+P)\frac{\mathrm{d}\Phi}{\mathrm{d}r}$. Substituting there the potential gradient from (20.6), we arrive at the **Tolman-Oppenheimer-Volkoff (TOV) equation**

$$\boxed{-\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{(\rho+P)(m+4\pi r^3 P)}{r(r-2m)}}\,. \tag{20.7}$$

Comparison with the classical equilibrium equation

$$-\frac{\mathrm{d}P}{\mathrm{d}r}\left(=\rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = \frac{\rho\, m}{r^2}$$

evinces that the GR equation involves three extra terms:

- There appears $(\rho+P)$ instead of just ρ. This comes from the $T_{\mu\nu}$ and we already know its meaning from the Euler equations of motion (7.33) – the pressure contributes to the density of inertial mass.

- There appears $(m + 4\pi r^3 P)$ instead of just m. This term comes from equation (20.6) for the gravitational field, from where it is clear that it tells that pressure also contributes to the density of gravitational mass – *pressure generates gravitation*. That has been expected actually, certainly from the equivalence principle, but "naively" as well: pressure is an effective term for mechanical interaction between elements of the fluid, and the corresponding interaction energy has to generate gravitation, similarly as any other kind of energy. A positive pressure corresponds to "repulsion" between the fluid particles (the corresponding interaction energy is positive), so it generates attractive gravity similarly as positive density. A negative pressure (tension) induces gravitational repulsion on the contrary.

- There appears $r(r-2m)$ instead of just r^2 in the denominator. This factor we already know from the Schwarzschild metric, it represents curvature of space – in changing the surface of a sphere $r = \mathrm{const}$, the sphere's proper radius changes $(1-2m/r)^{-1/2}$-times more than if the space were flat (see Section 14.1.2).

All the three terms make the right-hand side of the equation *bigger*, which means that

$\boxed{\text{the life of a relativistic star is } \textit{heavier} \text{ than that of a classical star}}$:

to balance its gravity, it needs larger pressure gradient. Not only that. We see that larger gradient implies larger *value* of pressure after every successive step towards the stellar interior – and larger value of pressure makes the pressure gradient still larger, etc. The pressure is sometimes said to be playing a *regenerative* role in the TOV equation (its value and gradient stimulate each other). Consequently, the pressure has to grow towards the star centre steeper than in classical physics. The question arises naturally whether the equilibrium is always possible at all. To answer it, one has to integrate the TOV equation and compare the result with the corresponding integration of the classical equilibrium equation. This typically is far from easy, because the TOV equation is in general coupled to other equations of stellar structure in a rather complicated way.

Integration of the TOV equation for uniform density

However, let us perform a very rough check: how the relativistic and classical equilibria differ for a *uniform density*, $\rho = \mathrm{const} > 0$ for $0 \leqslant r \leqslant R$ and $\rho = 0$ for $r > R$). In such a case, there is no coupling between equations, because the mass is found immediately as

$$m(r) = \frac{4}{3}\pi r^3 \rho\,, \qquad \text{especially} \quad M := m(R) = \frac{4}{3}\pi R^3 \rho\,,$$

which makes (20.7) read

$$-\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{(\rho+P)\left(\frac{4}{3}\pi r^3\rho + 4\pi r^3 P\right)}{r\left(r - \frac{8}{3}\pi r^3\rho\right)} = \frac{4}{3}\pi r\,\frac{(\rho+P)(\rho+3P)}{1-\frac{8}{3}\pi r^2\rho}$$

$$\Longrightarrow \quad -\frac{\rho\,\mathrm{d}P}{(\rho+P)(\rho+3P)} = \frac{\frac{4}{3}\pi r\rho\,\mathrm{d}r}{1-\frac{8}{3}\pi r^2\rho}\,. \tag{20.8}$$

After substitution

$$x^2 \equiv \frac{8}{3}\pi r^2\rho \qquad \text{and denoting} \quad x^2(r=R) = \frac{8}{3}\pi R^2\rho =: X^2\,,$$

the integration from the surface ($r = R$, given by $P = 0$) to a given radius ($0 \leqslant r \leqslant R$, $P \geqslant 0$) appears as

$$-\int_P^0 \frac{\rho\,\mathrm{d}P}{(\rho+P)(\rho+3P)} \equiv \int_0^P \frac{\rho\,\mathrm{d}P}{(\rho+P)(\rho+3P)} = \frac{1}{2}\int_x^X \frac{x\,\mathrm{d}x}{1-x^2}$$

and results in

$$\frac{1}{2}\ln\frac{\rho+3P}{\rho+P}=\frac{1}{4}\ln\frac{1-x^2}{1-X^2} \qquad \Longrightarrow \qquad P=\rho\,\frac{\sqrt{1-x^2}-\sqrt{1-X^2}}{3\sqrt{1-X^2}-\sqrt{1-x^2}}\,. \tag{20.9}$$

This result was reported to Einstein by K. Schwarzschild on 6th February 1916, still from Russia, after one of his "walks into your [Einstein's] land of ideas". Its most important feature is that *it may come out infinite*. The latter would happen if

$$P(r)\to\infty \qquad \Leftrightarrow \qquad 3\sqrt{1-X^2}=\sqrt{1-x^2} \qquad \Leftrightarrow \qquad 9X^2=8+x^2\,.$$

The most endangered is the star centre ($x=0$), where the condition gives

$$X^2=\frac{8}{9} \quad \Leftrightarrow \quad R=\frac{1}{\sqrt{3\pi\rho}}\,, \qquad \text{in physical units} \quad R=\frac{c}{\sqrt{3\pi G\rho}}\,. \tag{20.10}$$

Hence, a star with a given constant density cannot be arbitrarily big – if its radius exceeded the above value, it would not be possible to balance its gravitation by any finite pressure profile. We are not saying that exactly this value is crucial for real stars, because $\rho=\mathrm{const}$ is not a very realistic circumstance (besides other, it is called non-causal, since it corresponds to infinite speed of sound $\sqrt{\partial P/\partial\rho}$) and because it turns out that GR more affects stability that equilibrium itself, but it is interesting to check whether real stars at least "feel" this limit – whether it is not orders of magnitude larger than actual size of stars. It is not – for a neutron star, for example, the typical density is the nuclear one, $\rho\doteq 2.3\cdot 10^{14}\mathtt{g/cm^3}$, and the above limit reads

$$R\doteq 25\mathtt{km} \qquad \left(\Longrightarrow \quad M=\frac{4}{3}\pi R^3\rho\doteq 7.5M_\odot\right).$$

Real neutron stars have about $R\simeq 10\mathtt{km}$, $M\simeq 1.5M_\odot$. For normal stars the limit is very large, and even white dwarfs almost do not "feel" it – for their typical density $\rho\doteq 10^6\mathtt{g/cm^3}$, the critical radius is $R\doteq 3.8\cdot 10^5\mathtt{km}$ and the corresponding mass about $1.1\cdot 10^5 M_\odot$ (real white dwarfs have about $7000\,\mathtt{km}$ and $1M_\odot$).

Last but not least, we should show that the above limitation is really new in GR, that it does not happen in classical equilibria. Substituting $\rho=\mathrm{const}$ (thus $m=\frac{4}{3}\pi r^3\rho$) to the classical equilibrium condition $-P_{,r}=\rho\,m/r^2$, we have

$$-\frac{\mathrm{d}P}{\mathrm{d}r}=\frac{4}{3}\pi\rho^2 r \qquad \Longrightarrow \qquad P(r)=\frac{2}{3}\pi\rho^2(R^2-r^2)\,. \tag{20.11}$$

Therefore, inside a finite classical star, the equilibrium pressure is everywhere finite.

Anyway, a more serious limit for the mass of white dwarfs and neutron stars will be mentioned in Section 21.5 when we will better know how the pressure is being maintained in these extremely dense objects.

20.3.3 Equations of state and integration of the stellar-equilibrium equations

In reality, one cannot prescribe $\rho(r)$ (the less $\rho = \mathrm{const}$), it should be given as a part of solution of the equilibrium problem. One thus has three unknown functions – $\rho(r)$, $m(r)$ and $P(r)$ – while only two equations (the TOV equation and the equation for mass). The remaining input is standardly provided by an equation of state for P. However, besides ρ, the pressure also depends on n, s and Z_k, so all the thermodynamical side of the problem (eventually dependent on temperature) comes into the play. Let us have a look how to describe the thermodynamic state of matter in GR. It can be done in two basic ways:

- **Fundamental equation** $\rho = \rho(n, s, Z_k)$.
  This is the equation whose differential is the 1st law of thermodynamics. The latter's usual form

$$\mathrm{d}U = -P\mathrm{d}V + T\mathrm{d}S + \sum_k \mu_k \mathrm{d}A_k$$

can in terms of our quantities be written, for some macroscopic element containing $\delta A = \mathrm{const}$ baryons, as

$$\mathrm{d}\left(\rho\,\frac{\delta A}{n}\right) = -P\mathrm{d}\left(\frac{\delta A}{n}\right) + T\mathrm{d}(s\,\delta A) + \sum_k \mu_k \delta A\,\mathrm{d}Z_k\,,$$

where we have used that $\frac{1}{n}$ is the proper volume per one baryon, so $\frac{\delta A}{n}$ is the total volume of the element and $\rho\frac{\delta A}{n}$ is its total energy. Since δA is supposed to be constant, we may divide by it and obtain

$$\mathrm{d}\rho = \frac{\rho + P}{n}\,\mathrm{d}n + Tn\mathrm{d}s + n\sum_k \mu_k \mathrm{d}Z_k\,. \tag{20.12}$$

The first law implies $(B+2)$ equations

$$\left(\frac{\partial\rho}{\partial n}\right)_{s,Z_k} = \frac{\rho+P}{n}\,, \qquad \left(\frac{\partial\rho}{\partial s}\right)_{n,Z_k} = Tn\,, \qquad \left(\frac{\partial\rho}{\partial Z_i}\right)_{n,s,Z_{k\neq i}} = n\mu_i\,. \tag{20.13}$$

The first of these equations is tightly connected with **adiabatic indices**. Adiabatic index is defined as the ratio of the heat capacity at constant pressure to heat capacity at constant volume. In the stellar theory, definitions introduced by Chandrasekhar are usually employed (which all reduce to the ratio of heat capacities in the ideal-gas case),

$$\Gamma_1 := \left(\frac{\partial \ln P}{\partial \ln n}\right)_{s,Z_1,\ldots,Z_B} = \left(\frac{\partial \ln P}{\partial P}\frac{\partial P}{\partial n}\frac{\partial n}{\partial \ln n}\right)_{s,Z_1,\ldots,Z_B} = \frac{n}{P}\left(\frac{\partial P}{\partial n}\right)_{s,Z_1,\ldots,Z_B} \overset{\text{or}}{=}$$
$$\overset{\text{or}}{=} \frac{n}{P}\left(\frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial n}\right)_{s,Z_1,\ldots,Z_B} = \frac{\rho+P}{P}\left(\frac{\partial P}{\partial \rho}\right)_{s,Z_1,\ldots,Z_B}\,, \tag{20.14}$$

$$\Gamma_2 := \left[1 - \left(\frac{\partial \ln T}{\partial \ln P}\right)_{s,Z_1,\ldots,Z_B}\right]^{-1}, \tag{20.15}$$

$$\Gamma_3 := 1 + \left(\frac{\partial \ln T}{\partial \ln n}\right)_{s,Z_1,\dots,Z_B} = 1 + \left(\frac{\partial \ln T}{\partial T}\frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial n}\frac{\partial n}{\partial \ln n}\right)_{s,Z_1,\dots,Z_B} =$$
$$= 1 + \frac{\rho + P}{T}\left(\frac{\partial T}{\partial \rho}\right)_{s,Z_1,\dots,Z_B}.$$

- $(B+2)$ **equations of state**

$$P = P(\rho, n, s, Z_k)\,, \qquad T = T(\rho, n, s, Z_k)\,, \qquad \mu_i = \mu_i(\rho, n, s, Z_k)\,. \tag{20.16}$$

The thermodynamical equations are in general coupled to other stellar-structure equations in a complicated way. Sure: in particular, the $P = P(\rho, n, s, Z_k)$ equation of state can only be simplified to $P = P(\rho)$ if knowing the relations $\rho = \rho(n) = \rho[n, s(n), Z_k(n)]$ and $P = P[n, s(n), Z_k(n)]$, but that requires to know how exactly are connected the thermonuclear reactions, energy generation, energy transport and entropy distribution. This is given by equations for thermonuclear energy generation, for thermal equilibrium, for the transport of energy... Some of these we will address below.

However, from the GR point of view, the most interesting stars are dead stars, i.e. those whose nuclei have already finished their thermonuclear evolution. The lack of energy generation leads to a significant decrease of radiation pressure, which results in a considerable gravitational contraction of such a nucleus. The final destination of the nucleus mainly depends on its mass, but there appear to be three robust options – contraction to a white dwarf, collapse to a neutron star, or total collapse to a black hole. In white dwarfs as well as in neutron stars, basically no thermonuclear reactions take place, and the pressure-gradient which has halted further contraction is maintained, thanks to the Pauli exclusion principle, by a degenerate fermion gas (of electrons or neutrons) rather than by thermal motion of particles. In very short, in such a stage "temperature is not important", and thus "thermodynamical" equations become trivial (or at least decoupled from the other equations). This approximation is being called the state of **cold catalysed matter** – it is the matter which has exhausted all the energy resources available under given conditions and which resides in an energetic minimum. In such a matter, the relative baryon abundances are not changing ($Z_k = \mathrm{const}$, $\alpha_k = 0$), thus no thermonuclear energy is generated ($q = 0$, $q^{(\nu)} = 0$), so there is also no energy output ($L_r = 0$, $L_r^{(\nu)} = 0$); and provided that the Fermi-type energy resulting from compression of fermions dominates kT (which is largely satisfied in white dwarfs and neutron stars), one can also neglect temperature and entropy ($T = 0$, $s = 0$). Consequently, the above equations of state reduce to just $P = P(\rho)$ and the whole system of differential equations reduces to the TOV equation, the equation for mass and the $P(\rho)$ equation of state for the functions $P(r)$, $m(r)$ and $\rho(r)$ as unknowns.

Even in the limit approximation of cold matter, the problem remains rather difficult, because – mainly in conditions likely present in neutron-star interiors – the equation of state is not completely clear (in the very nuclei, it basically remains unknown). Anyway, the integration of the above three equations typically starts from the centre ($r = 0$) and is parameterized by central density; integration is finished when pressure drops to zero, which identifies the stellar surface. The sequence of the resulting equilibrium configurations is usually plotted in terms of the mass M against the radius R. The next step is to analyse stability of the obtained equilibria, in order to recognize which parts of the equilibrium curves are really relevant astrophysically. We postpone further discussion to the next chapter on degenerate objects.

20.3.4 TOV equation from a variational principle

Interestingly, the TOV equation can also be derived from a variational principle, namely by requiring that the total mass of the star $M = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r$ be stationary with respect to all variations of $\rho(r)$ which leave unchanged the total number of baryons $A = \int_0^R 4\pi r^2 n \sqrt{g_{rr}} \, \mathrm{d}r$ and which do not change s and A_k (adiabatic variation). Having two main constraints (on M and A), we use the Lagrange multiplier (λ) and write

$$\delta M - \lambda \delta A = \int_0^\infty 4\pi r^2 \delta\rho \, \mathrm{d}r - \lambda \int_0^\infty \frac{4\pi r^2 \delta n}{\sqrt{1-\frac{2m}{r}}} \, \mathrm{d}r - \lambda \int_0^\infty \frac{4\pi r^2 n}{\left(1-\frac{2m}{r}\right)^{3/2}} \frac{\delta m}{r} \, \mathrm{d}r \,. \tag{20.17}$$

For adiabatic variation, the 1st thermodynamical law (20.12) yields

$$\frac{\delta\rho}{\delta n} = \frac{\rho + P}{n} \qquad \Longrightarrow \qquad \delta n = \frac{n\,\delta\rho}{\rho + P} \,,$$

and from the mass equation we have

$$m(r) = \int_0^r 4\pi r'^2 \rho \, \mathrm{d}r' \qquad \Longrightarrow \qquad \delta m = \int_0^r 4\pi r'^2 \delta\rho \, \mathrm{d}r' \,,$$

which are now plugged to the main formula. The term containing δm can be integrated by parts. Indeed, despite we write the upper integration limit as ∞ for simplicity, all the integrals of course have compact support $r \leqslant R + \delta R$, so one finds

$$\int_0^\infty \frac{4\pi r n}{\left(1-\frac{2m}{r}\right)^{3/2}} \left[\int_0^r 4\pi r'^2 \delta\rho \, \mathrm{d}r'\right] \mathrm{d}r = \int_0^\infty 4\pi r^2 \delta\rho \left[\int_r^\infty \frac{4\pi r' n \, \mathrm{d}r'}{\left(1-\frac{2m}{r'}\right)^{3/2}}\right] \mathrm{d}r \tag{20.18}$$

by standard scheme, namely by using the per-partes formula

$$\int_0^\infty \frac{\mathrm{d}U}{\mathrm{d}r} V \, \mathrm{d}r = [UV]_0^\infty - \int_0^\infty U \frac{\mathrm{d}V}{\mathrm{d}r} \, \mathrm{d}r$$

for the functions

$$u(r) := \frac{4\pi r n(r)}{\left(1-\frac{2m(r)}{r}\right)^{3/2}} \,, \qquad U(r) := -\int_r^\infty u(r') \, \mathrm{d}r' \qquad \Longrightarrow \qquad u(r) \equiv \frac{\mathrm{d}U(r)}{\mathrm{d}r} \,,$$

$$v(r) := 4\pi r^2 \delta\rho(r) \,, \qquad V(r) := \int_0^r v(r') \, \mathrm{d}r' \qquad \Longrightarrow \qquad v(r) \equiv \frac{\mathrm{d}V(r)}{\mathrm{d}r}$$

(differentiation by the lower and upper integration limit, respectively). So introducing everything into the basic equation (20.17), we have

$$\delta M - \lambda\delta A = \int_0^\infty 4\pi r^2 \delta\rho \left\{ 1 - \lambda \left[\frac{n}{\rho+P} \frac{1}{\sqrt{1-\frac{2m}{r}}} + \int_r^\infty \frac{4\pi r' n\, \mathrm{d}r'}{\left(1-\frac{2m}{r'}\right)^{3/2}} \right] \right\} \mathrm{d}r\,.$$

Now, we want the above variation to vanish, which can only be the case if $1/\lambda$ equals the bracket [...]. This however means that the bracket has to be constant (independent of r), i.e. $[...]_{,r} = 0$, which we write out as

$$\frac{n_{,r}}{\rho+P} \frac{1}{\sqrt{1-\frac{2m}{r}}} - \frac{n}{(\rho+P)^2} \frac{\rho_{,r}+P_{,r}}{\sqrt{1-\frac{2m}{r}}} + \frac{n}{\rho+P} \frac{m_{,r} r - m}{r^2\left(1-\frac{2m}{r}\right)^{3/2}} - \frac{4\pi r^3 n}{r^2\left(1-\frac{2m}{r}\right)^{3/2}} = 0\,.$$

Multiplication by $\sqrt{1-\frac{2m}{r}}\,\frac{(\rho+P)^2}{n}$ and using the equation for mass $m_{,r} = 4\pi r^2\rho$ leads to

$$\cancel{\frac{\rho+P}{n}\frac{\mathrm{d}n}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r}} - \frac{\mathrm{d}P}{\mathrm{d}r} + \frac{\rho+P}{r(r-2m)}\left[\cancel{4\pi r^3\rho} - m - 4\pi r^3(\cancel{\rho} + P)\right] = 0\,.$$

Finally, the first two terms give zero due to adiabaticity, and the rest precisely yields the TOV equation,

$$-\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{(\rho+P)(m+4\pi r^3 P)}{r(r-2m)}\,.$$

20.3.5 Equation of thermal equilibrium

This equation follows from an energy balance for an infinitesimal spherical shell of the star. Let us call δV the proper volume of the shell, δa the number of baryons in the shell, τ the proper time of a clock staying at rest on the radius r (which marks the bottom surface of the shell), and δr the (small) radial thickness of the shell. Assuming that the volume of the shell changes slowly (quasi-statically), so that no energy is expended in acceleration of its matter, the energy conservation can be expressed as

$$\mathrm{d}(\epsilon\,\delta V) = q\,\delta a\,\mathrm{d}\tau - P\mathrm{d}(\delta V) - \left[L_r(r+\delta r)|_r^r - L_r(r)\right]\mathrm{d}\tau\,, \tag{20.19}$$

where d means the change of the energy contributions within an infinitesimal interval of τ. Clearly the left-hand side represents the change of internal energy, while on the right-hand side one places the energy generated (during $\mathrm{d}\tau$) in thermonuclear reactions, the work spent in the volume change, and the luminosity balance given by difference between the power which leaves the outer surface of the shell and the power entering the shell from inside. The clumsy notation $L_r(r+\delta r)|_r^r$ indicates that the power leaving the top surface, $L_r(r+\delta r)$, has to be converted with respect to the radius r (so that all the quantities were compared at r).

The main "issue" is to compute the luminosity term properly. Since L_r represents power (energy per time), the conversion of $L_r(r+\delta r)$ to r involves *two* corrections – hence the two r indicated at $L_r(r+\delta r)|_r^r$: (i) the time ticks differently at those two radii (usual dilation factor between two clocks at rest), and (ii) the energy is shifted between the two radii (in the same manner as frequency, i.e. reciprocally with respect to time). Proper times of clocks at rest at $r+\delta r$ and r are related by

$$\frac{\mathrm{d}\tau(r+\delta r)}{\mathrm{d}\tau(r)}=\sqrt{\frac{-g_{tt}(r+\delta r)}{-g_{tt}(r)}}=\frac{e^{\Phi(r+\delta r)}}{e^{\Phi(r)}}\,,$$

which means that

$$\frac{L_r(r+\delta r)|^r}{L_r(r+\delta r)}=\frac{\mathrm{d}\tau(r+\delta r)}{\mathrm{d}\tau(r)}=\frac{e^{\Phi(r+\delta r)}}{e^{\Phi(r)}}\,,$$

and the energy shift brings once more the same factor (energy transforms reciprocally to time, and it is in *numerator* of the definition of power), so, altogether,

$$\frac{L_r(r+\delta r)|_r^r}{L_r(r+\delta r)|_{r+\delta r}^{r+\delta r}}\equiv\frac{L_r(r+\delta r)|_r^r}{L_r(r+\delta r)}=\frac{e^{2\Phi(r+\delta r)}}{e^{2\Phi(r)}}=e^{2\Phi(r+\delta r)-2\Phi(r)}\,.$$

The luminosity term thus reads

$$L_r(r+\delta r)|_r^r-L_r(r)=L_r(r+\delta r)e^{2\Phi(r+\delta r)-2\Phi(r)}-L_r(r)\,.$$

Expanding the first part in δr, we have (without repeating the arguments, all will be taken at r from now)

$$L_r(r+\delta r)=L_r+\frac{\mathrm{d}L_r}{\mathrm{d}r}\delta r+O(\delta r^2)\,,$$
$$e^{2\Phi(r+\delta r)-2\Phi(r)}=e^{2\left[\Phi+\frac{\mathrm{d}\Phi}{\mathrm{d}r}\delta r+O(\delta r^2)-\Phi\right]}=e^{2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\delta r+O(\delta r^2)}=1+2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\delta r+O(\delta r^2)\,,$$

their product yields

$$\begin{aligned}L_r(r+\delta r)e^{2\Phi(r+\delta r)-2\Phi(r)}&=\left[L_r+\frac{\mathrm{d}L_r}{\mathrm{d}r}\delta r+O(\delta r^2)\right]\left[1+2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\delta r+O(\delta r^2)\right]=\\&=L_r+\frac{\mathrm{d}L_r}{\mathrm{d}r}\delta r+2L_r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\delta r+O(\delta r^2)\,,\end{aligned}$$

and hence the result

$$L_r(r+\delta r)|_r^r-L_r(r)=\left(\frac{\mathrm{d}L_r}{\mathrm{d}r}+2L_r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)\delta r+O(\delta r^2)\doteq e^{-2\Phi}\frac{\mathrm{d}}{\mathrm{d}r}\big(L_re^{2\Phi}\big)\,\delta r\,.$$

In the remaining terms of (20.19), we will assume that $\delta a=\mathrm{const}$ (number of baryons in the layer does not change). Writing $\delta V=\frac{\delta a}{n}$, the equation assumes the form

$$e^{-2\Phi}\frac{\mathrm{d}}{\mathrm{d}r}\big(L_re^{2\Phi}\big)\,\delta r\,\mathrm{d}\tau=\delta a\left[q\,\mathrm{d}\tau-\mathrm{d}\Big(\frac{\epsilon}{n}\Big)-P\,\mathrm{d}\Big(\frac{1}{n}\Big)\right].$$

Multiplying by $e^{2\Phi}/\mathrm{d}\tau$, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(L_r e^{2\Phi}\right)\delta r = e^{2\Phi}\frac{\delta a}{n}\left(qn - \frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} + \frac{\epsilon + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right).$$

Finally, we express δa using equation (20.3),

$$\delta a = \frac{\mathrm{d}a}{\mathrm{d}r}\,\delta r = \frac{4\pi r^2 n}{\sqrt{1-\frac{2m}{r}}}\,\delta r\,,$$

and divide by δr, thus obtaining

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(L_r e^{2\Phi}\right) = \frac{4\pi r^2 e^{2\Phi}}{\sqrt{1-\frac{2m}{r}}}\left(qn - \frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} + \frac{\epsilon + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right). \tag{20.20}$$

Still somewhat shorter version of the parenthesis can be achieved by employing $\rho \equiv \bar{m}_\mathrm{b} n + \epsilon$ instead of just ϵ,

$$\begin{aligned}\frac{\rho + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} - \frac{\mathrm{d}\rho}{\mathrm{d}\tau} &\equiv \frac{\bar{m}_\mathrm{b} n + \epsilon + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} - \frac{\mathrm{d}(\bar{m}_\mathrm{b} n + \epsilon)}{\mathrm{d}\tau} = \\ &= \cancel{\bar{m}_\mathrm{b}\frac{\mathrm{d}n}{\mathrm{d}\tau}} + \frac{\epsilon + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} + qn - \cancel{\bar{m}_\mathrm{b}\frac{\mathrm{d}n}{\mathrm{d}\tau}} - \frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = qn + \frac{\epsilon + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} - \frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\,,\end{aligned}$$

which puts the equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(L_r e^{2\Phi}\right) = \frac{4\pi r^2 e^{2\Phi}}{\sqrt{1-\frac{2m}{r}}}\left(\frac{\rho + P}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} - \frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right). \tag{20.21}$$

Under usual conditions, neutrinos do not interact with matter and, consequently, do not enter the thermodynamics. Their luminosity contribution is thus given solely by $q^{(\nu)}$, according to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(L_r^{(\nu)} e^{2\Phi}\right) = \frac{4\pi r^2 n e^{2\Phi}}{\sqrt{1-\frac{2m}{r}}}\,q^{(\nu)}\,. \tag{20.22}$$

20.3.6 Equation for energy transport

Energy is transported in a star by radiation (actually photon diffusion), conduction, convection and by neutrinos (the last one is "trivial" in that it is just an escape). If convection contributes comparably to the other channels, the analysis is very difficult, so we will only indicate the derivation in situations when convection is either negligible or dominant (which fortunately is far from rare in astrophysics).

- <u>The case when convection is negligible</u>

Crucial is the relation for radial gradient of the radiation pressure (which is almost isotropic, however). First, we know from special relativity already that the T^{0j} components of the energy-momentum tensor represent (energy-flux density)/c which is the same as pressure (momentum per time per area), and that also equals radiation intensity (also called irradiation) divided by speed of light = (luminosity per area)/c, that is,

$$\frac{\text{Poynting flux}}{c} = P_{\text{rad}} = \frac{I_{\text{rad}}}{c} = \frac{L_r^{\text{rad}}}{4\pi r^2 c} \; .$$

The pressure gradient is due to two reasons, gravitational field and interaction with matter (absorption). The first part is described by the standard equation of hydrostatic equilibrium,

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}r} = -(\rho_{\text{rad}} + P_{\text{rad}})\frac{\mathrm{d}\Phi}{\mathrm{d}r} \; ,$$

while the second is described by the equation of radiation transfer (with $c\equiv 1$)

$$\frac{\mathrm{d}P_{\text{rad}}}{\frac{\mathrm{d}r}{\sqrt{1-\frac{2m}{r}}}} = -\kappa_{\text{rad}}\rho \, P_{\text{rad}} \equiv -\kappa_{\text{rad}}\rho \, I_{\text{rad}} \; .$$

The third ingredient is the well known Stefan-Boltzmann law for power radiated by a black body, which in terms of energy density reads

$$\rho_{\text{rad}}(\doteq 3P_{\text{rad}}) = 4\sigma T^4 \qquad \Longrightarrow \qquad \frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}r} = \frac{16}{3}\,\sigma T^3 \,\frac{\mathrm{d}T}{\mathrm{d}r} \, ,$$

with σ the Stefan-Boltzmann constant and with the $\rho_{\text{rad}} \doteq 3P_{\text{rad}}$ relation valid for isotropic radiation (our flux is very nearly isotropic, with just tiny radiation gradient). Using the last relation, one obtains for the sum of the two contributions to the radiation-pressure gradient

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3}{16\sigma T^3}\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}r} = -\frac{3}{16\sigma T^3}\left[(\rho_{\text{rad}} + P_{\text{rad}})\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \kappa_{\text{rad}}\rho \, I_{\text{rad}} \,\frac{1}{\sqrt{1-\frac{2m}{r}}}\right] .$$

By substituting $(\rho_{\text{rad}} + P_{\text{rad}}) = \frac{4}{3}\,\rho_{\text{rad}} = \frac{16}{3}\sigma T^4$, one obtains the equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\sigma T^3}\left(\frac{16}{3}\sigma T^4\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \kappa_{\text{rad}}\rho \, I_{\text{rad}}\,\frac{1}{\sqrt{1-\frac{2m}{r}}}\right) = -T\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{3\kappa_{\text{rad}}\rho}{16\sigma T^3}\,\frac{I_{\text{rad}}}{\sqrt{1-\frac{2m}{r}}} \; ,$$

which can be written in a more concise way as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(Te^{\Phi}\right) = -\frac{3\kappa_{\text{rad}}\rho}{16\sigma T^3}\,\frac{I_{\text{rad}}e^{\Phi}}{\sqrt{1-\frac{2m}{r}}} \; . \tag{20.23}$$

However, the conduction (and neutrino) parts are yet to be included. In non-relativistic physics, the intensity (energy-flux density) due to the conductive luminosity, $I_{\text{cond}} \equiv$

$L_r^{\rm cond}/(4\pi r^2)$, is related to the temperature gradient through the thermal-conduction coefficient $\lambda_{\rm cond}$,

$$I_{\rm cond} = -\lambda_{\rm cond}\frac{{\rm d}T}{{\rm d}r} \; .$$

In GR, one takes radial proper-distance element instead of just ${\rm d}r$, and also has to account for the energy redshift. Actually, the Newtonian case of constant L_r (independent of r) corresponds, in GR, to constant $L_r\sqrt{-g_{tt}} \equiv L_r e^{\Phi}$ since energy is redshifted by e^{Φ}, and the same modification follows for temperature from the relation "${\rm energy} = kT$". Hence, the equation of conductive energy transport reads

$$\frac{\rm d}{{\rm d}r}\left(Te^{\Phi}\right) = -\frac{1}{\lambda_{\rm cond}}\,\frac{I_{\rm cond}e^{\Phi}}{\sqrt{1-\frac{2m}{r}}} \; . \tag{20.24}$$

Finally, introducing the conductive absorption coefficient $\kappa_{\rm cond}$ and the total absorption coefficient κ by

$$\kappa_{\rm cond} := \frac{16\sigma T^3}{3\rho\lambda_{\rm cond}} \, , \qquad \frac{1}{\kappa} := \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cond}} \, ,$$

and writing $L_r^{\rm rad} + L_r^{\rm cond} = L_r - L_r^{(\nu)}$, thus $I_{\rm rad} + I_{\rm cond} = I - I^{(\nu)}$ for the corresponding intensities, the sum of the equations (20.23) and (20.24) yields

$$\frac{\rm d}{{\rm d}r}\left(Te^{\Phi}\right) = -\frac{3\kappa\rho e^{\Phi}}{16\sigma T^3}\,\frac{I-I^{(\nu)}}{\sqrt{1-\frac{2m}{r}}} \quad\Longrightarrow\quad \frac{{\rm d}T}{{\rm d}r} = -T\,\frac{{\rm d}\Phi}{{\rm d}r} - \frac{3\kappa\rho}{16\sigma T^3}\,\frac{I-I^{(\nu)}}{\sqrt{1-\frac{2m}{r}}} \; . \tag{20.25}$$

Most important implication of this result is that if there is no other luminosity than the neutrino one (which does not interact with the star matter at all), $I = I^{(\nu)}$, there still remains a certain gradient of temperature in the star due to the gravitational redshift,

$$\frac{{\rm d}T}{{\rm d}r} = -T\,\frac{{\rm d}\Phi}{{\rm d}r} = -T\,\frac{m+4\pi r^3 P}{r(r-2m)} \; .$$

If $m(r)$ and $P(r)$ are known, one thus obtains $T(r)$ in the "cold-star" case.

- The case when convection is dominant

  Convection transport is difficult and is usually treated in some approximation. In the adiabatic approximation, the temperature gradient is related to the pressure gradient through the adiabatic coefficient (20.15)

$$\Gamma_2 = \left[1 - \left(\frac{\partial \ln T}{\partial \ln P}\right)_{s,Z_1,\ldots,Z_B}\right]^{-1} ,$$

  so if this coefficient is known, one can write

$$\frac{1}{\Gamma_2} = 1 - \left(\frac{\partial \ln T}{\partial \ln P}\right)_{s,Z_k} = 1 - \left(\frac{\partial \ln T}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial \ln P}\right)_{s,Z_k} = 1 - \frac{P}{T}\frac{\left(\frac{{\rm d}T}{{\rm d}r}\right)_{s,Z_k}}{\left(\frac{{\rm d}P}{{\rm d}r}\right)_{s,Z_k}} \, ,$$

from where

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{s,Z_k} = \frac{\Gamma_2 - 1}{\Gamma_2}\,\frac{T}{P}\,\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)_{s,Z_k} . \tag{20.26}$$

- *When* convection is important? -The question of convection stability

The equilibrium configuration is stable against convection if every its element, when perturbed from the equilibrium location, returns back. So let us shift an element from its equilibrium radius r_{A} to a nearby radius $r_{\mathrm{B}} > r_{\mathrm{A}}$, and wait until its pressure matches that in the surroundings. Denoting by A and B the equilibrium values at the respective radii, and leaving without any index the values which establish in the displaced element, the total force acting on the (unit-volume) element in the outward radial direction is

$$F_{\mathrm{B}} = F_{\mathrm{B}}^{\mathrm{grav}} + F_{\mathrm{B}}^{\mathrm{press}} = -(\rho + P)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)_{\mathrm{B}} - \left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)_{\mathrm{B}} ,$$

where however $P \equiv P_{\mathrm{B}}$ by assumption and the gradients are linked by the hydrostatic-equilibrium equation,

$$-\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)_{\mathrm{B}} = (\rho_{\mathrm{B}} + P_{\mathrm{B}})\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)_{\mathrm{B}} ,$$

so the force comes out as

$$F_{\mathrm{B}} = (\rho_{\mathrm{B}} - \rho)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)_{\mathrm{B}} \quad \ldots \;\; < 0 \quad \Longleftrightarrow \quad \rho > \rho_{\mathrm{B}} \quad \left(\text{since } \frac{\mathrm{d}\Phi}{\mathrm{d}r} > 0\right) . \tag{20.27}$$

Yes, we might have consulted Archimedes – it is nothing but standard answer about floating of bodies.

Let us suppose now that the thermodynamic process which leads to the matching of pressures ($P \to P_{\mathrm{B}}$) is *adiabatic*. Writing the above stability condition as

$$\rho \equiv \rho_{\mathrm{A}} + \delta\rho > \rho_{\mathrm{B}} \equiv \rho_{\mathrm{A}} + \delta\rho_{\mathrm{A}} , \qquad \text{i.e.} \quad \delta\rho > \delta\rho_{\mathrm{A}} ,$$

with $\delta\rho_{\mathrm{A}}$ given by expansion of the equilibrium profile,

$$\delta\rho_{\mathrm{A}} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{A}} \delta r ,$$

and with $\delta\rho$ given by adiabatic profile from the 1st law of thermodynamics (and using the definition (20.14) of the adiabatic index Γ_1)

$$\delta\rho = \left[\left(\frac{\partial\rho}{\partial P}\right)_{s,Z_1,\ldots,Z_B} \delta P\right]_{\mathrm{A}} = \left[\left(\frac{\partial\rho}{\partial P}\right)_{s,Z_1,\ldots,Z_B} \frac{\mathrm{d}P}{\mathrm{d}r}\right]_{\mathrm{A}} \delta r = \left[\frac{\rho + P}{P\Gamma_1}\frac{\mathrm{d}P}{\mathrm{d}r}\right]_{\mathrm{A}} \delta r ,$$

we obtain

$$\left(\frac{\rho + P}{P\Gamma_1}\frac{\mathrm{d}P}{\mathrm{d}r}\right)_{\mathrm{A}} > \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{A}} \quad \Longrightarrow \quad \left(\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{P\Gamma_1}{\rho + P}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{A}} =: S(r_{\mathrm{A}}) > 0 , \tag{20.28}$$

where $S(r)$ is called the **Schwarzschild discriminant** (yes, it is the same K. Schwarzschild, he derived the condition within the Newtonian theory in 1906).

The convection-stability condition $\delta\rho > \delta\rho_{\rm A}$ can also be written in terms of the temperature gradient. From the equation of state (for ideal gas) $P \sim \rho T$ it follows (at $r_{\rm A}$ again)

$$\delta P \sim T_{\rm A}\delta\rho + \rho_{\rm A}\delta T\,, \qquad \delta P_{\rm A} \sim T_{\rm A}\delta\rho_{\rm A} + \rho_{\rm A}\delta T_{\rm A}\,,$$

from where – because $\delta P \equiv \delta P_{\rm A}$ by assumption –

$$(\delta\rho - \delta\rho_{\rm A})\,T_{\rm A} \sim (\delta T_{\rm A} - \delta T)\,\rho_{\rm A}\,,$$

so the star is stable ($\delta\rho > \delta\rho_{\rm A}$) if $\delta T_{\rm A} > \delta T$. Remember now, from (20.25) (which exactly is valid if the star is convectively stable), that $\frac{{\rm d}T}{{\rm d}r} < -T\frac{{\rm d}\Phi}{{\rm d}r} < 0$, so both $\delta T_{\rm A}$ and δT are *negative* – hence the conclusion: stable is the star in which temperature decreases outwards more slowly than according to an adiabatic profile. And hence also the recipe how to decide between the non-convective and convective alternatives of the energy-transport equation: it is to be done according to whether (20.25) yields sub-adiabatic or super-adiabatic temperature gradient (respectively).

20.3.7 Boundary conditions

As a part of the stellar-equilibrium problem, we derived 7 first-order differential equations – equation for the number of baryons $a(r)$ (20.3), equation for mass $m(r)$ (20.5), equation for potential $\Phi(r)$ (20.6), TOV equation of hydrostatic equilibrium (20.7) which determines the pressure profile $P(r)$, equation of thermal equilibrium (20.21) which determines the luminosity profile $L_r(r)$, its counter-part (20.22) for neutrino part of the luminosity $L_r^{(\nu)}$, and equation for energy transport (20.25) which determines the temperature profile $T(r)$. These equations are being tackled under boundary conditions

$$a(0)=0,\ a(R)=:A;\quad m(0)=0,\ m(R)=:M;\quad \Phi(\infty)=0;\quad L_r(0)=0;\quad P(R)=0,\ T(R)=0\,.$$

It is worth to notice that the equations imply $\frac{{\rm d}a}{{\rm d}r} > 0$, $\frac{{\rm d}m}{{\rm d}r} > 0$, $\frac{{\rm d}\Phi}{{\rm d}r} > 0$, $\frac{{\rm d}P}{{\rm d}r} < 0$, $\frac{{\rm d}T}{{\rm d}r} < 0$. The luminosities L_r and $L_r^{(\nu)}$ have to grow with r at least for small r (since they are zero at the very centre).

You may have noted that there is no equation for the radial profile of density ρ. Indeed, density is obtained "indirectly" in integration of the equilibrium equations, usually from pressure via an equation of state. Normally it decreases from the centre towards the surface, as it is consistent with floating of a light fluid on a heavier one rather than vice versa. Hence, we may also add $\frac{{\rm d}\rho}{{\rm d}r} < 0$.

20.4 Radial oscillations and stability of stars

This section is on the response of a star to a small radial perturbation too, but the treatment will be *dynamical*. A star, similarly as every non-trivial system, may be subjected to many types of perturbations, and the problem of its stability is very difficult in general, even in case when the perturbations are very small (important in linear order only). The GR treatment, in addition, is considerably more complicated in that the perturbation of a source automatically brings on perturbation of the whole space-time as well, including generation of gravitational waves. The only case when one need not be solving for the exterior is a *purely radial perturbation of a spherically symmetric equilibrium*. Namely, in that case we know from Birkhoff that the exterior space-time remains Schwarzschild (irrespectively of the perturbation dynamics, it is sufficient if the star remains spherically symmetric). In addition, keeping the spherical symmetry *inside* the star ensures that also there the metric remains diagonal.

Besides the above geometrical constraint and limitation to linear perturbation order, we will assume that the perturbation is *adiabatic*, which in the ideal-fluid case means that it is *isentropic*, i.e. $s = 0$ and $\delta s = 0$. Note that such an assumption is *not* in general correct, as e.g. in the case of the most well known pulsating stars – cepheids.

- Parameterization of the problem

  We know from Section 12.1 that the metric of every spherically symmetric space-time may in spherical- type coordinates (t,r,θ,ϕ) be expressed in the form

  $$\mathrm{d}s^2 = g_{tt}(t,r)\mathrm{d}t^2 + g_{rr}(t,r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

  so we will naturally keep this as it is. Outside the star, such coordinates go over to the Schwarzschild ones. As in the treatment of equilibria, we will write $g_{tt} = -e^{2\Phi}$, and g_{rr} we will also write in the exponential form,

  $$g_{rr} = e^{2\lambda} \qquad \text{instead of} \qquad g_{rr} = \frac{1}{1-\frac{2m}{r}}\,.$$

- Perturbation equations

  Generally, we will perturb the relevant equilibrium equations, restrict to the linear perturbation order and subtract the equilibrium (unperturbed) part of the equations. Relevant quantities will be n, ρ, P, Φ and λ. From the baryon conservation, the 1st law of thermodynamics and the field equations we will obtain the so-called **initial-value equations** through which the perturbation of these quantities will be expressed in terms of the primary perturbation – the radial displacement ξ. Then, Euler equations for ideal fluid will provide **dynamical equations** for ξ as an independent degree of freedom.

20.4.1 Eulerian and Lagrangian perturbations

We will decompose all the relevant quantities $X(t,r)$ into their equilibrium profile $\bar{X}(r)$ (vertical-bar notation) plus a perturbation. The only exception from this notation rule will be the prime cause of perturbation of all the quantities – the spherically symmetric displacement of star elements which will be called $\xi(t,r)$ (so the displacement goes from the equilibrium position r to $r+\xi$).

Every continuous medium may be followed up by two basic family of observers:

- **Eulerian observers** are (and remain) at rest in some pre-defined sense, typically with respect to some space-time features (e.g. infinity), with respect to some special configuration of the continuum studied (e.g. the "equilibrium" one) or with respect to coordinates. In the case of our perturbation problem, they remain at rest in all the senses – with respect to infinity, with respect to the equilibrium configuration of the star and with respect to the coordinates (r, θ, ϕ). Perturbations measured by Eulerian observers will be denoted by δ, i.e. $X(t,r) = \bar{X}(r) + \delta X(t,r)$.

- **Lagrangian observers** comove together with matter, so – in our perturbation problem – they follow the displacement $r \to r+\xi(t,r)$ (and possible further evolution). Perturbations measured by Lagrangian observers will be denoted by Δ.

In order to write down, as usual, equations *at a given* r, we will generally try to use Eulerian perturbations, but sometimes Lagrangian approach will be necessary – typically in quantities which are based on derivative with respect to proper time of the fluid and when a quantity changes as a result of a thermodynamic process (the latter does not happen at given r, but along the world-line of a given fluid element). In the linear approximation, the two kinds of perturbations are related by

$$\begin{aligned}\Delta X(t,r) &:= X(t,r+\xi) - \bar{X}(r) \doteq X(t,r) + \frac{\partial X(t,r)}{\partial r}\,\xi(t,r) - \bar{X}(r) \\ &=: \delta X(t,r) + \frac{\partial X(t,r)}{\partial r}\,\xi(t,r) \doteq \delta X(t,r) + \frac{\mathrm{d}\bar{X}(r)}{\mathrm{d}r}\,\xi(t,r)\,. \qquad (20.29)\end{aligned}$$

20.4.2 Four-velocity of the perturbed fluid

With respect to the (r,θ,ϕ) coordinates, the perturbation gives the fluid a small radial component of velocity, so

$$u^\mu = (u^t, u^r, 0, 0), \qquad \text{where} \quad u^r = \frac{\mathrm{d}(r+\xi)}{\mathrm{d}\tau} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \xi_{,t}u^t\,.$$

The radial component being small of the $O(\xi)$ order, the four-velocity normalization

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 = g_{tt}(u^t)^2 + O(\xi^2)$$

yields

$$u^t \doteq \frac{1}{\sqrt{-g_{tt}}} = e^{-\Phi}, \qquad u^r = \xi_{,t}u^t \doteq \xi_{,t}e^{-\Phi}. \qquad (20.30)$$

At a given r, we thus have, to first order,

$$u^t = e^{-\bar{\Phi}-\delta\Phi} \doteq e^{-\bar{\Phi}}(1-\delta\Phi)\ \left(\equiv \bar{u}^t + \delta u^t\right), \qquad (20.31)$$

$$u^r = \xi_{,t}e^{-\bar{\Phi}-\delta\Phi} \doteq \xi_{,t}e^{-\bar{\Phi}}\ \left(\equiv \delta u^r\right). \qquad (20.32)$$

20.4.3 Initial-value equations

- <u>Determination of δn from baryon conservation</u>

  The conservation of baryons $(nu^\mu)_{;\mu}=0$ we write out as $n_{,\mu}u^\mu=-nu^\mu{}_{;\mu}$.
  On the left-hand side, equilibrium function $\bar{n}$ does not depend on τ, so we get, if substituting (20.31,20.32) for four-velocity and restricting to the first order,

$$n_{,\mu}u^\mu=\frac{\mathrm{d}n}{\mathrm{d}\tau}=\frac{\mathrm{d}(\bar{n}+\Delta n)}{\mathrm{d}\tau}=\frac{\mathrm{d}\Delta n}{\mathrm{d}\tau}=(\Delta n)_{,\mu}u^\mu\doteq(\Delta n)_{,t}\,u^t\doteq(\Delta n)_{,t}\,e^{-\bar{\Phi}} \tag{20.33}$$

  (Lagrangian perturbation has to be used, since the quantity is differentiated by τ!).
  On the right-hand side,

$$nu^\mu{}_{;\mu}=\frac{n}{\sqrt{-g}}\left(\sqrt{-g}\,u^\mu\right)_{,\mu}=\frac{n}{\sqrt{-g}}\left[\left(\sqrt{-g}\,u^t\right)_{,t}+\left(\sqrt{-g}\,u^r\right)_{,r}\right] \tag{20.34}$$

  is to be "equipped with"

$$\sqrt{-g}=e^{\Phi+\lambda}r^2\sin\theta\doteq e^{\bar{\Phi}+\bar{\lambda}}(1+\delta\Phi+\delta\lambda)\,r^2\sin\theta$$

  (perturbation is Eulerian here, *at a given location*, so we keep r rather than shifting to $r+\xi$). In the second term of (20.34) we only use the unperturbed part of $\sqrt{-g}$, because $u^r\equiv\delta u^r$ is itself of the $O(\xi)$ order,

$$\sqrt{-g}\,u^r\doteq(e^{\bar{\Phi}+\bar{\lambda}}r^2\sin\theta)\,\xi_{,t}e^{-\bar{\Phi}}=e^{\bar{\lambda}}r^2\xi_{,t}\sin\theta\,.$$

  The first term of $u^\mu{}_{;\mu}$ is $O(\xi)$ as well, because it is given by time derivative (and equilibrium values are time-independent),

$$\begin{aligned}\left(\sqrt{-g}\,u^t\right)_{,t}&\doteq\left[e^{\bar{\Phi}+\bar{\lambda}}(1+\delta\Phi+\delta\lambda)\,r^2\sin\theta\,e^{-\bar{\Phi}}(1-\delta\Phi)\right]_{,t}\doteq\\&\doteq e^{\bar{\lambda}}r^2\sin\theta\,[(1+\delta\Phi+\delta\lambda)(1-\delta\Phi)]_{,t}\doteq e^{\bar{\lambda}}r^2\sin\theta\,(\delta\lambda)_{,t}\,.\end{aligned}$$

  Therefore, the expression (20.34) has the structure $\frac{a\,\epsilon}{b+\epsilon}\doteq\frac{a\,\epsilon}{b}$, so up to the first order one is left with

$$\begin{aligned}\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,u^\mu\right)_{,\mu}&\doteq\frac{e^{\bar{\lambda}}r^2\sin\theta\,(\delta\lambda)_{,t}+(e^{\bar{\lambda}}r^2\xi_{,t}\sin\theta)_{,r}}{e^{\bar{\Phi}+\bar{\lambda}}\,r^2\sin\theta}\doteq\frac{e^{\bar{\lambda}}r^2(\delta\lambda)_{,t}+(e^{\bar{\lambda}}r^2\xi_{,t})_{,r}}{e^{\bar{\Phi}+\bar{\lambda}}\,r^2}=\\&=\frac{1}{e^{\bar{\Phi}}}\left[\delta\lambda+\frac{(e^{\bar{\lambda}}r^2\xi)_{,r}}{e^{\bar{\lambda}}r^2}\right]_{,t}=\frac{1}{e^{\bar{\Phi}}}\left(\delta\lambda+\bar{\lambda}_{,r}\xi+\frac{2\xi}{r}+\xi_{,r}\right)_{,t}\,.\end{aligned} \tag{20.35}$$

  Now we can plug both sides to the conservation equation $n_{,\mu}u^\mu=-nu^\mu{}_{;\mu}\doteq-\bar{n}u^\mu{}_{;\mu}$:

$$(\Delta n)_{,t}=-\bar{n}\left(\delta\lambda+\bar{\lambda}_{,r}\xi+\frac{2\xi}{r}+\xi_{,r}\right)_{,t}.$$

This can be integrated right away; the arbitrary integration function $f(r)$ we set to zero in order that Δn vanish for $\xi=0$, hence the result

$$\Delta n = -\bar{n}\left(\delta\lambda + \bar{\lambda}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right).$$

Using the relation (20.29), one also finds the corresponding Eulerian version,

$$\delta n = \Delta n - \bar{n}_{,r}\xi = -\bar{n}\left(\delta\lambda + \bar{\lambda}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right) - \bar{n}_{,r}\xi\,. \tag{20.36}$$

- Determination of δP from adiabatic character of the perturbation

  Under the assumption of adiabaticity, the pressure swing can be expressed easily in terms of Δn and the adiabatic index Γ_1 (20.14),

$$\Gamma_1 := \left(\frac{\partial \ln P}{\partial \ln n}\right)_s = \frac{n}{P}\left(\frac{\partial P}{\partial n}\right)_s \equiv \frac{n}{P}\frac{\Delta P}{\Delta n}$$

  (remember that the thermodynamic process happens along the trajectory of a given element, not at a given r). Hence, restricting to first order again,

$$\Delta P = \frac{P}{n}\,\Gamma_1 \Delta n \doteq \frac{\bar{P}}{\bar{n}}\,\Gamma_1 \Delta n\,,$$

  from where

$$\delta P = \Delta P - \bar{P}_{,r}\xi = -\bar{P}\Gamma_1\left(\delta\lambda + \bar{\lambda}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right) - \bar{P}_{,r}\xi\,. \tag{20.37}$$

- Determination of $\delta\rho$ from the first thermodynamic law

  In the absence of nuclear reactions, the first law (20.12) reads $\mathrm{d}\rho = \frac{\rho+P}{n}\,\mathrm{d}n$, i.e., to the first order in our case (and notation),

$$\Delta\rho \doteq \frac{\bar{\rho}+\bar{P}}{\bar{n}}\Delta n\,,$$

  hence

$$\delta\rho = \Delta\rho - \bar{\rho}_{,r}\xi = -(\bar{\rho}+\bar{P})\left(\delta\lambda + \bar{\lambda}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right) - \bar{\rho}_{,r}\xi\,. \tag{20.38}$$

- Determination of $\delta\lambda$ and $\delta\Phi_{,r}$ from Einstein equations

  Introducing our parameterization $g_{tt}\equiv -e^{2\Phi(t,r)}$, $g_{rr}\equiv e^{2\lambda(t,r)}$ to the spherically symmetric Einstein tensor (12.5), we obtain

$$G^t{}_t = \frac{1-e^{2\lambda}-2r\lambda_{,r}}{r^2 e^{2\lambda}}\,, \qquad G_{tr} = \frac{2}{r}\lambda_{,t}\,, \qquad G^r{}_r = \frac{1-e^{2\lambda}+2r\Phi_{,r}}{r^2 e^{2\lambda}}\,.$$

Expanding again up to the first order,

$$
\begin{aligned}
G^t{}_t &\doteq \frac{1-e^{2\bar\lambda}(1+2\delta\lambda)-2r\bar\lambda_{,r}-2r\,\delta\lambda_{,r}}{r^2e^{2\bar\lambda}(1+2\delta\lambda)} \doteq \\
&\doteq \frac{\left[1-e^{2\bar\lambda}(1+2\delta\lambda)-2r\bar\lambda_{,r}-2r\,\delta\lambda_{,r}\right](1-2\delta\lambda)}{r^2e^{2\bar\lambda}} \doteq \\
&\doteq \frac{1-2\delta\lambda-e^{2\bar\lambda}-2r\bar\lambda_{,r}+4r\bar\lambda_{,r}\delta\lambda-2r\,\delta\lambda_{,r}}{r^2e^{2\bar\lambda}} = \\
&\equiv \bar G^t_t-\frac{2\left[\delta\lambda-2r\bar\lambda_{,r}\delta\lambda+r\,\delta\lambda_{,r}\right]}{r^2e^{2\bar\lambda}} = \bar G^t_t-\frac{2}{r^2}\left(re^{-2\bar\lambda}\delta\lambda\right)_{,r}\;, && (20.39)\\
G_{tr} &\doteq \frac{2}{r}\,\cancel{\bar\lambda_{,t}}+\frac{2}{r}(\delta\lambda)_{,t}=\frac{2}{r}(\delta\lambda)_{,t} \qquad \left(\bar G_{tr}=0\right)\;, && (20.40)\\
G^r{}_r &\doteq \frac{1-e^{2\bar\lambda}(1+2\delta\lambda)+2r\bar\Phi_{,r}+2r\,\delta\Phi_{,r}}{r^2e^{2\bar\lambda}(1+2\delta\lambda)} \doteq \\
&\doteq \frac{\left[1-e^{2\bar\lambda}(1+2\delta\lambda)+2r\bar\Phi_{,r}+2r\,\delta\Phi_{,r}\right](1-2\delta\lambda)}{r^2e^{2\bar\lambda}} \doteq \\
&\doteq \frac{1-2\delta\lambda-e^{2\bar\lambda}+2r\bar\Phi_{,r}-4r\bar\Phi_{,r}\delta\lambda+2r\,\delta\Phi_{,r}}{r^2e^{2\bar\lambda}} = \\
&\equiv \bar G^r_r-\frac{2\left[\delta\lambda+2r\bar\Phi_{,r}\delta\lambda-r\,\delta\Phi_{,r}\right]}{r^2e^{2\bar\lambda}}\,. && (20.41)
\end{aligned}
$$

For the corresponding components of $T^\mu{}_\nu=(\rho+P)u^\mu u_\nu+P\delta^\mu_\nu$ we find, to the first order,

$$
\begin{aligned}
T^t{}_t &= (\rho+P)\,u^tu_t+P=(\rho+P)(-1-\cancel{u^ru_r})+P\doteq-(\rho+P)+P= \\
&= -\rho=-\bar\rho-\delta\rho\;\equiv\;\bar T^t{}_t-\delta\rho\,, && (20.42)\\
T_{tr} &= (\rho+P)\,u_tu_r=(\rho+P)\,g_{tt}g_{rr}u^tu^r\doteq(\bar\rho+\bar P)\,\bar g_{tt}\bar g_{rr}\bar u^t\delta u^r\doteq \\
&\doteq -(\bar\rho+\bar P)\,e^{2\bar\Phi}e^{2\bar\lambda}e^{-\bar\Phi}\,\xi_{,t}e^{-\bar\Phi}=-(\bar\rho+\bar P)\,e^{2\bar\lambda}\xi_{,t}\,, && (20.43)\\
T^r{}_r &= (\rho+P)\,\cancel{u^ru_r}+P\doteq P=\bar P+\delta P\;\equiv\;\bar T^r{}_r+\delta P\,. && (20.44)
\end{aligned}
$$

Plugging this to the Einstein equations $G_{\mu\nu}=8\pi T_{\mu\nu}$, one obtains three equations for the equilibrium state (from zeroth-order terms) and three perturbation equations (from the first-order terms). The equilibrium equations yield

$$
\begin{aligned}
\bar\lambda_{,r} &= \frac{1}{2r}\left(1-e^{2\bar\lambda}\right)+4\pi re^{2\bar\lambda}\bar\rho\,, && (20.45)\\
\bar\lambda_{,t} &= 0\,, && (20.46)\\
\bar\Phi_{,r} &= -\frac{1}{2r}\left(1-e^{2\bar\lambda}\right)+4\pi re^{2\bar\lambda}\bar P\,. && (20.47)
\end{aligned}
$$

The middle one is automatic and the remaining ones we of course know – these are the equations for mass and for potential, (20.5,20.6), just expressed in terms of $\lambda(r)$ instead of

$m(r)$. The perturbation equations read, in the same succession,

$$\begin{aligned}\frac{2}{r^2}\left(re^{-2\bar{\lambda}}\delta\lambda\right)_{,r} &= 8\pi\delta\rho\,,\\ \frac{2}{r}(\delta\lambda)_{,t} &= -8\pi(\bar{\rho}+\bar{P})\,e^{2\bar{\lambda}}\xi_{,t}\;,\\ -\frac{2\left[\delta\lambda+2r\bar{\Phi}_{,r}\delta\lambda-r\,\delta\Phi_{,r}\right]}{r^2e^{2\bar{\lambda}}} &= 8\pi\delta P\,. \end{aligned} \tag{20.48}$$

The perturbation $\delta\lambda$ can be obtained from the second one,

$$(\delta\lambda)_{,t} = \left[-4\pi r\,(\bar{\rho}+\bar{P})\,e^{2\bar{\lambda}}\xi\right]_{,t} \implies \delta\lambda = -4\pi r\,(\bar{\rho}+\bar{P})\,e^{2\bar{\lambda}}\xi = -(\bar{\lambda}+\bar{\Phi})_{,r}\xi\;, \tag{20.49}$$

where, first, we have chosen the free integration function of r zero, in order that $\delta\lambda$ vanish for $\xi=0$, and second, we have used the (sum of) equations (20.45) and (20.47),

$$\bar{\lambda}_{,r}+\bar{\Phi}_{,r} = 4\pi re^{2\bar{\lambda}}(\bar{\rho}+\bar{P})\,.$$

From the third perturbation equation, we derive

$$\delta\Phi_{,r} = \frac{1}{r}\left(1+2r\bar{\Phi}_{,r}\right)\delta\lambda+4\pi re^{2\bar{\lambda}}\delta P\,, \tag{20.50}$$

which should be further elaborated by submission of $\delta\lambda$ and δP. However, notice first that the "important parenthesis" which already accompanies us from Δn can be shortened by using the result (20.49), namely,

$$\left(\delta\lambda+\bar{\lambda}_{,r}\xi+\frac{2\xi}{r}+\xi_{,r}\right) = \left(-\bar{\Phi}_{,r}\xi+\frac{2\xi}{r}+\xi_{,r}\right). \tag{20.51}$$

Employing this and the equation for hydrostatic equilibrium $-\bar{P}_{,r} = (\bar{\rho}+\bar{P})\bar{\Phi}_{,r}$ in (20.37), we have

$$\begin{aligned}\delta P &= \bar{P}\Gamma_1\left(\bar{\Phi}_{,r}\xi-\frac{2\xi}{r}-\xi_{,r}\right)+(\bar{\rho}+\bar{P})\bar{\Phi}_{,r}\xi = \\ &= (\bar{\rho}+\bar{P}+\bar{P}\Gamma_1)\bar{\Phi}_{,r}\xi-\frac{1}{r}\bar{P}\Gamma_1(2\xi+r\xi_{,r})\,. \end{aligned} \tag{20.52}$$

Inserting now the first expression of $\delta\lambda$ from (20.49) into the first term of (20.50) and (20.52) into the second term of (20.50), we reach, after some shuffling,

$$\delta\Phi_{,r} = -4\pi e^{2\bar{\lambda}}\left[(\bar{\rho}+\bar{P}-\bar{P}\Gamma_1)(1+r\bar{\Phi}_{,r})\,\xi+\bar{P}\Gamma_1(3\xi+r\xi_{,r})\right]. \tag{20.53}$$

20.4.4 Dynamical equations for the displacement

Dynamical equations will be derived from the Euler equations of motion

$$(\rho+P)\,u_{\mu;\nu}u^{\nu} = -P_{,\nu}(\delta^{\nu}_{\mu}+u_{\mu}u^{\nu}).$$

For our four-velocity (20.31,20.32), the only non-trivial information is provided by their radial component; let us compute its left-hand and right-hand sides separately:

- On the left-hand side,

$$
\begin{aligned}
a_r &\equiv u_{r;\nu}u^\nu \equiv u_{r,\nu}u^\nu - \Gamma_{\rho\nu r}u^\rho u^\nu \equiv u_{r,\nu}u^\nu - \frac{1}{2}\left(g_{\rho\nu,r} + g_{r\rho,\nu} - g_{\nu r,\rho}\right)u^\rho u^\nu = \\
&= u_{r,\nu}u^\nu - \frac{1}{2}\,g_{\rho\nu,r}u^\rho u^\nu \doteq u_{r,t}\bar{u}^t - \frac{1}{2}\,g_{tt,r}(u^t)^2 = (g_{rr}u^r)_{,t}\bar{u}^t + \frac{1}{2}\left(e^{2\Phi}\right)_{,r}(u^t)^2 = \\
&= \left(e^{2\bar{\lambda}}\xi_{,t}e^{-\bar{\Phi}}\right)_{,t}e^{-\bar{\Phi}} + e^{2\Phi}\Phi_{,r}e^{-2\Phi} \doteq \xi_{,tt}\,e^{2\bar{\lambda}-2\bar{\Phi}} + \bar{\Phi}_{,r} + \delta\Phi_{,r}\ .
\end{aligned}
$$

- On the right-hand side,

$$
-P_{,\nu}(\delta^\nu_r + u_r u^\nu) = -P_{,r} - \cancel{P_{,t}u_r}u^t - P_{,r}\cancel{u_r u^r} \doteq -P_{,r} \doteq -\bar{P}_{,r} - \delta P_{,r}\ .
$$

Submitting the above to the Euler equation $(\rho + P)\,a_r = -P_{,\nu}(\delta^\nu_r + u_r u^\nu)$, we have

$$
(\bar{\rho} + \bar{P} + \delta\rho + \delta P)\left(\xi_{,tt}\,e^{2\bar{\lambda}-2\bar{\Phi}} + \bar{\Phi}_{,r} + \delta\Phi_{,r}\right) = -\bar{P}_{,r} - \delta P_{,r}\ .
$$

Now neglect the $O(\xi^2)$ terms $(\delta\rho + \delta P)\left(\xi_{,tt}\,e^{2\bar{\lambda}-2\bar{\Phi}} + \delta\Phi_{,r}\right)$ and subtract the equilibrium condition $(\bar{\rho} + \bar{P})\bar{\Phi}_{,r} = -\bar{P}_{,r}$. The equation thus assumes the form

$$
(\bar{\rho} + \bar{P})e^{2\bar{\lambda}-2\bar{\Phi}}\xi_{,tt} = -\delta P_{,r} - (\bar{\rho} + \bar{P})\delta\Phi_{,r} - (\delta\rho + \delta P)\bar{\Phi}_{,r}\ . \tag{20.54}
$$

Time for some ideal liquid.[1]

However, we need an equation containing, besides the equilibrium terms, only ξ. Sure, it is necessary to submit above δP, $\delta\Phi_{,r}$ and $\delta\rho$ from the "initial-value" equations (20.52), (20.53), (20.38) [with their "important parenthesis" rewritten according to (20.51)]. Equation (20.54) thus becomes less pretty, but – using the hydrostatic-equilibrium condition and the Einstein equations for $\bar{\lambda}_{,r}$ and $\bar{\Phi}_{,r}$ – it can still be arranged in several acceptable forms. One of them reads

$$
\left(\mathcal{P}\zeta_{,r}\right)_{,r} + \mathcal{Q}\zeta - \mathcal{W}\zeta_{,tt} = 0\,, \qquad \zeta := r^2 e^{-\bar{\Phi}}\xi\ , \tag{20.55}
$$

---

[1] Just one more remark. It may have seemed too careless to say, without a check, that the only non-trivial information is provided by radial component of the Euler equations. Let us compute the time component,

$$
\begin{aligned}
a_t \equiv u_{t;\nu}u^\nu \equiv u_{t,\nu}u^\nu - \Gamma_{\rho\nu t}u^\rho u^\nu &\equiv u_{t,\nu}u^\nu - \frac{1}{2}\left(g_{\rho\nu,t} + g_{t\rho,\nu} - g_{\nu t,\rho}\right)u^\rho u^\nu = \\
&= u_{t,\nu}u^\nu - \frac{1}{2}\,g_{\rho\nu,t}u^\rho u^\nu \doteq u_{t,t}\bar{u}^t + \bar{u}_{t,r}u^r - \frac{1}{2}\,g_{tt,t}(u^t)^2 = \\
&= -e^{\bar{\Phi}}\delta\Phi_{,t}e^{-\bar{\Phi}} - e^{\bar{\Phi}}\bar{\Phi}_{,r}e^{-\bar{\Phi}}\xi_{,t} + e^{2\Phi}\delta\Phi_{,t}e^{-2\Phi} = -\delta\Phi_{,t} - \bar{\Phi}_{,r}\xi_{,t} + \delta\Phi_{,t} = -\bar{\Phi}_{,r}\xi_{,t}\ .
\end{aligned}
$$

The right-hand side reduces to

$$
-P_{,\nu}(\delta^\nu_t + u_t u^\nu) = -P_{,t}(1-1) - P_{,r}u_t u^r = \bar{P}_{,r}e^{\bar{\Phi}}\xi_{,t}e^{-\bar{\Phi}} = \bar{P}_{,r}\xi_{,t}\ .
$$

Therefore, the time component of the Euler equations reads $(\bar{\rho}+\bar{P})\bar{\Phi}_{,r}\xi_{,t} = -\bar{P}_{,r}\xi_{,t}$, which holds automatically thanks to the unperturbed hydrostatic-equilibrium condition.

with "coefficients"

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}(r) := \bar{P}\Gamma_1 r^{-2} e^{\bar{\lambda}+3\bar{\Phi}}\,, \\
\mathcal{Q} &= \mathcal{Q}(r) := (\bar{\rho}+\bar{P})\, r^{-3} e^{\bar{\lambda}+3\bar{\Phi}} \left[r(\bar{\Phi}_{,r})^2 + 4\bar{\Phi}_{,r} - 8\pi r \bar{P} e^{2\bar{\lambda}} \right], \\
\mathcal{W} &= \mathcal{W}(r) := (\bar{\rho}+\bar{P})\, r^{-2} e^{3\bar{\lambda}+\bar{\Phi}}\,.
\end{aligned}$$

The main advantage of the new variable ζ is that it permits to write very concisely the "important parenthesis" of equations (20.36), (20.37) and (20.38),

$$\left(\delta\lambda + \bar{\lambda}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right) = \left(-\bar{\Phi}_{,r}\xi + \frac{2\xi}{r} + \xi_{,r}\right) = r^{-2} e^{\bar{\Phi}} \zeta_{,r}\,.$$

Boundary conditions: physically acceptable can only be solutions satisfying

$$\delta\rho < \infty,\ \delta P < \infty \quad \text{for}\ \ r \to 0 \quad \Longleftrightarrow \quad \left| \lim_{r\to 0} \frac{\xi}{r} \right| < \infty\,, \tag{20.56}$$

$$\Delta P = 0 \quad \text{for}\ \ r = R + \xi(t,R)\ \ \text{(surface)} \quad \Longleftrightarrow \quad \lim_{r\to R+\xi} \left(\bar{P}\Gamma_1 r^{-2} e^{\bar{\Phi}} \zeta_{,r}\right) \to 0\,. \tag{20.57}$$

The first two clearly follow from equations (20.37) and (20.38) by realizing that the only term dangerous at the very centre is ξ/r in the "important parenthesis", and the comoving (Lagrangian) condition for vanishing of P on the surface follows from (20.37) as well. Let us add that one can everywhere take just R instead of $R+\xi$ to specify the surface, because the quantities in question are themselves small, so the error thus made is $O(\xi^2)$.

20.4.5 Separated solution and the Sturm-Liouville problem

Let us look for the solution to (20.55) in the separated harmonic-oscillation form $\zeta(t,r) = \zeta(r)e^{-\mathrm{i}\omega t}$, with ω a constant representing frequency. One immediately obtains an equation for $\zeta(r)$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{P}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right) + \mathcal{Q}\,\zeta + \omega^2 \mathcal{W}\zeta = 0\,, \tag{20.58}$$

the so-called **Sturm-Liouville equation**; it often follows by solution of linear partial differential equations by separation of variables. (Actually, *all* second-order linear ordinary differential equations can be reduced to the Sturm-Liouville form.) It is assumed that within the applicable domain of r ($0 \leqslant r \leqslant R$ in our case) the three coefficient functions are real and sufficiently differentiable and that they satisfy $\mathcal{P} > 0$ and $\mathcal{W} > 0$ (which is clearly fulfilled in our case). The main aspect of the Sturm-Liouville problem is to find such values of ω for which the equation has a real solution $\zeta(r)$.

The search for the eigen-frequencies of the problem starts as a variational problem for the lowest, fundamental frequency. Multiplying by ζ and integrating over the stellar radius, one has[2]

$$\omega^2 = \frac{-\int\limits_0^R \zeta\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{P}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)\mathrm{d}r - \int\limits_0^R \mathcal{Q}\,\zeta^2\,\mathrm{d}r}{\int\limits_0^R \mathcal{W}\zeta^2\,\mathrm{d}r}\,.$$

The first term can be integrated per partes,

$$\int\limits_0^R \zeta\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{P}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)\mathrm{d}r = \cancel{\left[\zeta\mathcal{P}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right]_0^R} - \int\limits_0^R \mathcal{P}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)^2\mathrm{d}r\,,$$

where one can forget about the boundary term, because i) by condition (20.56), ξ must vanish at $r=0$ at least as r, hence ζ must vanish there at least as r^3 and thus $\left[\zeta\mathcal{P}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right] = \bar{P}\Gamma_1 e^{\bar{\lambda}+3\bar{\Phi}}\frac{\zeta}{r^2}\frac{\mathrm{d}\zeta}{\mathrm{d}r}$ at least as r^3 as well; ii) by condition (20.57), the same expression vanishes on the surface. In such a way, equation (20.58) implies

$$\omega^2 = \frac{\int\limits_0^R \left[\mathcal{P}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right)^2 - \mathcal{Q}\,\zeta^2\right]\mathrm{d}r}{\int\limits_0^R \mathcal{W}\zeta^2\,\mathrm{d}r}\,. \tag{20.59}$$

The fundamental frequency of stellar oscillations is obtained by minimization of this functional (which may physically be interpreted as the potential energy of the perturbation). Since the bottom integral is always positive, the crucial issue is the sign of the top integral. There are two alternatives:

- The minimal ω^2 is positive, so ω is real. In such a case, the star can oscillate, with this basic frequency, in a stable manner. Not only that. As every musical instrument, it also oscillates at higher frequencies called overtones (harmonic partials). The Sturm-Liouville (and Fourier) theory says that there in fact exists a whole *discrete infinite sequence* of eigen-frequencies, $(\omega_{\rm min})^2 \equiv (\omega_0)^2 < (\omega_1)^2 < (\omega_2)^2 < \ldots (\to \infty)$. The corresponding eigen-functions $\zeta_n(r)$ form, on the radial interval $\langle 0, R\rangle$, a complete system of functions, so any function satisfying the boundary conditions can be "decomposed into normal modes" there, $\zeta(t,r) = \sum \zeta_n(r)e^{-\mathrm{i}\omega_n t}$. The eigen-functions $\zeta_n(r)$ have exactly n nodes (zeros) within the range $(0,R)$.

- The minimal ω^2 is negative, so ω is pure imaginary. The corresponding solution then reads $\zeta(t,r)=\zeta(r)e^{\mathrm{Im}(\omega)t}$, so if $\mathrm{Im}(\omega)>0$, the displacement grows exponentially and the star is unstable, whereas if $\mathrm{Im}(\omega)<0$, the star is "damped" and cannot be excited to any musical performance. Astrophysically, these two options are not so interesting, because they do not imply any observational consequences: the damped stars do not provide any signal, and had there been any unstable ones, they are gone.

[2] Note that we can again take just R (instead of $R+\xi$) for the surface, because the integrands are $O(\xi^2)$.

Remark on string equation

So we have seen a star may behave like a musical instrument, hosting the whole infinite sequence of oscillation modes (fundamental plus "overtones"). Such a behaviour is best known from string – and indeed the string equation *is* of the same type. Specifically, small (both transversal and longitudinal) vibrations of a string can be described by equation (20.55), $(\mathcal{P}\zeta_{,r})_{,r} + \mathcal{Q}\zeta - \mathcal{W}\zeta_{,tt} = 0$, with $\mathcal{P}$ representing the string tension, $\mathcal{Q} = 0$ and $\mathcal{W}$ representing linear mass density of the string. (Gravity and other external forces acting upon the string are neglected.) For constant tension (perfectly elastic string) and constant density, obviously the one-dimensional wave equation remains, $\zeta_{,rr} - \frac{\mathcal{W}}{\mathcal{P}}\,\zeta_{,tt} = 0$.

CHAPTER 21

Degenerate fermion gas and final stages of stellar evolution

Star is a cosmic body in which, at least for a certain period, thermonuclear reactions run spontaneously. In these reactions heavier nuclei are being synthesized from lighter ones, with the strong-force potential energy of nucleons gradually converted into photons and neutrinos. The neutrinos directly escape from the star, whereas photons diffuse to the surface for $\sim 10^{4\div5}\texttt{years}$, undergoing scattering every $\lesssim 1\texttt{cm}$. In dependence on how high a temperature can be achieved in the stellar interior (which mainly depends on mass of the star), the fusion may proceed – as a source of energy – up to iron and nickel where the binding energy per nucleon reaches its minimum (it is maximally negative). The era of thermonuclear reactions is the longest part of stellar life; in astronomy, it is then said to be "on the main sequence" in the Hertzsprung-Russel diagram which shows luminosity against spectral temperature. The thermonuclear life of stars typically ends by rather violent processes in which the centre (where the reactions have ceased) contracts into a very dense object, while the outer layers expand or even explode into the interstellar space.

Hence, stars are kind of natural thermonuclear reactors which are driven by their own gravity. Gravitation puts the star together and through compression generates a sufficient temperature to ignite the reactions, then it also regulates the reaction rate (if the energy release decreases, gravitation overwhelms pressure and squeezes the star, which leads to the rise of temperature and thus acceleration of the reactions, which in turn rises the pressure and leads to expansion, thus making work and inducing temperature falloff which slows down the reactions, etc.), yet finally let the star explode anyway. From the GR point of view, interesting are the compact remnants which are left after the central regions of stars have depleted their thermonuclear resources and, consequently, underwent a substantial contraction. These remnants – **white dwarfs or neutron stars (if not black holes)** – are very dense and generate very strong and non-homogeneous gravitational field. In this chapter, we abstract from all the complex physics driving the stellar evolution, only focusing on these compact remnants, specifically on how and to what extent they can resist their own gravity. Yet even the complex physics of the compact object we largely ignore, only focusing on the role of **degenerate**

fermion gas in the support of their equilibria. (For more details, see e.g. [42, 4].)[1]

The fermion gas is called degenerate if the kinetic energy of its particles is dominated by the Fermi energy due to the Pauli exclusion principle rather than by thermal energy kT. Degeneration naturally appears in extremely dense astrophysical objects resulting as final evolutionary states of stars, because there the fermions (electrons and nucleons) are forced, by gravity, to assume very close positions in the configurations space – and thus to differentiate in the momentum direction accordingly. In the state of degeneracy, the properties of the gas only little depend on temperature, in particular, its pressure remains non-zero even at absolute zero temperature.

Basic parameters, notation

As in the previous chapter, we denote by A the total number of baryons (practically, of nucleons), by Z the number of protons (thus of electrons as well), and by N the number of degenerate fermions; these will be either electrons ($N = Z$), or neutrons ($N = A - Z$). In both cases, it often would not bring a large error to consider all these three numbers the same. However, we will try to distinguish them, if only for the reason that it is interesting to realize how they change during the life of a star (only A remains the same).

For early stars, the particle composition corresponded to abundances of elements provided by primordial big-bang nucleosynthesis, i.e. 75% of hydrogen and almost 25% of helium ($^4_2$He), plus very small amount of deuterium and $^3_2$He, plus negligible amount of lithium. Today (in fact in the corresponding past), thanks to the higher elements produced by stars, the abundances of baryon matter estimated in our Galaxy are 74% of hydrogen, 24% of helium, 1% of oxygen, 0.5% of carbon, 0.1% of neon, iron and nitrogen, etc. The above are *mass* fractions, so 75% of $^1_1$H and 25% of $^4_2$He correspond to 12 nuclei of hydrogen to 1 nucleus of helium, hence 7 protons to 1 neutron. Starting from $^4_2$He up to $^{56}_{26}$Fe, the fraction of protons to neutrons in the nuclei is very close to unity, so if the star ends as a white dwarf, it contains quite the same number of protons and neutrons, irrespectively of whether it has evolved up to helium, to carbon and oxygen, to oxygen + neon + magnesium, or even to a trace of iron. If the star ends as a neutron star, then, after experiencing wild changes of the proton/neutron ratio during the final collapse, the ratio (determined by equilibrium between the direct and inverse β decay) freezes at about 1/8. In short, evolution from the primordial matter to white dwarf and to neutron star means, respectively,

$$\frac{\text{protons}}{\text{neutrons}} \equiv \frac{Z}{A-Z} : \qquad \frac{7}{1} \to \frac{1}{1} \;\; (\text{white dwarf}) \, , \qquad \frac{7}{1} \to \frac{1}{8} \;\; (\text{neutron star}) \, .$$

However, the most important ratio will be that between the number of degenerate fermions and the number of baryons – this we will need to convert the quantities related to degenerate fermions and those related to baryons. The degenerate fermions being either

[1] May seem strange to speak of *gas* if its density may reach $10^{15}\,\mathtt{g/cm^3}$... The degenerate fermions are being described as **ideal fluid** (see section 21.2.3). It is called **gas** since it tends to fill all the available space (it can be squeezed while it can also expand to fill "larger space than before"). A **liquid**, on the contrary, does not adjust to the size of its container.

electrons or neutrons, the following two ratios are being employed

$$Y_{\rm e} := \frac{Z}{A} \quad \ldots \quad \frac{7}{8} \to \frac{1}{2}\,\text{(white dwarf)} \quad \left[\to \frac{1}{9}\,\text{(neutron star)}\right], \tag{21.1}$$

$$Y_{\rm n} := \frac{A-Z}{A} \quad \ldots \quad \frac{1}{8} \quad \left[\to \frac{1}{2}\,\text{(white dwarf)}\right] \quad \to \frac{8}{9}\,\text{(neutron star)}\,. \tag{21.2}$$

By definition, the above ratios satisfy $Y_{\rm e} + Y_{\rm n} = 1$ at any moment (for electrically neutral matter, of course), irrespectively of any possible degeneracy.

The conversion is mainly important in relating number density to mass density. Namely, below we will call n the proper number density *of degenerated fermions*, so the proper number density of baryons has to be written as n/Y_*; in particular, the relation $\rho = \bar{m}_{\rm b}n + \epsilon$ for total mass density has to be modified to $\rho = \bar{m}_{\rm b}n/Y_* + \epsilon$. Actually, it is not necessary to take into account electrons, since their mass contribution is just tiny. (In spite of this, the electron *pressure* can be crucial, as we will see below.)

21.1 Gravitational and thermal energy. Virial theorem

In this section we derive a relation between the total gravitational binding energy and kinetic energy of a fluid system, which is often very helpful in astrophysics – the **virial theorem**.

- Consider the classical hydrostatic-equilibrium condition

$$\frac{G\rho m(r)}{r^2} = -\frac{{\rm d}P}{{\rm d}r}\,.$$

Multiplying by r and integrating over the star (in a Euclidean way), we have

$$\int_0^R \frac{G\rho m(r)}{r}\,4\pi r^2\,{\rm d}r = -\int_0^R r\,\frac{{\rm d}P}{{\rm d}r}\,4\pi r^2\,{\rm d}r \stackrel{\rm p.p.}{=} -\cancel{\left[4\pi r^3 P\right]_0^R} + 3\int_0^R 4\pi r^2 P\,{\rm d}r\,, \tag{21.3}$$

where the boundary term vanishes for obvious reasons. On the left-hand side, we have (minus) total gravitational potential energy of the star ($=: A\bar{U}_{\rm g}$, with $\bar{U}_{\rm g}$ denoting its mean amount per baryon), while the right-hand side can be written in terms of a mean pressure $\bar{P}$ as $4\pi R^3\bar{P}$. Since the star volume can be expressed as $A/\bar{n}$, with $\bar{n}$ the mean number density of baryons (*baryons* indeed in this section, not degenerate fermions), we rewrite

$$4\pi R^3\bar{P} = \frac{3A}{\bar{n}}\bar{P}\,,$$

and hence (21.3) yields the "equation of state"

$$-\bar{U}_{\rm g} = \frac{3\bar{P}}{\bar{n}}\,.$$

- In Section 21.3, we will derive a significant relationship which gives the macroscopic gas pressure as a momentum integral over microscopic properties of its particles (momentum p, velocity v and momentum distribution function $n(p)$),

$$P = \frac{1}{3}\int_0^\infty p\, v\, n(p)\,\mathrm{d}p \quad \Longrightarrow \quad \bar{P} = \frac{1}{3}\,\overline{pvn}\,, \qquad \text{so, from above,} \quad -\bar{U}_\mathrm{g} = \overline{pv}$$

(here the mean values have been taken over the momentum distribution, yet one assumes they are not much different from the above, volume-averaged ones). For a non-relativistic gas, $v = p/m$, hence $\overline{pv} = \bar{p}p/m = 2\bar{E}_\mathrm{kin}$, while for an ultra-relativistic gas, $\bar{v} = c$, hence $\overline{pv} = \bar{p}c = \bar{E}_\mathrm{kin}$, and so we obtain

$$\bar{P} = \begin{cases} \frac{2}{3}\,\bar{n}\bar{E}_\mathrm{kin} & \text{... for a non-relativistic gas} \\ \frac{1}{3}\,\bar{n}\bar{E}_\mathrm{kin} & \text{... for an ultra-relativistic gas} \end{cases}. \tag{21.4}$$

Comparison of the above two results yields

$$\boxed{-\bar{U}_\mathrm{g} = 2\bar{E}_\mathrm{kin} \quad \text{for non-relativistic gas}\,, \quad -\bar{U}_\mathrm{g} = \bar{E}_\mathrm{kin} \quad \text{for ultra-relativistic gas}}\,. \tag{21.5}$$

This quite generic observation can be used at many places, yet let us only employ it here for a simple derivation of how the mean thermal kinetic energy of particles $\bar{E}_\mathrm{kin}$ depends on their number density. We simply find it for the corresponding binding energy $\bar{U}_\mathrm{g}$ instead: substituting in it

$$m(r) = \frac{4}{3}\pi r^3\bar{\rho}\,, \qquad M = \frac{4}{3}\pi R^3\bar{\rho} \quad \Rightarrow \quad R = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3},$$

we have

$$\begin{aligned} -\bar{U}_\mathrm{g} &= \frac{1}{A}\int_0^R \frac{G\rho m(r)}{r}\,4\pi r^2\,\mathrm{d}r = \frac{4\pi G}{A}\int_0^R \rho\, m\, r\,\mathrm{d}r = \frac{16\pi^2 G}{3A}\bar{\rho}^2\int_0^R r^4\,\mathrm{d}r = \frac{16\pi^2 G}{15A}\bar{\rho}^2 R^5 = \\ &= \frac{16\pi^2 G}{15A}\bar{\rho}^2\left(\frac{3M}{4\pi\bar{\rho}}\right)^{5/3} = \frac{(36\pi)^{1/3}}{5A}G\bar{\rho}^{1/3}M^{5/3} \simeq \frac{(36\pi)^{1/3}}{5A}G(\bar{m}_\mathrm{b}\bar{n})^{1/3}(A\bar{m}_\mathrm{b})^{5/3} = \\ &= \frac{(36\pi)^{1/3}}{5}GA^{2/3}\bar{m}_\mathrm{b}^2\bar{n}^{1/3} \doteq GA^{2/3}\bar{m}_\mathrm{b}^2\bar{n}^{1/3}. \end{aligned} \tag{21.6}$$

Therefore, the thermal kinetic energy of particles typically goes with density as $\bar{n}^{1/3}$. Remember that $\bar{n}$ denoted the *baryon* number density in this section, so if one needed to express the result in terms of the density of some specific fermions (electrons or neutrons in our case), as e.g. when comparing it with the other energy formulas given in terms of the electron or neutron density in the following sections, one would have to take n/Y_* instead of $\bar{n}$, with Y_* representing the respective abundance ratio (Y_e= electrons over baryons, Y_n= neutrons over baryons).

21.2 Pauli principle, Fermi momentum, and degeneracy

In 2020 COVID period, figures contrasting the behaviour of fermions and bosons in a potential well appeared, with captions like "keeping safe distance" [fermions] vs. "bunch of degenerates" [bosons]. Actually, **Pauli exclusion principle** says that in a single cell of the phase space, with $h^3 = (2\pi\hbar)^3$ volume, at most $(2s+1)$ fermions with spin s can live, $(2s+1)$ being the number of possible different projections of the spin to some fixed direction. If fermions are pushed together in space, they differentiate in momentum direction, populating all possible levels up to the so-called **Fermi momentum** p_F (we neglect *thermal* momentum now). From the total number of elementary cells in such a phase space (where, in the fundamental state, all momentum levels are occupied up to p_F, without vacancies), $(V\,\frac{4}{3}\pi p_\mathrm{F}^3)/(2\pi\hbar)^3$, it is easy to derive the number density of fermions $n \equiv N/V$ (N denoting the total number of the given type of fermions) and, vice versa, to express p_F in terms of the density:

$$n = \frac{N}{V} = (2s+1)\,\frac{\frac{4}{3}\pi p_\mathrm{F}^3}{(2\pi\hbar)^3} = \frac{2s+1}{6\pi^2}\,\frac{p_\mathrm{F}^3}{\hbar^3} \qquad \Longrightarrow \qquad \boxed{p_\mathrm{F} = \left(\frac{6\pi^2 n}{2s+1}\right)^{1/3}\hbar}\,. \tag{21.7}$$

For $s = 1/2$ ($\Rightarrow\ 2s+1 = 2$) which holds for all stable fermions – electrons, protons and neutrons (neutrinos as well), the Fermi momentum reads $p_\mathrm{F} = (3\pi^2 n)^{1/3}\hbar$. Note that the relation can also be obtained, without precise numerical factors, from the uncertainty relations, $p_\mathrm{F} \sim \hbar/\Delta x \sim \hbar n^{1/3}$. Similarly, regarding that the particle size is well approximated by its de Broglie wavelength h/p (i.e. by the length of the corresponding wave packet), we get, by comparing $\frac{h}{p} \sim \frac{1}{n^{1/3}}$, that $p \sim h n^{1/3}$.

If the momentum the particles have due to temperature, $p^2/(2m) \sim kT$, does not provide sufficient volume $\frac{4}{3}\pi p^3$ for their momentum differentiation, at least some of the fermions have to assume larger momentum than it would correspond to the local temperature. In such a situation the fermions start to become degenerate ($p_\mathrm{F}^2/2m \gtrsim kT$). The statement can also be voiced in terms of energy, with the energy corresponding to the Fermi momentum given by the usual special-relativity formula

$$E_\mathrm{F} = c\sqrt{m^2c^2 + p_\mathrm{F}^2} \qquad \begin{cases} \rightarrow \quad \frac{p_\mathrm{F}^2}{2m} = \frac{\hbar^2}{2m}\,(3\pi^2 n)^{2/3} & \ldots \text{ in non-relativistic limit} \\ \rightarrow \quad p_\mathrm{F} c = \hbar c\,(3\pi^2 n)^{1/3} & \ldots \text{ in ultra-relativistic limit} \end{cases}. \tag{21.8}$$

(Note that in the non-relativistic limit the rest energy mc^2 is omitted.) Naturally, the degree of degeneracy typically grows with density, and the non-relativistic relation (the $1/m$ dependence) also implies that, in compression, lighter fermions (electrons) degenerate earlier than more massive ones (nucleons).

Good to check *what* actually is the value of temperature when the gas becomes degenerate, and how much degenerate the fermions in white dwarfs and neutron stars are. We will use the fact that mean energy of non-relativistic degenerate fermions is $(3/5)E_\mathrm{F}$ [for ultra-relativistic fermions it is $(3/4)E_\mathrm{F}$] – cf. Section 21.4.

- For a non-relativistic electron gas, one compares $kT \sim \frac{3}{5}E_\mathrm{F} = \frac{3\hbar^2}{10m_\mathrm{e}}(3\pi^2 n_\mathrm{e})^{2/3}$ and expresses $n_\mathrm{e} = \frac{\rho Y_\mathrm{e}}{\bar{m}_\mathrm{b}}$, which yields, after evaluating the constants and $Y_\mathrm{e} = 1/2$,

$$\frac{T}{10^9\,\mathtt{K}} \sim \left(\frac{\rho}{10^6\,\mathtt{g/cm^3}}\right)^{2/3}. \tag{21.9}$$

One sees that inside the white dwarf (where $\rho \gtrsim 10^6\,\mathtt{g/cm^3}$), the Fermi energy is dominant up to temperatures above $5\cdot 10^9\,\mathtt{K}$. Such temperatures possibly may occur in the cores of white dwarfs just born, but typical central temperatures are $10^{6\div 7}\,\mathtt{K}$.

- For a non-relativistic neutron gas, one compares analogously $kT \sim \frac{3}{5}E_{\rm F} = \frac{3\hbar^2}{10m_{\rm n}}(3\pi^2 n_{\rm n})^{2/3}$, so, expressing $n_{\rm n} = \frac{\rho Y_{\rm n}}{\bar{m}_{\rm b}}$ and $Y_{\rm n} = 8/9$, one obtains

$$\frac{T}{10^{12}\,\mathtt{K}} \sim \left(\frac{\rho}{10^{15}\,\mathtt{g/cm^3}}\right)^{2/3}. \tag{21.10}$$

  At the typical density $\rho \sim 10^{14}\mathtt{g/cm^3}$, the dominance of degenerate neutrons could only be compromised by temperatures $T \gtrsim 2\cdot 10^{11}\,\mathtt{K}$; such may only occur, rather temporarily, in the core of a neutron star just born.

In Section 20.1 we noticed, from the uncertainty relations, that if a particle is confined in a cell of the order of its reduced Compton wavelength, its momentum reaches mc, so the particle becomes relativistic. Now when we have a formula for $p_{\rm F}$, we may check whether this corresponds to $p_{\rm F} = mc$:

$$(3\pi^2 n)^{1/3}\hbar = mc \qquad \Longleftrightarrow \qquad n = \frac{1}{3\pi^2}\frac{m^3c^3}{\hbar^3} = \frac{1}{3\pi^2}\frac{1}{\lambdabar_{\rm C}^3} = \frac{8\pi}{3\lambda_{\rm C}^3} =: n_0\,, \tag{21.11}$$

where $\lambda_{\rm C} = \frac{h}{mc} \equiv 2\pi\lambdabar_{\rm C}$ is the Compton wavelength of the given type of fermions.

21.2.1 Degeneracy occasions in stellar interiors

Let us stress again that the system of degenerate fermions behaves very differently from the "normal", non-degenerate system, namely, its properties only weakly depend on temperature. In astrophysical bodies, degeneracy mainly means that the system cannot cool efficiently (its particles cannot lose their kinetic energy, because the latter is kept on its level by Pauli). And, vice versa, if a system is being heated, its pressure does not grow, at least while the kinetic energy of its particles is dominated by Fermi energy, so a given region does *not* expand (and thus does not cool). In passing, the state of degeneracy does *not* only occur in white dwarfs and neutron stars. In some crystalline solids (metals), for instance, electrons are at least partially degenerate. In stars, degeneracy may also occur when a certain thermonuclear session is completed in a central region. If temperature is not enough for igniting a successive level of the thermonuclear chain, the core made of "ashes" of the preceding reaction level contracts, which may lead to at least partial degeneracy of electrons.

The most famous peripeteia of the above type concerns the end of the first thermonuclear stage, when hydrogen has been turned into helium in the stellar core. The cores of the lowest-mass stars just stop at that stage, slowly contracting to a white dwarf, whereas high-mass stars can provide enough temperature to right ignite the following, helium → carbon thermonuclear stage. In moderate-mass stars ($M \lesssim 2M_\odot$), the helium ash is not ignited immediately, they rather gradually condense towards the centre, down to a density when its electrons become more or less degenerate. Yet when the helium core grows to about $0.45M_\odot$, its central density reaches some $10^{4\div5}\mathrm{g/cm^3}$ and the kinetic energy of its electrons reaches the equivalent of $10^8\mathrm{K}$, enough to start the helium → carbon burning. Thus generated energy is supplied to the stellar core which however is chiefly supported by degenerate-electron pressure (like in white dwarfs), so the energy goes to heating of the other particles, without any significant change of pressure (and thus without subsequent expansion accompanied by cooling). Since the 3α process which leads to the synthesis of carbon is extremely sensitive to temperature (its rate is proportional to the 40th power of temperature!), the helium burning quickly turns into a runaway process called **helium flash** in which all the helium turn to carbon in a few seconds, generating peak power of some $10^{8\div10}L_\odot$, i.e. comparable to a supernova. Still basically none of that is enjoyed by an external observer: all the energy is consumed to making the core non-degenerate again (dominated by thermal pressure) and, subsequently, expanding back to roughly the original size.

Still more violent is an analogous process which may happen "one level higher" on a thermonuclear ladder – the **carbon detonation**. It consists in an ignition and runaway burning of carbon core in a carbon-oxygen white dwarf which has previously been slowly cooling, but whose central temperature rose above the critical temperature due to an accretion of material (typically from a binary companion). In seconds, the carbon (and oxygen) fusion generates enough energy for the white dwarf to explode as the **type Ia supernova**. Since the critical temperature is tightly bound to a certain critical mass (of about $1.44M_\odot$, fairly close to the Chandrasekhar limit mentioned at the end of this chapter), the type-Ia supernovas are believed to produce very uniform peak luminosity (of about $4\cdot10^9L_\odot$) and are thus used as one of astronomical "standard candles". (This property can e.g. be employed to infer the time profile of cosmic expansion from the statistics of the Ia-supernova distance-redshift relations. In 1998, such a line of research culminated in the discovery that the cosmic expansion accelerates rather than decelerates – see Chapter 13.)

21.2.2 Intermezzo: the 3α-process story

At the beginning of 1950s, it was a mystery how any carbon can exist in the Universe, because no process of non-negligible yield was known that might produce it in stars. Even worse, carbon is also a key to the synthesis of heavier elements, so without such a process, all stars would have to end at helium (beryllium is unstable). A propos, beryllium: it originates from two heliums (2α) and decays back in $8\cdot10^{-17}\mathrm{s}$, only if a third helium fuses with it by that time, it might form "kind of carbon" (A and Z are correct). Yet the probability of such a process – **the** 3α **process** – is extremely low, since the fusion of the third α particle does not yield carbon in any of the then known energy states (including resonant states). In 1953, F. Hoyle, the founder of the modern theory of stellar nucleosynthesis, took it the other way round – there *is* carbon, and there appears to be no other way how to produce it than out of three heliums in stars; so there *must* exist a special resonant state of the carbon nucleus, through which the stable, base carbon arises. He predicted the values of the energy, of the nuclear spin and of the parity of that state. The state was there, at 7.656 MeV. Almost always decaying back to three alpha particles, but once in about 2421 times "cooling" into a stable carbon. The road to heavier elements was open.

21.2.3 Is the stellar Fermi gas ideal? –Coulomb and nuclear interactions

Before embarking on equations of state for the degenerate gas, we should check whether the gas which specifically exists in compact astrophysical objects is ideal, that is, non-interacting otherwise than "mechanically" (thus generating pressure), on the basis of kinetic energy of its particles. Such a check consists in comparing the typical kinetic energy (the Fermi energy in the case of the degenerate gas) with the energy of relevant interaction acting between the particles.

- In the case of the degenerate electron gas, the strongest interaction is the electromagnetic one of the electrons with nuclei and with other electrons. Both can be expected to have similar effect, since the interaction with nuclei containing Z protons is Z times stronger than the electron-electron interaction, but such nuclei are Z times rarer at the same time. For the electron-nucleus electrostatic (Coulomb) contribution, one has

$$E_{\rm C} \sim \frac{-Ze^2}{\text{typical distance}} \sim \frac{-Ze^2}{\left(\frac{Z}{n}\right)^{1/3}} = -Z^{2/3}e^2n^{1/3}\,, \tag{21.12}$$

where we have expressed the typical distance between the nuclei as (volume per nucleus)$^{1/3}$ = $(Z/n)^{1/3}$, n being number density of electrons (thus also of protons). A more thorough calculation also including the electron-electron contribution leads to almost the same result,

$$E_{\rm C} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3}e^2n^{1/3} \doteq -1.45\cdot Z^{2/3}e^2n^{1/3}\,. \tag{21.13}$$

It grows with $n^{1/3}$, whereas the Fermi energy grows with $n^{2/3}$ in the non-relativistic regime, so, in this regime, the gas is the more ideal the higher is the density (!). In the ultra-relativistic regime, both the energies grow with $n^{1/3}$ (recall that $E_{\rm F} = \hbar c\,(3\pi^2 n)^{1/3}$ then), so it is necessary to compare the coefficients:

$$|E_{\rm C}| \simeq E_{\rm F} \quad\Longleftrightarrow\quad Z^{2/3} \simeq \frac{\frac{10}{9}\left(\frac{9\pi}{4}\right)^{1/3}}{\frac{e^2}{\hbar c}} \doteq \frac{2.13}{\alpha} \doteq 292.2 \quad\Longleftrightarrow\quad Z \simeq 4995\,.$$

Z of the nuclei really present in white dwarfs is 1000 times lower, so $E_{\rm F}$ also dominates in the ultra-relativistic limit.

- In the case of the degenerate neutron gas, the strongest interaction is the strong nuclear force. It acts attractively between the nucleons from about $3\cdot 10^{-13}\texttt{cm} \equiv 3\,\texttt{Fermi}$ of distance, peaked (or "bottomed") at about $1\,\texttt{Fermi}$, and below some $0.8\,\texttt{Fermi}$ sharply switching strongly repulsive. (Note that the radius of nucleons is about $0.8\,\texttt{Fermi}$ and that in normal situation corresponding to "nuclear" density $\rho \simeq 2.3\cdot 10^{14}\texttt{g/cm}^3$ their centres are about $1.9\,\texttt{Fermi}$ from each other. A classical "density of a nucleon" amounts to some $6\cdot 10^{14}\texttt{g/cm}^3$.) The nuclear binding energy per nucleon is about $8\,\texttt{MeV}$, while the Fermi kinetic energy, as given by the non-relativistic formula, is about $50\,\texttt{MeV}$ at nuclear density. The degenerate gas of non-interacting neutrons dominates the pressure within the density range $5 \cdot 10^{13}\mathtt{g/cm^3} \lesssim \rho \lesssim 10^{15}\mathtt{g/cm^3}$; the above numbers indicate that at several times the nuclear density (i.e. around $10^{15}\mathtt{g/cm^3}$), the (repulsive) strong force becomes significant and the situation turns very uncertain – also because nucleons "touch each other" at such a density, so the image of a gas of particles itself ceases to have good sense.

21.3 The pressure integral

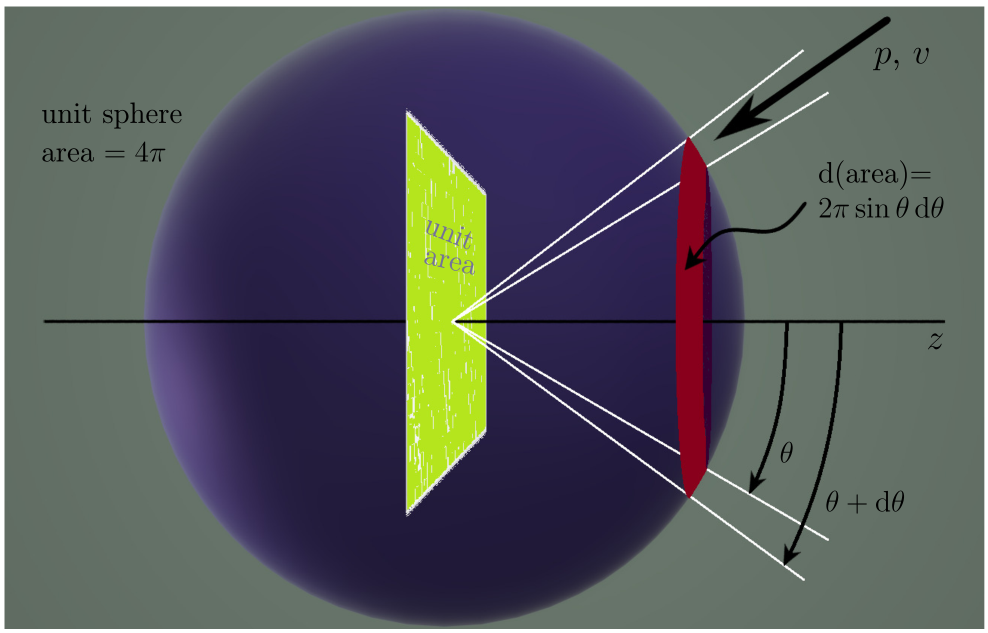

Figure 21.1 *An outline of parameterization of the pressure-integral computation. Computed is the momentum which the particles arriving from one half-space pass on, in a unit time, to a unit-area panel through an elastic bounce. Total number of incoming particles is determined, under the assumption of isotropy, from the fraction which the $(\theta, \theta + \mathrm{d}\theta)$ interval accounts for within the 4π of all possible directions.*

In order to find the desired equations of state, we will first derive the **pressure integral**, a generic relation which evaluates the pressure from particles' momentum, velocity and distribution function. We will simply start from definition: pressure is the force with which the gas pushes a unit-area panel (from one side), as a result of particles' hitting the panel and transferring to it their momentum. So, have a generic ensemble of free particles and immerse such a panel in it in an arbitrary way (assuming that the pressure is isotropic). Denote the Cartesian axis normal to the panel as z and the spherical-type angle measuring deviation from this axis as θ (Figure 21.1). Denote by $n(p, \theta)$ the number density of particles which move with momentum magnitude p in the direction θ, thus with $n(p, \theta)\mathrm{d}p\mathrm{d}\theta$ counting the number density of particles whose momentum is in the $(p, p + \mathrm{d}p)$ interval and which are arriving in directions lying between the cones $(\theta, \theta + \mathrm{d}\theta)$.

Then, the number of such particles hitting the panel from one half-space in unit time is given by the corresponding flux $v^z n(p,\theta)\mathrm{d}p\mathrm{d}\theta$, where, naturally, $v^z = v\cos\theta$. In an elastic reflection, each particle imparts to the panel the momentum $2p^z = 2p\cos\theta\,(>0)$ (because, in the elastic case, p^z of the particle turns to $-p^z$). Hence,

$$P = \int_0^\infty \int_0^{\pi/2} 2p\cos\theta\, v\cos\theta\; n(p,\theta)\; \mathrm{d}\theta\mathrm{d}p\,.$$

Simple, yet crucial observation: the momentum distribution being (assumed) isotropic, the particles are coming uniformly from all directions, hence the number of particles having momentum in $(p, p+\mathrm{d}p)$ and approaching within $(\theta, \theta+\mathrm{d}\theta)$ is, to the *total* number of particles which have momentum in $(p, p+\mathrm{d}p)$ (and arbitrary direction), in the same proportion as the area of the spherical annulus delimited on the unit sphere by the cones $(\theta, \theta+\mathrm{d}\theta)$ ("spherical annulus", because it is the difference between two spherical caps rather than between two discs) to the total area of the unit sphere,

$$\frac{\text{number of } (p, p+\mathrm{d}p) \text{ particles approaching from } (\theta, \theta+\mathrm{d}\theta)}{\text{total number of } (p, p+\mathrm{d}p) \text{ particles (coming from any direction)}} = \\ = \frac{n(p,\theta)\,\mathrm{d}p\mathrm{d}\theta}{n(p)\,\mathrm{d}p} = \frac{2\pi\sin\theta\,\mathrm{d}\theta}{4\pi}\,,$$

where $n(p)$ is the density of particles with momentum magnitude p of course. Expressing from here $n(p,\theta)\,\mathrm{d}p\mathrm{d}\theta = \frac{1}{2}n(p)\sin\theta\,\mathrm{d}\theta\mathrm{d}p$ and using it in the pressure integral, we have

$$P = \int_0^\infty \int_0^{\pi/2} 2pv\cos^2\theta\, n(p,\theta)\; \mathrm{d}\theta\mathrm{d}p = \int_0^\infty \int_0^{\pi/2} 2pv\cos^2\theta\, \frac{1}{2}\, n(p)\sin\theta\; \mathrm{d}\theta\mathrm{d}p = \\ = \int_0^{\pi/2} \cos^2\theta\sin\theta\; \mathrm{d}\theta \int_0^\infty p\, v\, n(p)\,\mathrm{d}p = \frac{1}{3}\int_0^\infty p\, v\, n(p)\,\mathrm{d}p\ .$$

Hence the important formula

$$\boxed{P = \frac{1}{3}\int_0^\infty p\, v\, n(p)\,\mathrm{d}p}\,. \tag{21.14}$$

We may check whether it really yields what expected in some notorious case – say of a classical Maxwellian distribution, i.e. for a classical ideal gas of free particles described by the Boltzmann distribution

$$n(p) = (2s+1)\,\frac{4\pi p^2}{h^3}\,\exp\!\left(-\frac{E-\mu}{kT}\right) \qquad [\text{where } E-\mu \gg kT \text{ is assumed}]\,, \tag{21.15}$$

specifically when the chemical potential is totally negligible, $\mu = 0$. "Classical" and "free" is consistent with $E = p^2/(2m)$ and $v = p/m$ relations, so, taking $s = 1/2$, we have

$$n(p) = \frac{8\pi p^2}{h^3} \exp\left(-\frac{p^2}{2mkT}\right).$$

The pressure integral gives

$$P = \frac{8\pi}{3mh^3} \int_0^\infty p^4 \exp\left(-\frac{p^2}{2mkT}\right) \mathrm{d}p = \frac{\pi^{3/2}}{mh^3} (2mkT)^{5/2},$$

while

$$n = \int_0^\infty n(p)\, \mathrm{d}p = \frac{8\pi}{h^3} \int_0^\infty p^2 \exp\left(-\frac{p^2}{2mkT}\right) \mathrm{d}p = \frac{2\pi^{3/2}}{h^3} (2mkT)^{3/2},$$

which really are in the ideal-gas relation $P = nkT$.

21.4 Equations of state of a degenerate fermion gas

After the star-nuclear resources languish, the pertinent region of the star contracts, because the radiation pressure falls down and the weight of the material overwhelms the buoyant force. The matter pressure, normally linked to the thermal motion of particles, gradually (or quite quickly) becomes dominated by quantum-mechanical resistance of fermions against compression, because the corresponding characteristic energy (the Fermi energy $E_{\rm F}$) grows with density faster (as $n^{2/3}$) than the thermal energy kT (which grows as $n^{1/3}$ as we know from (21.6)) – the fermions become *degenerate*. Below, we focus on this state of matter whose pressure of fundamental origin is the only able to resist gravity (at least to a certain extent) at this stage of stellar evolution.

Have an ideal gas of fermions / bosons – i.e., a grand-canonical ensemble of free particles with half-integer / integer spin. The number density of particles with momentum magnitude p is given by the Fermi-Dirac / Bose-Einstein distribution

$$n(p) = (2s+1)\, \frac{4\pi p^2}{h^3}\, \frac{1}{\exp\left(\frac{E-\mu}{kT}\right) \pm 1}, \tag{21.16}$$

where E is the energy of individual particles and μ is their chemical potential. For $E - \mu \gg kT$, the formula goes over to the Boltzmann distribution for a classical ideal gas (21.15). Yet we are now rather interested in the limit $T \to 0$ of the Fermi-Dirac distribution,

$$n(p) = \begin{cases} (2s+1)\, \frac{4\pi p^2}{h^3} & \text{for } \; E < \mu =: E_{\rm F} \\ 0 & \text{for } \; E > E_{\rm F} \end{cases}, \tag{21.17}$$

i.e. when all the states with $E < \mu =: E_{\rm F}$ are occupied by $(2s+1)$ particles and the states with $E > E_{\rm F}$ are unoccupied. In such a case, the ensemble is said to represent a *totally degenerate fermion gas in a fundamental state*.

Approximating the white-dwarf electrons or neutron-star neutrons as the totally degenerate fermion gas with $s = 1/2$, i.e. taking

$$n(p) = \frac{8\pi p^2}{h^3} \quad \text{for} \;\; p < p_{\rm F} \qquad \text{and } n(p) = 0 \quad \text{for} \;\; p > p_{\rm F}\,,$$

we thus obtain for pressure

$$P = \frac{8\pi}{3h^3}\int_0^{p_{\rm F}} p^3 v\,{\rm d}p = \frac{1}{3\pi^2\hbar^3}\int_0^{p_{\rm F}} p^3 v\,{\rm d}p\,. \tag{21.18}$$

In a non-relativistic case we substitute $v = p/m$, while in the ultra-relativistic case $v = c$, so by integration and insertion of the Fermi momentum $p_{\rm F} = (3\pi^2 n)^{1/3}\hbar$, we have

$$P = \begin{cases} \frac{1}{3\pi^2\hbar^3 m}\int_0^{p_{\rm F}} p^4{\rm d}p = \frac{p_{\rm F}^5}{15\pi^2\hbar^3 m} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\,n^{5/3} = \frac{2}{5}\,nE_{\rm F} \\ \frac{c}{3\pi^2\hbar^3}\int_0^{p_{\rm F}} p^3{\rm d}p = \frac{p_{\rm F}^4 c}{12\pi^2\hbar^3} = \frac{(3\pi^2)^{1/3}\hbar c}{4}\,n^{4/3} = \frac{1}{4}\,nE_{\rm F} \end{cases}, \tag{21.19}$$

where finally the expressions (21.8) for $E_{\rm F}$ have been used respectively. By comparison of these relations with the mean-values relations (21.4), we confirm that the Fermi energy is a *typical* rather than exceptional peak energy of the particles – specifically, $\bar{E}_{\rm kin} = (3/5)E_{\rm F}$ in the non-relativistic gas and $\bar{E}_{\rm kin} = (3/4)E_{\rm F}$ in the ultra-relativistic gas. (Let us emphasize that in the non-relativistic case $E_{\rm kin}$ and $E_{\rm F}$ really concern just the kinetic part of energy, the rest energy is not included.)

Besides mentioning the limits, let us give a precise and generic result. Actually, using the general momentum-velocity relation

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \Longrightarrow \qquad v = \frac{pc}{\sqrt{m^2c^2+p^2}}$$

in the pressure integral for degenerate gases (21.18), it is still possible to integrate it analytically,

$$\boxed{P = \frac{8\pi c}{3h^3}\int_0^{p_{\rm F}} \frac{p^4\;{\rm d}p}{\sqrt{m^2c^2+p^2}} = \frac{\pi m^4 c^5}{3h^3}\left[\chi\sqrt{1+\chi^2}\,(2\chi^2-3) + 3\,{\rm arcsinh}\chi\right]}\,, \tag{21.20}$$

where $\chi := \frac{p_{\rm F}}{mc}$ quantifies how much relativistic the fermions are. The above non-relativistic limit corresponds to $\chi \ll 1$; restricting to the leading term of the expansion at $\chi = 0$,

$$\left[\chi\sqrt{1+\chi^2}\,(2\chi^2-3) + 3\,{\rm arcsinh}\chi\right] \sim \frac{8\chi^5}{5}\,,$$

one really obtains the limit expression. Similarly, the ultra-relativistic situation $\chi \gg 1$ really yields the latter limit expression if restricting to the leading term of the asymptotic expansion,

$$\left[\chi\sqrt{1+\chi^2}\,(2\chi^2-3) + 3\,{\rm arcsin}\,\chi\right] \sim 2\chi^4\,.$$

Finally, in order to express the obtained equations of state in terms of the total density ρ (instead of the number density of degenerate fermions n), we will neglect the internal energy $\sim kT$ and the mass contribution of electrons, and then use the relations for $n(p_\mathrm{F})$ (21.7) and n_0 (21.11) to write

$$\frac{n}{n_0} = \frac{\frac{p_\mathrm{F}^3}{3\pi^2\hbar^3}}{\frac{m^3c^3}{3\pi^2\hbar^3}} = \frac{p_\mathrm{F}^3}{m^3c^3} \equiv \chi^3 \,.$$

Therefore, the total density can be expressed as $\left[\text{recall that } Y \equiv \frac{\text{density of degenerate fermions}}{\text{total density of baryons}}\right]$

$$\rho = \bar{m}_\mathrm{b}\frac{n}{Y} = \frac{\bar{m}_\mathrm{b}}{Y}\frac{n}{n_0}\,n_0 = \frac{\bar{m}_\mathrm{b}n_0}{Y}\,\chi^3 =: \rho_0\,\chi^3\,, \tag{21.21}$$

where, explicitly,

$$\rho_0 := \frac{\bar{m}_\mathrm{b}n_0}{Y} = \frac{\bar{m}_\mathrm{b}}{3\pi^2 Y}\frac{m^3c^3}{\hbar^3}$$

is the total (in fact baryon) density corresponding to the typical number density n_0 of degenerate fermions of the relevant type (electrons or neutrons). In astrophysical reality, this density amounts to

$$\rho_0 \simeq 2.0\cdot 10^6\,\mathrm{g/cm^3} \qquad \text{for a degenerate electron gas} \left[m=m_\mathrm{e},\ Y\equiv Y_\mathrm{e}=\frac{1}{2}\right], \tag{21.22}$$

$$\rho_0 \simeq 6.8\cdot 10^{15}\,\mathrm{g/cm^3} \qquad \text{for a degenerate neutron gas} \left[m=m_\mathrm{n},\ Y\equiv Y_\mathrm{n}=\frac{8}{9}\right] \tag{21.23}$$

(we have inserted values of Y typical for the white dwarfs and for neutron stars, respectively). The former value is really deemed typical for white dwarfs, whereas the latter value rather seems to correspond to a peak core value for neutron stars, at least for baryon stars (not dominated by quark fluid in the core).

The limit equations of state (21.19) gain by the change from n to ρ. Writing out in them, respectively,

$$n^{5/3} = n\left(\frac{n}{n_0}\right)^{2/3} n_0^{2/3} = \frac{\rho Y}{\bar{m}_\mathrm{b}}\left(\frac{\rho}{\rho_0}\right)^{2/3}\frac{m^2c^2}{(3\pi^2)^{2/3}\hbar^2}\,,$$
$$n^{4/3} = n\left(\frac{n}{n_0}\right)^{1/3} n_0^{1/3} = \frac{\rho Y}{\bar{m}_\mathrm{b}}\left(\frac{\rho}{\rho_0}\right)^{1/3}\frac{mc}{(3\pi^2)^{1/3}\hbar}\,,$$

we arrive at

$$\boxed{\frac{P}{\rho c^2} = \begin{cases} \frac{mY}{5\bar{m}_\mathrm{b}}\left(\frac{\rho}{\rho_0}\right)^{2/3} & \ldots \text{ in the non relativistic limit} \\ \frac{mY}{4\bar{m}_\mathrm{b}}\left(\frac{\rho}{\rho_0}\right)^{1/3} & \ldots \text{ in the ultra-relativistic limit} \end{cases}}\,. \tag{21.24}$$

Let us repeat again that m is the mass of the degenerate fermion; for white dwarfs, it is the electron mass (and $Y\equiv Y_\mathrm{e}=1/2$), while for neutron stars, it is the neutron mass (and $Y\equiv Y_\mathrm{n}=8/9$).

21.5 Cold-star equilibria and the Chandrasekhar limit

Arising by such a fundamental and inescapable reason, the degenerate-fermion pressure might seem to be able to withstand any burden. If *under pressure*, the gas just adjusts its own pressure by stretching over the momentum space accordingly. Still it is not always sufficient. Later, we will mention *practical* limitations, but even if those did not work, one might expect a *fundamental* limit to exist. Indeed, larger momentum means larger energy – and energy is the source of gravity. In other words, *pressure* contributes to gravity, as we know from the equation for potential. When the typical speed of the fermions approaches that of light, their own weight grows quickly, so their pressure gradually loses efficiency as an opponent of gravitation. And really, we saw that in the ultra-relativistic regime the adiabatic index is lower (4/3) than in the non-relativistic regime (5/3), so with rising momentum range, the gas gradually becomes softer (less resistant to compression). Besides that, gravity has two inherent aces on its side: it is long-range, and it is *universal*. The more massive the object, the more weight it feels (because gravity goes as $m(r)$), and the smaller it is, the yet more weight it feels (because gravity goes as $1/r^2$). Actually, we saw that the gravitational binding energy (21.6) grows with $A^{2/3}$ and *in addition* with $n^{1/3}$.

We will derive the "fundamental core" of the Chandrasekhar mass limit, simply from the balance between the gravitational binding energy and the kinetic energy, the former being dominated by baryons, while the latter by degenerate fermions. The binding energy per baryon $\bar{U}_{\rm g}$ was computed in (21.6), while the kinetic (Fermi) energy $E_{\rm F}$ in (21.8). Remember that the binding energy was expressed as a function of the mean number density of *baryons*, wheres the Fermi energy naturally involves the number density of *degenerate fermions*, so, in comparing them, the latter should be multiplied by the respective Y ratio. Also, in the relativistic case, the mean energy of degenerate gas is $(3/4)E_{\rm F}$. However, the binding-energy calculation anyway stemmed from a *classical* equation of hydrostatic equilibrium, which may not be fully adequate in our problem of finding the *maximal* possible mass of a degenerate object, so, in order to identify the fundamental part of the result only, let us just omit *all* numerical factors. In the ultra-relativistic regime, the balance $-\bar{U}_{\rm g} \sim E_{\rm F}$ thus means

$$GA^{2/3}\bar{m}_{\rm b}^2\, n^{1/3} \sim \hbar c\, n^{1/3}\,, \quad \text{i.e. (take 3/2-th power and } \bar{m}_{\rm b} \doteq m_{\rm p}) \quad Am_{\rm p}^3 \sim \left(\frac{\hbar c}{G}\right)^{3/2}$$

$$\Longrightarrow \quad \boxed{M \equiv Am_{\rm p} \sim \left(\frac{\hbar c}{Gm_{\rm p}^2}\right)^{3/2} m_{\rm p} =: \frac{m_{\rm p}}{\alpha_{\rm g}^{3/2}} =: M_{\rm C}}\,, \tag{21.25}$$

where

$$\alpha_{\rm g} := \frac{Gm_{\rm p}^2}{\hbar c} \qquad \left(\leftrightarrow \quad \alpha = \frac{e^2}{\hbar c}\right)$$

is the gravitational analog of the fine-structure constant of electrodynamics. The values are

$$\alpha_{\rm g} \doteq 5.9 \cdot 10^{-39} \qquad \Longrightarrow \qquad M_{\rm C} \doteq 2.2 \cdot 10^{57} m_{\rm p} \doteq 1.85 M_{\odot}\,.$$

–Nice result, just given by fundamental constants. In particular, it does not depend on the mass of degenerate fermions, so it indicates the existence of a certain universal limit for mass of an object supported against gravity by pressure of degenerate fermions.

What about non-relativistic regime? Making similar comparison with the non-relativistic alternative of (21.8) and omitting small numerical factors again,

$$GA^{2/3}\bar{m}_{\mathrm{b}}^2\, n^{1/3} \sim \frac{\hbar^2}{m}\, n^{2/3}\,, \quad \text{i.e.}$$
$$Am_{\mathrm{p}}^3 \sim \frac{\hbar^3\, n^{1/2}}{G^{3/2}m^{3/2}} = \frac{\hbar^3\, n_0^{1/2}}{G^{3/2}m^{3/2}}\left(\frac{n}{n_0}\right)^{1/2} \sim \frac{\hbar^3}{G^{3/2}\cancel{m^{3/2}}}\left(\frac{\cancel{m}c}{\hbar}\right)^{3/2}\left(\frac{n}{n_0}\right)^{1/2} = \left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{n}{n_0}\right)^{1/2}$$
$$\Longrightarrow \quad M \equiv Am_{\mathrm{p}} \sim \left(\frac{\hbar c}{Gm_{\mathrm{p}}^2}\right)^{3/2} m_{\mathrm{p}}\left(\frac{n}{n_0}\right)^{1/2} = M_{\mathrm{C}}\left(\frac{n}{n_0}\right)^{1/2} = M_{\mathrm{C}}\left(\frac{\rho}{\rho_0}\right)^{1/2}. \qquad (21.26)$$

The story can now be read as follows: have an object supported by degenerate fermions and look for its equilibrium configuration. With density rising, the resulting mass of the equilibrium grows as $\sqrt{\rho}$. However, as the gas gradually becomes relativistic, the density dependence weakens, that is, it is more and more difficult to reach higher mass. Finally the mass saturates to some limit value which is proportional to the above fundamental result M_{C}. Taken in the other way round, increasing the *mass* of a degenerate object makes its density higher, but when the object already contains more than about $2.2 \cdot 10^{57}$ baryons, it can no longer stay in equilibrium – it has to either collapse or explode (?). Of course, this has been “fundamental estimate” only, details have to be added if aiming for realistic predictions. With the details incorporated, the limit mass comes out slightly below $1.5M_{\odot}$ for white dwarfs (degenerate electrons), while for neutron stars the actual limit is less certain and may probably lie higher (about $2 \div 3\ M_{\odot}$), mainly due to the possibly important effects of spin and of magnetic field.

21.5.1 Stable and unstable branches of the cold-matter equilibria

Results of integration of the TOV equation together with the equation of mass and equation of state $P = P(\rho)$ are being plotted in the (R, M) diagram, as equilibrium curves parameterized by central density. The main issue is the equation of state, mainly in the heavier neutron-star case when “strong-field” chromodynamics becomes crucial. However, it has been observed that the resulting equilibrium curves $M = M(R; \rho_{\mathrm{c}})$ typically show three main features, relatively independently of the state equation (see Figure 21.2):

- For a non-degenerate matter, the equilibrium objects (“planets”) have bigger mass and bigger radius with growing central density. When the density reaches about $10^{4\div 5}\mathtt{g}/\mathtt{cm}^3$, electrons gradually become degenerate and the curve sharply bends – with rising density, the equilibrium objects are more massive but *smaller* now. This behaviour also applies – under much greater densities – to objects supported by degenerate neutron gas.

- There exist two regions where the degenerate-gas dominated equilibria are stable, one at the central-density range $(10^5\div 10^9)\mathtt{g}/\mathtt{cm}^3$ (white dwarfs, dominated by degenerate electron gas) and one at the range $(4\cdot 10^{13} \div 3\cdot 10^{15})\mathtt{g}/\mathtt{cm}^3$ (neutron stars, dominated by degenerate neutron gas). For other central densities, the degenerate equilibria are unstable, which practically means that they have no physical relevance (if a star does have such a central density, it is not in equilibrium – it is undergoing contraction or expansion, likely a very

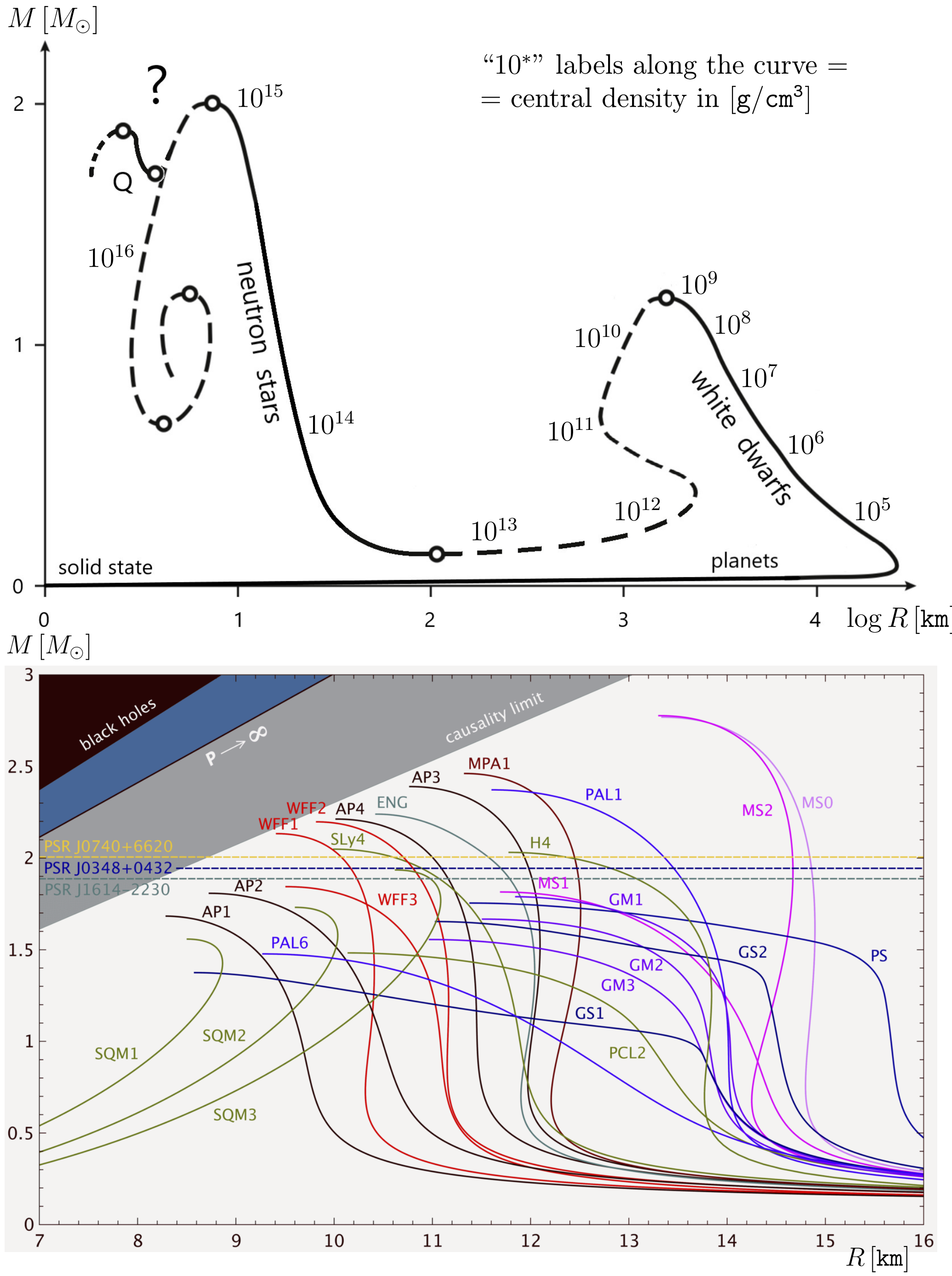

Figure 21.2 *Top:* *Typical map of the cold-matter equilibria in the radius-mass (R, M) axes and parameterized by central density. Besides the standard stable branches (solid line) of white dwarfs and neutron stars, a possible quark-star branch is also indicated.* ***Bottom:*** *Detail of the neutron-star branch, computed for different equations of state (marked by different abbreviations). The "causality limit" is where sound speed exceeds that of light. Possible stars with quark cores are also included – see the curves denoted by "SQM". Masses of three well studied pulsars are indicated.*

dynamical one). The stars with quark cores seem to be a likely third stable possibility which might/should occur at central densities $\gtrsim 10^{15}\mathtt{g/cm^3}$.

- All the two/three branches of stable equilibria culminate at mass values close to the Chandrasekhar limit. Indeed, it is not necessarily so that white dwarfs are lighter than neutron stars and these in turn are lighter than quark stars – the three kinds of cold objects rather differ in central density and radius (the neutron and quark stars only slightly, however).

Theoretical **minimum mass** is about $0.05M_\odot$ for white dwarfs and about $0.1M_\odot$ for neutron stars. For white dwarfs, smaller mass simply would not ensure sufficient electron degeneracy. For neutron stars, smaller mass would mean such a low density that neutrons *could* β decay, because electrons would not be so highly degenerate, so the extra ones arising from the decay would be able to find their niche in the phase space (Pauli blocking is not that strict then). Therefore, the matter would not get sufficiently neutronized. Let us emphasize that this is not to claim that, say, $0.5M_\odot$ of matter should always *give rise* to a white dwarf or a neutron star; it only says that an *already made, static* configuration of that mass should be able to stay as a white dwarf or as a neutron star. Good to mention observations as well: white dwarfs are known with masses between $0.17M_\odot$ and $1.33M_\odot$, with majority lying in $(0.5\div 0.7)M_\odot$. Their radii are mostly estimated at $(5500 \div 14000)\mathtt{km}$, with the two parameters inversely proportional (in the non-relativistic regime, they are related as $M \sim 1/R^3$). Known (and precisely measured) neutron stars have masses between $1.17M_\odot$ and $2.14M_\odot$, with majority lying around $1.4M_\odot$. Radii are being estimated at $(9.5 \div 12)\mathtt{km}$. Again the two parameters should be inversely proportional. Hence, there exists an overlap in mass ranges of the two types of degenerate objects.

21.6 Limits of dominance of the degenerate fermion gas

We were emphasizing that the Chandrasekhar limit represents a fundamental message rather than an accurate value. The maximal mass of real degenerate objects clearly *is* close to the Chandrasekhar limit, but several physical circumstances make the *practical* reach slightly lower than would correspond to a total dominance of the degenerate ideal fermion gas.

Generally, close to the surface of the objects the density is much lower and the matter tends to condense to a rigid structure with a considerably different equation of state. On the other hand, in the direction towards higher densities where the fermion Fermi energy increases rapidly, there typically open interaction channels through which the relevant fermions are spent, their energy being converted to some form in which it no longer generates pressure effectively or even completely escapes from the star (neutrinos). It is however very difficult to assess the importance of different possible processes, the more that there is very little laboratory experience with similar states of matter (energy is within reach, but density only very, very temporarily).

21.6.1 The electron-gas case

- We saw that most of the white-dwarf volume is very well dominated by the degenerate electron gas. Typically only a thin surface layer (about 1/100 of radius) is non-degenerate.

In the surface layer, **Coulomb interaction** is important as well, so the gas is not ideal there and its pressure is lower accordingly.

- A different story are the **nuclei**. They are far from degenerate, their kinetic energy is dominated by kT, but mainly the Coulomb interaction *between them* is important, due to which the nuclei **tend to arrange in a rigid lattice**. When, in the direction of decreasing density, the electron gas finally loses degeneracy (at about $10^{4\div 5}\,\mathrm{g/cm^3}$), the pressure falls rapidly and the curve of equilibria turns towards *smaller* radii as well as masses (the behaviour of "planetary" type). The turn-over roughly corresponds to $(10^{-3} \div 10^{-2})M_\odot$, which may thus be regarded as the minimal mass of white dwarfs. For lower masses (and densities), one approaches the realm of condensed-matter physics. When the density falls to some $500\,\mathrm{g/cm^3}$, electrons gradually get bound to the nuclei (atoms recombine) and the substance differentiates chemically. At densities around $50\,\mathrm{g/cm^3}$, even the valence orbitals are populated, so all the electrons are bound and the matter properties are chiefly determined by chemical elements involved.

- There is one clear limit for the dominance of the degenerate electron gas: when its Fermi energy exceeds a certain value, electrons start to be spent in the **inverse β decay** (**electron capture**, or "neutronization")

$$\mathrm{p} + \mathrm{e}^- \longrightarrow \mathrm{n} + \nu_\mathrm{e}\,.$$

For a mixture of *free* protons, electrons and neutrons, the threshold value is simply given by the difference between the rest energies of proton and neutron,

$$(m_\mathrm{n} - m_\mathrm{p})c^2 = c\sqrt{m_\mathrm{e}^2 c^2 + p_\mathrm{F}^2} = c\sqrt{m_\mathrm{e}^2 c^2 + \hbar^2\left(\frac{3\pi^2\rho Y_\mathrm{e}}{\bar{m}_\mathrm{b}}\right)^{2/3}}$$
$$\Longleftrightarrow \qquad \rho \doteq \frac{1.2}{Y_\mathrm{e}}\cdot 10^7\,\mathrm{g/cm^3}$$

(with the neutrino energy gone), but in the conditions really existing in the white dwarf the question is much more complicated: for a mixture of certain nuclei ${}^A_Z\mathrm{X}$ and free electrons, is it energetically more favourable (= lower) to keep status quo, or to go for ${}^{\;\,A}_{Z-1}\mathrm{Y}$ by electron capture and release the corresponding neutrino energy,

$${}^A_Z\mathrm{X} + \mathrm{e}^- \longrightarrow {}^{\;\,A}_{Z-1}\mathrm{Y} + \nu_\mathrm{e}\,,$$

or to fuse some "higher", neutron-rich nuclei (while typically spending some electrons and releasing neutrino energy)?

Just to have an idea of how strongly the answer depends on the white-dwarf composition, we give several typical nuclei ${}^A_Z\mathrm{X}$ and [approximate density threshold for their neutronization]: for ${}^4_2\mathrm{He}$ [$\gtrsim 10^{11}\mathrm{g/cm^3}$], ${}^{12}_{\;6}\mathrm{C}$ [$4\cdot 10^{10}\mathrm{g/cm^3}$], ${}^{16}_{\;8}\mathrm{O}$ [$2\cdot 10^{10}\mathrm{g/cm^3}$], ${}^{20}_{10}\mathrm{Ne}$ [$6\cdot 10^{9}\mathrm{g/cm^3}$], ${}^{24}_{12}\mathrm{Mg}$ [$3\cdot 10^{9}\mathrm{g/cm^3}$], ${}^{28}_{14}\mathrm{Si}$ [$2\cdot 10^{9}\mathrm{g/cm^3}$], ${}^{32}_{16}\mathrm{S}$ [$\gtrsim 10^{8}\mathrm{g/cm^3}$], ${}^{56}_{26}\mathrm{Fe}$ [$10^{9}\mathrm{g/cm^3}$]. Generally, however, up to some $\rho \sim 4\cdot 10^{11}\,\mathrm{g/cm^3}$, advantageous are bigger nuclei, even as big as $A \doteq 120$, $Z \doteq 40$.

A remark: combination of the inverse β decay with the direct one, $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}$, yields the so-called URCA process, which normally may cool the stars efficiently through releasing neutrinos and antineutrinos. However, in the "bath" of degenerate electrons, it is not easy for "the (N+1)-st" electron to "return from the nucleus", because there is no space for it in the extremely densely populated phase space (the return is said to be "Pauli-blocked").

- We have reached densities where stable equilibrium is not possible ($\gtrsim 10^{9\div10}\mathrm{g/cm^3}$). Besides some of the inverse-β-decay channels, another serious issue in the life of an ultra-dense star is the **neutron drip**. Namely, when the density rises to some $\rho \sim 4.3 \cdot 10^{11}$ $\mathrm{g/cm^3}$, the big nuclei formed by the inverse β decay start releasing neutrons. The abundance of neutrons rises quickly to about $(A-Z)/Z \simeq 400$ at $\rho \sim 10^{12}$ $\mathrm{g/cm^3}$. With the density growing further, the ratio decreases back and finally approaches 8. Good to add that the contribution of neutrons to pressure behaves in quite an *opposite* way: around "drip" it is negligible, which induces strong instability (large contribution to density and, at the same time, small contribution to pressure means an easily compressible object), but then it rises rapidly as the neutrons start to become degenerate. Indeed, at $\rho \sim 4 \cdot 10^{11}$ $\mathrm{g/cm^3}$ the degenerate electrons still dominate, whereas at $\rho \sim 10^{13}$ $\mathrm{g/cm^3}$ almost all pressure is due to degenerate free neutrons. But this is already a situation applying to neutron stars...

  Hence, whereas the mass ranges for white dwarfs and neutron stars overlap, in central density there exists quite a broad "forbidden zone" between $10^{9\div10}$ and 10^{13} $\mathrm{g/cm^3}$. The above mentioned processes of neutronization and of neutron drip thus happen in seconds, while the object rapidly collapses.

- In higher densities, white dwarfs are *general* relativistic objects as well – their interior pressure contributes to gravitation non-negligibly, which restricts the equilibrium prospects according to the TOV equation. However, more sensitive to **GR effects** than equilibrium is *stability*. An analysis similar to what we performed in Section 20.4 shows that, contrary to the Newtonian treatment, the white dwarfs denser than roughly $2.65 \cdot 10^{10} \left(\frac{1/2}{Y_\mathrm{e}}\right)^2$ $\mathrm{g/cm^3}$ at their centre may be prone to a collapse.

- If an object possesses **spin and/or magnetic field**, it may intuitively be more protected against collapse, and papers that take them into account really conclude that the limit mass may then be as high as about $3M_\odot$. However, details remain uncertain, for example, it is not known how strong magnetic fields really occur inside white dwarfs.

21.6.2 The neutron-gas case

- It is estimated that density is roughly higher than the nuclear "saturation" value $\rho \sim 2.3 \cdot 10^{14}\mathrm{g/cm^3}$ in the central half of the neutron-star radius. The behaviour of matter at densities $\gtrsim 4 \cdot 10^{14}\mathrm{g/cm^3}$ remains largely unknown because of uncertain behaviour of strong nuclear force at distances where it sharply turns negative. Let us briefly mention some further issues occurring there (besides this basic uncertainty).

- With the strong force turned extremely strong, the nucleon gases cease to be ideal. At hyper-nuclear densities, another types of baryons may occur, namely hyperons and baryon resonances (which under normal conditions decay quickly, but here they might live longer). Also possible is the analog of the β decay with π-mesons (pions) in place of electrons, $\mathrm{n} \to \mathrm{p} + \pi^-$. Actually, mesons have zero spin, so they are *bosons* and do not block each other in the phase space. Under very low temperatures, they may even hand over of all their momentum and settle down to a Bose-Einstein condensate. Pions also may enter further interactions, for example a cooling cycle analogous to the URCA process, in which neutrons convert into pions and back while generating neutrinos and antineutrinos. (Under the given high densities, neutrinos already do slightly scatter on matter, mainly on some of the possible hyperons.) Also, by interaction with hyperons, the π-mesons may generate K-mesons (kaons) which have non-zero strangeness. The appearance of such extra particles generally softens the equation of state.

- When the nucleons start to touch each other, the model of "particles" moving in a certain interaction potential loses sense – it should be replaced with the model of a nucleon liquid dominated by strong interaction. The neutron liquid may be superfluid (zero viscosity) and the proton liquid may be superconducting (zero resistance).

- Around $10^{15}\mathrm{g/cm^3}$ when nucleons start to coalesce, the nuclear liquid begins to turn into the quark phase; one can call this "quark drip" in analogy with the neutron one. Quarks are asymptotically free, so they only achieve freedom when under extreme pressure. The "quark-gluon plasma" likely preceded the formation of hadrons in the early Universe; this state of matter has already been identified in collision experiments with heavy nuclei for tiny instants. The quark-matter equation of state turns out (from quantum chromodynamics) to be rather similar to that of non-interacting massless particles, $P \sim \rho/3$, which means considerably higher compressibility than has the hadronic liquid. Therefore, neutron stars with quark core are smaller then the pure-hadron ones. It is believed that the quark phase occurs in the core of stars heavier than about $1.2M_\odot$.

21.7 Stellar fates

An observation exists (made in Amsterdam) that *Good girls go to heaven, bad girls go to Amsterdam*. With stars it is not that simple. Traditionally, we were taught that the cores of the lightest stars ($\lesssim 10M_\odot$) end as white dwarfs, those of moderate-mass stars ($10 \div 30\ M_\odot$) become neutron stars, and the heaviest stars ($\gtrsim 30M_\odot$) give rise to black holes. In reality, the stellar fates rather interlace in a complicated (in fact chaotic) pattern, only very approximately following the above tendency. Relevant are various subtleties of stellar evolution and mass loss (chiefly dependent on mass and nuclear composition), the star's spin and magnetic field, and surely profound is the influence of a possible involvement of the star in a binary or multiple system.[2] However, modern simulations mainly point to the extreme complexity

[2] Astronomical statistics indicates that solar-mass stars almost in 50% exist in multiple systems (mostly binaries); for $\gtrsim 16M_\odot$ stars it is even more that 80%.

of **supernova explosion** – the spectacular ejection of outer stellar layers which typically accompanies the collapse of the core into a neutron star and sometimes also into a black hole.[3] Several mechanisms are supposed to perturb the above simplistic scheme:

- A neutron star need not only be formed directly – it may also arise from a white dwarf if the latter accretes enough matter to exceed the Chandrasekhar limit. However, it is believed that such an accretion mostly leads to a violent (carbon) fusion which completely disrupts the white dwarf in a thermal runaway (in supernova Ia) *before* the mass limit is reached. Neither a black hole needs to arise directly – if an explosion leaving a neutron star down there is not "successful" enough and a significant amount of the expelled material falls back, the neutron-star remnant may collapse into a black hole (of course, this may also happen due to a later accretion episode).

- In dependence on how massive is the burnt-out nucleus of the star, some massive stars may collapse to a black hole without significant explosion of the outer layers (astronomers speak of "failed supernovae").

- In very massive stars ($140 \div 250\ M_\odot$), the electron-capture picture of neutronization and collapse is supplemented or even substituted by instability against the production of electron-positron pairs. Actually, the core of such massive stars being extremely hot ($\gtrsim 3 \cdot 10^8\mathtt{K}$), photons generated there are mostly in the gamma band and thus able to produce electron-positron pairs. Such a production may become unstable, because more pairs generated $\Rightarrow$ less photons left to support the stellar core $\Rightarrow$ contraction $\Rightarrow$ rise of temperature $\Rightarrow$ nuclear burning speeds up + still harder photons generated $\Rightarrow$ still more pairs created $\Rightarrow$ etc. The e^- and e^+ also annihilate back of course, but if the instability grows quickly, they simply do not make it and the runaway completely disrupts the star, i.e. *no remnant is left*.

- In the most massive stars ($\gtrsim 250 M_\odot$), the radiation generated in the core is so hard that it is even able to photo-disintegrate the nuclei. Similarly as pair-creation, this process consumes energy and can also turn into a runaway which however seems to result in a core collapse into a black hole rather than disintegration; behaviour of the outer layers remains rather unclear.

- If the dying star has significant spin, its collapsing core may form a disc which accretes onto a just-born central compact object. Such a system may evolve similarly as accreting compact objects known from X-ray binaries and microquasars, i.e. it may produce jets ejected along the rotational axis and colliding with the outer stellar envelopes, thus generating hard radiation mainly emitted in the jet direction. This is a standard scenario for **gamma-ray bursts**, with the object sometimes called a **collapsar** and the whole event a **hypernova**.

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[3] These supernovae are called the **core-collapse supernovae**, in contrast to the **thermal-runaway (type Ia) supernovae** caused by a nuclear-fusion outburst in a white dwarf subsequently burdened with an accreted material.

- A remark: a supernova-like event could also result from the conversion of a neutron star into a quark star (quark deconfinement in the core), most likely due to a gradual spin-down (and thus larger weight experienced by the core).

CHAPTER 22

Linearized theory of gravitation and gravitational waves

It is common in physics that a problem *would be simple(r) if this or that parameter were negligible*. It is thus crucial to identify "difficult" dimensions of a problem and, assessing their factual significance, consider whether a suitable approximation could help to overcome theoretical complications. A usual approach is the perturbation theory, in which a certain parameter is assumed to be small and the equations are expanded in it.

In GR, there may occur various such parameters linked to specific situations, but one aspect is *always* difficult, at least in a sufficiently generic situation: the non-linearity of the theory in the "field" (affine connection, i.e. metric derivatives). Consequently, a very important position within the theory has the approach which assumes the field to be weak and *linearizes* the equations in it. Since the field non-linearity fully shows itself in extreme situations only, such a **linearized theory** can adequately describe most of the problems arising in astrophysics.

One particular application of the linearised theory is worth a special mention: *gravitational waves*. Namely, imagine a generic space-time, possibly curved in a complicated way, and a gravitational wave propagating in it. It is a very difficult idea in fact, because how to uniquely distinguish the curvature belonging to the "background" and the one carried by the wave? In the linearised theory it is simple, because *the background is flat*.

22.1 Metric, affine connection, curvature, and conservation laws

We know how to parameterize the assumption of a "weak field" from Section 3.7 on the Newtonian limit of geodesics. There have to exist such coordinates in which the metric assumes an almost-Minkowski form,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \text{where } h_{\mu\nu} \text{ are very small (against 1), including derivatives} . \tag{22.1}$$

"Very small" means that $h_{\mu\nu}$ and their arbitrary derivatives will be left in the equations up to linear order only, $O(h)$. For the contravariant metric perturbation we found

$$h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}\,,$$

observing that $h_{\mu\nu}$ behaves like a tensor field *living in the Minkowski space-time*. And also simple was to learn that Christoffel symbols are $O(h)$, specifically

$$\Gamma^{\mu}{}_{\kappa\lambda} = \frac{1}{2}\,\eta^{\mu\sigma}(h_{\sigma\kappa,\lambda} + h_{\lambda\sigma,\kappa} - h_{\kappa\lambda,\sigma})\,. \tag{22.2}$$

When performing the Newtonian limit, we assumed, besides the weakness of the field, its stationarity, and also slow motion (for the special relativistic effects to be negligible). The linearized theory does *not* restrict speeds, although the point is in fact a delicate one, because the derivatives of the metric have to remain small. (We will more comment on it later.) Neither stationarity of the metric will be assumed, though, again, the time derivatives of the metric have to remain small as well.

The Riemann-tensor formula (6.8), yields, to the linear order,

$$\begin{aligned} R_{\mu\nu\kappa\lambda} &= \frac{1}{2}\left(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}\right) + g_{\pi\rho}\left(\cancel{\Gamma^{\pi}{}_{\mu\lambda}\Gamma^{\rho}{}_{\nu\kappa}} - \cancel{\Gamma^{\pi}{}_{\mu\kappa}\Gamma^{\rho}{}_{\nu\lambda}}\right) = \\ &= \frac{1}{2}\left(h_{\mu\lambda,\nu\kappa} + h_{\nu\kappa,\mu\lambda} - h_{\mu\kappa,\nu\lambda} - h_{\nu\lambda,\mu\kappa}\right)\,, \end{aligned} \tag{22.3}$$

so the Ricci tensor, Ricci scalar and Einstein tensor read

$$R_{\nu\lambda} \equiv \eta^{\mu\kappa}R_{\mu\nu\kappa\lambda} = \frac{1}{2}\left(h^{\kappa}_{\lambda,\kappa\nu} + h^{\kappa}_{\nu,\kappa\lambda} - h_{,\nu\lambda} - \Box\, h_{\nu\lambda}\right), \tag{22.4}$$

$$R \equiv \eta^{\nu\lambda}R_{\nu\lambda} = h^{\kappa\lambda}{}_{,\kappa\lambda} - \Box\, h\,, \tag{22.5}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R\,\eta_{\mu\nu} = \frac{1}{2}\left[h^{\kappa}_{\nu,\kappa\mu} + h^{\kappa}_{\mu,\kappa\nu} - h_{,\mu\nu} - \Box\, h_{\mu\nu} - \eta_{\mu\nu}(h^{\kappa\lambda}{}_{,\kappa\lambda} - \Box\, h)\right], \tag{22.6}$$

where clearly $\Box$ is the *flat-space* d'Alembert. Acting by divergence on the last guy, we get zero,

$$G_{\mu\nu}{}^{,\nu} = \frac{1}{2}\left[\cancel{h^{\kappa\nu}{}_{,\kappa\nu\mu}} + \cancel{\Box\, h^{\kappa}_{\mu,\kappa}} - \Box\, h_{,\mu} - \cancel{\Box\, h_{\mu\nu}{}^{,\nu}} - \cancel{h^{\kappa\lambda}{}_{,\kappa\lambda\mu}} + \Box\, h_{,\mu})\right] = 0\,,$$

so the Einstein equations (we will not take into account the cosmological term)

$$h^{\kappa}_{\nu,\kappa\mu} + h^{\kappa}_{\mu,\kappa\nu} - h_{,\mu\nu} - \Box\, h_{\mu\nu} - \eta_{\mu\nu}(h^{\kappa\lambda}{}_{,\kappa\lambda} - \Box\, h) = 16\pi T_{\mu\nu} \tag{22.7}$$

imply that the energy-momentum tensor satisfies conservation laws *in the special relativistic form*,

$$G_{\mu\nu}{}^{,\nu} = 0 \qquad \Longrightarrow \qquad T_{\mu\nu}{}^{,\nu} = 0\,.$$

Note that the above should have been expected: similarly as it is clear that $h_{\mu\nu}$ has its indices handled by $\eta_{\mu\nu}$ (because in fact it is being done by $g_{\mu\nu}$, yet the "hh" term is already $O(h^2)$ and thus omitted), the conservation laws may also be understood as "actually" involving covariant divergence, yet since $G_{\mu\nu}$ and (hence) $T_{\mu\nu}$ are $O(h)$, the "$\Gamma\, T$" terms are $O(h^2)$ and thus omitted.

22.2 Applicability of the linearized theory

When trying to understand, physically, the non-linearity of the Einstein equations (Section 8.2), we spoke about the effect of a source on itself through its own field. Exactly this information can be expected to be missing in the linearized description. Imagine, for example, a binary system in which two massive bodies orbit about a common centre of mass. Such a system generates gravitational waves; these carry away energy, which makes the bodies' orbital radii shrink. The field strength thus increases, and thus the gravitational emission – so the radii shrink faster; etc. Unless the bodies are very compact and we enquire about their close vicinity, the linearized theory can be expected to provide a good approximation of how the metric looks in the whole space-time, including its non-stationary, wave component. However, one does not expect to obtain information about the "back-consequences" of the emission for the source motion.

Technically, such an image should follow from conservation laws (because these constrain the behaviour of sources), and it is indeed so. In Section 7.4.2 we showed, for example, that for the ideal fluid the covariant conservation laws $T^{\mu\nu}{}_{;\nu}=0$ imply the Euler equation of motion. If pressure in the fluid can be neglected, the equation reduces to the geodesic equation $a_\alpha \equiv u_{\alpha;\beta}u^\beta \equiv \frac{\mathrm{D}u_\alpha}{\mathrm{d}\tau}=0$. This means that the fluid elements move freely in the total gravitational field that the fluid itself generates (were some other sources present around, they would also contribute to the field, of course). So it *is* a self-consistent description: the sources move in the field that they themselves generate. The linearized conservation laws only contain partial divergence, so in the same situation (without pressure) they yield $a_\alpha \equiv u_{\alpha,\beta}u^\beta \equiv \frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau}=0$, i.e. $u_\alpha=\mathrm{const}$ – in the nearly-Lorentz system there is *no gravitational influence*.

A query might arise here: how is it possible that (even) in the Newtonian limit of the geodesic equation we did obtain some gravitational effect (described by $\Phi^{,i}$), whereas now it seems we have switched off the gravitation completely? When studying geodesic motion, we considered a *test* particle reacting on an *external* field whose origin was not part of the problem, it was simply *given* (by $\Gamma^\mu{}_{\kappa\lambda}$). In the Euler equations, on the contrary, the fluid is *not test*, every its element contributes to the field – while, at the same time, reacting on the total field generated by the whole fluid body. However, the field equations say that in order that the curvature be small, the mass of every source element (the density) has to be such – and hence in this case the "Γ-terms" of the equation have to be omitted, since "$\rho\cdot\Gamma$" is already $O(h^2)$.

From the Newtonian limit of the geodesic equation, we obtained an estimate that a "weak field" means that the dimensionless Newtonian potential is much smaller than unity. Let us check whether such an estimate also follows from the linearised field equations, symbolically written as $\partial\partial h \approx \frac{16\pi G}{c^4}T$. Estimating, in the spirit of the mean-value theorem, that $|\partial\partial h| \approx |h|/R^2$, where R is the characteristic size of the source, and approximating the energy-momentum tensor using its dominant, density term, $T \approx T_{00} \approx \rho c^2 \approx Mc^2/R^3$, we have

$$\frac{|h|}{R^2} \approx \frac{16\pi G}{c^4}\frac{Mc^2}{R^3} \qquad \Longrightarrow \qquad |h| \approx \frac{16\pi GM}{c^2 R} \approx \frac{16\pi\Phi}{c^2}\,,$$

which confirms the previous guess.

A special note are worth the time derivatives. In their case, one should rather write $|\partial\partial h| \approx |h|/(c\Delta t)^2$, with Δt a characteristic time-scale of the source (in which the source can significantly change). However, since any change can only spread within the source with subluminal speed, $v \approx R/\Delta t \lesssim c$, it must hold $c\Delta t \gtrsim R$, which means that the time derivatives are typically $\lesssim$ the spatial derivatives. Let us stress that this is an expectation, it is *not* an assumption, so we are not otherwise restricting the velocities occurring in the source, in particular, we are not requiring that the time derivatives be *much* smaller than the spatial ones.

In dynamical situations, especially in the case of gravitational radiation, which we will address mainly using the linearised theory, the sources often change periodically or quasi-periodically, typically in connection with their rotation. Such cases can be parameterized by characteristic frequency (ω) instead of the time interval Δt. The gravitational radiation is of quadrupole level (dipole component does not exist due to conservation of total momentum), so, in analogy with electromagnetism, we expect a reasonable estimate for radiative $|h|$ to be analogous to the quadrupole term in the expansion of EM potential,

$$|h| \approx \frac{1}{c^4 r}\,\frac{\mathrm{d}^2 D}{\mathrm{d}t^2} \approx \frac{1}{c^4 r}\,\frac{GMR^2}{(\Delta t)^2} \approx \frac{GM}{c^2 r}\,\frac{R^2\omega^2}{c^2}\,,$$

where r is the distance from the source. Substituting there values describing the Crab-nebula pulsar, for example, one gets $|h| \lesssim 10^{-23}$. This roughly corresponds to the "design" sensitivity of current top interferometric detectors of gravitational waves, with the strongest astrophysical signals (from compact-binary mergers) reaching about 100 times that level.

The question of time change can also be discussed on the energy-momentum tensor. For that of incoherent dust, for example, $T^{\mu\nu} = \rho u^\mu u^\nu$, it holds $|T^{ij}| < c\,|T^{0j}| < c^2 T^{00}$ since $|u^i| = |v^i||u^t| < c|u^t|$, but it is not necessary to *neglect* any of the components. The considerations are also valid for an ideal fluid, because pressure P has to be less then energy density ρc^2, in order that the sound speed remain below the speed of light, $P/\rho \approx \mathrm{d}P/\mathrm{d}\rho \equiv v_\mathrm{s}^2 < c^2$. Actually, the most "stiff" equation of state in usage reads $P = \rho c^2$; an isotropic radiation (or ultra-relativistic gas) has $P = \rho c^2/3$.

22.3 Covariance properties of the linearised theory

One of the most important properties of a theory is its (possible) invariance under a certain family (group) of transformations. Actually, such symmetries have turned out to probably reach down to a deep "underground" of the physical world, similarly as e.g. the extremum properties on which the variational principles build. Mainly in microphysics it is common to characterize, compare and classify a theory by identifying (or right by starting from) its symmetry group. Sure that in relativity – a theory in which so much attention goes to distinguishing "absolute" from "relative", i.e. to the transformation properties – it all the more so cannot be otherwise.

22.3.1 Infinitesimal-diffeomorphism freedom

In full GR, the "issue" is right from the beginning answered by the principle of general covariance: the theory has to be invariant under any *diffeomorphism*, i.e. any map – from (a certain region of) the manifold to itself or to (certain region of) some other manifold – which is a C^∞ bijection together with its inverse, bijection meaning that it is "one-to-one" (injective) and "onto" (surjective). The two manifolds connected by a diffeomorphism are "the same (*isomorphic*) as concerns their differential structure". (Note that they still may/may not be equipped with different affine connections and/or different metrics.) However, although "differential structure" standardly means complete atlas of coordinate charts, the diffeomorphisms of GR need not only be understood as *coordinate* transformations, i.e. in a "passive" sense (as "leaving everything as it is, just changing the coordinates"). They may equally well be understood in an *"active"* sense, as indeed *shifting points* (one thus speaks of *flow*), while providing, automatically, a natural way how to *pull back* functions (to its domain ← from its range, simply by composition) and how to *push forward* vectors (from tangent spaces of the domain → to tangent spaces of the range), which already is sufficient to know how to transport any tensor. We followed the active view in Chapter 11 on Lie derivative, since it more naturally leads to a coordinate-independent picture, yet the two perspectives are equivalent. In particular, the components of the transported tensors in coordinates *there* are, technically, just the components of tensors in the "new" coordinates of the passive view. Or, in different words, the matrix of the tangent map (of the active view) just equals the Jacobi matrix of the corresponding coordinate transformation. We have stopped for a while at this point, because to grasp the above equivalence was once important in understanding the lower floors of GR (philosophers, you may look at *Einstein's hole argument*).

However, it is also important in understanding how *the linearised theory differs from the full GR*. The linearised theory cannot have *general* diffeomorphism covariance, because under a generic transformation the components $h_{\mu\nu}$ might not remain small. They only do if the transformation is "infinitesimal" (take the "active" view now),

$$x^\mu \;\to\; x'^\mu = x^\mu + \xi^\mu(x)\,, \qquad \text{where } \; \xi_{\mu,\nu} \lesssim O(h)\,. \tag{22.8}$$

The sense of this smallness requirement should be clear, since the tensor components transform via the Jacobi matrix

$$\begin{aligned}
&\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + \xi^\mu{}_{,\alpha}\;, \\
&\frac{\partial x^\alpha}{\partial x'^\lambda} = \delta^\alpha_\lambda - \xi^\alpha{}_{,\iota}\,\frac{\partial x^\iota}{\partial x'^\lambda} = \delta^\alpha_\lambda - \xi^\alpha{}_{,\iota}\left(\delta^\iota_\lambda - \xi^\iota{}_{,\kappa}\,\frac{\partial x^\kappa}{\partial x'^\lambda}\right) = \delta^\alpha_\lambda - \xi^\alpha{}_{,\lambda} + O(\mathrm{d}\xi^2)\,.
\end{aligned} \tag{22.9}$$

The metric thus transforms as

$$\begin{aligned}
g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\,g_{\alpha\beta} &= (\delta^\alpha_\mu - \xi^\alpha{}_{,\mu})(\delta^\beta_\nu - \xi^\beta{}_{,\nu})\,g_{\alpha\beta} = \\
&= g_{\mu\nu} - g_{\mu\beta}\xi^\beta{}_{,\nu} - g_{\alpha\nu}\xi^\alpha{}_{,\mu} + O(\mathrm{d}\xi^2)\,,
\end{aligned} \tag{22.10}$$

of which the first smallness order reads

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \eta_{\mu\beta}\xi^\beta{}_{,\nu}(x) - \eta_{\alpha\nu}\xi^\alpha{}_{,\mu}(x) := h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\;, \tag{22.11}$$

$$\Longrightarrow \quad h' \;\equiv\; h'^\mu_\mu = h - 2\xi^\mu{}_{,\mu}\;. \tag{22.12}$$

It's worth noticing that the smallness of ξ^μ itself has actually nowhere been employed, only that of its *derivatives* (and that of $h_{\mu\nu}$, of course).

From the transformation behaviour of the metric, one easily derives

$$\begin{aligned}
\Gamma'^{\mu}{}_{\kappa\lambda} &= \frac{1}{2}\,\eta^{\mu\sigma}(h_{\sigma\kappa,\lambda} - \xi_{\sigma,\kappa\lambda} - \cancel{\xi_{\kappa,\sigma\lambda}} + h_{\lambda\sigma,\kappa} - \cancel{\xi_{\lambda,\sigma\kappa}} - \xi_{\sigma,\lambda\kappa} - h_{\kappa\lambda,\sigma} + \cancel{\xi_{\kappa,\lambda\sigma}} + \cancel{\xi_{\lambda,\kappa\sigma}}) = \\
&= \Gamma^{\mu}{}_{\kappa\lambda} - \eta^{\mu\sigma}\xi_{\sigma,\kappa\lambda} = \Gamma^{\mu}{}_{\kappa\lambda} - \xi^{\mu}{}_{,\kappa\lambda}\ , \qquad (22.13)\\
R'_{\mu\nu\kappa\lambda} &= R_{\mu\nu\kappa\lambda} - \cancel{\xi_{\mu,\lambda\nu\kappa}} - \cancel{\xi_{\lambda,\mu\nu\kappa}} - \cancel{\xi_{\nu,\kappa\mu\lambda}} - \underline{\xi_{\kappa,\nu\mu\lambda}} + \cancel{\xi_{\mu,\kappa\nu\lambda}} + \underline{\xi_{\kappa,\mu\nu\lambda}} + \cancel{\xi_{\nu,\lambda\mu\kappa}} + \cancel{\xi_{\lambda,\nu\mu\kappa}} = \\
&= R_{\mu\nu\kappa\lambda}\,. \qquad (22.14)
\end{aligned}$$

So the Riemann tensor is *invariant* with respect to the infinitesimal diffeomorphisms (it is "gauge invariant"), and thus all the quantities computed from it. This however is not any special property of Riemann – in transformation of *any* tensorial quantity which is itself $O(\xi)$, only the Kronecker-delta term of the transformation matrices (22.9) contribute to that order, so the quantity remains unchanged. (Gammas do change, because they are not tensors.)

22.3.2 Understanding the infinitesimal diffeomorphisms

Worth to realize that the above paragraph actually was about the Lie derivative. Indeed, the Lie derivative was *derived* from the flow defined by an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ in Section 11.3.1 – there, however, written as $x_0^\mu \to x_0^\mu + \epsilon\xi^\mu$, with ϵ ensuring its infinitesimality (because ξ^μ itself did not have to be small there).[1] Subsequently, we found in Section 11.4 – equation (11.23) – that the metric changes under such a flow according to

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu,\iota}\xi^\iota + \xi^\iota{}_{,\mu}g_{\iota\nu} + \xi^\iota{}_{,\nu}g_{\mu\iota} = \xi_{\nu;\mu} + \xi_{\mu;\nu}\ .$$

Comparison with (22.10) reveals that[2]

$$g'_{\mu\nu}(x') - g_{\mu\nu}(x) = -\pounds_\xi g_{\mu\nu} + g_{\mu\nu,\iota}\xi^\iota\,.$$

After the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small including derivatives, one obtains

$$\begin{aligned}
&\cancel{\eta_{\mu\nu}} + h'_{\mu\nu}(x') - \cancel{\eta_{\mu\nu}} - h_{\mu\nu}(x) = -\pounds_\xi\eta_{\mu\nu} - \cancel{\pounds_\xi h_{\mu\nu}} + \cancel{\eta_{\mu\nu,\iota}}\xi^\iota + \cancel{h_{\mu\nu,\iota}\xi^\iota} \\
&\implies\quad h'_{\mu\nu}(x') - \cancel{\eta_{\mu\nu}} - h_{\mu\nu}(x) = -\pounds_\xi\eta_{\mu\nu} = -\xi_{\nu,\mu} - \xi_{\mu,\nu}\ ,
\end{aligned}$$

where the terms crossed out ~~this way~~ have been cancelled because they are "small squared" (being given by "$h\xi$" products).

[1] We apologize for such a notation difference. Namely, in the section on Lie derivative, it is necessary to keep ϵ *explicitely*, as a primary infinitesimal shift of parameter along the flow of a vector field ξ^μ. Actually, ϵ sometimes appeared *alone* there, without ξ^μ, as e.g. in the very definition of the Lie derivative (11.3). In the present chapter, on the contrary, ϵ would everywhere appear together with ξ^μ, so it is simpler to omit it and to bear in mind, instead, that ξ^μ has to be small, including all its derivatives.

[2] Yes, the Lie-derivative term is with minus in front, because in the Lie derivative one made the difference between the object pulled from $x + \xi$ *"back"* to x and the object "existing" at x, see (11.3), whereas in the present chapter the primed quantities mean those pushed *forward* from x to $x + \xi$ and the unprimed quantities mean those at x.

This in fact is a special case of a theorem describing the behaviour, under infinitesimal transformations, of tensor perturbations induced by small perturbation of the host space-time (with our perturbation specifically being described by $\eta \to \eta + h$): if, as a result of such a perturbation, some tensor T changes as $\underline{T} \to \underline{T} + \delta T$ (indices omitted), then in the transformation (22.8) the perturbation part changes according to

$$\delta T' = \delta T \pm \pounds_\xi \underline{T} \,,$$

with $\underline{T}$ denoting the tensor in the original space-time (in our case, $\underline{g_{\mu\nu}} \equiv \eta_{\mu\nu}$, $\delta g_{\mu\nu} \equiv h_{\mu\nu}$ and $h'_{\mu\nu} = h_{\mu\nu} - \pounds_\xi \eta_{\mu\nu}$, for example). The $\pm$ signs just distinguish whether one understands the transformation in the "active" way, as a shift (upper sign), or in the "passive" way, as a coordinate change (lower sign) – cf. footnote.

The behaviour of quantities under the infinitesimal coordinate translations – but also under "intrinsic" variations, independent of the coordinates – will again be important in Chapters 23 and 28.

22.3.3 Global Lorentz covariance

The linearised theory lies somewhere "between special and general relativity", so it may be expected to also involve some of the symmetries of special relativity. Actually, it possesses the *full*, global Lorentz (or in fact Poincaré) covariance. Applying to the metric tensor the Poincaré transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu \,, \quad x^\alpha = \Lambda_\mu{}^\alpha (x'^\mu - b^\mu) \,, \qquad \text{where} \quad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \,,$$

we have

$$\begin{aligned} &\eta_{\alpha\beta} + h'_{\alpha\beta} \;\equiv\; g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\, g_{\mu\nu} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu\,(\eta_{\mu\nu} + h_{\mu\nu}) = \eta_{\alpha\beta} + \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu\, h_{\mu\nu} \\ &\Longrightarrow \quad h'_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu\, h_{\mu\nu} \,. \end{aligned} \tag{22.15}$$

Often not stressed enough in textbooks is the precise meaning of the above. Namely, it is in fact possible to decompose *any* metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, even globally – and irrespectively of whether $h_{\mu\nu}$ is small or not. Also always possible is to make a Lorentz or even Poincaré transformation (satisfying the orthogonality relations). Such transformation indeed holds between different possible locally orthonormal frames of observers moving through *some* (one and the same) space-time point. To be stressed here is that, in the linearised theory, the special-relativity properties are *global*, not just local as in GR (as ensured there by the equivalence principle): the quasi-Minkowskian coordinates – those in which the metric is almost Minkowskian (i.e. in which h's are small) – are global, and the Poincaré transformations between different such coordinates are global as well (not specific for every point).

Since $h_{\mu\nu}$ behaves like a flat-space tensor, it is also true for the curvature tensors, thus for the energy-momentum tensor, which in turn requires that the matter and non-gravitational fields behave like in special relativity as well. (Note that even affine connection behaves in a tensorial manner under Lorentz transformations, as under any linear transformation.) This feature offers an alternative way how to understand the theory of weak gravitational fields:

not as an approximation to some non-linear theory (GR), but as an autonomous theory of a symmetric tensor field $h_{\mu\nu}$ on the Minkowski background. Such a view is sometimes called **field-theoretical**, as opposed to **geometrical**, since the gravitational field has no relation to the space-time geometry in it. The geometrical view is natural when one wants to employ the tools of the full GR, mainly when wishing to compare these theories, while the field-theoretical view is useful for comparison with electrodynamics (to which linearised theory is similar in many respects), and also in the "field" approach towards quantization of the theory.

22.4 Einstein equations as a wave equation

The practical value of the diffeomorphism freedom is that one is often able to "transform out" spurious, purely coordinate-dependent features of the theory, arriving thus at its simple and physically sound account. In electromagnetism, one makes use of the gauge freedom which consists in the possibility to add to the four-potential a gradient of any scalar function, $A_\alpha \to A_\alpha + \chi_{,\alpha}$. In order to obtain, from the 1st set of Maxwell equations, a simple form of the wave equation, $\Box A^\alpha = -\mu J^\alpha$, the gauge freedom is most often being fixed by requiring the **Lorenz condition** $A^\alpha{}_{,\alpha} = 0$; this can be ensured by taking such a χ which satisfies the equation $\Box \chi = -A^\alpha{}_{,\alpha}$. The last equation shows immediately that the Lorenz condition does not fix the gauge freedom completely, specifically, one may still change A_α by gradient of any function whose d'Alembertian vanishes, $\Box \chi = 0$.

22.4.1 Harmonic condition, Lorenz condition

In GR, the Lorenz condition is the linearized version of the **harmonic coordinate condition**. Harmonic coordinates are such which satisfy $\Box x^\mu = 0$, where x^μ are understood here as a set of four scalars, hence

$$(x^\mu)^{;\alpha} = g^{\alpha\beta}(x^\mu)_{;\beta} \equiv g^{\alpha\beta}(x^\mu)_{,\beta} = g^{\alpha\beta}\delta^\mu_\beta = g^{\alpha\mu} \,.$$

Now attention is at place: further covariant derivative might simply seem to give zero on the metric, $g^{\alpha\mu}{}_{;\alpha} = 0$, but this is *not* the case here, because *thus obtained* $g^{\alpha\mu}$ should *not* be considered a tensor, specifically, μ is *not* a tensor index – it only labelled the coordinates, it does *not* transform properly. Hence, one does *not* know how to perform a covariant derivative of such a quantity. However, a *partial* derivative certainly works "on anything", so we simply write the d'Alembert in terms of the partial derivative and continue,

$$0 = \Box x^\mu \equiv (x^\mu)^{;\alpha}{}_\alpha = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}\,(x^\mu)^{;\alpha}\right]_{,\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{\alpha\mu}\right)_{,\alpha}$$
$$\Longrightarrow \quad \left(\sqrt{-g}\,g^{\alpha\mu}\right)_{,\alpha} = 0 \,.$$

Finding the linear approximation of the metric determinant (directly from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$),

$$\begin{aligned} g &= (-1 + h_{00})(1 + h_{11})(1 + h_{22})(1 + h_{33}) + O(h^2) = \\ &= -1 + h_{00} - h_{11} - h_{22} - h_{33} + O(h^2) = \\ &= -1 - h^0_0 - h^1_1 - h^2_2 - h^3_3 + O(h^2) \;\equiv\; -1 - h^\iota_\iota + O(h^2) \;\equiv\; -1 - h \,, \end{aligned} \tag{22.16}$$

we have

$$\sqrt{-g}\, g^{\alpha\mu} = \sqrt{1+h}\,(\eta^{\alpha\mu} - h^{\alpha\mu}) = \left(1 + \frac{h}{2}\right)(\eta^{\alpha\mu} - h^{\alpha\mu}) + O(h^2) =$$

$$= \eta^{\alpha\mu} - h^{\alpha\mu} + \frac{h}{2}\,\eta^{\alpha\mu} + O(h^2)\,,$$

hence the harmonic condition yields

$$\left(h^{\alpha\mu} - \frac{h}{2}\,\eta^{\alpha\mu}\right)_{,\alpha} = 0\,, \qquad \text{i.e.} \qquad \gamma^{\alpha\mu}{}_{,\alpha} = 0, \quad \text{where} \quad \gamma^{\alpha\mu} := h^{\alpha\mu} - \frac{h}{2}\,\eta^{\alpha\mu}\,. \tag{22.17}$$

In GR, the harmonic condition is also called the de Donder or the Hilbert condition. In the linearized theory, it is usually called the Lorenz condition due to its form clearly similar to that of $A^\alpha{}_{,\alpha} = 0$ known from electrodynamics. The condition was first used by Einstein himself in his *Zürich Notebook* likely dated August 1912. Einstein in fact derived the *correct* linearized theory there, the same we are presenting here (at that time testing the Ricci tensor as the left-hand side of the field equations). Later, however, he abandoned this direction due to a seeming disagreement with the Newtonian limit of the theory; he only returned to it in 1915, shortly before finishing GR. See *The Collected Papers of Albert Einstein*, Vol. 4, p. 201 – document 10 (Princeton Univ. Press, online thanks to the Einstein Papers Project).

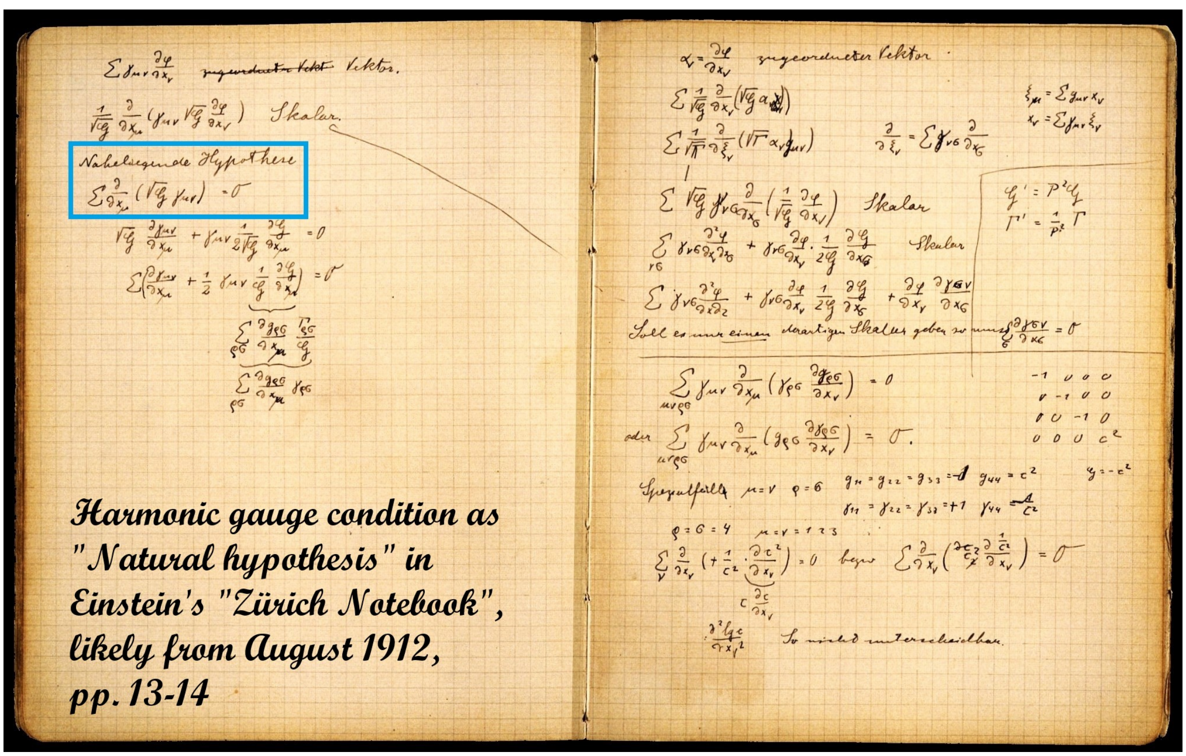

Figure 22.1: The harmonic condition, as first written by Einstein, in almost "our" notation.

22.4.2 Simplification of the Einstein equations

With the hint provided by the above Lorenz condition, let us check how the Einstein equations (22.7) look in terms of $\gamma_{\mu\nu}$. Substituting

$$h = -\gamma, \qquad h^{\mu\nu} = \gamma^{\mu\nu} + \frac{h}{2}\,\eta^{\mu\nu} = \gamma^{\mu\nu} - \frac{\gamma}{2}\,\eta^{\mu\nu}$$

into (22.7), we obtain, on the left-hand side,

$$\gamma^{\kappa}_{\nu,\kappa\mu} - \cancel{\frac{1}{2}\gamma_{,\nu\mu}} + \gamma^{\kappa}_{\mu,\kappa\nu} - \cancel{\frac{1}{2}\gamma_{,\mu\nu}} + \cancel{\gamma_{,\mu\nu}} - \Box\,\gamma_{\mu\nu} + \cancel{\frac{1}{2}\,\eta_{\mu\nu}\Box\,\gamma} - \eta_{\mu\nu}\left(\gamma^{\kappa\lambda}{}_{,\kappa\lambda} - \cancel{\frac{1}{2}\Box\,\gamma} + \cancel{\Box\,\gamma}\right),$$

so we have

$$\Box\,\gamma_{\mu\nu} - \gamma^{\kappa}_{\nu,\kappa\mu} - \gamma^{\kappa}_{\mu,\kappa\nu} + \eta_{\mu\nu}\gamma^{\kappa\lambda}{}_{,\kappa\lambda} = -16\pi T_{\mu\nu}\ . \tag{22.18}$$

The left-hand side would simplify to just $\Box\,\gamma_{\mu\nu}$ if the Lorenz condition $\gamma^{\kappa}_{\mu,\kappa} = 0$ held. The only possibility to ensure that is to make a suitable infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu$. Regarding that $h_{\mu\nu}$ changes in such a transformation according to (22.11), we have for $\gamma_{\mu\nu}$

$$\gamma'_{\mu\nu} \equiv h'_{\mu\nu} - \frac{h'}{2}\,\eta_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{h}{2}\,\eta_{\mu\nu} + \xi^{\iota}{}_{,\iota}\eta_{\mu\nu} =$$

$$= \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\iota}{}_{,\iota} \tag{22.19}$$

$$\Longrightarrow \quad \gamma'^{\kappa}_{\mu,\kappa} = \gamma^{\kappa}_{\mu,\kappa} - \Box\,\xi_\mu - \cancel{\xi^{\kappa}{}_{,\mu\kappa}} + \cancel{\delta^{\kappa}_{\mu}\xi^{\iota}{}_{,\iota\kappa}}\ , \tag{22.20}$$

so, quite like in electrodynamics, the Lorenz condition is achieved by the transformation with ξ^μ satisfying the wave equation

$$\Box\,\xi_\mu = \gamma^{\kappa}_{\mu,\kappa}\ . \tag{22.21}$$

We have thus simplified the Einstein equations into the wave-equation form

$$\boxed{\Box\,\gamma_{\mu\nu} = -16\pi T_{\mu\nu}}\ . \tag{22.22}$$

- The analogy with electrodynamics is pretty clear now. The solutions will undoubtedly have very similar properties as their electromagnetic counterparts. In particular, if there are no incoming waves, the solution to the above equation can in analogy with electrodynamics be written as

$$\gamma_{\mu\nu}(t,\vec{x}) = 4\int \frac{T_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'\,, \tag{22.23}$$

  where $(t,\vec{x})$ fixes location where the field is being computed and $\vec{x}'$ scans through the source volume (support of $T_{\mu\nu}$).

- In particular, it is clear now that weak gravitational disturbances propagate with the speed of light. One might query whether the "generic" Einstein equations (22.18) (not involving the Lorenz condition) do not yield solutions with different speeds of propagation (we will fix this in detail later), yet one can give a coordinate-independent answer now already: in a vacuum, the Einstein equations reduce to $\Box\gamma_{\mu\nu}=0$, thus also $\Box\gamma=0$, and so the Riemann tensor (22.3) satisfies $\Box R_{\mu\nu\kappa\lambda}=0$ as well. Since Riemann is a Lorentz tensor and since it is gauge independent, this wave equations manifests that the curvature disturbances propagate with the speed of light. Note that in the flat-space-time source-free electrodynamics, the EM-field tensor also satisfies $\Box F_{\mu\nu}=0$.

- At this stage, without any further restrictions, the metric has 6 degrees of freedom: as a symmetric tensor, $\gamma_{\mu\nu}$ has 10 independent components, and the Lorenz condition $\gamma^{\kappa}_{\mu,\kappa}=0$ represents 4 constraints. The residual freedom (to change the coordinate system while still complying with the Lorenz condition) is as wide as the space of solutions of the equation $\Box\xi_\mu=0$. Such a solution can be written as an integral linear combination of $\xi_\mu=\hat{\xi}_\mu\sin(k_\sigma x^\sigma)$ over all possible null k^μ vectors, where the amplitude $\hat{\xi}_\mu$ may be chosen arbitrarily. This means 4 free components. Therefore, we expect to be able to fix 4 more degrees of freedom.

- Note the difference between the nature of the gauge freedom in electrodynamics and the diffeomorphism freedom in gravitation: in electrodynamics, it is the potential A_α which can be "gauged", while in gravitation the "potential" (the metric) is only "gauged" indirectly by an infinitesimal adjustment of coordinates. We will more think about it in Section 28.2.3.

22.4.3 Field of a quasi-Newtonian finite stationary source

As an example of how to evaluate the solution (22.23), and, at the same time, as a supplement to what we learned by Newtonian limit of the geodesic equation (that $g_{00}=-1-2\Phi$), we will find the linearised metric generated by a quasi-Newtonian finite stationary source (a "star"). Assuming that stationarity means that the source is unchanging in the nearly-Minkowskian coordinates consistent with the Lorenz condition, we can omit the time dependence and neglect the retardation in (22.23),

$$\gamma_{\mu\nu}(\vec{x})=4\int\frac{T_{\mu\nu}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'\,.$$

Consider now a quasi-Newtonian body, i.e. a weak source (thus within the scope of the linearised theory) whose all elements move, with respect to the $(t,\vec{x})$ coordinates, so slowly that the terms of the order $O(v^2)$ can be neglected. Neglecting also stresses in the source, we have for its energy-momentum tensor the incoherent-dust form $T_{\mu\nu}=\rho u_\mu u_\nu$, i.e. (see Section 7.2.1, with quadratic terms in velocity neglected and with hats omitted)

$$T_{00}=\rho\,,\qquad T_{0j}=-\rho v_j\,,\qquad T_{ij}=\rho v_i v_j=O(v^2)\,,$$

so the integral(s) yield

$$\gamma_{00} = 4\int \frac{\rho(x')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' = -4\Phi\,, \quad \gamma_{0j} = -4\int \frac{\rho(x')v_j(x')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' =: -4A_j\,, \quad \gamma_{ij} = O(v^2)\,,$$

where the A_j notation has been employed in analogy with electrodynamics. Computing

$$\gamma \equiv \eta^{\mu\nu}\gamma_{\mu\nu} = -\gamma_{00} = 4\Phi$$

and returning to $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{\gamma}{2}\,\eta_{\mu\nu}$, we find

$$\begin{aligned} &h_{00} = \gamma_{00} + \frac{\gamma}{2} = -2\Phi\,, \qquad h_{0j} = \gamma_{0j} = -4A_j\,, \qquad h_{ij} = -\frac{\gamma}{2}\,\eta_{ij} = -2\Phi\delta_{ij}\,, \qquad \text{i.e.} \\ &\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 - 8A_j\mathrm{d}t\,\mathrm{d}x^j + (1-2\Phi)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\,. \end{aligned} \tag{22.24}$$

Besides confirming the already known Newtonian-limit relation $g_{00} = -1-2\Phi$, we have thus newly learned how the remaining metric components are perturbed from their Minkowskian values.

At large distances from the source, where $|\vec{x}| \gg |\vec{x}'|$, we may resort to a monopole + dipole approximation

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + O(1/r^3)\ ,$$

plugging the monopole term in the electrostatic-like scalar potential,

$$\text{monopole:}\quad \frac{1}{r} \quad\Longrightarrow\quad \Phi \sim -\frac{1}{r}\int \rho(x')\,\mathrm{d}^3x' = -\frac{M}{r}\ , \tag{22.25}$$

and the dipole term in the magnetostatic-like vector potential (we assume to be in the centre-of-mass system where the monopole term does not contribute to $\vec{A}$),

$$\text{dipole:}\quad \frac{\vec{x}\cdot\vec{x}'}{r^3} \quad\Longrightarrow\quad \vec{A} \sim \frac{1}{r^3}\int (\vec{x}\cdot\vec{x}')\,\rho(x')\vec{v}(x')\,\mathrm{d}^3x' \sim \frac{\vec{J}\times\vec{x}}{2r^3}\,, \tag{22.26}$$

where $\vec{J} = \int \vec{x}'\times\rho(x')\vec{v}(x')\,\mathrm{d}^3x'$ is the rotational angular momentum ("spin") of the source, as evaluated in the latter's centre-of-mass coordinates.

To be on the safe side, let us support the dipole formula for $\vec{A}$ by the derivation known from computation of the field of a magnetic dipole moment. Rewrite first the integrand using the "BAC-CAB" rule,

$$\vec{x}\times(\vec{x}'\times\vec{\jmath}') = \vec{x}'(\vec{x}\cdot\vec{\jmath}') - \vec{\jmath}'\,(\vec{x}\cdot\vec{x}') \quad\Longrightarrow\quad (\vec{x}\cdot\vec{x}')\,\vec{\jmath}' - \vec{x}'(\vec{x}\cdot\vec{\jmath}') = (\vec{x}'\times\vec{\jmath}')\times\vec{x}\ ,$$

where $\vec{\jmath}' := \rho(x')\vec{v}(x')$. The left-hand side actually yields twice the original integral,

$$\int (\vec{x}\cdot\vec{x}')\,\vec{\jmath}'\ \mathrm{d}^3x' - \int \vec{x}'(\vec{x}\cdot\vec{\jmath}')\ \mathrm{d}^3x' = 2\int (\vec{x}\cdot\vec{x}')\,\vec{\jmath}'\ \mathrm{d}^3x'\ ,$$

while the right-hand-side term yields the announced result, so

$$\int (\vec{x}\cdot\vec{x}')\,\vec{\jmath}'\,\mathrm{d}^3x' = \frac{1}{2}\int (\vec{x}'\times\vec{\jmath}')\times\vec{x}\,\mathrm{d}^3x' = \frac{1}{2}\int (\vec{x}'\times\vec{\jmath}')\,\mathrm{d}^3x'\times\vec{x} \equiv \frac{1}{2}\vec{J}\times\vec{x}\,.$$

It remains to prove the former assertion. Consider the divergence

$$\begin{aligned}\vec{\nabla}'\cdot(x'^i x'^j \vec{\jmath}') &\equiv \frac{\partial}{\partial x'^k}(x'^i x'^j j'^k) = \\ &= \delta^i_k x'^j j'^k + x'^i \delta^j_k j'^k + x'^i x'^j \frac{\partial j'^k}{\partial x'^k} = x'^j j'^i + x'^i j'^j + x'^i x'^j (\vec{\nabla}'\cdot\vec{\jmath}')\,.\end{aligned}$$

Assume now conservation of charge (here mass) as in magnetostatics (stationary Ampère's law $\vec{\nabla}\times\vec{B} = 4\pi\vec{J}$ implies $\vec{\nabla}\cdot\vec{J}=0$), i.e. $\vec{\nabla}'\cdot\vec{\jmath}'=0$, and integrate the rest over d^3x', using the Gauss law for the first term,

$$\int \vec{\nabla}'\cdot(x'^i x'^j \vec{\jmath}')\,\mathrm{d}^3x' = \oint (x'^i x'^j \vec{\jmath}')\cdot\mathrm{d}\vec{S}' = 0$$

(it vanishes for any finite body since the integration surface may always be chosen far enough, where $\vec{\jmath}'=0$). Finally, multiply by x_j the relation thus found, $\int (x'^j j'^i + x'^i j'^j)\,\mathrm{d}^3x' = 0$, obtaining

$$\int (x_j x'^j j'^i + x'^i x_j j'^j)\,\mathrm{d}^3x' = 0\,, \quad \text{i.e.} \quad \int (\vec{x}\cdot\vec{x}')\,\vec{\jmath}'\,\mathrm{d}^3x' = -\int \vec{x}'(\vec{x}\cdot\vec{\jmath}')\,\mathrm{d}^3x'\,,$$

which is exactly what needed to be proved.

To summarize, in the centre-of-mass coordinates, the far field of the stationary source can approximately be described by

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 - \frac{4}{r^3}\,\epsilon_{jkl}J^k x^l\,\mathrm{d}t\,\mathrm{d}x^j + \left(1+\frac{2M}{r}\right)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\,. \quad (22.27)$$

We may check this result against the far-field of the Kerr source (Section 16.2). At large radii, $r \gg M\ (\sim a)$, one has, in the Boyer-Lindquist coordinates (far away becoming spherical),

$$\begin{aligned} g_{tt} &= -1 + \frac{2Mr}{r^2+a^2\cos^2\theta} \sim -1 + \frac{2M}{r}\,, \qquad g_{t\phi} = -\frac{2Jr\sin^2\theta}{r^2+a^2\cos^2\theta} \sim -\frac{2J}{r}\sin^2\theta\,, \\ g_{rr} &= \frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2} \sim \frac{r^2}{r^2-2Mr} = \frac{1}{1-\frac{2M}{r}} \sim 1+\frac{2M}{r}\,. \end{aligned}$$

To properly compare the non-diagonal term $-\frac{4}{r^3}\,\epsilon_{jkl}J^k x^l$, one adjusts the Cartesian coordinates so that $\vec{J} = (0,0,J)$ and then transforms to the corresponding spherical coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ (in terms of which $x\mathrm{d}y - y\mathrm{d}x = r^2\sin^2\theta\,\mathrm{d}\phi$):

$$-\frac{4}{r^3}\,\epsilon_{jzl}J^z x^l \mathrm{d}x^j = -\frac{4}{r^3}\,J\epsilon_{zxy}(x\mathrm{d}y - y\mathrm{d}x) = -\frac{4}{r^3}\,Jr^2\sin^2\theta\,\mathrm{d}\phi = -\frac{4J}{r}\sin^2\theta\,\mathrm{d}\phi\,.$$

22.5 Plane harmonic gravitational waves

The wave solution of the linearised gravitational law also proceeds in the same manner as in electrodynamics. Considering the vacuum situation, $T_{\mu\nu}=0$, one starts from the ansatz for a monochromatic plane harmonic wave,

$$\gamma_{\mu\nu} = \hat{\gamma}_{\mu\nu}\cos(k_\sigma x^\sigma)\,, \qquad \text{with } \ \hat{\gamma}_{\mu\nu} \text{ (amplitude) and } k^\alpha \text{ (wave vector) constant}$$
$$\Longrightarrow \quad \gamma_{\mu\nu,\alpha} = -\hat{\gamma}_{\mu\nu}\sin(k_\sigma x^\sigma)\,k_\alpha\,, \quad \gamma_{\mu\nu,\alpha\beta} = -\gamma_{\mu\nu}k_\alpha k_\beta\,, \quad \Box\,\gamma_{\mu\nu} = -\gamma_{\mu\nu}\eta^{\alpha\beta}k_\alpha k_\beta\,.$$

Therefore, the wave equation dictates that the wave vector has to be null, while the Lorenz conditions dictates that it has to be orthogonal to $\hat{\gamma}_{\mu\nu}$,

$$\Box\,\gamma_{\mu\nu} = 0 \quad \Longrightarrow \quad \eta_{\mu\nu}k^\mu k^\nu = 0\,, \qquad \gamma_{\mu\kappa}{}^{,\kappa} = 0 \quad \Longrightarrow \quad \hat{\gamma}_{\mu\kappa}k^\kappa = 0\,. \tag{22.28}$$

The wave thus propagates with the speed of light and is transversal with respect to k^α.

From the covariant equation $\Box\,R_{\mu\nu\kappa\lambda}=0$ we inferred that only the waves travelling with the speed of light can have physical relevance. On the other hand, it is clear that without the Lorenz condition the gravitational law does *not* reduce to the d'Alembert equation, so it should also have solutions propagating with different speeds. Let us check what exactly is the nature of those solutions. Plugging $\gamma_{\mu\nu}=\hat{\gamma}_{\mu\nu}\cos(k_\sigma x^\sigma)$ into the *generic* form (22.18) of the linearised field equations (not involving Lorenz condition), we get, in the vacuum case again,

$$-\gamma_{\mu\nu}\eta^{\alpha\beta}k_\alpha k_\beta + \gamma^\kappa_\nu k_\kappa k_\mu + \gamma^\kappa_\mu k_\kappa k_\nu - \eta_{\mu\nu}\gamma_{\kappa\lambda}k^\kappa k^\lambda = 0 \qquad \ldots \quad \text{multiply by } \eta^{\mu\nu} \tag{22.29}$$
$$\Longrightarrow \quad -\gamma^\mu_\mu\,\eta^{\alpha\beta}k_\alpha k_\beta - 2\gamma_{\kappa\lambda}k^\kappa k^\lambda = 0\,. \tag{22.30}$$

- With k^α null, (22.30) implies $\gamma_{\kappa\lambda}k^\kappa k^\lambda=0$ and thus (22.29) implies $\gamma^\kappa_\nu k_\kappa k_\mu + \gamma^\kappa_\mu k_\kappa k_\nu = 0$. The former relation can hold if $\gamma_{\kappa\lambda}k^\kappa$ is either zero or proportional to k_λ. The latter is impossible, however, because the second relation $\gamma^\kappa_\nu k_\kappa k_\mu + \gamma^\kappa_\mu k_\kappa k_\nu = 0$ would in such a case require $k_\mu k_\nu = 0$, which only has trivial solution $k_\mu=0$. Hence, if $k_\alpha k^\alpha=0$, there also holds $\gamma^\kappa_\mu k_\kappa=0$, i.e., the light-speed solution satisfies the Lorenz condition *automatically*.

- If k^α is not null (the wave does not propagate with the speed of light), it is still possible to ensure the Lorenz condition. Actually, regarding that this requires to perform a transformation $x^\mu \to x^\mu + \xi^\mu$ with ξ^μ satisfying (22.21), i.e. $\Box\,\xi^\mu = \gamma^{\mu\kappa}{}_{,\kappa}$, and that the wave form of $\gamma^{\mu\kappa}$ satisfies $\Box\,\gamma^{\mu\kappa} = -\gamma^{\mu\kappa}\eta^{\alpha\beta}k_\alpha k_\beta$, we see that the right generator reads $\xi^\mu = -\frac{\gamma^{\mu\kappa}{}_{,\kappa}}{\eta^{\alpha\beta}k_\alpha k_\beta}$. Now look how $\gamma_{\mu\nu}$ itself behaves under such a transformation: according to the rule (22.19) and thanks to the above relations

$$\gamma_{\mu\nu,\alpha\beta} = -\gamma_{\mu\nu}k_\alpha k_\beta \qquad \Longrightarrow \qquad \xi_{\mu,\nu} = -\frac{\gamma^\kappa_{\mu,\kappa\nu}}{\eta^{\alpha\beta}k_\alpha k_\beta} = \frac{\gamma^\kappa_\mu\,k_\kappa k_\nu}{\eta^{\alpha\beta}k_\alpha k_\beta}\,,$$

one finds

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\iota{}_{,\iota} = \gamma_{\mu\nu} - \frac{\gamma^\kappa_\mu\,k_\nu k_\kappa + \gamma^\kappa_\nu\,k_\mu k_\kappa - \eta_{\mu\nu}\gamma^{\iota\kappa}k_\iota k_\kappa}{\eta^{\alpha\beta}k_\alpha k_\beta}\,.$$

Multiplying this by $\eta^{\alpha\beta}k_\alpha k_\beta$ and using the field equations (22.29) on the right-hand side, we obtain

$$\gamma'_{\mu\nu}\eta^{\alpha\beta}k_\alpha k_\beta = 0$$

Since $\eta^{\alpha\beta}k_\alpha k_\beta \neq 0$ by assumption, $\gamma'_{\mu\nu}$ must vanish – the waves have been transformed out. The waves which do not travel at the speed of light are thus of mere coordinate nature, they have no physical significance. (Which also confirms that the Lorenz condition can be posed without losing any important solutions.)

Identically vanishing $\gamma_{\mu\nu}$ obviously implies vanishing Riemann tensor (which is gauge invariant, so if it has been found zero in one system, it has to be such in general). The situation is similar to that in electromagnetism: the waves which do not travel with the speed of light can be transformed out by a suitable gauge transformation (of potential), and their $F_{\mu\nu}$ tensor vanishes.

22.5.1 Physical degrees of freedom: fixing the coordinates

Since the coordinate freedom still remaining after imposing the Lorenz condition is spanned by solutions to the equation $\Box\xi^\mu = 0$, it is *four-dimensional*. A canonical option how to fix it is through the so-called **transverse and traceless (TT) condition** – an analogy of the **radiation gauge** (or Coulomb gauge) $A_0 = 0$ from electrodynamics.

- *Transverse* means that the potentials satisfy $\gamma_{0\nu} = 0$; in a covariant way, this is expressed in terms of projection on *some* time-like vector field, say u^μ: let an observer with four-velocity u^μ exist such that $\gamma_{\mu\nu}u^\mu = 0$. Since one projection of $\gamma_{\mu\nu}$ is already fixed by the Lorenz condition, $\gamma_{\mu\nu}k^\nu = 0$, the new requirement only represents *three* independent constraints ($\gamma_{\mu\nu}u^\mu = 0$ are four, but they are bound by the relation $\gamma_{\mu\nu}u^\mu k^\nu = 0$).

- *Traceless* means that the tensor $\gamma_{\mu\nu}$ has zero trace, $\eta^{\mu\nu}\gamma_{\mu\nu} = 0$. This is clearly one constraint. From the transformation behaviour (22.19) of $\gamma_{\mu\nu}$, i.e. $\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\iota{}_{,\iota}$, we easily see that to achieve tracelessness $\gamma'^{\mu}_{\mu} = 0$, we need ξ^μ which fulfils $\xi^\iota{}_{,\iota} = -\gamma/2$. Plugging this to the transversality requirement $\gamma'_{\mu\nu}u^\mu = 0$, one obtains for ξ_μ the equation

  $$(\xi_{\mu,\nu} + \xi_{\nu,\mu})u^\mu = (\gamma_{\mu\nu} + \eta_{\mu\nu}\xi^\iota{}_{,\iota})\,u^\mu = \left(\gamma_{\mu\nu} - \frac{\gamma}{2}\,\eta_{\mu\nu}\right)u^\mu \equiv h_{\mu\nu}u^\mu\,.$$

  Multiplying this by k^ν and using the Lorenz condition $\gamma_{\mu\nu}k^\nu = 0$ (supposed to be satisfied by the original $\gamma_{\mu\nu}$), one has

  $$(\xi_{\mu,\nu} + \xi_{\nu,\mu})u^\mu k^\nu = -\frac{\gamma}{2}\,k_\mu u^\mu\,.$$

- Explicit derivation of how ξ^μ has to look uses the fact that it must satisfy $\Box\xi^\mu = 0$. Choosing it in the form $\xi^\mu = \hat{\xi}^\mu\sin(k_\sigma x^\sigma)$, with $\hat{\xi}^\mu$ the amplitude again, one has $\xi_{\mu,\nu} = \hat{\xi}_\mu k_\nu\cos(k_\sigma x^\sigma)$, so the transformation (22.19) of $\gamma_{\mu\nu}$ assumes, in terms of the corresponding amplitudes, the form

  $$\hat{\gamma}'_{\mu\nu} = \hat{\gamma}_{\mu\nu} - \hat{\xi}_\mu k_\nu - \hat{\xi}_\nu k_\mu + \eta_{\mu\nu}\hat{\xi}^\iota k_\iota\,,$$

and the above derived conditions thus appear as

$$\hat{\xi}^\iota k_\iota = -\frac{\hat{\gamma}}{2} \text{ (tracelessness)} \qquad \longrightarrow \qquad (\hat{\xi}_\mu k_\nu + \hat{\xi}_\nu k_\mu)u^\mu = \hat{h}_{\mu\nu}u^\mu \text{ (transversality)}\,.$$

Multiplying the second relation by u^ν, one obtains prescription for

$$\hat{\xi}_\mu u^\mu = \frac{\hat{h}_{\mu\nu}u^\mu u^\nu}{2k_\lambda u^\lambda}\,,$$

so substituting this back to the same transversality relation yields

$$\hat{\xi}_\nu = \frac{\hat{h}_{\mu\nu}u^\mu - k_\nu \hat{\xi}_\mu u^\mu}{k_\iota u^\iota} = \frac{\hat{h}_{\mu\beta}u^\mu}{k_\iota u^\iota}\left(\delta^\beta_\nu - \frac{u^\beta k_\nu}{2u^\lambda k_\lambda}\right)\,. \tag{22.31}$$

- If ξ^μ is not the question, one may achieve the TT-condition metric from a more geometrical perspective:

  $\boxed{\text{Lemma}}$ A transverse and traceless metric satisfying the Lorenz condition can be obtained, from a generic metric $\gamma_{\alpha\beta}$, by

$$\gamma^{\rm TT}_{\mu\nu} = \left(P^\alpha_\mu P^\beta_\nu - \frac{1}{2}P_{\mu\nu}P^{\alpha\beta}\right)\gamma_{\alpha\beta}\,, \tag{22.32}$$

$$\text{where}\quad P^\alpha_\mu := \delta^\alpha_\mu + u^\alpha u_\mu - n^\alpha n_\mu\,,\quad n^\alpha := \frac{(\delta^\alpha_\lambda + u^\alpha u_\lambda)k^\lambda}{\sqrt{g_{\rho\sigma}\,(\delta^\rho_\iota + u^\rho u_\iota)k^\iota\,(\delta^\sigma_\kappa + u^\sigma u_\kappa)k^\kappa}}\,.$$

  In order to prove the claim, consider that the above realizes a straightforward geometrical way how to satisfy the three requirements

$$\gamma^{\rm TT}_{\mu\nu}k^\nu = 0 \text{ (Lorenz condition)}\,,\ \gamma^{\rm TT}_{\mu\nu}u^\nu = 0 \text{ (transversality)}\,,\ \eta^{\mu\nu}\gamma^{\rm TT}_{\mu\nu} = 0 \text{ (tracelessness)}\,.$$

  For a given k^μ and a chosen u^μ, one first defines the unit spatial vector n^μ along the projection of k^μ to the 3D space orthogonal to u^μ (the square-root denominator just ensures that $g_{\rho\sigma}n^\rho n^\sigma = 1$); n^μ thus represents a direction (of wave propagation) in the space of the observer with four-velocity u^μ. To be normal to both u^μ and k^μ means to be normal to the plane spanned by u^μ and n^μ, hence the projection of $\gamma_{\alpha\beta}$ by the corresponding two-metric P^α_μ. Finally, the standard operation how to make a tensor traceless is to simply subtract its trace multiplied by the pertinent metric,

$$(\text{traceless tensor})_{\mu\nu} = \text{tensor}_{\mu\nu} - \frac{\text{trace of the tensor}}{\text{trace of the metric}}\,\text{metric}_{\mu\nu}\,.$$

  From the automatic properties

$$u_\alpha u^\alpha = -1,\ \ n_\alpha n^\alpha = 1,\ \ u_\alpha n^\alpha = 0,\ \ P^\alpha_\mu P^{\beta\mu} = P^{\alpha\beta},\ \ P^\beta_\nu u^\nu = 0,\ \ P^\beta_\nu n^\nu = 0,\ \ P^\alpha_\alpha = 2\,,$$

  one immediately confirms the tracelessness and transversality, and by expressing (from the definition of n^α)

$$k^\nu = \sqrt{\ldots}\;n^\nu - u^\nu u_\lambda k^\lambda\,,$$

  one also sees that $P^\beta_\nu k^\nu = 0$, so the Lorenz condition is satisfied as well. □

Since the traces of $\gamma_{\mu\nu}$ and $h_{\mu\nu}$ satisfy $\gamma^{\iota}_{\iota} = -h^{\iota}_{\iota}$, in the *traceless* case $\gamma_{\mu\nu}$ directly represents deviation from the Minkowski metric ($h_{\mu\nu}$), $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} = h_{\mu\nu}$. The usual parameterization of the remaining, physical two degrees of freedom (two "polarizations") is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{22.33}$$

This form represents a plane harmonic wave travelling in the z-direction. Actually, for such a wave the wave vector reads $k^{\mu} = \omega(1, 0, 0, 1)$, so the Lorenz condition $h_{\mu\nu}k^{\nu} = 0$ is clearly satisfied. The "TT" properties are obvious as well, in particular, the transversality $h_{\mu\nu}u^{\mu} = 0$ holds (e.g.) for the time-like vector field $u^{\mu} = (u^t, 0, 0, 0)$. Let us add, from (22.3), that there are only two non-trivial and independent components of the Riemann tensor (we perform the calculation in the TT coordinates, yet recall that linearised Riemann is gauge independent!),

$$R_{i0j0} = \frac{1}{2}\left(\cancel{h^{\rm TT}_{i0,0j}} + \cancel{h^{\rm TT}_{0j,i0}} - h^{\rm TT}_{ij,00} - \cancel{h^{\rm TT}_{00,ij}}\right) = -\frac{1}{2}\, h^{\rm TT}_{ij,00} \,, \tag{22.34}$$

$$R_{i3j3} = \frac{1}{2}\left(\cancel{h^{\rm TT}_{i3,3j}} + \cancel{h^{\rm TT}_{3j,i3}} - h^{\rm TT}_{ij,33} - \cancel{h^{\rm TT}_{33,ij}}\right) = -\frac{1}{2}\, h^{\rm TT}_{ij,33} = -\frac{1}{2}\, h^{\rm TT}_{ij,00} = R_{i0j0} \,, \tag{22.35}$$

$$R_{i0j3} = \frac{1}{2}\left(\cancel{h^{\rm TT}_{i0,3j}} + \cancel{h^{\rm TT}_{3j,i0}} - h^{\rm TT}_{ij,03} - \cancel{h^{\rm TT}_{03,ij}}\right) = -\frac{1}{2}\, h^{\rm TT}_{ij,03} \,, \tag{22.36}$$

$$R_{i3j0} = \frac{1}{2}\left(\cancel{h^{\rm TT}_{i3,0j}} + \cancel{h^{\rm TT}_{0j,i3}} - h^{\rm TT}_{ij,30} - \cancel{h^{\rm TT}_{30,ij}}\right) = R_{i0j3} \,, \tag{22.37}$$

where in the second line the wave equation was used,

$$\eta^{\alpha\beta}h^{\rm TT}_{\mu\nu,\alpha\beta} = -h^{\rm TT}_{\mu\nu,00} + \cancel{h^{\rm TT}_{\mu\nu,11}} + \cancel{h^{\rm TT}_{\mu\nu,22}} + h^{\rm TT}_{\mu\nu,33} = 0 \,,$$

with the middle terms vanishing since $h^{\rm TT}_{\mu\nu,1} \sim \hat{h}_{\mu\nu}k_1$, $h^{\rm TT}_{\mu\nu,2} \sim \hat{h}_{\mu\nu}k_2$, and k_{μ} does not have the k_1 and k_2 components. The Riemann-tensor wave equation $\Box R_{\mu\nu\kappa\lambda} = 0$ clearly holds, because $\Box h^{\rm TT}_{\mu\nu} = 0$.

TT condition and real gravitational fields

So far, we have everywhere been talking about *monochromatic* wave. However, thanks to the linearity of the theory, the conclusions also apply more generally, because one can decompose a generic wave to monochromatic plane waves, namely to perform the Fourier decomposition

$$\gamma_{\mu\nu}(x) = \frac{1}{4\pi^2}\iint \gamma_{\mu\nu}(\omega, \vec{k}) \cos(k_{\rho}x^{\rho})\, {\rm d}^3k\, {\rm d}\omega \,,$$

and then to apply the procedure accordingly for every mode (for every frequency) and sum the results. Yet the $h_{\mu\nu}$ tensor cannot be arranged in a simple form (22.33) if the superposition contains waves with different directions of propagation.

This is not to claim, however, that *every* gravitational field could be transformed to the TT coordinates. Actually, the TT condition is very useful for *radiating* fields (typically reducing to plane waves far from isolated sources), but it may not be reached for rather stationary fields. We know already that for such fields the deviation from Minkowski is represented by terms of the $2M/r$ and $2J/r$ types, which already brings a problem on the level of Fourier integral, since $1/r$ is not an integrable function. In particular, it is not possible to write Schwarzschild solution in the TT coordinates.

Finally, let us stress that everything we have done above – and the feasibility of the TT/radiation condition in particular – holds for *source-free regions* only, i.e. those where the waves are described by the equation $\Box\gamma_{\mu\nu}=0$. It is similar with the radiation gauge in electrodynamics.

22.6 Effect of gravitational waves on test particles

It is common to say that waves sway the matter which they pass through. Such a statement may be natural within the field-theoretical presentation of the linearised theory, relying on fixed Minkowski "barracks",[3] where it is clear, in any inertial frame, what it means to "stay at a given place". Within the geometrical story, one should listen to Mach: that we have no "rigid" reference at our disposal, so we can only speak of *relative* relations between more-than-one bodies, and that one thing is *coordinate* picture and the other is what someone really could measure. Let us try to contrast these views.

Suppose to have some TT-condition metric (22.33); it actually suffices to have any metric of the form $\mathrm{d}s^2 = -\mathrm{d}t^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j$, i.e. with $g_{00} = -1$, $g_{0j} = 0$, hence $g^{00} = -1$ and $g^{0k} = 0$ as well. Consider now to have two free test particles; before the wave arrival, let they sit at rest ($u^i = 0$) at $x^i = 0$ and at some nearby $x^i = \delta x^i$ (in the coordinates fixed by the requirements of the TT condition). In order to learn the wave effect, look at the geodesic equation. At any moment when particles are at rest, this reduces to

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = -\Gamma^\mu{}_{\kappa\lambda}u^\kappa u^\lambda = -\Gamma^\mu{}_{00}(u^0)^2\,,$$

where, however, for the above metric we have

$$\Gamma^\mu{}_{00} = \frac{1}{2}\,g^{\mu\nu}(g_{\nu0,0} + g_{0\nu,0} - g_{00,\nu}) = 0\,,$$

so the four-velocity $u^\mu = (u^t, 0, 0, 0)$ of the particles does *not* change even if the wave has arrived – *the particles stay at fixed coordinate locations anyway*.

In order to see that still *proper distance of the particles does change*, suppose to tie a local inertial frame (LIFE) to the first and inspect the position of the second with respect to it. The proper equation to describe such a situation is the equation of geodesic deviation – see Section 6.4 and in particular the LIFE-form of the equation (hatted indices), (6.27), i.e.

$$\frac{\mathrm{d}^2\delta x^{\hat{\imath}}}{\mathrm{d}\tau^2} = -R^{\hat{\imath}}{}_{\hat{0}\hat{\jmath}\hat{0}}\delta x^{\hat{\jmath}}\,.$$

[3] H. Weyl is said to once have thus expressed the constraint of the rigid, non-dynamical flat *background*.

Since the Riemann tensor is gauge invariant, we can use its TT components found in the previous section (vertical position of the index $\hat{\imath}$ is irrelevant),

$$\frac{\mathrm{d}^2\delta x^{\hat{\imath}}}{\mathrm{d}\tau^2} = \frac{1}{2}\, h_{\hat{\imath}\hat{\jmath},\hat{0}\hat{0}}\delta x^{\hat{\jmath}} = \frac{1}{2}\,\frac{\partial^2 h_{\hat{\imath}\hat{\jmath}}}{\partial\tau^2}\,\delta x^{\hat{\jmath}}\,, \tag{22.38}$$

where we have also regarded that $x^{\hat{0}} \equiv \tau$. So the relative acceleration of the particles is really non-zero (if the curvature has arrived). The equation closely corresponds to the Newtonian tidal-force formula

$$\frac{\mathrm{d}^2\delta x^{i}}{\mathrm{d}t^2} = -\Phi_{,ij}\delta x^{j}\ .$$

Looking at the TT metric (22.33), one further crucial point can be seen: the particles' relative acceleration is zero in the $\hat{z}$-direction, because $h_{\hat{3}\hat{\jmath}} \equiv h_{\hat{z}\hat{\jmath}} = 0$, whereas it is non-zero in the directions $\hat{x}$ and $\hat{y}$. The wave is thus **transversal** – only acting in the plane perpendicular to $\vec{k}$, similarly as the plane EM wave. Indeed, by integrating equation (22.38), one has

$$\delta x^{\hat{\imath}}(\tau) = \delta x^{\hat{\imath}}(\tau_{\mathrm{in}}) + \frac{1}{2}\, h_{\hat{\imath}\hat{\jmath}}(\tau)\delta x^{\hat{\jmath}}(\tau_{\mathrm{in}})\,, \tag{22.39}$$

which, for the TT wave (22.33), only means change in the $\hat{x}$ and $\hat{y}$ directions. In general each of $\delta\hat{x}(\tau)$, $\delta\hat{y}(\tau)$ depends on both $\delta\hat{x}(\tau_{\mathrm{in}})$ and $\delta\hat{y}(\tau_{\mathrm{in}})$, yet it is more illustrative to decouple them by decomposing the wave into two (linear) polarization states.

22.6.1 Decomposition into independent polarization states

The two physical degrees of freedom of the waves correspond to two independent *polarizations* in which they can be decomposed. Two types of such a decomposition are being considered standardly, the one into two *linear* polarized modes and the one into *circular* polarized modes. The first case reads

$$h_{\mu\nu} = h^{+}_{\mu\nu} + h^{\times}_{\mu\nu}\,, \qquad \text{with} \quad h^{+}_{\mu\nu} = \mathrm{diag}(0, h_{xx}, -h_{xx}, 0), \quad h^{\times}_{\mu\nu} = \mathrm{antidiag}(0, h_{xy}, h_{xy}, 0),$$

where the individual modes can be written explicitly as

$$h^{+}_{\mu\nu} = \mathrm{Re}\left\{\hat{h}^{+} e^{\mathrm{i}k_\sigma x^\sigma}\left[(\vec{e}_{(x)})_\mu(\vec{e}_{(x)})_\nu - (\vec{e}_{(y)})_\mu(\vec{e}_{(y)})_\nu\right]\right\},$$
$$h^{\times}_{\mu\nu} = \mathrm{Re}\left\{\hat{h}^{\times} e^{\mathrm{i}k_\sigma x^\sigma}\left[(\vec{e}_{(x)})_\mu(\vec{e}_{(y)})_\nu + (\vec{e}_{(y)})_\mu(\vec{e}_{(x)})_\nu\right]\right\},$$

$\hat{h}^{+}$ and $\hat{h}^{\times}$ denoting the respective two amplitudes and $\vec{e}_{(x)}$ and $\vec{e}_{(y)}$ standing for two mutually orthogonal unit vectors spanning the plane normal to $\vec{k}$. The decomposition into two circular polarized modes works similarly, just involving the unit vectors

$$\vec{e}_{\mathrm{L}} = \frac{1}{\sqrt{2}}(\vec{e}_{(x)} + \mathrm{i}\,\vec{e}_{(y)})\,, \qquad \vec{e}_{\mathrm{R}} = \frac{1}{\sqrt{2}}(\vec{e}_{(x)} - \mathrm{i}\,\vec{e}_{(y)})$$

instead of the vectors $\vec{e}_{(x)}$, $\vec{e}_{(y)}$.

Let us check how the above polarization modes affect test particles. For the linear-polarization "plus" mode (solely described by h_{xx}), equation (22.39) gives

$$\delta\hat{x}(\tau) = \left[1 + \frac{1}{2}\,h_{xx}(\tau)\right]\delta\hat{x}(\tau_{\text{in}})\,, \qquad \delta\hat{y}(\tau) = \left[1 - \frac{1}{2}\,h_{xx}(\tau)\right]\delta\hat{y}(\tau_{\text{in}})\,.$$

The effect is better seen on a *circle* of test particles (rather than on just two of them). Imagine to have a circle of particles, initially at rest, given by

$$\delta\hat{x}(\tau_{\text{in}}) = a\cos\phi\,, \qquad \delta\hat{y}(\tau_{\text{in}}) = a\sin\phi \qquad \Longrightarrow \quad [\delta\hat{x}(\tau_{\text{in}})]^2 + [\delta\hat{y}(\tau_{\text{in}})]^2 = a^2\,.$$

Plug these to the above evolution equations, solve the latter for $\cos\phi$ and $\sin\phi$, and then use these in the identity $\cos^2\phi + \sin^2\phi = 1$:

$$\left[\frac{\delta\hat{x}(\tau)}{a\left(1+\frac{1}{2}h_{xx}(\tau)\right)}\right]^2 + \left[\frac{\delta\hat{y}(\tau)}{a\left(1-\frac{1}{2}h_{xx}(\tau)\right)}\right]^2 = 1\,. \tag{22.40}$$

This is an *ellipse* with changing semi-axes $a\left(1 \pm \frac{1}{2}h_{xx}(\tau)\right)$, centred at the origin. Still more specifically, let us we choose, in

$$h_{xx} = \text{Re}\left(\hat{h}^+ e^{\mathrm{i}\omega\tau}\right) = (\text{Re}\,\hat{h}^+)\cos(\omega\tau) + (\text{Im}\,\hat{h}^+)\sin(\omega\tau)$$

(remember that $\vec{e}_{(x,y)}$ and $\vec{k}$ are orthogonal, so $\vec{k}\cdot\vec{r}=0$), the amplitude $\hat{h}^+$ having $\text{Re}(\hat{h}^+) = 0$ and $\text{Im}(\hat{h}^+) \equiv \hat{h} \neq 0$. In such a case, the semi-axes of the ellipse evolve according to $a[1 \pm \hat{h}\sin(\omega\tau)]$, so the ellipse periodically pulsates, in a counter-phase, quadrupole manner, along the x and y axes – see Figure 22.2.

The "cross" mode (fully described by h_{xy}) has the same effect, only that the pattern is rotated by $\pi/4$, so the ellipse pulsates along the diagonals between the x and the y axes. Finally, the circularly polarized modes *rotate* the ellipses with constant angular velocity $\omega/2$, the L mode in the counter-clockwise sense and the R mode in the clockwise sense with respect to $\vec{k}$.

22.6.2 Helicity: a window to the quantum realm

Have a generic long-range field (i.e. such that after quantization yields massless particles) ψ and consider its plane-wave configuration. The **helicity** of the field is a number h (this is a standard notation, it has nothing in common with the $h_{\mu\nu}$ tensor) with such a value that the plane wave behaves, under rotation by an angle ϕ about its direction of propagation, according to the relation

$$\psi' = e^{\mathrm{i}h\phi}\psi\,.$$

Helicity of a given field can thus be read off from the symmetry of its plane waves – they are symmetric with respect to rotation by $2\pi/h$. The EM wave is symmetric under rotation by 2π, so the EM-field helicity is $h_{\text{EM}} = 1$. The gravitational wave – as best seen from Figure

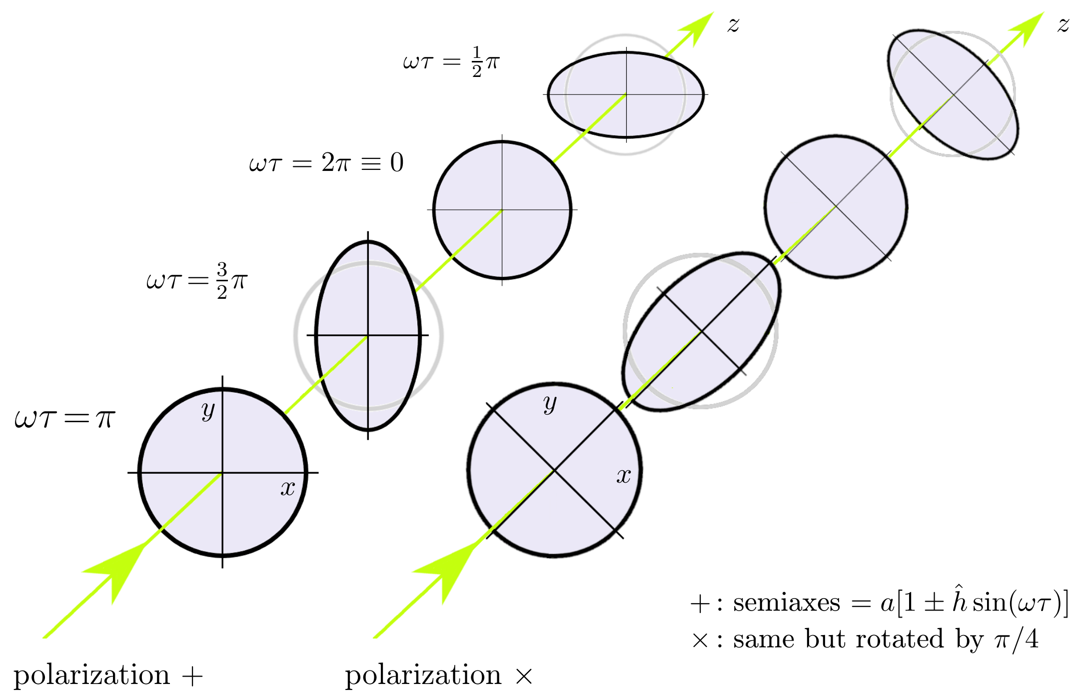

Figure 22.2 *Effect on a circle of test particles of the two linearly polarized components ($+$ and $\times$) of a plane harmonic gravitational wave propagating in the z direction.*

22.2 – is symmetric under rotation by π, so the helicity of the gravitational field is $h_{\rm g} = 2$. It also holds that if the plane wave is decomposed into two linearly polarized components, their polarization directions make an angle of $\pi/(2h)$ (thus $\pi/2$ in the EM case, while $\pi/4$ in the case of gravitation).

Interestingly, although given by purely "classical" properties of the field, the helicity brings information about an important quantum property, namely *it equals spin of the particles obtained by quantization of the field*, more accurately, it equals the component of spin along the particle's direction of flight. In fact, such an interconnection is usually deemed obvious, because, in a quantum theory, spin is represented by Pauli (spin) matrices, which stand for generators of the rotation group SO(3) (or, in the spin-1/2 case, of SU(2)). However, as notably pointed out by [39] (section 9.1.1), such a view is only adequate for non-relativistic particles actually, whereas for light-like ones (like photon or graviton), spin is *not* described by a rotation group but by the group E(2) of the Euclidean-plane isometries. The helicity of massless particles is thus associated with the rotational degree of freedom of E(2) and rather follows from the gauge invariance than from the spin-matrices O(3)-type symmetry.

22.7 Gravitoelectromagnetism

We saw the linearized GR can (in the "field-theoretical account") be formulated as a theory of the symmetric tensor field $h_{\mu\nu}$ (or $\gamma_{\mu\nu}$) in the Minkowski space-time, i.e. without ever mentioning space-time curvature. Such a theory has to necessarily behave in a special relativistic way, *like electrodynamics*, and we have really confirmed at many places that the analogy is very strong. There in fact exist *two* levels of analogy between gravitation and electrodynamics – one starting from the correspondence between the metric and the electromagnetic potentials, in general valid *exactly* for weak fields only (linearized theory), and one involving curvature on the contrary, namely the similarity between the Riemann tensor and the EM-field tensor $F_{\mu\nu}$ (thus between quantities given by *different* order of derivatives of the potentials – by the 2nd derivatives in gravitation whereas by the 1st derivatives in electrodynamics). In this section, we describe the first analogy, arising on the metric level and very useful in weak gravity.

In Section 22.4.3, in describing a weak quasi-stationary source, we denoted, on the basis of writing the energy-momentum tensor as

$$T_{00} = \rho\,, \qquad T_{0j} = -\rho v_j\,, \qquad T_{ij} = \rho v_i v_j = O(v^2)\,,$$

the components of the corresponding solution $\gamma_{\mu\nu}$ as

$$\gamma_{00} = -4\Phi\,, \quad \gamma_{0j} = -4A_j \quad (\text{hence } \gamma^{0j} = 4A^j)\,, \quad \gamma_{ij} \approx 0 \qquad \Longrightarrow \qquad \gamma = 4\Phi\,,$$

where Φ and $\vec{A}$ were defined in analogy with the EM potentials. If the source is confined around the coordinate origin, then far from it the potentials are given by

$$\Phi \approx -\frac{M}{r}\,, \qquad \vec{A} \approx \frac{\vec{J}\times\vec{x}}{2r^3}\,.$$

Although the above followed from solution of the linearized field equations in special coordinates (ensuring the Lorenz condition), and although, in addition, the potentials Φ and $\vec{A}$ were introduced to describe such a solution in a special, stationary case, we will show now that they are useful in general. The point is that they behave similarly as the EM potentials in all the basic respects.

22.7.1 Wave equations

Standardly, one obtains wave equation(s) from the field equation(s), but here it is natural to do it in the opposite way, since the Einstein equations are equations for the metric (and metric plays the role of potentials). In a generic coordinates, they were presented in (22.18); let us repeat it for convenience,

$$\Box\,\gamma_{\mu\nu} - \gamma^{\kappa}_{\nu,\kappa\mu} - \gamma^{\kappa}_{\mu,\kappa\nu} + \eta_{\mu\nu}\gamma^{\kappa\lambda}{}_{,\kappa\lambda} = -16\pi T_{\mu\nu}\,.$$

Substituting the above potentials for the components of $\gamma_{\mu\nu}$, we obtain, on the left-hand sides,

$$\begin{aligned}
&\Box\,\gamma_{00} - 2\gamma^{0}_{0,00} - \cancel{2\gamma^{k}_{0,k0}} - \gamma^{00}{}_{,00} - \cancel{2\gamma^{0l}{}_{,0l}} - \xcancel{\gamma^{kl}{}_{,kl}} = \Box\,\gamma_{00} - \gamma_{00}{}^{,0}{}_{0} = \Delta\gamma_{00} = -4\Delta\Phi\,,\\
&\Box\,\gamma_{0j} - \gamma^{0}_{j,00} - \xcancel{\gamma^{k}_{j,k0}} - \gamma^{0}_{0,0j} - \gamma^{k}_{0,kj} = \Delta\gamma_{0j} + (\gamma_{00,0} - \gamma^{k}_{0,k})_{,j} =\\
&\qquad = 4\left[-\Delta A_j - (\Phi_{,0} - \mathrm{div}\vec{A})_{,j}\right]\,,
\end{aligned}$$

$$\cancel{\Box\gamma_{ij}} - \gamma^{0}_{j,0i} - \cancel{\gamma^{k}_{j,ki}} - \gamma^{0}_{i,0j} - \cancel{\gamma^{k}_{i,kj}} + \delta_{ij}\gamma^{00}{}_{,00} + 2\delta_{ij}\gamma^{0l}{}_{,0l} + \cancel{\delta_{ij}\gamma^{kl}{}_{,kl}} =$$
$$= -4\left(A_{j,i} + A_{i,j} + \delta_{ij}\Phi_{,0} - 2\delta_{ij}\mathrm{div}\vec{A}\right)_{,0} ,$$

so the field equations yield

$$\Delta\Phi = 4\pi\rho\,, \qquad -\Delta A_j - \Phi_{,0j} + \mathrm{div}\vec{A}_{,j} = 4\pi\rho v_j\ , \tag{22.41}$$
$$\left(A_{j,i} + A_{i,j} + \delta_{ij}\Phi_{,0} - 2\delta_{ij}\mathrm{div}\vec{A}\right)_{,0} = 0\,.$$

By contraction of the last equation, one has

$$(3\Phi_{,0} - 4\,\mathrm{div}\vec{A})_{,0} = 0\,. \tag{22.42}$$

22.7.2 Lorenz condition

The Lorenz condition $\gamma^{\mu\nu}{}_{,\nu} = 0$ yields

$$\mu = 0: \quad -\Phi_{,0} + \mathrm{div}\vec{A} = 0\,, \qquad \mu = i: \quad \vec{A}_{,0} = 0\,.$$

The first relation differs from its EM counter-part in the sign of Φ (in electrodynamics, $\Phi = Q/r$, whereas here we take $\Phi = -M/r$). The second relation just reflects our assumption of a quasi-stationary situation. Under the Lorenz condition, the field equations of the preceding subsection reduce to

$$\Delta\Phi = 4\pi\rho\,, \qquad \Delta A_j = -4\pi\rho v_j\ , \tag{22.43}$$
$$\left(A_{j,i} + A_{i,j} - \delta_{ij}\mathrm{div}\vec{A}\right)_{,0} = 0 \quad \Longrightarrow \quad (\mathrm{div}\vec{A})_{,0} = 0\,. \tag{22.44}$$

22.7.3 Gravitoelectric and gravitomagnetic fields

Having the potentials at one's disposal, we can define **the gravitoelectric and the gravitomagnetic fields** in analogy with how the electric and the magnetic fields are related to the potentials in electrodynamics,

$$\vec{E} := -\mathrm{grad}\,\Phi - \vec{A}_{,0}\,, \qquad \vec{B} := \mathrm{rot}\vec{A}\,.$$

22.7.4 Field equations

The field equations can now be written in a Maxwellian form,

$$\mathrm{rot}\vec{B} = \mathrm{grad}(\mathrm{div}\vec{A}) - \Delta\vec{A} = \mathrm{grad}\,\Phi_{,0} - \Delta\vec{A} = -\vec{E}_{,0} - \cancel{\vec{A}_{,00}} + 4\pi\rho\vec{v}\,, \tag{22.45}$$
$$\mathrm{div}\vec{E} = -\Delta\Phi - \cancel{\mathrm{div}\vec{A}_{,0}} = -4\pi\rho\,, \tag{22.46}$$
$$\mathrm{rot}\vec{E} = -\mathrm{rot}\vec{A}_{,0} = -\vec{B}_{,0}\,, \tag{22.47}$$
$$\mathrm{div}\vec{B} = 0\,. \tag{22.48}$$

Good to recapitulate what we have (or have not) neglected and why. First, in Section 22.2 we saw that the energy-momentum tensor typically satisfies $|T_{ij}| < |T_{0j}| < T_{00} = \rho$. Therefore, it should hold $|\gamma_{ij}| < |\gamma_{0j}| < |\gamma_{00}|$. In all of the above, we in fact *neglected* the γ_{ij} components and left just γ_{00} and γ_{0j}. Also, we neglected $\mathrm{div}\vec{A}_{,0}$ and $\vec{A}_{,00}$ finally (they are crossed out explicitly), whereas we did *not* neglect $\mathrm{rot}\vec{A}_{,0}$ (although $\vec{A}_{,0}$ was claimed negligible by the Lorenz condition). Admittedly, the "GEM" analogy is often being restricted to the strictly stationary case, when *all* these terms are omitted, and possibly even $\vec{E}_{,0}$. However, notice how the terms combine in the equations: $\mathrm{div}\vec{A}_{,0}$ appears next to the "classical" $4\pi\rho$ which is the largest term of all, and $\vec{A}_{,00}$ appears next to $\vec{E}_{,0}$ which is only linearly small (similarly as $4\pi\rho\vec{v}$); on the contrary, $\mathrm{rot}\vec{A}_{,0}$ is *the only* term of the equation for $\mathrm{rot}\vec{E}$, so it is not *necessary* to neglect it with respect to anything larger (though one might of course say that $\mathrm{rot}\vec{E}=\vec{0}$).

Needless to say, from the first two equations it follows the continuity equation: by divergence of the first one with $\mathrm{div}\vec{E}_{,0}$ substituted from the time derivative of the second one, one has

$$\rho_{,0} + \mathrm{div}(\rho\vec{v}) = 0\,.$$

22.7.5 Lorentz-like equation of motion

Let us check how the equation of motion of a free test particle in a gravitational field (the geodesic equation) appears in terms of the GEM quantities. Omitting the terms quadratic in spatial velocity components (but *leaving* the terms linear in spatial velocity, in contrast to when the Newtonian limit was made in Section 3.7), and restricting to a *stationary* situation ($h_{\mu\nu,0}=0$), the geodesic equation reads

$$\begin{aligned}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} &= -\Gamma^\mu{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - 2\Gamma^\mu{}_{0j}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = \\ &= -\frac{1}{2}\,\eta^{\mu\nu}(h_{\nu0,0}+h_{0\nu,0}-h_{00,\nu})\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \eta^{\mu\nu}(h_{\nu0,j}+h_{j\nu,0}-h_{0j,\nu})\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = \\ &= \frac{1}{2}\,h_{00}{}^{,\mu}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \eta^{\mu\nu}(h_{\nu0,j}-h_{0j,\nu})\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\,.\end{aligned}$$

The zeroth component reduces to

$$\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = h_{00,j}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\,,$$

and so the left-hand side of the spatial components can easily be rewritten in terms of the t-derivative,

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + h_{00,j}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\cancel{\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}t}}\,.$$

Dividing the spatial components of the geodesic equation by $(u^t)^2$, they thus read (denoting $\frac{\mathrm{d}x^i}{\mathrm{d}t} =: v^i$)

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \frac{1}{2}\,h_{00}{}^{,i} - \delta^{ik}(h_{k0,j}-h_{0j,k})\,v^j\,.$$

Now introduce the GEM potentials, $h_{00}=\gamma_{00}+\frac{\gamma}{2}=-2\Phi$ and $h_{0j}=\gamma_{0j}=-4A_j$, to get

$$\frac{\mathrm{d}v^i}{\mathrm{d}t}=-\Phi^{,i}+4\delta^{ik}(A_{k,j}-A_{j,k})\,v^j=E^i+4\delta^{ik}\epsilon_{jkl}B^l v^j=E^i-4\epsilon^i{}_{jl}\,v^j B^l$$
$$\longleftrightarrow\quad \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=\vec{E}-4\,\vec{v}\times\vec{B} \tag{22.49}$$

(remember that $\vec{A}_{,0}=0$ in a stationary situation, so $\vec{E}=-\mathrm{grad}\,\Phi$).

22.7.6 Gravitomagnetic moment and the Lense-Thirring precession

The GEM analogy is not only helpful conceptually, in understanding the levels of non-Newtonian behaviour. It may allow one to predict phenomena which might otherwise remain hidden within the GR complexity. As an example, consider the **Larmor precession** – the precession of a body having **magnetic (dipole) moment** if placed in a magnetic field. Denoting by $\vec{\mu}$ the magnetic moment and by $\vec{B}$ the magnetic field, the body experiences the torque (moment of force) $\vec{\tau}=\vec{\mu}\times\vec{B}$, similarly as an electric dipole experiences a torque in an electric field. In this "passive" way, the magnetic moment scales how strong a magnetic field an object itself generates at a dipole level, i.e. at the level corresponding to two equal but opposite magnetic poles at an infinitesimal separation. Among objects which possess magnetic moment are permanent magnets, particles (electrons, atomic nuclei, atoms, molecules), and also current loops – and thus various astronomical objects (stars, planets, pulsars, etc.).

The magnetic moment due to a current-density distribution $\vec{j}$ is defined as

$$\vec{\mu}=\frac{1}{2}\int\vec{r}\times\vec{j}\,\mathrm{d}V\,.$$

Specifically for a planar circular current loop, we place the spherical-coordinate origin at its centre and set its plane at $\theta=\pi/2$, so $\vec{r}=(r,0,0)$, $\vec{j}=(0,0,(I/a)\delta(r-a)\delta(\theta-\pi/2))$, and hence

$$\vec{\mu}=\frac{I\vec{n}}{2a}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r\,\delta(r-a)\delta(\theta-\pi/2)\,r^2\sin\theta\;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi=\pi a^2 I\vec{n}\,,$$

where $\vec{n}$ is the unit normal to the loop defined so that the current is "clock-wise" if you look along $\vec{n}$.

If taking mass current instead of the electric one, the magnetic moment clearly becomes angular momentum. Let us fix the relation exactly. Consider an axially symmetric body spinning about its symmetry axis. Take its element of mass m and charge q orbiting on a circle of radius r with speed v. The time average of current due to such a motion equals charge over period, i.e. $qv/(2\pi r)$, so the corresponding magnetic moment is

$$\mu=\pi r^2 I=\pi r^2\,\frac{qv}{2\pi r}=\frac{qvr}{2}=\frac{qL}{2m}\qquad\Longrightarrow\qquad\vec{\mu}=\frac{q}{2m}\,\vec{L}\,, \tag{22.50}$$

where $L=mvr$ is the orbital angular momentum of the element and the coefficient $q/(2m)$ is called the **gyromagnetic ratio**. The relation applies to any classical body or a classical system of particles, provided that the ratio is the same for every its element (for every particle). For a single body, the sum of all orbital angular momenta of its elements represents its **spin** (rotational angular momentum) $\vec{s}$, so the relation is then written as $\vec{\mu}=q\vec{s}/(2m)$.

For a classical body, the angular frequency of the **Larmor precession** – the Larmor frequency – amounts just to $\frac{q}{2m}B$, but for relativistic or/and quantum systems it is different usually. The deviation from the classical formula is being expressed in terms of the gyromagnetic factor (g-factor) – the factor by which the gyromagnetic ratio has to be multiplied in order to reach the correct Larmor-type formula, $g\cdot\frac{q}{2m}\,B$. For an isolated electron, for example, this factor is known to be about -2.00231930436, for an isolated neutron it is -3.82608545, while for an isolated proton it is $+5.58569469$. For atomic nuclei, it has various values of the same order, depending on the nuclear spin. Interestingly, the gyromagnetic factor of the Kerr-Newman solution (Section 16.6) is 2, so it is very close to that of electron (which has lead to its suggestion as a "classical model" of an electron). Indeed, in a flat space-time, the dipole magnetic field – the leading component of the magnetic field – is known to read

$$\vec{B}=\frac{1}{r^3}\left[3(\vec{\mu}\cdot\vec{e}_{(r)})\,\vec{e}_{(r)}-\vec{\mu}\right], \qquad \vec{r}=:r\vec{e}_{(r)}\ .$$

Remembering the value of the Kerr-Newman EM invariant (see Section 16.6)

$$F^{\mu\nu}F_{\mu\nu}=-\frac{2Q^2}{\Sigma^4}\left[(r^2-a^2\cos^2\theta)^2-4r^2a^2\cos^2\theta\right] \qquad \dots\ \equiv\ 2\hat{B}^2-2\hat{E}^2$$

($\hat{E}_\mu=F_{\mu\nu}\hat{u}^\nu$ and $\hat{B}_\mu=-{}^*F_{\mu\nu}\hat{u}^\nu$ are field magnitudes measured by some physical observer – see Section 7.3.1), we see that for a stationary observer

$$2\hat{B}^2\ \longleftrightarrow\ \frac{2Q^2}{\Sigma^4}\,4r^2a^2\cos^2\theta \qquad \Longrightarrow \qquad \hat{B}=\frac{2Qar}{\Sigma^2}\,|\cos\theta|\ \ \stackrel{r\to\infty}{\longrightarrow}\ \ \frac{2Qa}{r^3}\,|\cos\theta|\,.$$

Comparing this with the above generic dipole formula on the symmetry axis ($\theta=0$), where $\vec{\mu}\cdot\vec{e}_{(r)}=\mu$ (the magnetic moment exactly points in that direction), we have

$$\left[3(\vec{\mu}\cdot\vec{e}_{(r)})\,\vec{e}_{(r)}-\vec{\mu}\right]^r=2\mu \quad\Longrightarrow\quad B(\theta\!=\!0)=B^r(\theta\!=\!0)=\frac{2\mu}{r^3}\quad\longleftrightarrow\quad\frac{2Qa}{r^3}\ .$$

Hence, for Kerr-Newman,

$$\mu=Qa\ \equiv\ \frac{Q}{M}\,J\ \equiv\ g\,\frac{Q}{2M}\,J \qquad\Longrightarrow\qquad g=2\,.$$

So far, everything has concerned electrodynamics. Concluding from above that the gravitational analogue of the magnetic dipole is half the body's spin, $\vec{s}/2$, and remembering the extra factor of 4 at the magnetic term in the Lorentz-like equation of motion, we infer from the GEM analogy that a test gyroscope of spin $\vec{s}$ should experience, in a *gravito*magnetic field $\vec{B}$, the torque

$$\vec{\tau}=\frac{\vec{s}}{2}\times 4\vec{B}=2\vec{s}\times\vec{B}\,.$$

In GR, the Larmor-like precession is called the **Lense-Thirring precession** (see Section 18.4). For a massive spinning body (with spin $\vec{J}$), the gravitomagnetic field reads, again in analogy with the above formula from electrodynamics (with $\vec{\mu} \to \vec{J}/2$)

$$\vec{B} = \frac{1}{2r^3}\left[3(\vec{J}\cdot\vec{e}_{(r)})\,\vec{e}_{(r)} - \vec{J}\right].$$

From the torque equation, we see that its double stands for the Larmor-like (Lense-Thirring) angular frequency of precession. In the Kerr space-time, for example, the formula yields

$$\theta = 0: \quad \Omega_{\rm gyro} = 2B^r = \frac{2J}{r^3}\,, \qquad\qquad \theta = \frac{\pi}{2}: \quad \Omega_{\rm gyro} = 2B^\theta = -\frac{J}{r^3}\,.$$

These values really correspond to the weak-field (i.e. large-radius) limit of what we found in Sections 18.4.1 and 18.4.2, respectively. In the first case (gyro on the symmetry axis), it is the limit of (18.9), namely of the dragging angular velocity, $\Omega_{\rm gyro} = \omega \approx 2J/r^3$, and in the second case (gyro "at rest" in the equatorial plane), it is the limit of the result (18.12) obtained for a gyro fixed to a ZAMO, as well as of the result (18.13) obtained for a gyro fixed to a static observer (with the minus sign indicating that the precession is retrograde).

CHAPTER 23

Lagrangian (variational) formulation of Einstein equations

It's one of the wonders of nature that physical processes generally happen in a way which has extremal properties – in which a certain quantity is minimized or maximized. Most of the physical problems thus can be posed and solved in a variational way, that is, by finding such a behaviour of relevant quantities which minimizes or maximizes a certain functional. Variational (Lagrangian) formulation, in connection with symmetry requirements, has been particularly successful in the analysis and classification of different classes of field theories describing fundamental interactions. Standardly employed is the **Hamilton's variational principle**, applicable for conservative systems subject to holonomic constraints. According to it, the evolution of a system is identified by a stationary value of **action** – a time integral of the Lagrangian (or space-time integral of the Lagrangian density).

To any real thinker on physics and gravitation, one should recommend J. L. Anderson's *Principles of Relativity Physics* [1]. The variational approach is thoroughly treated there (symmetry groups and variations begin on p. 84 [!]), and the author also fittingly cite, in this respect, from another notable account on the relativity theory [34] by W. Pauli. When mentioning Hilbert's variational derivation of Einstein's equations (see below), Pauli – then in 1921 – added: "His presentation, though, would not seem to be acceptable to physicists, for two reasons. First, the existence of a variational principle is introduced as an axiom." Anderson remarks to this (in 1960s): "Today most physicists would be not only willing to accept as axiomatic the existence of a variational principle but would be also loath to accept any dynamical equations that were not derivable from such a principle." The more today, in 3rd millennium, it is taken as a merit if a theory can be derived from a nice Lagrangian. Actually, æsthetic sentiments have proved to be "unreasonably effective" as criteria for truthfulness. Below, we present two main versions of the variational formulation of GR, the historically first one due to D. Hilbert (supplemented by the part of Lagrangian which yields the energy-momentum tensor), and the one by A. Einstein (yet often called Palatini's) which provides the metric-connection relation in addition.

23.1 Action and the functional derivative

The Lagrangian formulation, as well as the Hamiltonian formulation, are too important to leave them on intuitive grounds and immediately start varying. We will at least summarize first how the problem stands in general (we use the same language and notation as [50]).

- Imagine to have a theory represented by a certain collection of tensor fields – let us symbolically denote it by just one letter ψ (many various indices may be around, but we omit them now) – living on a given manifold M. (The metric field of the underlying manifold is included in the set of fields ψ.)

- Let ψ_p denote a smooth family of configurations of ψ satisfying certain prescribed conditions on a boundary of some fixed region of M which is under consideration (will denote it by Ω), with p denoting a smooth parameter which distinguishes between those different configurations. (The configurations need not satisfy any particular equations.)

- Infinitesimal smooth deformation of the configuration ψ is called the **variation** of ψ, formally $\bar{\delta}\psi := \frac{\mathrm{d}\psi_p}{\mathrm{d}p}(p\!=\!p_0)$, where p_0 corresponds to some particular, selected configuration.

- Let a functional exist $S[\psi]$, i.e. a map from the configurations of ψ to real numbers, for which $\frac{\mathrm{d}S}{\mathrm{d}p}$ exists (at least at $p\!=\!p_0$) for all relevant families of configurations ψ_p, and suppose that for all these it holds

  $$\frac{\mathrm{d}S}{\mathrm{d}p} = \int_\Omega \chi\, \bar{\delta}\psi \quad \text{(natural measure on } M \text{ employed tacitly)}\,,$$

  where all the possible indices at $\bar{\delta}\psi$ are contracted against the same (dual) indices occurring at χ (supposed to be a smooth tensor field). Then S is said to be *functionally differentiable* at $\psi_0 \equiv \psi(p\!=\!p_0)$, with χ called the **functional derivative** (or **variational derivative**) of S with respect to ψ and denoted as

  $$\chi := \frac{\bar{\delta}S}{\bar{\delta}\psi}(\psi\!=\!\psi_0)\,.$$

- Consider now, specifically, a real functional of the form

  $$S[\psi] = \int_\Omega \mathcal{L}[\psi]\,,$$

  where $\mathcal{L}$ is a local function of the field variables ψ and of a finite number of their (partial or covariant) derivatives ("local" meaning that all the fields including their derivatives are taken at the same space-time point). Suppose that S *is* functionally differentiable and that the configurations ψ which make S stationary, i.e. for which $\frac{\bar{\delta}S}{\bar{\delta}\psi}\!=\!0$, exactly correspond to the solutions of the pertinent field equations (known from the theory describing ψ on the given M). Then $\mathcal{L}$ is called the **Lagrangian density** of the theory, S is the corresponding **action**, and the whole construction reproducing (or just yielding – perhaps even *predicting*) the field equations is called the **Lagrangian formulation** of the theory.

23.1.1 What kind of change does the "variation" represent?

The Lagrangian densities depend on coordinate location x^μ and on the fields of a given theory ψ together with their derivatives by x^μ. Two kinds of transformations may occur: **coordinate diffeomorphisms** ($x^\mu \to x'^\mu(x)$) (which naturally induce a certain transformation of the fields, according to the latter's mathematical type), and "direct" transformations of the fields (usually called **gauge transformations**). In principle, both may act simultaneously in changing a configuration of a given system to some other close configuration. Considering both the coordinate diffeomorphisms and the gauge transformations to be only *infinitesimal*, it is useful to introduce two types of variations. First,

$$\delta\psi(x) := \psi'(x') - \psi(x) \qquad \text{...} \ \ \textbf{total change of the field value}\,.$$

This is an extension of the notation we use everywhere else: $\psi'(x')$ etc. we generally denote the result of a coordinate transformation "alone", whereas here it possibly stands for the result of both the coordinate and the gauge shifts. Besides the total variation $\delta\psi(x)$, it is useful to also define

$$\bar{\delta}\psi(x) := \psi'(x) - \psi(x) \qquad \text{...} \ \ \text{field-value change due to the } \textbf{change of the field form}\,.$$

This is *also* caused by *both* shifts, yet it is determined *at the same coordinate values*. Worth to realize that such a definition does *not* represent the change of the field *at a given point* (had it any clear meaning), but rather the difference between the original components of ψ at x and the transformed components ψ' at the point that is mapped onto x by the coordinate shift.

Independently of how transform the fields, the two variations are related by shift between the points in which the final field is evaluated,

$$\begin{aligned}\bar{\delta}\psi \;&\equiv\; \psi'(x) - \psi(x) = \psi'(x' - \delta x) - \psi(x) \doteq \psi'(x') - \psi'_{,\alpha}(x')\,\delta x^\alpha - \psi(x) \equiv \\ &\equiv\; \delta\psi - \psi'_{,\alpha}(x')\,\delta x^\alpha \doteq \delta\psi - \psi_{,\alpha}(x)\,\delta x^\alpha \;\equiv\; \delta\psi + \epsilon\,\psi_{,\alpha}\xi^\alpha \end{aligned} \tag{23.1}$$

This is the well known relation between the **Lagrangian and Eulerian variations** (if understanding the diffeomorphism as a passive shift of coordinates, it is natural to view $\bar{\delta}\psi$ as a Lagrangian variation and $\delta\psi$ as an Eulerian variation).[1] Exactly the same characterization we will repeat in Chapter 28 (Section 28.3) where a clear distinction between $\delta\psi(x)$ and $\bar{\delta}\psi(x)$ will be especially important.

In the present chapter, all the variations mean those *at the same coordinate position*, so we denote them by $\bar{\delta}$ (as it is much more usual than the opposite convention). Since we will *only* consider this type of variations, it would make no harm to call them simply $\delta\psi$ (or however), yet we have decided to keep consistency with Chapter 28. Anyway, really important will be one point: since $\bar{\delta}\psi(x)$ is defined at the same x^μ, it *commutes with partial derivative* (whereas $\delta\psi$ does not). This will be used at several places below.

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[1] We have met similar language in the analysis of stellar pulsations in Section 20.4. However, the variations considered now are "off-shell" ("virtual") in the sense that they need not lead along the actual evolution of the system. In the pulsation problem, on the contrary, the changes of the quantities described real motion of the star's fluid. Also, the attributes "Lagrangian" and "Eulerian" (denoted by Δ and δ, respectively) meant the opposite than in the present chapter, because the shift $r \to r + \xi$ was *active* (thus with the plus sign at ξ), following the real fluid motion. Hence, the Lagrangian deflection was tied to $r + \xi(t, r)$, whereas the Eulerian one was measured at fixed r.

23.2 Euler-Lagrange equations

To approach our specific task, take the 4D space-time and consider – as an example yet – an invariant Lagrangian density depending on a vector field $\psi^\mu(x)$ and its first (partial) derivatives. (The field in general depends on the space-time metric $g_{\mu\nu}$.) We wish to find when the corresponding action is stationary with respect to the variation $\bar{\delta}\psi^\mu$ of the field,

$$\bar{\delta}S = \bar{\delta}\int_\Omega \mathcal{L}\sqrt{-g}\,\mathrm{d}^4x = \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu}\,\bar{\delta}\psi^\mu + \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\,\bar{\delta}{\psi^\mu}_{,\alpha}\right]\mathrm{d}^4x\,. \tag{23.2}$$

In *problems with "fixed boundary"* – when the field variation $\bar{\delta}\psi^\mu$ is fixed to vanish on the boundary of Ω (denoted by $\partial\Omega$) – a standard method is to rewrite the second term as total divergence minus the term thus extra added, to use the Gauss theorem to express the total divergence as the flux of its vector argument over $\partial\Omega$, which however vanishes due to the fixed boundary, and so to be left with the subtracted "extra added" term:

$$\int_\Omega \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\,\bar{\delta}{\psi^\mu}_{,\alpha}\,\mathrm{d}^4x = \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\,\bar{\delta}\psi^\mu\right]_{,\alpha}\mathrm{d}^4x - \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\right]_{,\alpha}\bar{\delta}\psi^\mu\,\mathrm{d}^4x =$$
$$= \int_{\partial\Omega} \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\,\cancel{\bar{\delta}\psi^\mu}\,n_\alpha\,\mathrm{d}^3x - \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\right]_{,\alpha}\bar{\delta}\psi^\mu\,\mathrm{d}^4x\,.$$

The variation of S thus comes out as

$$\bar{\delta}S \equiv \int_\Omega \frac{\bar{\delta}(\sqrt{-g}\,\mathcal{L})}{\bar{\delta}\psi^\mu}\,\bar{\delta}\psi^\mu\,\mathrm{d}^4x = \int_\Omega \left\{\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\right]_{,\alpha}\right\}\bar{\delta}\psi^\mu\,\mathrm{d}^4x\,.$$

Demanding that the action remain stationary, $\bar{\delta}S = 0$, under *arbitrary* field variation $\bar{\delta}\psi^\mu$, we thus arrive at the **Euler-Lagrange equations** of the variational problem,

$$\frac{\bar{\delta}(\sqrt{-g}\,\mathcal{L})}{\bar{\delta}\psi^\mu} = \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha}}\right]_{,\alpha} =: [\mathrm{EL}(\sqrt{-g}\,\mathcal{L})]^\mu = 0\,,$$

where $[\mathrm{EL}]$ with the appropriate index is called the **Euler operator**. These equations provide (or, are equivalent to) the field equations of the given field theory. They are standardly required to contain at most the second derivatives of the fields, because the theories with higher derivatives may yield non-causal propagation.

If the Lagrangian density depends on higher derivatives of the field(s), the above recipe generalizes straightforwardly, at least if the variations *of all the derivatives of the field except the highest ones* vanish on the boundary $\partial\Omega$ (this is of course not necessarily true, this is just the simplest case, because than the "divergence trick" can also be employed for the higher terms accordingly). For example, consider the next level of a Lagrangian density also depending on the second derivatives of ψ^μ. Then, besides the above, one also has in (23.2) the third term

$$\int_\Omega \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha\beta}}\,\bar{\delta}{\psi^\mu}_{,\alpha\beta}\,\mathrm{d}^4x = \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha\beta}}\,\bar{\delta}{\psi^\mu}_{,\alpha}\right]_{,\beta}\mathrm{d}^4x - \int_\Omega \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial{\psi^\mu}_{,\alpha\beta}}\right]_{,\beta}\bar{\delta}{\psi^\mu}_{,\alpha}\,\mathrm{d}^4x =$$

$$= \int\limits_{\partial\Omega} \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\,\cancel{\bar{\delta}\psi^\mu{}_{,\alpha}}\,n_\beta\,\mathrm{d}^3x - \int\limits_{\Omega}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\beta}\bar{\delta}\psi^\mu{}_{,\alpha}\,\mathrm{d}^4x =$$

$$= -\int\limits_{\Omega}\left\{\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\beta}\bar{\delta}\psi^\mu\right\}_{,\alpha}\mathrm{d}^4x + \int\limits_{\Omega}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\beta\alpha}\bar{\delta}\psi^\mu\,\mathrm{d}^4x =$$

$$= -\int\limits_{\partial\Omega}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\beta}\cancel{\bar{\delta}\psi^\mu}\,n_\alpha\,\mathrm{d}^3x + \int\limits_{\Omega}\left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\beta\alpha}\bar{\delta}\psi^\mu\,\mathrm{d}^4x\,.$$

If the variations $\bar{\delta}\psi^\mu$ as well as $\bar{\delta}\psi^\mu{}_{,\alpha}$ vanish on the boundary, the variational problem thus leads to the Euler-Lagrange equations

$$\frac{\bar{\delta}(\sqrt{-g}\,\mathcal{L})}{\bar{\delta}\psi^\mu} = \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha}}\right]_{,\alpha} + \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi^\mu{}_{,\alpha\beta}}\right]_{,\alpha\beta} =: [\mathrm{EL}(\sqrt{-g}\,\mathcal{L})]^\mu = 0\,. \tag{23.3}$$

23.2.1 Euler-Lagrange equations in terms of covariant derivatives

In cases when the field variables ψ of the Lagrangian density $\mathcal{L}$ does *not* depend on the space-time metric, it is often appropriate to use their *covariant* derivatives in the procedure.[2] Besides others, this is the straightest way how to obtain an invariant $\mathcal{L}$, as required.

Imagine again the case of $\mathcal{L}$ depending on a vector field ψ^μ and its first derivatives, but now the covariant ones. The variation of ψ^μ then induces the variation of the derivatives

$$\psi^\mu{}_{;\alpha} \to \psi^\mu{}_{;\alpha} + \bar{\delta}\psi^\mu{}_{;\alpha}\,,$$

where the variation is again supposed to commute with the derivative, similarly as for the partial derivative. Hence $\bar{\delta}(\psi^\mu{}_{;\alpha}) = (\bar{\delta}\psi^\mu)_{;\alpha}$. The corresponding variation of the action *with respect to the field* ψ^μ reads

$$\bar{\delta}S = \bar{\delta}\int_\Omega \mathcal{L}\sqrt{-g}\,\mathrm{d}^4x = \int_\Omega\left[\frac{\partial\mathcal{L}}{\partial\psi^\mu}\,\bar{\delta}\psi^\mu + \frac{\partial\mathcal{L}}{\partial\psi^\mu{}_{;\alpha}}\,\bar{\delta}\psi^\mu{}_{;\alpha}\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where the $\Gamma^\mu{}_{\alpha\lambda}\psi^\lambda$ terms involved in $\psi^\mu{}_{;\alpha}$ are *not* to be differentiated partially with respect to ψ^μ within the first term (here ψ^μ and the *whole* $\psi^\mu{}_{;\alpha}$ are the independent variables). In order to reach the Euler-Lagrange equations, one has to "liberate" $\bar{\delta}\psi^\mu$ from the derivative in the second term. Proceeding similarly as in the partial-derivative case, one employs the divergence trick ("per partes") as

$$\int\limits_\Omega \frac{\partial\mathcal{L}}{\partial\psi^\mu{}_{;\alpha}}\,\bar{\delta}\psi^\mu{}_{;\alpha}\sqrt{-g}\,\mathrm{d}^4x = \int\limits_\Omega\left[\frac{\partial\mathcal{L}}{\partial\psi^\mu{}_{;\alpha}}\,\bar{\delta}\psi^\mu\right]_{;\alpha}\sqrt{-g}\,\mathrm{d}^4x - \int\limits_\Omega\left[\frac{\partial\mathcal{L}}{\partial\psi^\mu{}_{;\alpha}}\right]_{;\alpha}\bar{\delta}\psi^\mu\sqrt{-g}\,\mathrm{d}^4x\,,$$

[2] $\mathcal{L}$ may still depend on $g_{\mu\nu}$ (and possibly its partial derivatives), just that this dependence is not coupled with the dependence on ψs.

rewriting the first (total-divergence) guy in terms of partial divergence,

$$\int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \psi^\mu{}_{;\alpha}}\,\bar{\delta}\psi^\mu\right]_{;\alpha} \sqrt{-g}\,\mathrm{d}^4x = \int_\Omega \frac{1}{\cancel{\sqrt{-g}}}\left[\sqrt{-g}\,\frac{\partial \mathcal{L}}{\partial \psi^\mu{}_{;\alpha}}\,\bar{\delta}\psi^\mu\right]_{,\alpha} \cancel{\sqrt{-g}}\,\mathrm{d}^4x\,,$$

and cancelling it, as above, on the basis of the Gauss theorem and of the fixed-boundary assumption. We thus end with

$$\bar{\delta}S \equiv \int_\Omega \frac{\bar{\delta}\mathcal{L}}{\bar{\delta}\psi^\mu}\,\bar{\delta}\psi^\mu \sqrt{-g}\,\mathrm{d}^4x = \int_\Omega \left\{\frac{\partial \mathcal{L}}{\partial \psi^\mu} - \left[\frac{\partial \mathcal{L}}{\partial \psi^\mu{}_{;\alpha}}\right]_{;\alpha}\right\}\bar{\delta}\psi^\mu \sqrt{-g}\,\mathrm{d}^4x\,,$$

so the Euler-Lagrange equations now read

$$\frac{\bar{\delta}\mathcal{L}}{\bar{\delta}\psi^\mu} = \frac{\partial \mathcal{L}}{\partial \psi^\mu} - \left[\frac{\partial \mathcal{L}}{\partial \psi^\mu{}_{;\alpha}}\right]_{;\alpha} \equiv [\mathrm{EL}(\mathcal{L})]^\mu = 0\,. \tag{23.4}$$

Mathematically, the difference clearly is that when using covariant derivatives, one differentiates the Lagrangian density $\mathcal{L}$ alone which is an invariant, whereas when using partial derivatives, one differentiates the expression $\sqrt{-g}\,\mathcal{L}$ which is a scalar density of weight $+1$. Now, after the above introductory summary, let us directly jump to our central gravitational problem.

23.3 The theorem

- Einstein equations are Euler-Lagrange equations for the action

$$\boxed{S = \int_\Omega (\mathcal{L}_\mathrm{g} + 16\pi\mathcal{L}_\mathrm{ng})\sqrt{-g}\,\mathrm{d}^4x\,, \qquad \text{where}\;\; \mathcal{L}_\mathrm{g} = R - 2\Lambda} \tag{23.5}$$

  is the Lagrangian density for a gravitational field[3] and $\mathcal{L}_\mathrm{ng}(g_{\mu\nu}, g_{\mu\nu,\iota})$ is a scalar representing Lagrangian density of matter and non-gravitational fields. Ω is a chosen space-time region and Λ is (cosmological) constant.

- $\mathcal{L}_\mathrm{ng}$ yields the energy-momentum tensor by prescription

$$\frac{1}{2}\sqrt{-g}\,T_{\mu\nu} \equiv -\frac{\partial(\mathcal{L}_\mathrm{ng}\sqrt{-g})}{\partial g^{\mu\nu}} + \frac{\partial}{\partial x^\iota}\left[\frac{\partial(\mathcal{L}_\mathrm{ng}\sqrt{-g})}{\partial g^{\mu\nu}{}_{,\iota}}\right], \tag{23.6}$$

  which reduces to $T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\mathrm{ng} - 2\frac{\partial \mathcal{L}_\mathrm{ng}}{\partial g^{\mu\nu}}$ in case when $\mathcal{L}_\mathrm{ng}$ does not depend on derivatives of $g_{\mu\nu}$.

- From the diffeomorphism invariance of $\mathcal{L}_\mathrm{ng}$ (which one demands) it follows that the energy-momentum tensor satisfies conservation laws $T^{\mu\nu}{}_{;\nu} = 0$.

[3] The crucial part $\int R\sqrt{-g}\,\mathrm{d}^4x$ is being called the **Hilbert action**, since D. Hilbert was the first to derive from it the vacuum part of the Einstein equations.

Remarks:
• Basic quantities to be varied independently are $g_{\mu\nu}$ a $g_{\mu\nu,\iota}$ (similarly as x^i and p^i in classical mechanics), with variations of both assumed to vanish on the boundary of Ω.
• Sure that the "non-gravitational Lagrangian" $\mathcal{L}_{\rm ng}$ also depends on the non-gravitational fields. Variation with respect to these (while taking the metric as a fixed background) would yield equations governing the respective fields (e.g. Maxwell equations). Here, however, we focus on the behaviour given by the space-time geometry.

23.4 The proof

23.4.1 Variation of the metric determinant

Let us first remind the lemma (5.12) for the differentiation of determinant of a square matrix (M),

$$\frac{(\det M)_{,\lambda}}{\det M} = \mathrm{Tr}\left(M^{-1}\cdot M_{,\lambda}\right) .$$

The derivative indicated by "λ" can actually represent differentiation with respect to any variable (not necessarily a coordinate), so we may employ the relation for the derivative (of the covariant-metric determinant) with respect to $g_{\mu\nu}$,

$$\frac{\partial(-g)}{\partial g_{\mu\nu}} = (-g)g^{\iota\kappa}\frac{\partial g_{\iota\kappa}}{\partial g_{\mu\nu}} = (-g)g^{\iota\kappa}\delta^\mu_\iota\delta^\nu_\kappa = (-g)g^{\mu\nu} \ , \tag{23.7}$$

which permits to calculate the variation

$$\bar\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\frac{\partial(-g)}{\partial g_{\mu\nu}}\,\bar\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\;g^{\mu\nu}\bar\delta g_{\mu\nu} \, . \tag{23.8}$$

The relations can also be expressed in terms of the inverse metric:

$$g^{\mu\nu}g_{\beta\nu} = \delta^\mu_\beta \quad\Rightarrow\quad (\bar\delta g^{\mu\nu})g_{\nu\beta} + g^{\mu\nu}(\bar\delta g_{\nu\beta}) = 0\,\Big|\cdot g_{\alpha\mu} \quad\Rightarrow\quad \bar\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\bar\delta g^{\mu\nu} \, . \tag{23.9}$$

Mind the different signs – the indices of the metric variation can *not* be simply risen and lowered by the original metric! (Sure: metric is not constant with respect to variation.) We thus arrive at alternative expression

$$\bar\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\;g_{\mu\nu}\bar\delta g^{\mu\nu} \quad\Rightarrow\quad \frac{\partial(-g)}{\partial g^{\mu\nu}} = -(-g)g_{\mu\nu}\,, \quad \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\;g_{\mu\nu} \, . \tag{23.10}$$

23.4.2 Variation of curvature: the Palatini equation

We vary the Riemann tensor straightforwardly,

$$\bar\delta R^\kappa{}_{\mu\lambda\nu} = \bar\delta\Gamma^\kappa{}_{\nu\mu,\lambda} - \bar\delta\Gamma^\kappa{}_{\lambda\mu,\nu} + \bar\delta\left(\Gamma^\kappa{}_{\lambda\iota}\Gamma^\iota{}_{\nu\mu}\right) - \bar\delta\left(\Gamma^\kappa{}_{\nu\iota}\Gamma^\iota{}_{\lambda\mu}\right) ,$$

and after switching variation and partial derivatives, one easily verifies to obtain the **Palatini equation**

$$\bar{\delta}R^{\kappa}{}_{\mu\lambda\nu} = (\bar{\delta}\Gamma^{\kappa}{}_{\nu\mu})_{;\lambda} - (\bar{\delta}\Gamma^{\kappa}{}_{\lambda\mu})_{;\nu}\,. \tag{23.11}$$

Indeed, "from definition",

$$\begin{aligned}(\bar{\delta}\Gamma^{\kappa}{}_{\nu\mu})_{;\lambda} - (\bar{\delta}\Gamma^{\kappa}{}_{\lambda\mu})_{;\nu} = {} & \bar{\delta}\Gamma^{\kappa}{}_{\nu\mu,\lambda} + \Gamma^{\kappa}{}_{\lambda\iota}\bar{\delta}\Gamma^{\iota}{}_{\nu\mu} - \cancel{\Gamma^{\iota}{}_{\lambda\nu}\bar{\delta}\Gamma^{\kappa}{}_{\iota\mu}} - \Gamma^{\iota}{}_{\lambda\mu}\bar{\delta}\Gamma^{\kappa}{}_{\nu\iota} \\ & - \bar{\delta}\Gamma^{\kappa}{}_{\lambda\mu,\nu} - \Gamma^{\kappa}{}_{\nu\iota}\bar{\delta}\Gamma^{\iota}{}_{\lambda\mu} + \cancel{\Gamma^{\iota}{}_{\nu\lambda}\bar{\delta}\Gamma^{\kappa}{}_{\iota\mu}} + \Gamma^{\iota}{}_{\nu\mu}\bar{\delta}\Gamma^{\kappa}{}_{\lambda\iota}\end{aligned}$$

equals $\bar{\delta}R^{\kappa}{}_{\mu\lambda\nu}$ exactly (even with index order at Gammas, so it can also be used with torsion). By contraction in $(\kappa\lambda)$, one finds the Ricci-tensor variation

$$\bar{\delta}R_{\mu\nu} = (\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu})_{;\lambda} - (\bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu})_{;\nu}\,. \tag{23.12}$$

Important remark: although the covariant-derivative rules can of course be applied "mechanically", it might seem improper to apply that derivative to $\bar{\delta}\Gamma^{\kappa}{}_{\mu\nu}$, because the affine-connection components are not tensors. However, the variation $\bar{\delta}\Gamma^{\kappa}{}_{\mu\nu}$ means the difference between the original and varied Gammas *at a given point*, and at a given point the second, non-tensorial term in the transformation relation (2.17) is the same for both, so it drops out by subtraction – the variations $\bar{\delta}\Gamma^{\kappa}{}_{\nu\mu}$ *are tensors*.

23.4.3 Variation of the gravitational part of action

Employing the above skills, let us tackle the gravitational integral,

$$\begin{aligned}&\bar{\delta}\int(R-2\Lambda)\sqrt{-g}\;\mathrm{d}^4x \;\equiv\; \bar{\delta}\int(g^{\mu\nu}R_{\mu\nu}-2\Lambda)\sqrt{-g}\;\mathrm{d}^4x = \\ &= \int\left[\bar{\delta}g^{\mu\nu}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\bar{\delta}R_{\mu\nu}\sqrt{-g} + (R-2\Lambda)\,\bar{\delta}\sqrt{-g}\right]\mathrm{d}^4x = \\ &= \int\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)\bar{\delta}g^{\mu\nu}\sqrt{-g}\;\mathrm{d}^4x + \int g^{\mu\nu}\bar{\delta}R_{\mu\nu}\sqrt{-g}\;\mathrm{d}^4x\,.\end{aligned} \tag{23.13}$$

In the second integral, substitute for $\bar{\delta}R_{\mu\nu}$ and then try to rewrite it as a partial divergence of the vector $\sqrt{-g}\left(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} - g^{\mu\lambda}\bar{\delta}\Gamma^{\nu}{}_{\nu\mu}\right)$:

$$\begin{aligned}&\int g^{\mu\nu}\left[(\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu})_{;\lambda} - (\bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu})_{;\nu}\right]\sqrt{-g}\;\mathrm{d}^4x = \int\left[(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu})_{;\lambda} - (g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu})_{;\nu}\right]\sqrt{-g}\;\mathrm{d}^4x = \\ &= \int(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} - g^{\mu\lambda}\bar{\delta}\Gamma^{\nu}{}_{\nu\mu})_{;\lambda}\sqrt{-g}\;\mathrm{d}^4x = \int\left[\sqrt{-g}\,(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} - g^{\mu\lambda}\bar{\delta}\Gamma^{\nu}{}_{\nu\mu})\right]_{,\lambda}\mathrm{d}^4x\,.\end{aligned} \tag{23.14}$$

By the Gauss theorem, this equals the integral of the flow of that vector over the boundary $\partial\Omega$, which however is zero since $\bar{\delta}\Gamma$ vanish there by assumption. Hence, variation of the gravitational part of the action yields

$$\bar{\delta}S_{\mathrm{g}} \equiv \bar{\delta}\int_{\Omega}\mathcal{L}_{\mathrm{g}}\sqrt{-g}\;\mathrm{d}^4x = \int_{\Omega}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)\bar{\delta}g^{\mu\nu}\sqrt{-g}\;\mathrm{d}^4x\,. \tag{23.15}$$

23.4.4 Variation of the non-gravitational part of action

Now let us vary the non-gravitational action,

$$\bar\delta\int\mathcal{L}_{\rm ng}\big(g^{\mu\nu},g^{\mu\nu}{}_{,\iota}\big)\sqrt{-g}\,\mathrm{d}^4x=\int\left[\frac{\partial\left(\mathcal{L}_{\rm ng}\sqrt{-g}\right)}{\partial g^{\mu\nu}}\,\bar\delta g^{\mu\nu}+\frac{\partial\left(\mathcal{L}_{\rm ng}\sqrt{-g}\right)}{\partial g^{\mu\nu}{}_{,\iota}}\,\bar\delta g^{\mu\nu}{}_{,\iota}\right]\mathrm{d}^4x=$$
$$=\int\left\{\frac{\partial(\mathcal{L}_{\rm ng}\sqrt{-g})}{\partial g^{\mu\nu}}\,\bar\delta g^{\mu\nu}+\cancel{\left[\frac{\partial(\mathcal{L}_{\rm ng}\sqrt{-g})}{\partial g^{\mu\nu}{}_{,\iota}}\bar\delta g^{\mu\nu}\right]_{,\iota}}-\left[\frac{\partial(\mathcal{L}_{\rm ng}\sqrt{-g})}{\partial g^{\mu\nu}{}_{,\iota}}\right]_{,\iota}\bar\delta g^{\mu\nu}\right\}\mathrm{d}^4x\,.$$

The middle term has been crossed out, because it can be expressed, due to Gauss, as the integral of $\frac{\partial(\mathcal{L}_{\rm ng}\sqrt{-g})}{\partial g^{\mu\nu}{}_{,\iota}}\,\bar\delta g^{\mu\nu}$ over the boundary $\partial\Omega$, where however $\bar\delta g^{\mu\nu}=0$. Therefore, if defining a symmetric tensor $T_{\mu\nu}$ according to (23.6), we can write

$$\bar\delta S_{\rm ng}\equiv 16\pi\,\bar\delta\int_\Omega\mathcal{L}_{\rm ng}\sqrt{-g}\,\mathrm{d}^4x=-8\pi\int_\Omega T_{\mu\nu}\bar\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\,, \tag{23.16}$$

which is what is wanted for the Einstein equations. In particular, if $\mathcal{L}_{\rm ng}$ does *not* depend on $g^{\mu\nu}{}_{,\iota}$, the formula (23.6) reduces to

$$\frac{1}{2}\sqrt{-g}\,T_{\mu\nu}=-\frac{\partial\mathcal{L}_{\rm ng}}{\partial g^{\mu\nu}}\sqrt{-g}-\mathcal{L}_{\rm ng}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\,,$$

from where it follows, by (23.10),

$$T_{\mu\nu}=g_{\mu\nu}\mathcal{L}_{\rm ng}-2\frac{\partial\mathcal{L}_{\rm ng}}{\partial g^{\mu\nu}}\,. \tag{23.17}$$

Example 1: scalar (Klein-Gordon) field

For a scalar field (in general a massive one), for instance, the Lagrangian density reads

$$\mathcal{L}_{\rm scalar}=-\frac{1}{2}\left(g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}+m^2\psi^2\right),$$

so, according to the above recipe,

$$T_{\mu\nu}=\psi_{,\mu}\psi_{,\nu}+g_{\mu\nu}\mathcal{L}_{\rm ng}\,. \tag{23.18}$$

Example 2: electromagnetic field

For a source-free EM field, the Lagrangian density reads

$$\mathcal{L}_{\rm EM}=-\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta}=-\frac{1}{16\pi}g^{\alpha\kappa}g^{\beta\lambda}F_{\kappa\lambda}F_{\alpha\beta}\,,$$

so

$$T_{\mu\nu}=-\frac{1}{16\pi}g_{\mu\nu}\,F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{8\pi}g^{\beta\lambda}F_{\nu\lambda}F_{\mu\beta}+\frac{1}{8\pi}g^{\alpha\kappa}F_{\kappa\nu}F_{\alpha\mu}=$$
$$=\frac{1}{4\pi}\left(F_{\mu\beta}F_\nu{}^\beta-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)\,.$$

Example 3: ideal fluid

For an ideal fluid, the Lagrangian density is at times said to be given by pressure, $\mathcal{L}_{\rm fluid} = P$. It is not that simple (for dust it would then be zero, right?), at least not in connection with the above "canonical" prescription. However, it is simple to verify that

$$\mathcal{L}_{\rm fluid} = -\frac{1}{2}(\rho + P)g^{\alpha\beta}u_\alpha u_\beta - \frac{\rho}{2} + \frac{P}{2} \qquad \Longrightarrow \qquad T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}\,.$$

So the Lagrangian has to be taken "off-shell" in the sense that one may only use $g^{\alpha\beta}u_\alpha u_\beta = -1$ after making derivatives.

23.4.5 Conservation laws for T-mu-nu

The variation of the metric has yet been generic, we only demanded that it vanish, along with derivatives, on the boundary of the relevant space-time region. Let us consider now a *special case* – a variation purely caused by an infinitesimal coordinate change. Such a variation cannot (in fact *must not*) induce any physical change, so, in particular, the action has to remain the same. Interestingly, this requirement leads to the conservation equation for the energy-momentum tensor.

First, from (23.9) we know that $T_{\mu\nu}\bar\delta g^{\mu\nu} = -T^{\mu\nu}\bar\delta g_{\mu\nu}$, so (23.16) can also be written as

$$\bar\delta S_{\rm ng} = 8\pi \int_\Omega T^{\alpha\beta}\bar\delta g_{\alpha\beta}\sqrt{-g}\,{\rm d}^4x\,. \tag{23.19}$$

Now, in Section 11.4 we saw that under a shift $x'^\mu = x^\mu + \xi^\mu$ the metric changes according to $g'_{\mu\nu} = g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$, which means that the corresponding variation reads

$$\bar\delta g_{\mu\nu} := g'_{\mu\nu} - g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}\,.$$

Submitting this to $\bar\delta S_{\rm ng}$ above, we obtain 8π times

$$-\int T^{\alpha\beta}(\xi_{\alpha;\beta} + \xi_{\beta;\alpha})\sqrt{-g}\,{\rm d}^4x = -2\int T^{\alpha\beta}\xi_{\alpha;\beta}\,\sqrt{-g}\,{\rm d}^4x =$$
$$= \cancel{-2\int (T^{\alpha\beta}\xi_\alpha)_{;\beta}\sqrt{-g}\,{\rm d}^4x} + 2\int T^{\alpha\beta}{}_{;\beta}\xi_\alpha\,\sqrt{-g}\,{\rm d}^4x\,.$$

In the first term, one rewrites the covariant divergence to the partial one and expresses the integral, by the Gauss theorem, as a surface integral of $\sqrt{-g}\,T^{\alpha\beta}\xi_\alpha$ over the boundary; there, however, the coordinate variation ξ_α has to vanish (in order to be consistent with the general formulation of the variational problem), so we are only left with the second integral.

Finally, a coordinate shift cannot change the physics, so the variation $\bar\delta S_{\rm ng}$ induced by an arbitrary ξ_α has to vanish,

$$\bar\delta S_{\rm ng} = 16\pi \int_\Omega T^{\alpha\beta}{}_{;\beta}\xi_\alpha\,\sqrt{-g}\,{\rm d}^4x = 0 \qquad \Longleftrightarrow \qquad T^{\alpha\beta}{}_{;\beta} = 0\,. \tag{23.20}$$

This concludes the proof. □

Note that the same result can also be obtained for the Einstein tensor $G^{\mu\nu}$ (of course, we know it from the Bianchi identities, but here we do *not* employ the latter). Actually, writing, similarly, (23.15) as

$$\bar{\delta}S_{\rm g}=\int_\Omega (G_{\mu\nu}+\Lambda g_{\mu\nu})\,\bar{\delta}g^{\mu\nu}\sqrt{-g}\,{\rm d}^4x=-\int_\Omega (G^{\mu\nu}+\Lambda g^{\mu\nu})\,\bar{\delta}g_{\mu\nu}\sqrt{-g}\,{\rm d}^4x$$

and using $\bar{\delta}g_{\mu\nu}=-\xi_{\mu;\nu}-\xi_{\nu;\mu}$, we have

$$\bar{\delta}S_{\rm g}=\int_\Omega (G^{\mu\nu}+\Lambda g^{\mu\nu})(\xi_{\mu;\nu}+\xi_{\nu;\mu})\sqrt{-g}\,{\rm d}^4x=2\int_\Omega (G^{\mu\nu}+\Lambda g^{\mu\nu})\,\xi_{\mu;\nu}\sqrt{-g}\,{\rm d}^4x=$$
$$=\cancel{2\int [(G^{\mu\nu}+\Lambda g^{\mu\nu})\,\xi_\mu]_{;\nu}\sqrt{-g}\,{\rm d}^4x}-2\int (G^{\mu\nu}{}_{;\nu}+\Lambda \cancel{g^{\mu\nu}{}_{;\nu}})\,\xi_\mu\sqrt{-g}\,{\rm d}^4x\,,$$

so should $\bar{\delta}S_{\rm g}$ vanish (no change of physics by the coordinate shift), it has to hold $G^{\mu\nu}{}_{;\nu}=0$.

23.4.6 Boundary term

The derivation can also be performed in a more natural case when solely the metric itself is kept fixed on the boundary of Ω, but the metric *derivatives* are left unconstrained ($\bar{\delta}g_{\mu\nu,\iota}\neq 0$). The integral $\int g^{\mu\nu}\bar{\delta}R_{\mu\nu}\sqrt{-g}\,{\rm d}^4x$ in variation of the gravitational action (23.15) – the one which we translated into an integral (23.14) from partial divergence of a vector and threw out thanks to Gauss and vanishing of $\bar{\delta}g_{\mu\nu,\iota}$ (and thus $\bar{\delta}\Gamma^\lambda{}_{\nu\mu}$) on a boundary – is not zero then. Its contribution thus has to be *subtracted* from $S_{\rm g}$ in order that the field equations come out correct. Let us treat this term more properly here.

- Starting from (23.14), we first use the Gauss law,

$$\int_\Omega g^{\mu\nu}\bar{\delta}R_{\mu\nu}\sqrt{-g}\,{\rm d}^4x=\int_\Omega \left(g^{\mu\nu}\bar{\delta}\Gamma^\lambda{}_{\nu\mu}-g^{\mu\lambda}\bar{\delta}\Gamma^\nu{}_{\nu\mu}\right)_{;\lambda}\sqrt{-g}\,{\rm d}^4x=$$
$$=\int_\Omega \left[\sqrt{-g}\,(g^{\mu\nu}\bar{\delta}\Gamma^\lambda{}_{\nu\mu}-g^{\mu\lambda}\bar{\delta}\Gamma^\nu{}_{\nu\mu})\right]_{,\lambda}{\rm d}^4x=\epsilon\oint_{\partial\Omega}\left(g^{\mu\nu}\bar{\delta}\Gamma^\lambda{}_{\nu\mu}-g^{\mu\lambda}\bar{\delta}\Gamma^\nu{}_{\nu\mu}\right)n_\lambda\sqrt{h}\,{\rm d}^3y\,,$$

  where n_λ is the unit normal to the boundary $\partial\Omega$ (with $\epsilon:=g^{\mu\nu}n_\mu n_\nu=\mp 1$ its norm; we suppose the boundary is nowhere null), h is the determinant of the 3D metric induced on the boundary and y^i are some 3D coordinates there.

- Now we write out the variation of the Christoffel symbols in more detail:

$$\bar{\delta}\Gamma_{\iota\nu\mu}=\frac{1}{2}\,\bar{\delta}(g_{\iota\nu,\mu}+g_{\mu\iota,\nu}-g_{\nu\mu,\iota})=\frac{1}{2}\left[(\bar{\delta}g_{\iota\nu})_{,\mu}+(\bar{\delta}g_{\mu\iota})_{,\nu}-(\bar{\delta}g_{\nu\mu})_{,\iota}\right]=$$
$$=\frac{1}{2}\left[(\bar{\delta}g_{\iota\nu})_{;\mu}+(\bar{\delta}g_{\mu\iota})_{;\nu}-(\bar{\delta}g_{\nu\mu})_{;\iota}\right]+$$
$$+\frac{1}{2}\left(\cancel{\Gamma^\sigma{}_{\mu\iota}\bar{\delta}g_{\sigma\nu}}+\Gamma^\sigma{}_{\mu\nu}\bar{\delta}g_{\iota\sigma}+\Gamma^\sigma{}_{\nu\mu}\bar{\delta}g_{\sigma\iota}+\cancel{\Gamma^\sigma{}_{\nu\iota}\bar{\delta}g_{\mu\sigma}}-\cancel{\Gamma^\sigma{}_{\iota\nu}\bar{\delta}g_{\sigma\mu}}-\cancel{\Gamma^\sigma{}_{\iota\mu}\bar{\delta}g_{\nu\sigma}}\right)=$$
$$=\frac{1}{2}\left[(\bar{\delta}g_{\iota\nu})_{;\mu}+(\bar{\delta}g_{\mu\iota})_{;\nu}-(\bar{\delta}g_{\nu\mu})_{;\iota}\right]+\Gamma^\sigma{}_{\nu\mu}\bar{\delta}g_{\sigma\iota}\,,\tag{23.21}$$

hence for the second-kind Gammas we obtain

$$\begin{aligned}
\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} &= \bar{\delta}g^{\lambda\iota}\Gamma_{\iota\nu\mu} + g^{\lambda\iota}\bar{\delta}\Gamma_{\iota\nu\mu} = \\
&= \cancel{-\bar{\delta}g_{\rho\sigma}\, g^{\rho\lambda} g^{\sigma\iota}\Gamma_{\iota\nu\mu}} + \frac{1}{2}g^{\lambda\iota}\left[(\bar{\delta}g_{\iota\nu})_{;\mu} + (\bar{\delta}g_{\mu\iota})_{;\nu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right] + \cancel{g^{\lambda\iota}\Gamma^{\sigma}{}_{\nu\mu}\bar{\delta}g_{\sigma\iota}} = \\
&= \frac{1}{2}g^{\lambda\iota}\left[(\bar{\delta}g_{\iota\nu})_{;\mu} + (\bar{\delta}g_{\mu\iota})_{;\nu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right], \qquad (23.22)
\end{aligned}$$

where in the first term we have used the relation (23.9).

- Thanks to the last formula, we can simplify

$$\begin{aligned}
&\left(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} - g^{\mu\lambda}\bar{\delta}\Gamma^{\nu}{}_{\nu\mu}\right) = \frac{1}{2}(g^{\mu\nu}g^{\lambda\iota} - g^{\mu\lambda}g^{\nu\iota})\left[(\bar{\delta}g_{\iota\nu})_{;\mu} + (\bar{\delta}g_{\mu\iota})_{;\nu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right] = \\
&= \frac{1}{2}(g^{\mu\nu}g^{\lambda\iota} - g^{\mu\lambda}g^{\nu\iota})\left[(\bar{\delta}g_{\iota\nu})_{;\mu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right] = (g^{\mu\nu}g^{\lambda\iota} - g^{\mu\lambda}g^{\nu\iota})\,(\bar{\delta}g_{\iota\nu})_{;\mu}
\end{aligned}$$

(only employed has been the anti-symmetry in $[\mu, \iota]$ of the parenthesis and, then, that of the bracket as well). Therefore, in the above Gauss-integral argument, we get

$$\begin{aligned}
&(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} - g^{\mu\lambda}\bar{\delta}\Gamma^{\nu}{}_{\nu\mu})\, n_{\lambda} = (n^{\iota}g^{\mu\nu} - n^{\mu}g^{\nu\iota})(\bar{\delta}g_{\iota\nu})_{;\mu} = \\
&= \left[n^{\iota}(h^{\mu\nu} + \epsilon^{-1}n^{\mu}n^{\nu}) - n^{\mu}(h^{\nu\iota} + \epsilon^{-1}n^{\nu}n^{\iota})\right](\bar{\delta}g_{\iota\nu})_{;\mu} = \\
&= n^{\iota}\,\cancel{h^{\mu\nu}(\bar{\delta}g_{\iota\nu})_{;\mu}} - n^{\mu}h^{\nu\iota}(\bar{\delta}g_{\iota\nu})_{;\mu}\ ,
\end{aligned}$$

where $h_{\mu\nu} = g_{\mu\nu} - \epsilon^{-1}n_{\mu}n_{\nu}$ stands for the 3D metric of $\partial\Omega$, with $\epsilon := g_{\mu\nu}n^{\mu}n^{\nu}$ (note that the boundary has different character at different parts, so ϵ generally does not have the same sign everywhere on it). The term $h^{\mu\nu}(\bar{\delta}g_{\iota\nu})_{;\mu}$ dropped out regarding that i) on $\partial\Omega$ the covariant derivatives in fact reduce to partial derivatives, because their Γ-terms are proportional to $\bar{\delta}g_{\iota\nu}$ which vanish on $\partial\Omega$, and that ii) $\bar{\delta}g_{\iota\nu}$ vanishes *everywhere* on $\partial\Omega$, hence also vanishes its partial derivative in the direction tangent to $\partial\Omega$, given by $h^{\mu\nu}(\bar{\delta}g_{\iota\nu})_{,\mu}$.

- Now we show that the above result can be expressed in terms of variation on $\partial\Omega$ of the quantity

$$K := K^{\nu}_{\nu} := h^{\nu}_{\kappa}n^{\kappa}{}_{;\nu} = n^{\nu}{}_{;\nu}$$

($K_{\mu\nu}$ will be called the extrinsic curvature [and its trace K the mean curvature] and more thoroughly discussed in Chapter 25; it will play a crucial role in Chapter 27). Since $\bar{\delta}g_{\mu\nu} = 0$ on $\partial\Omega$, the same must also hold for $\bar{\delta}h_{\mu\nu}$, for $\bar{\delta}n^{\alpha}$ and for the latter's tangent derivative $h^{\nu}_{\kappa}\bar{\delta}(n^{\kappa}{}_{,\nu}) = h^{\nu}_{\kappa}(\bar{\delta}n^{\kappa})_{,\nu}$ (the region Ω is fixed, it does not change under variation, so the metric decomposition is also not varied). Hence, we have

$$\begin{aligned}
\bar{\delta}K &= \bar{\delta}(h^{\nu}_{\kappa}n^{\kappa}{}_{;\nu}) = \bar{\delta}(h^{\nu}_{\kappa}n^{\kappa}{}_{,\nu} + h^{\nu}_{\kappa}\Gamma^{\kappa}{}_{\nu\mu}n^{\mu}) = h^{\nu}_{\kappa}(\bar{\delta}\Gamma^{\kappa}{}_{\nu\mu})\, n^{\mu} = \\
&= \frac{1}{2}\, h^{\nu}_{\kappa}g^{\kappa\iota}\left[(\bar{\delta}g_{\iota\nu})_{;\mu} + (\bar{\delta}g_{\mu\iota})_{;\nu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right] n^{\mu} = \\
&= \frac{1}{2}\, h^{\nu\iota}\left[(\bar{\delta}g_{\iota\nu})_{;\mu} + (\bar{\delta}g_{\mu\iota})_{;\nu} - (\bar{\delta}g_{\nu\mu})_{;\iota}\right] n^{\mu} = \frac{1}{2}\, h^{\nu\iota}(\bar{\delta}g_{\iota\nu})_{;\mu}n^{\mu}\ .
\end{aligned}$$

We have thus learned that

$$\int_{\Omega} g^{\mu\nu}\bar{\delta}R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = -\epsilon\oint_{\partial\Omega} h^{\nu\iota}(\bar{\delta}g_{\iota\nu})_{;\mu}n^{\mu}\,\sqrt{h}\,\mathrm{d}^3y = -2\epsilon\oint_{\partial\Omega}\bar{\delta}K\,\sqrt{h}\,\mathrm{d}^3y\,. \tag{23.23}$$

Footnote: why, in computing $\bar{\delta}K$, we have employed the expression $K = h^{\nu}_{\kappa}n^{\kappa}{}_{;\nu}$ rather than the simpler one $K = n^{\nu}{}_{;\nu}$? The latter would lead to $\bar{\delta}K = \bar{\delta}(n^{\nu}{}_{;\nu}) = \bar{\delta}(n^{\nu}{}_{,\nu} + \Gamma^{\nu}{}_{\nu\mu}n^{\mu})$, which *is* simpler but its first term $\bar{\delta}(n^{\nu}{}_{,\nu}) = (\bar{\delta}n^{\nu})_{,\nu}$ can *not* be claimed to vanish, because here the gradient of $\bar{\delta}n^{\nu}$ is taken in *general* direction, rather than in the direction tangent to $\partial\Omega$ as in the term $h^{\nu}_{\kappa}n^{\kappa}{}_{,\nu}$ occurring above.

The above result is known as the **Gibbons-Hawking-York boundary term**. Since it is *extra* to what has been obtained from (23.5), the corresponding part of the Lagrangian has to be subtracted from the original $\mathcal{L}$ in order that the latter yield the correct field equations. The action thus has to be modified to

$$\boxed{S = \int_{\Omega}(\mathcal{L}_{\mathrm{g}} + 16\pi\mathcal{L}_{\mathrm{ng}})\sqrt{-g}\,\mathrm{d}^4x + 2\epsilon\oint_{\partial\Omega}K\,\sqrt{h}\,\mathrm{d}^3y\,, \qquad \text{where } \ \mathcal{L}_{\mathrm{g}} = R - 2\Lambda}\,. \tag{23.24}$$

Boundary-term normalization

One unfavourable feature still remains. Imagine the host manifold is Minkowski, take spherical coordinates (t, r, θ, ϕ) there and choose Ω to be a 3D "cylinder" with some radius $r = R$ and with bases at $t = \mathrm{const} = t_{\mathrm{in}}$ and $t = \mathrm{const} = t_{\mathrm{fin}}$. The 4D metric and the 3D metric induced on the cylinder (actually a history of the spherical surface $r = R$) are

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \qquad \Longrightarrow \qquad -g = r^4\sin^2\theta\,,$$
$$\mathrm{d}\sigma^2\left(\equiv h_{ij}\mathrm{d}y^i\mathrm{d}y^j\right) = -\mathrm{d}t^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \qquad \Longrightarrow \qquad h = R^4\sin^2\theta\,.$$

The unit normal to any $r = \mathrm{const}$ surface and the corresponding mean curvature read

$$n_{\lambda} = r_{,\lambda} = \delta^r_{\lambda}\,, \quad n^{\kappa} = g^{\kappa\lambda}\delta^r_{\lambda} = g^{\kappa r} = \delta^{\kappa}_{r} \qquad \Longrightarrow \qquad \epsilon \equiv n_{\lambda}n^{\lambda} = \delta^r_{\lambda}\delta^{\lambda}_{r} = 1\,,$$
$$K \equiv n^{\kappa}{}_{;\kappa} = \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,n^{\kappa})_{,\kappa} = \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,\delta^{\kappa}_{r})_{,\kappa} = \frac{1}{r^2\sin\theta}\,(r^2\sin\theta)_{,r} = \frac{2}{r}\,.$$

The bases of the cylinder do not contribute to the integral, because $K = 0$ on them, so the surface term amounts to

$$2\epsilon\oint_{\partial\Omega}K\,\sqrt{h}\,\mathrm{d}^3y = 4\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_0^{2\pi}\int_0^{\pi}R\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}t = 16\pi R(t_{\mathrm{fin}} - t_{\mathrm{in}})\,.$$

Unfavourable is that this diverges for $R \to \infty$. This is *not* an issue for the variation and for the field equations, nevertheless the action itself behaves badly in infinite (non-compact) spaces, even if taken between two finite times.

The above numerical feature is simply being remedied by subtracting the divergent term from the result, i.e. by taking, instead of (23.24),

$$S=\int_{\Omega}(\mathcal{L}_{\rm g}+16\pi\mathcal{L}_{\rm ng})\sqrt{-g}\,{\rm d}^4x+2\epsilon\oint_{\partial\Omega}(K-K_{\rm flat})\,\sqrt{h}\,{\rm d}^3y\,,\quad\text{where}\;\;\mathcal{L}_{\rm g}=R-2\Lambda\,,\tag{23.25}$$

where $K_{\rm flat}$ is the K term computed for $\partial\Omega$ embedded in flat space-time. This behaves as fixed in variation, so it does not affect the field equations (the additional $K_{\rm flat}$ term is thus being called a **non-dynamical term**).

23.5 "Palatini's" variational principle [due to Einstein, 1925]

A. Palatini obtained his formula for variation of the Ricci tensor in 1919. Several people were asking, subsequently, whether it was possible to form an action only depending on the dynamical field (the metric) and its *first* (not second) derivatives, similarly as it is in electrodynamics, for example. (There, the Lagrangian depends on the EM-field tensor, i.e. on the first derivatives of the four-potential.) An elegant way how to achieve this was finally provided by Einstein in 1925:[4] one must release the link between the metric and connection – then actually the metric only enters the Lagrangian without any derivatives. A more general formulation of the variational problem thus arises, in which it is *not* supposed that the affine connection is of the Levi-Civita type (represented through Christoffel symbols). In fact metric and connection are supposed to be totally unrelated (the connection is only assumed to be torsion-free), and the variation is made with respect to them as independent variables. If the connection only appears in the curvature term, which is the case if the non-gravitational part of action does not depend on $g_{\mu\nu,\iota}$, we may omit the treatment of the cosmological and source terms (no change occurs there) and only show the novelty of the present formulation on the Hilbert's term.

So consider the vacuum-case action

$$S_{\rm vac}(g_{\mu\nu},\Gamma^{\iota}{}_{\mu\nu},\Gamma^{\iota}{}_{\mu\nu,\kappa})\equiv\int R\,\sqrt{-g}\,{\rm d}^4x\equiv\int g^{\mu\nu}R_{\mu\nu}(\Gamma^{\iota}{}_{\mu\nu},\Gamma^{\iota}{}_{\mu\nu,\kappa})\,\sqrt{-g}\,{\rm d}^4x\,.$$

By varying with respect to the dynamical variable (the metric), one obtains (not even necessary to evaluate it in detail)

$$\bar{\delta}S_{\rm vac}=\int\bar{\delta}(g^{\mu\nu}\sqrt{-g})\,R_{\mu\nu}\,{\rm d}^4x\qquad\ldots\;\;=0\quad\Longleftrightarrow\quad R_{\mu\nu}=0\,.\tag{23.26}$$

One thus immediately reaches the field equations (for vacuum).

[4] "Einheitliche Feldtheorie von Gravitation und Elektrizität", Sitzungsberichte der Preußischen Akademie der Wissenschaften XXII (1925) 414. We are giving full reference since *this* is the crucial paper, despite the approach is incorrectly being referred to as "Palatini's variational method" in most textbooks. See M. Ferraris, M. Francaviglia & C. Reina, Gen. Rel. Grav. 14 (1982) 243.

23.5.1 Variation with respect to connection (as an independent variable)

Varying, on the other hand, with respect to the affine connection, one obtains

$$\bar{\delta} S_{\mathrm{vac}} = \int g^{\mu\nu} \bar{\delta} R_{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x\,.$$

One can substitute here the Palatini equation (23.12) as we did already since that was derived just using affine connection (not metric),

$$\bar{\delta} S_{\mathrm{vac}} = \int g^{\mu\nu} \left[(\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu})_{;\lambda} - (\bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu})_{;\nu} \right] \sqrt{-g}\, \mathrm{d}^4 x\,, \tag{23.27}$$

but then one can*not* “absorb” the metric inside the covariant derivative as automatically as in (23.14). Neither is one allowed to employ the relation $\sqrt{-g}\, V^{\lambda}{}_{;\lambda} = \left(\sqrt{-g}\, V^{\lambda}\right)_{,\lambda}$, since that we derived for the Levi-Civita connection.

In the paper mentioned in the footnote, Einstein writes: “After a very hard research during the last two years I now think I have got the correct solution...” So it must be *worth a theorem*.

$\boxed{\text{Theorem}}$ Vanishing of the variation of S_{vac} with respect to $\Gamma^{\lambda}{}_{\nu\mu}$ (as an independent variable) implies that if the affine connection is torsion-free, it has to be of the Levi-Civita type.

Proof:

- We can certainly rewrite the above variation, just using the Leibniz rule, as

$$\bar{\delta} S_{\mathrm{vac}} = \int \left(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu}\sqrt{-g}\right)_{;\lambda} \mathrm{d}^4 x - \int \bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} \left(g^{\mu\nu}\sqrt{-g}\right)_{;\lambda} \mathrm{d}^4 x - \\ - \int \left(g^{\mu\nu}\bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu}\sqrt{-g}\right)_{;\nu} \mathrm{d}^4 x + \int \bar{\delta}\Gamma^{\lambda}{}_{\lambda\mu} \left(g^{\mu\nu}\sqrt{-g}\right)_{;\nu} \mathrm{d}^4 x\,. \tag{23.28}$$

- Consider now the formula (A.22), $(V^{\rho}\sqrt{-g})_{;\rho} = (V^{\rho}\sqrt{-g})_{,\rho}$, valid for any torsion-free connection. It implies that the integrals of covariant divergences in (23.28) drop out, since by Gauss they equal the corresponding fluxes over the boundary, which are zero since they contain $\bar{\delta}\Gamma$.

- The remaining terms of (23.28) can be rewritten

$$\bar{\delta} S_{\mathrm{vac}} = \int \bar{\delta}\Gamma^{\lambda}{}_{\nu\mu} \left[\delta^{\nu}_{\lambda} \left(g^{\mu\kappa}\sqrt{-g}\right)_{;\kappa} - \left(g^{\mu\nu}\sqrt{-g}\right)_{;\lambda} \right] \mathrm{d}^4 x\,.$$

  Assuming symmetric connection (without torsion) as above, this vanishes, for a general variation $\bar{\delta}\Gamma^{\lambda}{}_{\nu\mu}$, if and only if the symmetric-in-(μ,ν) part of the bracket vanishes,

$$\frac{1}{2}\,\delta^{\nu}_{\lambda}\left(g^{\mu\kappa}\sqrt{-g}\right)_{;\kappa} + \frac{1}{2}\,\delta^{\mu}_{\lambda}\left(g^{\nu\kappa}\sqrt{-g}\right)_{;\kappa} - \left(g^{\mu\nu}\sqrt{-g}\right)_{;\lambda} = 0\,.$$

Contracting this in ${}^{\mu}/_{\lambda}$ yields[5]

$$\frac{1}{2}\left(g^{\nu\kappa}\sqrt{-g}\right)_{;\kappa}+2\left(g^{\nu\kappa}\sqrt{-g}\right)_{;\kappa}-\left(g^{\mu\nu}\sqrt{-g}\right)_{;\mu}\;\equiv\;\frac{3}{2}\left(g^{\nu\kappa}\sqrt{-g}\right)_{;\kappa}=0\,,$$

which implies, when used back in the equation before, $\left(g^{\mu\nu}\sqrt{-g}\right)_{;\lambda}=0$.

- Applying now the relation $(\sqrt{-g})_{;\rho}=\frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta;\rho}$ derived in (A.21), we thus have condition

  $$0=\frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\right)_{;\lambda}=g^{\mu\nu}{}_{;\lambda}+\frac{1}{2}\,g^{\mu\nu}g^{\alpha\beta}g_{\alpha\beta;\lambda}\,.$$

  Multiplication of this equation by $g_{\mu\nu}$ tells that its second term has to vanish,

  $$0=g_{\mu\nu}g^{\mu\nu}{}_{;\lambda}+2g^{\alpha\beta}g_{\alpha\beta;\lambda}=-g_{\mu\nu;\lambda}g^{\mu\nu}+2g^{\alpha\beta}g_{\alpha\beta;\lambda}\;\equiv\;g^{\alpha\beta}g_{\alpha\beta;\lambda}\,,$$

  so one finally arrives at the condition $g^{\mu\nu}{}_{;\lambda}=0$ ... the connection has to be of the Levi-Civita type. □

The strength of this variational method – using such a simple Lagrangian (R)! – thus has been confirmed: not only that it yields the field equations, but it even prescribes what is the correct affine connection of the theory.

[5] Gratefully acknowledged is the suggestion by Jonáš Dujava of how to best proceed here.

CHAPTER 24

Vector fields and their integral congruences

It is often beneficial to study the space-time on the properties of its lower-dimensional sections. As a classical geometrical problem, the study of submanifolds is a solid part of every differential-geometry textbook. In fact a given family of lower-dimensional sections may *not* be integrable into a family of global submanifolds, with the exception of the 1D case – there does exist a congruence of integral curves to every smooth (and nowhere vanishing) vector field. This is the topic of the present chapter. We will first describe a vector field (a congruence) in a suitable geometrical language, while emphasizing the difference between the time-like and light-like cases, and then will derive how the space-time geometry enters the properties of congruences. From physical side, the topic is mainly useful in problems involving continuous media (hydrodynamics).

24.1 Kinematics of time-like congruences

Let's have a congruence of time-like world-lines, parameterized in the same way as in Section 6.4 on geodesic deviation: $x^\mu = x^\mu(l;\tau)$, where l and τ are real parameters; l "numbers" the curves (but it is continuous), while τ runs along them. Two vectors can thus be introduced immediately,

$$u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \qquad \ldots \quad \text{tangent field (four-velocity: we assume } u_\mu u^\mu = -1)\,, \tag{24.1}$$

$$\delta x^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}l} \qquad \ldots \quad \text{relative position vector;} \;\; u_\mu \delta x^\mu = 0\,. \tag{24.2}$$

The relative position (between "neighbouring" world-lines) is *prescribed* as being orthogonal to u^μ, because exactly this corresponds to an actual measurement (the position on the neighbouring world-line is measured *at given time* τ of a certain selected "reference" world-line). In the chapter on geodesic deviation, we first reminded that the relative-position vectors ("connecting vectors") have zero Lie derivative along the reference world-line,

$$(\pounds_{\mathbf{u}}\delta\mathbf{x})^\mu = (\delta x^\mu)_{,\nu}u^\nu - u^\mu{}_{,\nu}\delta x^\nu = (\delta x^\mu)_{;\nu}u^\nu - u^\mu{}_{;\nu}\delta x^\nu = 0\,, \tag{24.3}$$

in other words,

$$\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}u^\mu}{\mathrm{d}l}\,, \qquad \text{or} \qquad \frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{D}u^\mu}{\mathrm{d}l} = u^\mu{}_{;\nu}\delta x^\nu\,. \tag{24.4}$$

For the evolution of the *transversal* properties of the congruence, the crucial quantity is thus the tensor $u^\mu{}_{;\nu}$ – it says how the transport of δx^μ along the reference world-line differs from parallel transport (under which the right-hand side would vanish). The tensor $u^\mu{}_{;\nu}$ is going to be the main character of this whole chapter.

Let us denote, as already previously, by $h_{\mu\nu}$ the tensor

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \tag{24.5}$$

which at any point projects on the three-space orthogonal to the local u^μ, and by $a_\mu := u_{\mu;\nu}u^\nu$ the four-acceleration of the tangent field u^μ. By double projection of the $u_{\mu;\nu}$, one has

$$h^\kappa_\mu h^\lambda_\nu u_{\kappa;\lambda} = (\delta^\kappa_\mu + u^\kappa u_\mu)(\delta^\lambda_\nu + u^\lambda u_\nu)u_{\kappa;\lambda} = (\delta^\lambda_\nu + u^\lambda u_\nu)u_{\mu;\lambda} = u_{\mu;\nu} + a_\mu u_\nu\,,$$

because $u^\kappa u_{\kappa;\lambda} = 0$ due to the normalization of u^μ.

Now the main point: in analogy with the decomposition of the velocity field in classical hydrodynamics, we can write

$$\boxed{u_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - a_\mu u_\nu}\,, \tag{24.6}$$

where

$$\omega_{\mu\nu} \equiv h^\kappa_\mu h^\lambda_\nu u_{[\kappa;\lambda]} = h^\kappa_{[\mu}h^\lambda_{\nu]}u_{\kappa;\lambda} = u_{[\mu;\nu]} + a_{[\mu}u_{\nu]} \tag{24.7}$$

is the antisymmetric **vorticity** (or **twist**) tensor, and

$$\Theta_{\mu\nu} \equiv h^\kappa_\mu h^\lambda_\nu u_{(\kappa;\lambda)} = h^\kappa_{(\mu}h^\lambda_{\nu)}u_{\kappa;\lambda} = u_{(\mu;\nu)} + a_{(\mu}u_{\nu)} \quad \left(= \frac{1}{2}\pounds_{\mathbf{u}}h_{\mu\nu}\right) \tag{24.8}$$

is the symmetric **expansion** tensor, its trace

$$\Theta \equiv h^{\mu\nu}\Theta_{\mu\nu} = u^\mu{}_{;\mu} \tag{24.9}$$

being called the **expansion scalar** and its traceless part

$$\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3}\Theta h_{\mu\nu} \tag{24.10}$$

the **shear** tensor. All the terms of the decomposition (24.6) are individually normal to u^μ (naturally, since they are defined by the $h_{\mu\nu}$ projection),

$$u^\mu a_\mu u_\nu = u^\mu\omega_{\mu\nu} = u^\mu\sigma_{\mu\nu} = u^\mu h_{\mu\nu} = 0\,. \tag{24.11}$$

Note also that the multiplication of (24.6) by u^ν just confirms $a_\mu = a_\mu$.

The decomposition (24.6) is covariant, its terms characterize the transversal properties of the congruence independently of the coordinate system. And they have a clear geometric meaning. Submitting the decomposition to the evolution equation (24.4), we get

$$\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau} = \left(\omega^\mu{}_\nu + \Theta^\mu{}_\nu\right)\delta x^\nu = \left(\omega^\mu{}_\nu + \sigma^\mu{}_\nu\right)\delta x^\nu + \frac{1}{3}\Theta\,\delta x^\mu\,. \tag{24.12}$$

Imagine a 3D element of the flow (3D means "occupying a certain volume in the three-space of u^μ"). The scalar Θ describes the isotropic expansion/contraction of the element, while the remaining two terms do not change its volume – $\omega_{\mu\nu}$ describes vorticity (how world-lines "entwine" like fibres of a rope within the element) and $\sigma_{\mu\nu}$ describes shear (deformation ball → ellipsoid due to a different speed of the flow in different directions).

The tensor $\omega_{\mu\nu}$ is antisymmetric (it is a bivector) and it fulfils 3 constraints $\omega_{\mu\nu}u^\nu = 0$ (only 3 of them are independent, because due to the antisymmetry they are automatically bound by $\omega_{\mu\nu}u^\mu u^\nu = 0$). Therefore, it has 6-3=3 independent components and its content may be fully represented by a (pseudo-)vector – the **vorticity (twist) vector**

$$\omega^\mu := \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}\omega_{\nu\iota}u_\lambda = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}u_{\nu;\iota}u_\lambda = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}u_{\nu,\iota}u_\lambda \quad\Longleftrightarrow\quad \omega_{\nu\iota} = \epsilon_{\nu\iota\lambda\mu}\omega^\lambda u^\mu\,. \tag{24.13}$$

Clearly this vector, lying in the three-space orthogonal to u^μ ($\omega^\iota u_\iota = 0$), is a direct counterpart of the angular-velocity vector $\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{v}$ known from classical mechanics.

Similarly as for the expansion, one can also find scalars for acceleration, vorticity and shear,

$$\kappa^2 := a_\mu a^\mu\,, \tag{24.14}$$

$$\omega^2 := \frac{1}{2}\,\omega_{\mu\nu}\omega^{\mu\nu} = \omega_\iota\omega^\iota = \frac{1}{2}\left(u_{[\mu;\nu]} + a_{[\mu}u_{\nu]}\right)\left(u^{\mu;\nu} + a^\mu u^\nu\right) = \frac{1}{2}\,u_{[\mu;\nu]}u^{\mu;\nu} + \frac{\kappa^2}{4}, \tag{24.15}$$

$$\Theta := u^\mu{}_{;\mu}\,, \tag{24.16}$$

$$\sigma^2 := \frac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}\left(\Theta_{\mu\nu} - \frac{1}{3}\Theta h_{\mu\nu}\right)\left(\Theta^{\mu\nu} - \frac{1}{3}\Theta h^{\mu\nu}\right) = \frac{1}{2}\,u_{(\mu;\nu)}u^{\mu;\nu} + \frac{\kappa^2}{4} - \frac{\Theta^2}{6} \tag{24.17}$$

(κ, the magnitude of a^μ, plays the role of the first curvature of a world-line in the Frenet-Serret formalism of the intrinsic geometry of curves).

Remarks

- One would actually need *three* parameters (instead of just l) to parameterize a congruence penetrating the whole 3D volume, yet we in fact consider a "sheet-filling" sub-congruence with the δx^μ vector always being its space-like tangent.
- Thanks to the basic property $\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{D}u^\mu}{\mathrm{d}l}$, one finds

  $$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\mu\delta x^\mu) = \frac{\mathrm{D}}{\mathrm{d}\tau}(u_\mu\delta x^\mu) = \frac{\mathrm{D}u_\mu}{\mathrm{d}\tau}\,\delta x^\mu + u_\mu\,\frac{\mathrm{D}\delta x^\mu}{\mathrm{d}\tau} = a_\mu\delta x^\mu + \cancel{u_\mu\frac{\mathrm{D}u^\mu}{\mathrm{d}l}}\,.$$

  So δx^μ remains orthogonal to u^μ if and only if it is *also* normal to a^μ. (Or if $a^\mu = 0$ as in Section 6.4.)

- In introducing $\Theta_{\mu\nu}$, we added $\frac{1}{2}\pounds_{\mathbf{u}} h_{\mu\nu}$ rather quickly as an alternative expression. Really,

$$\begin{aligned}\pounds_{\mathbf{u}} h_{\mu\nu} &= (h_{\mu\nu;\iota}u^\iota + u^\iota{}_{;\mu}h_{\iota\nu} + u^\iota{}_{;\nu}h_{\mu\iota}) = (u_\mu u_\nu)_{;\iota}u^\iota + u_{\iota;\mu}(\delta^\iota_\nu + \cancel{u^\iota}u_\nu) + u_{\iota;\nu}(\delta^\iota_\mu + \cancel{u^\iota}u_\mu)\\ &= a_\mu u_\nu + u_\mu a_\nu + u_{\nu;\mu} + u_{\mu;\nu} = 2\Theta_{\mu\nu}\,,\end{aligned}$$

  where the cancellations do not apply to u^ι itself, but they indicate that $u_{\iota;\mu}u^\iota = 0$.

- In Section 24.4, we will see that $\omega_{\mu\nu} = 0$ if and only if u_μ is proportional to a gradient of some scalar, i.e. if u_μ is *orthogonal to hypersurfaces* given as isosurfaces of that scalar. The expression $u_{\mu;\nu} + a_\mu u_\nu = \Theta_{\mu\nu}$ is then symmetric automatically; it becomes the so-called **extrinsic curvature** $K_{\mu\nu}$ of the hypersurfaces, which will be important in Chapter 25 on foliation of space-time by hypersurfaces and on the thus induced "3+1" splitting.

- The expansion scalar Θ evidently vanishes for Killing vector fields. For non-accelerated congruences ($a_\mu = 0$), this also holds in the opposite direction: $\Theta_{\mu\nu} = 0$ implies that u^μ is a Killing field.

24.2 Kinematics of light-like congruences

Consider a congruence of light-like (null) world-lines now. Denote the tangent field by k^μ; it is null, so $k_\mu k^\mu = 0$. (This property implies $k_{\mu;\nu}k^\mu = 0$, as for any vector normalized to a constant.) We will assume the congruence is geodesic (and affinely parameterized), $k_{\mu;\nu}k^\nu = 0$, as it is usual in the case of photon world-lines.

Null congruences are somewhat less comfortable in that they do not fix uniquely the "tangent" and "normal" directions. Actually, the relation $k_\mu \delta x^\mu = 0$ does *not* exclude that δx^μ has a component proportional to k^μ. If we introduced the projector to the normal space in the same way as in the time-like case, $h_{\mu\nu} = g_{\mu\nu} + k_\mu k_\nu$, we would have $h^\mu_\nu k^\nu = (\delta^\mu_\nu + k^\mu k_\nu)k^\nu = k^\mu \neq 0$, so it would not work.[1]

A standard recipe how to remedy the situation is to consider one more null vector, call it l^μ, which is not normal/parallel to k^μ (which in the null case means that it is "linearly independent", so the two vectors locally form a plane); one may always normalize it so that $k_\mu l^\mu = -1$. "Purely transversal" are then considered such vectors which are orthogonal to both k^μ and l^μ. The projection to that plane ensures the tensor

$$h_{\mu\nu} = g_{\mu\nu} + k_\mu l_\nu + l_\mu k_\nu\,, \tag{24.18}$$

which (really) satisfies

$$h^\mu_\sigma h^\sigma_\nu = h^\mu_\nu\,, \qquad h^\mu_\nu k^\nu = h^\mu_\nu l^\nu = 0\,, \qquad h^\sigma_\sigma = 2\,. \tag{24.19}$$

[1] With null directions one should also be careful on an intuitive and pictorial level. If, for example, drawing a (2D or 1D) light cone in Minkowski as usual, its normal should *not* be drawn perpendicular to it. Really, having a vector $k^\mu = (1,1,0,0)$ in such a plot, its space-time normal does *not* read $l^\mu = (1,-1,0,0)$ – will you check that $\eta_{\mu\nu}k^\mu l^\nu = -2$. Unfortunately :-), the normal of k^μ is k^μ. (Well, it is not *the only* normal – one may play with the remaining components arbitrarily.)

The last relation confirms that it represents a *2D* metric (this will be one of important differences from the time-like case where $h^\sigma_\sigma = \delta^\sigma_\sigma + u^\sigma u_\sigma = 3$, naturally). To conclude the introductory part, let us summarize all the properties we have got up to now:

$$k_\mu k^\mu = 0 \quad (\Rightarrow)\ k_{\mu;\nu}k^\mu = 0\,, \qquad k_{\mu;\nu}k^\nu = 0\,, \qquad k_\mu \delta x^\mu = 0\,, \tag{24.20}$$
$$l_\mu l^\mu = 0 \quad (\Rightarrow)\ l_{\mu;\nu}l^\mu = 0\,, \qquad k_\mu l^\mu = -1\,, \qquad h^\mu_\nu k^\nu = h^\mu_\nu l^\nu = 0\,. \tag{24.21}$$

Intending to describe the transversal properties of a null congruence, we again use the evolution equation $\delta x^\mu{}_{;\sigma}k^\sigma = k^\mu{}_{;\sigma}\delta x^\sigma$, yet in order that the latter contain purely transversal quantities only (which it does not, because $k^\mu{}_{;\sigma}k_\mu = 0$, but $k^\mu{}_{;\sigma}l_\mu \neq 0$), we have to use $h_{\mu\nu}$ at several places:

$$\begin{aligned} h^\mu_\alpha(h^\alpha_\rho \delta x^\rho)_{;\sigma}k^\sigma &= h^\mu_\alpha(h^\alpha_{\rho;\sigma}\delta x^\rho + h^\alpha_\rho \delta x^\rho{}_{;\sigma})\,k^\sigma = \\ &= h^\mu_\alpha(k^\alpha{}_{;\sigma}l_\rho + k^\alpha l_{\rho;\sigma} + l^\alpha k_{\rho;\sigma} + l^\alpha{}_{;\sigma}k_\rho)\,\delta x^\rho k^\sigma + h^\mu_\rho \delta x^\rho{}_{;\sigma}k^\sigma = \\ &= h^\mu_\rho \delta x^\rho{}_{;\sigma}k^\sigma\,, \end{aligned}$$

from where it follows, by plugging the original (unprojected) equation $\delta x^\rho{}_{;\sigma}k^\sigma = k^\rho{}_{;\sigma}\delta x^\sigma$,

$$h^\mu_\alpha(h^\alpha_\rho \delta x^\rho)_{;\sigma}k^\sigma = h^\mu_\rho k^\rho{}_{;\sigma}\delta x^\sigma = h^\mu_\rho k^\rho{}_{;\sigma}(h^\sigma_\iota \delta x^\iota) = h^\mu_\rho k^\rho{}_{;\sigma}h^\sigma_\nu(h^\nu_\iota \delta x^\iota)\,. \tag{24.22}$$

Evolution of the transversal separation $h^\alpha_\rho \delta x^\rho$ is thus determined by the tensor $h^\mu_\rho h^\sigma_\nu k^\rho{}_{;\sigma}$. Explicitly,

$$h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma} = k_{\mu;\nu} + k_\mu l^\rho k_{\rho;\nu} + k_\nu l^\sigma k_{\mu;\sigma} + k_\mu k_\nu k_{\rho;\sigma}l^\rho l^\sigma \tag{24.23}$$

(it is often convenient to realize that the 2nd-4th terms are all orthogonal to the first).

Now we are ready to introduce similar kinematical quantities as in the time-like case. They are going to contain the "auxiliary" vector field l^μ which is by far not fixed uniquely by the relations $l^\mu l_\mu = 0$, $k^\mu l_\mu = -1$, but the scalars we will define at the end will not depend on it. So let us decompose $h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma}$ like we did with $u_{\mu;\nu}$ in the preceding section:

$$h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{2}\Theta h_{\mu\nu}\,, \tag{24.24}$$

where the **vorticity (twist)** tensor $\omega_{\mu\nu}$ is the skew-symmetric part,

$$\begin{aligned} \omega_{\mu\nu} := h^\rho_{[\mu}h^\sigma_{\nu]}k_{\rho;\sigma} &= k_{[\mu;\nu]} + l^\rho k_{\rho;[\nu}k_{\mu]} + l^\rho k_{[\nu}k_{\mu];\rho} = \\ &= k_{[\mu;\nu]} + k_\mu k_{[\rho;\nu]}l^\rho + k_\nu k_{[\mu;\rho]}l^\rho\,, \end{aligned} \tag{24.25}$$

while the **expansion** tensor $\Theta_{\mu\nu}$ is the symmetric part,

$$\begin{aligned} \Theta_{\mu\nu} := h^\rho_{(\mu}h^\sigma_{\nu)}k_{\rho;\sigma} &= k_{(\mu;\nu)} + l^\rho k_{\rho;(\nu}k_{\mu)} + l^\rho k_{(\nu}k_{\mu);\rho} + k_\mu k_\nu k_{\rho;\sigma}l^\rho l^\sigma = \\ &= k_{(\mu;\nu)} + k_\mu k_{(\rho;\nu)}l^\rho + k_\nu k_{(\mu;\rho)}l^\rho + k_\mu k_\nu k_{\rho;\sigma}l^\rho l^\sigma\,; \end{aligned} \tag{24.26}$$

the **expansion scalar** is again given by trace

$$\Theta := h^{\iota\rho}h^\sigma_\iota k_{\rho;\sigma} = h^{\rho\sigma}k_{\rho;\sigma} = k^\sigma{}_{;\sigma} \tag{24.27}$$

and, finally, the **shear** tensor represents the traceless part of $\Theta_{\mu\nu}$,

$$\sigma_{\mu\nu} := \Theta_{\mu\nu} - \frac{1}{2}\Theta h_{\mu\nu}\,. \tag{24.28}$$

Naturally, $\omega_{\mu\nu}k^\mu = \sigma_{\mu\nu}k^\mu = h_{\mu\nu}k^\mu = 0$, and also $\omega_{\mu\nu}l^\mu = \sigma_{\mu\nu}l^\mu = h_{\mu\nu}l^\mu = 0$. (Hence, on purely spatial quantities the projector $h_{\mu\nu}$ again acts as a metric.)

Remark: in the time-like case, we saw $\Theta_{\mu\nu}$ could also be expressed as $\Theta_{\mu\nu} = \frac{1}{2}\pounds_{\mathbf{u}} h_{\mu\nu}$. In the null case, one also finds straightforwardly that

$$\frac{1}{2}\pounds_{\mathbf{k}} h_{\mu\nu} = k_{(\mu;\nu)} + k_{(\mu}\pounds_{\mathbf{k}} l_{\nu)}\,, \quad \text{hence} \quad \frac{1}{2}h^\mu_\alpha h^\nu_\beta\, \pounds_{\mathbf{k}} h_{\mu\nu} = h^\mu_\alpha h^\nu_\beta k_{(\mu;\nu)} \equiv \Theta_{\alpha\beta}\,. \tag{24.29}$$

Similarly as for the time-like congruence, it is possible to introduce the **vorticity (twist) vector**,

$$\omega^\mu = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}\omega_{\nu\iota}k_\lambda = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}k_{\nu,\iota}k_\lambda \qquad \Longleftrightarrow \qquad \omega_{\nu\iota} = \epsilon_{\nu\iota\lambda\mu}\omega^\lambda k^\mu\,, \tag{24.30}$$

obviously normal to k^μ, and also to form scalars

$$\omega^2 := \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = \omega_\iota\omega^\iota = \frac{1}{2}k_{[\mu;\nu]}k^{\mu;\nu}\,, \qquad \sigma^2 := \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}k_{(\mu;\nu)}k^{\mu;\nu} - \frac{1}{4}\Theta^2\,. \tag{24.31}$$

For the null congruence, these are (together with the expansion scalar Θ) often called the **optical scalars**. You may check that the only differences with respect to the time-like case are in the absence of acceleration and in the metric trace now being 2 instead of 3 (this yields the factor $-1/4$ instead of $-1/6$ in front of Θ^2 in the shear scalar). The scalars are clearly independent of the vector l^μ.

24.3 Congruences as probes of the space-time geometry

The properties of congruences are surely connected with the geometry of the host space-time (they are even fully *determined* by the latter in the unaccelerated, geodesic case), which can be used for implications in both directions. We will now try to learn how the transversal characteristics introduced in previous paragraphs evolve, in a given space-time, along the given congruence. The answer will be derived by differentiating, along the congruence, the crucial tensor $u_{\mu;\nu}$ or $h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma}$, while decomposing the latter as in (24.6) or (24.24). The equation for the derivative of this whole tensor is typically quite long, but the formulas it yields for the individual kinematical quantities (mainly for the scalar ones) are very useful.

24.3.1 The time-like case

From the Ricci identities and by definition of acceleration $a_\mu \equiv u_{\mu;\lambda}u^\lambda$, we have

$$\frac{\mathrm{D}u_{\mu;\nu}}{\mathrm{d}\tau} = u_{\mu;\nu\lambda}u^\lambda = u_{\mu;\lambda\nu}u^\lambda + R^\iota{}_{\mu\nu\lambda}u_\iota u^\lambda = a_{\mu;\nu} - u_{\mu;\lambda}u^\lambda{}_{;\nu} - R_{\mu\iota\nu\lambda}u^\iota u^\lambda\,. \tag{24.32}$$

Substituting the decomposition (24.6) to the left-hand side and to the second (quadratic) term on the right-hand side, we get

$$\frac{\mathrm{D}\omega_{\mu\nu}}{\mathrm{d}\tau}+\frac{\mathrm{D}\Theta_{\mu\nu}}{\mathrm{d}\tau}-\frac{\mathrm{D}}{\mathrm{d}\tau}(a_\mu u_\nu)=a_{\mu;\nu}-\omega_{\mu\lambda}\omega^\lambda{}_\nu-\Theta_{\mu\lambda}\Theta^\lambda{}_\nu-\omega_{\mu\lambda}\Theta^\lambda{}_\nu-\Theta_{\mu\lambda}\omega^\lambda{}_\nu+ \\ +\omega_{\mu\lambda}a^\lambda u_\nu+\Theta_{\mu\lambda}a^\lambda u_\nu-R_{\mu\iota\nu\lambda}u^\iota u^\lambda\,, \tag{24.33}$$

or, after also decomposing $\Theta_{\mu\nu}=\sigma_{\mu\nu}+\frac{1}{3}\Theta h_{\mu\nu}$,

$$\frac{\mathrm{D}\omega_{\mu\nu}}{\mathrm{d}\tau}+\frac{\mathrm{D}\sigma_{\mu\nu}}{\mathrm{d}\tau}+\frac{1}{3}\frac{\mathrm{D}\Theta}{\mathrm{d}\tau}h_{\mu\nu}+\frac{1}{3}\Theta(\cancel{a_\mu u_\nu}+u_\mu a_\nu)-\frac{\mathrm{D}}{\mathrm{d}\tau}(a_\mu u_\nu)= \\ =a_{\mu;\nu}-\omega_{\mu\lambda}\omega^\lambda{}_\nu-\sigma_{\mu\lambda}\sigma^\lambda{}_\nu-\omega_{\mu\lambda}\sigma^\lambda{}_\nu-\sigma_{\mu\lambda}\omega^\lambda{}_\nu-\frac{1}{9}\Theta^2h_{\mu\nu}-\frac{2}{3}\Theta\omega_{\mu\nu}-\frac{2}{3}\Theta\sigma_{\mu\nu}+ \\ +\omega_{\mu\lambda}a^\lambda u_\nu+\sigma_{\mu\lambda}a^\lambda u_\nu+\frac{1}{3}\Theta\,\cancel{a_\mu u_\nu}-R_{\mu\iota\nu\lambda}u^\iota u^\lambda\,. \tag{24.34}$$

Before proceeding further, it is good to notice that the terms $(\omega_{\mu\lambda}\omega^\lambda{}_\nu)$ and $(\sigma_{\mu\lambda}\sigma^\lambda{}_\nu)$ are symmetric in (μ,ν), e.g.

$$\omega_{\mu\lambda}\omega^\lambda{}_\nu=-\omega_{\lambda\mu}\omega^\lambda{}_\nu=-\omega^\lambda{}_\mu\omega_{\lambda\nu}=\omega^\lambda{}_\mu\omega_{\nu\lambda}\equiv\omega_{\nu\lambda}\omega^\lambda{}_\mu\ ,$$

and also symmetric is the curvature term $R_{\mu\iota\nu\lambda}u^\iota u^\lambda$, whereas the term[2]

$$-\omega_{\mu\lambda}\sigma^\lambda{}_\nu-\sigma_{\mu\lambda}\omega^\lambda{}_\nu=\omega_{\lambda\mu}\sigma^\lambda{}_\nu-\sigma_{\mu\lambda}\omega^\lambda{}_\nu=\omega^\lambda{}_\mu\sigma_{\lambda\nu}-\sigma_{\mu\lambda}\omega^\lambda{}_\nu=\sigma_{\nu\lambda}\omega^\lambda{}_\mu-\sigma_{\mu\lambda}\omega^\lambda{}_\nu$$

is anti-symmetric.

- By tracing the equation (24.34), i.e. multiplying it by $g^{\mu\nu}$ (remember that the metric can pass through the absolute derivatives), we obtain the (Landau-)**Raychaudhuri equation**

$$\boxed{\frac{\mathrm{D}\Theta}{\mathrm{d}\tau}\equiv\frac{\mathrm{d}\Theta}{\mathrm{d}\tau}=a^\nu{}_{;\nu}+2\omega^2-2\sigma^2-\frac{1}{3}\Theta^2-R_{\iota\lambda}u^\iota u^\lambda}\,. \tag{24.35}$$

  This equation is useful at many places, most notably it is crucial in the *singularity theorems*.

- Before extracting the remaining information, it may be good to write, in (24.34),

$$\frac{\mathrm{D}}{\mathrm{d}\tau}(a_\mu u_\nu)+a_{\mu;\nu}=a_{\mu;\lambda}u^\lambda u_\nu+a_\mu a_\nu+a_{\mu;\nu}=h^\lambda_\nu a_{\mu;\lambda}+a_\mu a_\nu\,.$$

  Then the symmetric and traceless part of (24.34), i.e. $(24.34)_{(\mu\nu)}-\frac{1}{3}h_{\mu\nu}\,(24.34)^\kappa_\kappa$, yields

$$\frac{\mathrm{D}\sigma_{\mu\nu}}{\mathrm{d}\tau}=h^\lambda_{(\nu}a_{\mu);\lambda}+a_\mu a_\nu-\frac{1}{3}\Theta u_{(\mu}a_{\nu)}-\frac{1}{3}h_{\mu\nu}a^\kappa{}_{;\kappa}+u_{(\nu}\omega_{\mu)\lambda}a^\lambda+u_{(\nu}\sigma_{\mu)\lambda}a^\lambda- \\ -\omega_{\mu\lambda}\omega^\lambda{}_\nu-\sigma_{\mu\lambda}\sigma^\lambda{}_\nu-\frac{2}{3}\Theta\sigma_{\mu\nu}+\frac{2}{3}h_{\mu\nu}(\sigma^2-\omega^2)-$$

[2] This is, indeed, brutely slow, how the indices we low...
Are you thick with all that dough? -Hang up, go to under-row...

$$- R_{\mu\iota\nu\lambda}u^\iota u^\lambda + \frac{1}{3}h_{\mu\nu}R_{\iota\lambda}u^\iota u^\lambda \,, \tag{24.36}$$

while the anti-symmetric part of (24.34) appears as

$$\frac{\mathrm{D}\omega_{\mu\nu}}{\mathrm{d}\tau} = h^\lambda_{[\nu}a_{\mu];\lambda} - \frac{1}{3}\Theta u_{[\mu}a_{\nu]} + u_{[\nu}\omega_{\mu]\lambda}a^\lambda + u_{[\nu}\sigma_{\mu]\lambda}a^\lambda - \\ - \omega_{\mu\lambda}\sigma^\lambda{}_\nu - \sigma_{\mu\lambda}\omega^\lambda{}_\nu - \frac{2}{3}\Theta\omega_{\mu\nu} \,. \tag{24.37}$$

These equations are rewarding for an exam, but otherwise they are only used in the acceleration-free case when their first rows vanish (the second one is fairly short in that case!). Note also that only in the acceleration-free case all their terms are orthogonal to u^μ. Apparently, curvature does not explicitly enter the equation for the evolution of $\omega_{\mu\nu}$, yet it still does affect vorticity through other quantities (Θ and $\sigma_{\mu\nu}$), because the three equations are coupled.

- However, both the equations are much simpler in the scalar version: multiplying (24.34), or the equations (24.36) and (24.37), by $\frac{1}{2}\sigma^{\mu\nu}$ and by $\frac{1}{2}\omega^{\mu\nu}$, one finds, respectively,

$$\sigma\,\frac{\mathrm{D}\sigma}{\mathrm{d}\tau}\left(= \frac{1}{2}\frac{\mathrm{D}\sigma^2}{\mathrm{d}\tau} = \frac{1}{2}\,\sigma^{\mu\nu}\frac{\mathrm{D}\sigma_{\mu\nu}}{\mathrm{d}\tau}\right) = \\ = -\frac{2}{3}\Theta\sigma^2 + \frac{1}{2}\sigma^{\mu\nu}\left(a_{\mu;\nu} + a_\mu a_\nu - \omega_{\mu\lambda}\omega^\lambda{}_\nu - \sigma_{\mu\lambda}\sigma^\lambda{}_\nu - R_{\mu\iota\nu\lambda}u^\iota u^\lambda\right), \tag{24.38}$$

$$\omega\,\frac{\mathrm{D}\omega}{\mathrm{d}\tau}\left(= \frac{1}{2}\frac{\mathrm{D}\omega^2}{\mathrm{d}\tau} = \frac{1}{2}\,\omega^{\mu\nu}\frac{\mathrm{D}\omega_{\mu\nu}}{\mathrm{d}\tau}\right) = -\frac{2}{3}\Theta\omega^2 + \frac{1}{2}\omega^{\mu\nu}\left(a_{\mu;\nu} - 2\omega_{\mu\lambda}\sigma^\lambda{}_\nu\right). \tag{24.39}$$

24.3.2 The light-like case

The procedure is similar in the light-like case. The only differences are that i) the acceleration $k_{\mu;\nu}k^\nu$ is zero now (it would correspond to a_μ present in the time-like case), and ii) the trace of $h_{\mu\nu}$ is 2 rather than 3 now, so instead of $u_{(\mu;\nu)}u^{\mu;\nu} = 2\sigma^2 + \frac{1}{3}\Theta^2$ one has $k_{(\mu;\nu)}k^{\mu;\nu} = 2\sigma^2 + \frac{1}{2}\Theta^2$ for the shear deformation. Therefore, it should suffice to change all the factors 1/3 to 1/2 in the equations, and to delete the acceleration. In particular, the Raychaudhuri equation really has the form

$$\boxed{\Theta_{,\mu}k^\mu = 2\omega^2 - 2\sigma^2 - \frac{1}{2}\Theta^2 - R_{\iota\lambda}k^\iota k^\lambda} \tag{24.40}$$

now. However, let us better derive the equations, rather than just guessing. Recall that the transversal properties of the congruence are determined by the tensor $h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma}$ (24.23) in the light-like case (a counterpart of $u_{\mu;\nu}$ from the time-like case), so we are now interested in evolution of that tensor along the congruence. Using again the commutator of covariant derivatives, we have

$$\left(h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma}\right)_{;\lambda} k^\lambda = \left(h^\rho_{\mu;\lambda}h^\sigma_\nu k_{\rho;\sigma} + h^\rho_\mu h^\sigma_{\nu;\lambda}k_{\rho;\sigma} + h^\rho_\mu h^\sigma_\nu k_{\rho;\sigma\lambda}\right) k^\lambda = \\ = \left(k^\rho{}_{;\lambda}l_\mu + k^\rho l_{\mu;\lambda} + l^\rho{}_{;\lambda}k_\mu + l^\rho k_{\mu;\lambda}\right) k^\lambda h^\sigma_\nu k_{\rho;\sigma} +$$

Does not seem to be a diffeomorphism.

$$
\begin{aligned}
&+ \left(k^{\sigma}{}_{;\lambda} l_{\nu} + k^{\sigma} l_{\nu;\lambda} + l^{\sigma}{}_{;\lambda} k_{\nu} + l^{\sigma} k_{\nu;\lambda}\right) k^{\lambda} h^{\rho}_{\mu} k_{\rho;\sigma} + \\
&+ h^{\rho}_{\mu} h^{\sigma}_{\nu} \left(k_{\rho;\lambda\sigma} k^{\lambda} + R^{\iota}{}_{\rho\sigma\lambda} k_{\iota} k^{\lambda}\right) = \\
= {} & \left(l^{\rho}{}_{;\lambda} k^{\lambda} k_{\mu} h^{\sigma}_{\nu} + l^{\sigma}{}_{;\lambda} k^{\lambda} k_{\nu} h^{\rho}_{\mu}\right) k_{\rho;\sigma} + \\
&+ h^{\rho}_{\mu} h^{\sigma}_{\nu} \left[(\cancel{k_{\rho;\lambda} k^{\lambda}})_{;\sigma} - k_{\rho;\lambda} k^{\lambda}{}_{;\sigma}\right] - h^{\rho}_{\mu} h^{\sigma}_{\nu} R_{\rho\iota\sigma\lambda} k^{\iota} k^{\lambda} \, .
\end{aligned}
$$

On the left-hand side, we just substitute the decomposition $h^{\rho}_{\mu} h^{\sigma}_{\nu} k_{\rho;\sigma} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{2}\Theta h_{\mu\nu}$, while on the right-hand side we compute, with the same substitution,

$$
\begin{aligned}
h^{\rho}_{\mu} h^{\sigma}_{\nu} k_{\rho;\lambda} k^{\lambda}{}_{;\sigma} &= (h^{\rho}_{\mu} h^{\iota}_{\lambda} k_{\rho;\iota})(h^{\lambda}_{\kappa} h^{\sigma}_{\nu} k^{\kappa}{}_{;\sigma}) = \\
&= \left(\omega_{\mu\lambda} + \sigma_{\mu\lambda} + \frac{1}{2}\Theta h_{\mu\lambda}\right)\left(\omega^{\lambda}{}_{\nu} + \sigma^{\lambda}{}_{\nu} + \frac{1}{2}\Theta h^{\lambda}_{\nu}\right) = \\
&= \omega_{\mu\lambda}\omega^{\lambda}{}_{\nu} + \sigma_{\mu\lambda}\sigma^{\lambda}{}_{\nu} + \omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} + \sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu} + \Theta(\omega_{\mu\nu} + \sigma_{\mu\nu}) + \frac{1}{4}\Theta^2 h_{\mu\nu} \, ,
\end{aligned}
$$

thus arriving at

$$
\begin{aligned}
&\omega_{\mu\nu;\lambda} k^{\lambda} + \sigma_{\mu\nu;\lambda} k^{\lambda} + \frac{1}{2}\Theta_{,\lambda} k^{\lambda} h_{\mu\nu} + \Theta k_{(\mu} l_{\nu);\lambda} k^{\lambda} - \left(l^{\rho}{}_{;\lambda} k^{\lambda} k_{\mu} h^{\sigma}_{\nu} + l^{\sigma}{}_{;\lambda} k^{\lambda} k_{\nu} h^{\rho}_{\mu}\right) k_{\rho;\sigma} = \\
&= -\omega_{\mu\lambda}\omega^{\lambda}{}_{\nu} - \sigma_{\mu\lambda}\sigma^{\lambda}{}_{\nu} - \omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} - \sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu} - \Theta(\omega_{\mu\nu} + \sigma_{\mu\nu}) - \frac{1}{4}\Theta^2 h_{\mu\nu} - h^{\rho}_{\mu} h^{\sigma}_{\nu} R_{\rho\iota\sigma\lambda} k^{\iota} k^{\lambda} \, .
\end{aligned}
$$

Note the symmetries: as in the time-like case, the term $[\omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} + \sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu}]$ is anti-symmetric in $(\mu\nu)$, whereas $(\omega_{\mu\lambda}\omega^{\lambda}{}_{\nu})$, $(\sigma_{\mu\lambda}\sigma^{\lambda}{}_{\nu})$ and $(h^{\rho}_{\mu} h^{\sigma}_{\nu} R_{\rho\iota\sigma\lambda} k^{\iota} k^{\lambda})$ are symmetric. Trace of the equation (thus) easily yields (24.40) indeed.

Before extracting evolutions of $\omega_{\mu\nu}$ and $\sigma_{\mu\nu}$, it is suitable to somewhat fix the choice of the auxiliary null vector l^μ, namely to demand that it parallel transports along k^μ, i.e. that $l_{\mu;\lambda}k^\lambda = 0$. Such a choice is natural actually, since k^μ (of course) transports parallelly along itself (it is geodesic), and parallel transport conserves scalar product, so it keeps satisfied our normalization $k_\mu l^\mu = -1$. With that choice, the terms explicitly containing l^μ drop out,[3] so the symmetric trace-free and antisymmetric parts of the equation read, respectively,

$$\begin{aligned}\sigma_{\mu\nu;\lambda}k^\lambda &= -\omega_{\mu\lambda}\omega^\lambda{}_\nu - \sigma_{\mu\lambda}\sigma^\lambda{}_\nu + (\sigma^2 - \omega^2)h_{\mu\nu} - \Theta\sigma_{\mu\nu} \\ &\quad - h^\rho_\mu h^\sigma_\nu R_{\rho\iota\sigma\lambda}k^\iota k^\lambda + \frac{1}{2}h_{\mu\nu}R_{\iota\lambda}k^\iota k^\lambda\,, \qquad (24.41)\\ \omega_{\mu\nu;\lambda}k^\lambda &= -\omega_{\mu\lambda}\sigma^\lambda{}_\nu - \sigma_{\mu\lambda}\omega^\lambda{}_\nu - \Theta\omega_{\mu\nu}\,. \qquad (24.42)\end{aligned}$$

The correspondence with (24.36) and (24.37) is obvious – it is indeed sufficient to replace 1/3 with 1/2 and delete acceleration in the time-like equations.

2D character of the light-like case revisited

Still there is one point to fully exploit: recall that both $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are normal to both k^μ and l^μ, that is, they "live" on a certain 2D surface (in adapted coordinates, they would be 2x2). In 2D, a symmetric traceless matrix has only two independent components, and an anti-symmetric matrix even has but one. In other words, a symmetric matrix in 4D has 10 components, but the conditions $\sigma_{\mu\nu}k^\nu = 0$ (four), $\sigma_{\mu\nu}l^\nu = 0$ (three, because now already $k^\mu\sigma_{\mu\nu}l^\nu = 0$ holds automatically) and $\sigma^\mu_\mu = 0$ (one) reduce them to two; similarly, an anti-symmetric matrix in 4D has 6 components, but the conditions $\omega_{\mu\nu}k^\nu = 0$ (three, because $\sigma_{\mu\nu}k^\mu k^\nu = 0$ holds automatically due to anti-symmetry) and $\omega_{\mu\nu}l^\nu = 0$ (two, because $\omega_{\mu\nu}l^\mu l^\nu = 0$ holds automatically due to anti-symmetry and now already $k^\mu\omega_{\mu\nu}l^\nu = 0$ holds automatically as well) reduce them to a single scalar. Besides that, of all the coordinates adapted to the surface orthogonal to both k^μ and l^μ, we can choose such in which the 2D metric is diagonal (a 2x2 matrix is always diagonalisable). In these, the tensors could be written $(A, B = 1, 2)$

$$\begin{aligned}h^{AB} &= \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad \sigma_{AB} = \begin{pmatrix} h_2\sigma_+ & \sigma_\times \\ \sigma_\times & -h_1\sigma_+ \end{pmatrix}, \qquad \omega_{AB} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix},\\ \sigma^C{}_B &= h^{AC}\sigma_{AB} = \begin{pmatrix} h_1h_2\sigma_+ & h_1\sigma_\times \\ h_2\sigma_\times & -h_1h_2\sigma_+ \end{pmatrix}, \qquad \omega^C{}_B = h^{AC}\omega_{AB} = \begin{pmatrix} 0 & h_1\Omega \\ -h_2\Omega & 0 \end{pmatrix},\\ \sigma^{CD} &= h^{BD}\sigma^C{}_B = \begin{pmatrix} h_2h_1^2\sigma_+ & h_1h_2\sigma_\times \\ h_2h_1\sigma_\times & -h_1h_2^2\sigma_+ \end{pmatrix}, \qquad \omega^{CD} = h^{BD}\omega^C{}_B = \begin{pmatrix} 0 & h_1h_2\Omega \\ -h_1h_2\Omega & 0 \end{pmatrix},\end{aligned}$$

so one would have

$$\begin{aligned}\sigma_{\mu\lambda}\sigma^\lambda{}_\nu \quad &\longrightarrow \quad \sigma_{AD}\sigma^D{}_B = \begin{pmatrix} h_1h_2^2\sigma_+^2 + h_2\sigma_\times^2 & 0 \\ 0 & h_2h_1^2\sigma_+^2 + h_1\sigma_\times^2 \end{pmatrix},\\ \omega_{\mu\lambda}\omega^\lambda{}_\nu \quad &\longrightarrow \quad \omega_{AD}\omega^D{}_B = \begin{pmatrix} -h_2\Omega^2 & 0 \\ 0 & -h_1\Omega^2 \end{pmatrix},\end{aligned}$$

[3] Exactly those terms, in addition, are proportional to k_μ or k_ν, so they are not transversal (i.e. not relevant).

$$\omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} \quad \longrightarrow \quad \omega_{AD}\sigma^{D}{}_{B} = \begin{pmatrix} h_2\Omega\sigma_\times & -h_1h_2\Omega\sigma_+ \\ -h_1h_2\Omega\sigma_+ & -h_1\Omega\sigma_\times \end{pmatrix},$$

$$\sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu} \quad \longrightarrow \quad \sigma_{AD}\omega^{D}{}_{B} = \begin{pmatrix} -h_2\sigma_\times\Omega & h_1h_2\sigma_+\Omega \\ h_1h_2\sigma_+\Omega & h_1\sigma_\times\Omega \end{pmatrix},$$

from which, finally,

$$\sigma^2 \equiv \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \quad \longrightarrow \quad \frac{1}{2}\,\sigma_{AD}\sigma^{D}{}_{B}h^{AB} = h_1^2h_2^2\sigma_+^2 + h_1h_2\sigma_\times\ ,$$

$$-\,\sigma_{\mu\lambda}\sigma^{\lambda}{}_{\nu} + \sigma^2 h_{\mu\nu} \quad \longrightarrow \quad -\sigma_{AD}\sigma^{D}{}_{B} + \sigma^2 h_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\omega^2 \equiv \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} \quad \longrightarrow \quad -\frac{1}{2}\,\omega_{AD}\omega^{D}{}_{B}h^{AB} = h_1h_2\Omega^2\ ,$$

$$-\,\omega_{\mu\lambda}\omega^{\lambda}{}_{\nu} - \omega^2 h_{\mu\nu} \quad \longrightarrow \quad -\omega_{AD}\omega^{D}{}_{B} - \omega^2 h_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and also

$$-\omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} - \sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu} \quad \longrightarrow \quad -\omega_{AD}\sigma^{D}{}_{B} - \sigma_{AD}\omega^{D}{}_{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have thus found that in the case of light-like congruences, it is always possible to "transform out" the terms

$$-\sigma_{\mu\lambda}\sigma^{\lambda}{}_{\nu} + \sigma^2 h_{\mu\nu}\ , \qquad -\omega_{\mu\lambda}\omega^{\lambda}{}_{\nu} - \omega^2 h_{\mu\nu}\ , \qquad -\omega_{\mu\lambda}\sigma^{\lambda}{}_{\nu} - \sigma_{\mu\lambda}\omega^{\lambda}{}_{\nu}$$

in equations (24.41) and (24.41) by choosing suitable coordinates. These terms are thus *unnecessary*, so we may conclude that the equations are in fact very short,

$$\sigma_{\mu\nu;\lambda}k^{\lambda} = -\Theta\sigma_{\mu\nu} - h^{\rho}_{\mu}h^{\sigma}_{\nu}R_{\rho\iota\sigma\lambda}k^{\iota}k^{\lambda} + \frac{1}{2}h_{\mu\nu}R_{\iota\lambda}k^{\iota}k^{\lambda}\,, \tag{24.43}$$

$$\omega_{\mu\nu;\lambda}k^{\lambda} = -\Theta\omega_{\mu\nu}\,. \tag{24.44}$$

Last point to notice: from the definition (8.5) of the Weyl tensor, i.e. (with the same arrangement of indices as the above Riemann bears)

$$C_{\rho\iota\sigma\lambda} = R_{\rho\iota\sigma\lambda} - \frac{1}{2}\left(g_{\rho\sigma}R_{\iota\lambda} + g_{\iota\lambda}R_{\rho\sigma} - g_{\rho\lambda}R_{\iota\sigma} - g_{\iota\sigma}R_{\rho\lambda}\right) - \frac{R}{6}\left(g_{\rho\sigma}g_{\iota\lambda} - g_{\rho\lambda}g_{\iota\sigma}\right),$$

one finds by multiplication by $h^{\rho}_{\mu}h^{\sigma}_{\nu}k^{\iota}k^{\lambda}$ that

$$h^{\rho}_{\mu}h^{\sigma}_{\nu}R_{\rho\iota\sigma\lambda}k^{\iota}k^{\lambda} - \frac{1}{2}h_{\mu\nu}R_{\iota\lambda}k^{\iota}k^{\lambda} = h^{\rho}_{\mu}h^{\sigma}_{\nu}C_{\rho\iota\sigma\lambda}k^{\iota}k^{\lambda}\ ,$$

so the final form of the shear equation is

$$\sigma_{\mu\nu;\lambda}k^{\lambda} = -\Theta\sigma_{\mu\nu} - h^{\rho}_{\mu}h^{\sigma}_{\nu}C_{\rho\iota\sigma\lambda}k^{\iota}k^{\lambda}\,. \tag{24.45}$$

For evolution of the scalars ω^2 and σ^2, we multiply the above equations by $\frac{1}{2}\sigma^{\mu\nu}$ and $\frac{1}{2}\omega^{\mu\nu}$, respectively:

$$\boxed{\sigma_{,\lambda}k^{\lambda} = -\Theta\sigma - \frac{1}{2\sigma}\,\sigma^{\mu\nu}C_{\mu\iota\nu\lambda}k^{\iota}k^{\lambda}\,, \qquad \omega_{,\lambda}k^{\lambda} = -\Theta\omega}\,. \tag{24.46}$$

Together with the Raychaudhuri equation (24.40), they are often called the **Sachs equations**. Note that, in contrast to the time-like case, the equations (24.44) and (24.45) are not directly coupled (only through Θ, thus through the Raychaudhuri equation). The equation for ω is obviously independent of l^{μ}. In order to see this for the σ equation as well, recall that $\sigma^{\mu\nu} = h^{(\mu}_{\rho}h^{\nu)}_{\sigma}k^{\rho;\sigma} - \frac{1}{2}\Theta h^{\mu\nu}$, and that due to the anti-symmetries of Weyl, the only non-zero contributions to $\sigma^{\mu\nu}C_{\mu\iota\nu\lambda}k^{\iota}k^{\lambda}$ may arise from those terms of $\sigma^{\mu\nu}$ which do *not* contain k^{μ} or k^{ν}; there are two such terms only, $k^{\mu;\nu}$ from the first part of $\sigma^{\mu\nu}$ and $-\frac{1}{2}\Theta g^{\mu\nu}$ from the second part. But the Weyl tensor is trace-free, $g^{\mu\nu}C_{\mu\iota\nu\lambda} = 0$, so the equation actually reads

$$\sigma_{,\lambda}k^{\lambda} = -\Theta\sigma - \frac{1}{2\sigma}\,k^{\mu;\nu}C_{\mu\iota\nu\lambda}k^{\iota}k^{\lambda}\,. \tag{24.47}$$

24.3.3 Remark on the singularity theorems

As an illustration of the congruence-knowledge usage, let us take the Raychaudhuri equation (24.35)

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = a^{\nu}{}_{;\nu} + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\Theta^2 - R_{\iota\lambda}u^{\iota}u^{\lambda}$$

determining the evolution of the expansion scalar along a time-like congruence. Substituting for the Ricci tensor from the Einstein equations (8.4),

$$R_{\iota\lambda}u^{\iota}u^{\lambda} = 8\pi\left(T_{\iota\lambda} - \frac{1}{2}\,Tg_{\iota\lambda}\right)u^{\iota}u^{\lambda} + \Lambda g_{\iota\lambda}u^{\iota}u^{\lambda} = 8\pi\left(T_{\iota\lambda}u^{\iota}u^{\lambda} + \frac{T}{2}\right) - \Lambda\,, \tag{24.48}$$

we can see that if $\Lambda = 0$ and if the strong energy condition $T_{\iota\lambda}u^{\iota}u^{\lambda} \geqslant -\frac{T}{2}$ holds, the curvature term will contribute negatively. Especially in the case when the congruence is geodesic and non-twisting (thus hypersurface orthogonal), such as e.g. the flow of the cosmic fluid in the FLRW cosmologies, negative will surely be the whole right-hand side of the equation, because one is left then with

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} + \frac{1}{3}\Theta^2 = -2\sigma^2 - R_{\iota\lambda}u^{\iota}u^{\lambda} \leqslant 0\,.$$

This tells that the expansion of a twist-free geodesic congruence always *diminishes* in time: a diverging congruence diverges slower and slower (perhaps even becoming convergent one day), while a converging congruence converges faster and faster. It is consistent with intuition, namely with the universal attractive nature of gravity (assumed here through the energy condition) – imagine, typically, a congruence outgoing from a gravitating body as the divergent one, and that ingoing towards the body as the convergent one.

We may rewrite the equation as

$$-\Theta^2\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\Theta}\right)-\frac{1}{3}\right]\leqslant 0 \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\Theta}\right)\geqslant\frac{1}{3} \quad\Longrightarrow\quad \frac{1}{\Theta(\tau)}\geqslant\frac{1}{\Theta(0)}+\frac{\tau}{3}\,.$$

If the congruence is converging initially, $\Theta(0)<0$, then its Θ^{-1} necessarily reaches zero value (so Θ reaches minus ∞) within the time $\tau\leqslant\frac{3}{|\Theta(0)|}$. Similarly, if a congruence is diverging at some time, $\Theta(0)>0$, its Θ must have been infinite at some *past* moment ($\tau<0$). Therefore, in both cases, some kind of singularity ("caustic") occurs in the congruence. This does not necessarily mean that the space-time itself becomes singular there, but if such a conclusion is obtained for *any* congruence, or for some sufficiently representative one, it also indicates something about the space-time background. The first (converging) situation is crucial for the black-hole singularity theorems (inside every trapped surface, there exists a singularity of some kind), while the second (diverging) situation is the case in cosmology (the past singularity is the big bang).

However, in the actual Universe $\Lambda>0$ and the above argumentation may not work, because then (24.48) need not be positive. On the scale of black holes, the realistic Λ only plays a marginal role, but one guesses that the positive Λ may allow for cosmological models *without* the initial singularity. Well, in fact we know this already from Chapter 13.

24.3.4 Remark on cosmology

In Chapter 13, we showed that in the FLRW universes the cosmic-fluid flow is geodesic ($a^\mu=0$), vorticity-free ($\omega=0$), shear-free ($\sigma=0$), yet of course expanding – in (13.15) we computed that its expansion is $\Theta\equiv u^\nu{}_{;\nu}=3H$, with H the Hubble constant. Substituting this into the Raychaudhuri equation (24.35), one has

$$3\,\frac{\mathrm{d}H}{\mathrm{d}t}+3H^2=-R_{\iota\lambda}u^\iota u^\lambda\,.$$

From Einstein's equations $R_{\iota\lambda}=8\pi\left(T_{\iota\lambda}-\frac{1}{2}Tg_{\iota\lambda}\right)+\Lambda g_{\iota\lambda}$, we have, for ideal fluid,

$$R_{\iota\lambda}u^\iota u^\lambda=4\pi(\rho+3P)-\Lambda\,,$$

and expressing here the density term from the Friemann equation (13.35), i.e.

$$4\pi\rho=\frac{3}{2}H^2+\frac{3}{2}\frac{K}{a^2}-\frac{\Lambda}{2}\,,$$

we obtain

$$3\,\frac{\mathrm{d}H}{\mathrm{d}t}+3H^2=-\frac{3}{2}H^2-\frac{3}{2}\frac{K}{a^2}+\frac{\Lambda}{2}-12\pi P+\Lambda\,,$$

which after multiplication by $2/3$ yields

$$2\,\frac{\mathrm{d}H}{\mathrm{d}t}+3H^2+\frac{K}{a^2}=\Lambda-8\pi P\,.$$

This is exactly the deceleration equation (13.36) (only that there we wrote it in standard units).

24.4 Hypersurface-orthogonal fields; the Frobenius theorem

Have a smooth vector field defined in a certain space-time region. To such a field, there always exists a unique congruence of integral curves – each point is passed by exactly one curve which has the field as its tangent, i.e. exactly one solution of the equation $\frac{\mathrm{d}x^\mu}{\mathrm{d}p} = V^\mu$. Higher-dimensional counter-parts of this problem are not any more trivial, and they may not have a solution.

Have now *two* independent smooth vector fields. At every point, it is possible to rise a plane spanned by the corresponding two vectors – a 2D subspace of the local tangent space. Do there exist integral surfaces to such local planes, i.e. 2D submanifolds which at every point have the given planes as their local tangent planes? In the 2D case, the answer can still be imagined: if you make a commutator of the vector sum of the fields, i.e. you add them in two opposite orders, the difference between the two results has to be a vector again lying in the plane spanned by the two fields. In particular, if the integral congruences had non-zero vorticity, they probably could not span 2D surfaces globally.

Consider the above exercise in a small neighbourhood of some point. So have two infinitesimal vector fields, v^μ, w^μ. Their integral curves will, in a neighbourhood of the chosen point, form a surface if and only if the commutator of their vector sum will also lie in the plane defined by v^μ and w^μ, i.e. if

$$[v^\mu(x^\alpha) + w^\mu(x^\alpha + v^\alpha)] - [w^\mu(x^\alpha) + v^\mu(x^\alpha + w^\alpha)]$$

is a linear combination of v^μ and w^μ. Expanding this difference to the first order, we have

$$\begin{aligned}&[\cancel{v^\mu(x^\alpha)} + \cancel{w^\mu(x^\alpha)} + (w^\mu{}_{,\nu}v^\nu)(x^\alpha)] - [\cancel{w^\mu(x^\alpha)} + \cancel{v^\mu(x^\alpha)} + (v^\mu{}_{,\nu}w^\nu)(x^\alpha)] = \\ &= w^\mu{}_{,\nu}v^\nu - v^\mu{}_{,\nu}w^\nu = (\pounds_{\mathbf{v}}\mathbf{w})^\mu = [\mathbf{v}, \mathbf{w}]^\mu .\end{aligned}$$

Hence, the commutator of the fields has to be a linear combination of the same fields,

$$\epsilon_{\mu\nu\rho\sigma} v^\nu w^\rho \, (\pounds_{\mathbf{v}}\mathbf{w})^\sigma = 0 . \tag{24.49}$$

The vector fields which do have this property are called **surface-forming**.

Within the surface-forming fields, especially "nice" are those which commute, of course. In such a case, it is totally same if going, from a given point, first along $v^\mu(x^\alpha)$ and then along $w^\mu(x^\alpha + v^\alpha)$, or first along $w^\mu(x^\alpha)$ and then along $v^\mu(x^\alpha + w^\alpha)$ – one *anyway* arrives at the same point. This feature remains also true integrally, i.e. if you make *finite* shifts along the fields. In particular, commuting have to be the coordinate fields ($\partial/\partial x^\mu$) – in their case, the vanishing commutator means interchangeable partial derivatives.

The above finding naturally generalizes to higher dimensions. Let's have a manifold M of dimension d and $n(<d)$ independent smooth vector fields on it. At every point $m \in M$, the fields determine an n-dimensional subspace ${}^nT_mM$ of the local tangent space T_mM. The union of all these subspaces, ${}^nTM \equiv \bigcup_{m\in M} {}^nT_mM$, is called the smooth **distribution** (or **subbundle**) on the manifold M (obviously it is a subset of the tangent bundle TM of the manifold). The distribution is called **involutive** if a commutator of arbitrary pair of its fields

"belongs" to it, i.e. if any such commutator can be expressed as a linear combination of *some* fields of the distribution.[4]

Coming to the central question: when does a family of integral submanifolds exist to a given smooth n-dimensional distribution on a smooth manifold M?

Frobenius theorem A smooth n-dimensional distribution on a smooth manifold M is tangent to a family of integral submanifolds of M if and only if the distribution is involutive. Every point of the manifold is then passed by exactly one integral submanifold – the manifold is said to be **foliated** by integral submanifolds.[5] □

Now we return to space-time ($d = 4$) and focus on the $n = 3$ case.

Above, we have been considering the n-dimensional submanifolds as defined by an (involutive) smooth distribution spanned by certain n independent vector fields. However, besides defining the (n-dimensional) tangent subbundle, such vector fields also define the subbundle (of dimension $d - n$) which is *normal* to all of them. Actually, one may equally well pose the integrability problem in the sense of orthogonal complement: does there exist a family of $(d{-}n)$-dimensional submanifolds which are everywhere *orthogonal* to a given distribution of n smooth vector fields? We have in fact met such a formulation already – it was in the issue of *orthogonal transitivity* in stationary and axially symmetric space-times (Section 19.3.1), when we asked whether integral meridional planes exist, i.e. such 2D integral submanifolds which are everywhere orthogonal to both the Killing vector fields t^μ and ϕ^μ (the problem lead there to the requirement of *circularity*). Below, we consider the "(1+3)"-dimensional analogue of this problem (in a *generic* space-time).

Let a smooth vector field V^μ be defined in some space-time. At every space-time point m, a 3D hyperplane is thus defined in the local tangent space $T_m M$ which has V^μ as its normal. When do integral submanifolds (hypersurfaces) exist to a 3D distribution given in this way? Quite naturally, such a vector field V^μ for which the 3D foliation does exist is called the **hypersurface-orthogonal** vector field. An example of a field with such a property was the four-velocity field of the ZAMO congruence in circular space-times (the orthogonal submanifolds were given there by $t = \mathrm{const}$, with t the Killing time). Intuitively, the crucial property should be the vanishing vorticity of the field...

Theorem The following three statements are equivalent:

- (i) The vector field V^μ is hypersurface-orthogonal, i.e. there exists such a scalar function $\Phi(x^\alpha)$ whose isosurfaces are orthogonal to V^μ,

$$V_\mu = -f\Phi_{,\mu}\,, \tag{24.50}$$

[4] Adjectives *integrable* or *holonomic* are often used as synonyms of *involutive*, although they are in fact *consequences* of the latter. *Integer* is a Latin word, while *holos* is a Greek word, both meaning *whole*. Hence, as already mentioned in Section 6.3.1, integrable or holonomic means "makeable whole", "completeable". In GR, these adjectives are however often reserved for the special case of *vanishing* commutator, when going along different paths makes *no* difference at all, like in the case of coordinate fields.

[5] Clearly *foliation* is a higher-dimensional analogue of the 1D term *congruence*.

where the "coefficient" $f(x^\alpha)$ is a scalar function as well.

- (ii) It holds $V_{\{[\mu;\nu]}V_{\rho\}} = 0$, where

$$2V_{\{[\mu;\nu]}V_{\rho\}} \equiv 6V_{[\mu;\nu}V_{\rho]} \equiv (V_{\mu;\nu}-V_{\nu;\mu})V_\rho + (V_{\rho;\mu}-V_{\mu;\rho})V_\nu + (V_{\nu;\rho}-V_{\rho;\nu})V_\mu \,. \quad (24.51)$$

- (iii) The field V^μ has zero vorticity,

$$\omega_{\mu\nu} \equiv h^\rho_{[\mu}h^\sigma_{\nu]}V_{\rho;\sigma} = 0\,. \quad (24.52)$$

Proof (i) ⇒ (ii)

- From (i) we compute $V_{\mu;\nu} - V_{\nu;\mu} = V_{\mu,\nu} - V_{\nu,\mu} = f_{,\mu}\Phi_{,\nu} - f_{,\nu}\Phi_{,\mu}$ and submit it to the explicit form of (ii):

$$2V_{\{[\mu;\nu]}V_{\rho\}} = (f_{,\nu}\Phi_{,\mu} - f_{,\mu}\Phi_{,\nu})f\Phi_{,\rho} + (f_{,\mu}\Phi_{,\rho} - f_{,\rho}\Phi_{,\mu})f\Phi_{,\nu} + (f_{,\rho}\Phi_{,\nu} - f_{,\nu}\Phi_{,\rho})f\Phi_{,\mu} = \\ = 0\,.$$

Proof (ii) ⇒ (iii)

- Assume first that V^μ is time-like, call it u^μ, so $u_\mu u^\mu = -1$ and $\omega_{\mu\nu} = u_{[\mu;\nu]} + a_{[\mu}u_{\nu]}$, where $a_\mu \equiv u_{\mu;\sigma}u^\sigma$ is the acceleration of u^μ. Writing out $0 = 2u_{\{[\mu;\nu]}u_{\rho\}}$ and multiplying it by u^ρ, one has

$$0 = (u_{\mu;\nu} - u_{\nu;\mu})u_\rho u^\rho + (u_{\rho;\mu} - u_{\mu;\rho})u_\nu u^\rho + (u_{\nu;\rho} - u_{\rho;\nu})u_\mu u^\rho = \\ = -u_{\mu;\nu} + u_{\nu;\mu} - a_\mu u_\nu + a_\nu u_\mu = -2\omega_{\mu\nu}\,.$$

- Assume now that the field $V^\mu \equiv k^\mu$ is null, $k_\rho k^\rho = 0$ (and geodesic, $k_{\rho;\sigma}k^\sigma = 0$). In Section 24.2 on null congruences we introduced the second, "auxiliary" null field l^ρ normalized by $k_\rho l^\rho = -1$. Multiplying $0 = 2k_{\{[\mu;\nu]}k_{\rho\}}$ by l^ρ this time, we have

$$0 = (k_{\mu;\nu} - k_{\nu;\mu})k_\rho l^\rho + (k_{\rho;\mu} - k_{\mu;\rho})k_\nu l^\rho + (k_{\nu;\rho} - k_{\rho;\nu})k_\mu l^\rho = \\ = -2\left(k_{[\mu;\nu]} + k_{\rho;[\nu}k_{\mu]}l^\rho + k_{[\nu}k_{\mu];\rho}l^\rho\right) = -2\omega_{\mu\nu}$$

according to (24.23) and (24.25).

Proof (iii) ⇒ (i)

- If $\omega_{\nu\iota} = 0$, the vorticity vector $\omega^\mu = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}\omega_{\nu\iota}V_\lambda = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}V_{\nu;\iota}V_\lambda = \frac{1}{2}\,\epsilon^{\mu\nu\iota\lambda}V_{\nu,\iota}V_\lambda$ vanishes as well. This vector represents rotation (curl) of V_ν within the (hyper)surface orthogonal to V_λ; if that rotation is zero, the field V_ν has to be proportional to a gradient of some scalar field. Writing it out in components, $\omega^\mu = 0$ means $V_{\nu,\iota} = V_{\iota,\nu}$, which are exactly the integrability conditions for the equation $-f\mathrm{d}\Phi = V_\nu \mathrm{d}x^\nu$, i.e. $-f\Phi_{,\nu} = V_\nu$. (see Section 6.3.1). □

24.4.1 Geometric identification of stationary and static space-times

Quite some times already, we have needed to distinguish between *stationary* and *static* situation. Intuitively, we specified *staticity* as such a special case of stationarity (i.e., independence of a certain time, $g_{\mu\nu,t}=0$) when, in addition, the direction of time does not matter (so $g_{ti}=0$). This picture is completely correct and complete, yet let us characterize these two properties geometrically.

Stationarity of course means that there exists a non-trivial Killing vector field which at least in some region (typically in the asymptotic one) is time-like. The static subcase is the one in which that Killing field is hypersurface-orthogonal. From equation (11.25) we know that the existence of a time-like Killing field is equivalent to the existence of such a time coordinate on which the metric does not depend. It thus remains to be shown that the property $g_{ti}=0$ is equivalent to the hypersurface orthogonality of $t^\mu=\partial x^\mu/\partial t$. Assume first we have $g_{ti}=0$. Then $t_\alpha=g_{\alpha\beta}t^\beta=g_{\alpha t}$ only has the $\alpha=t$ component, which however means that it can be written as $t_\alpha = g_{tt}t_{,\alpha}$ – so it is orthogonal to the hypersurfaces $t=\mathrm{const}$.

Conversely, assume that t^μ is hypersurface orthogonal. From the preceding section, we know this implies that it satisfies

$$t_{\mu;\nu}t_\rho - t_{\nu;\mu}t_\rho + t_{\rho;\mu}t_\nu - t_{\mu;\rho}t_\nu + t_{\nu;\rho}t_\mu - t_{\rho;\nu}t_\mu = 0\,.$$

Now use the Killing equation, $t_{\nu;\mu} = -t_{\mu;\nu}$, in the second and the third pair of the $2t_{[\mu;\nu]}t_\rho$ terms,

$$(t_{\mu;\nu} - t_{\nu;\mu})t_\rho + 2t_{\rho;\mu}t_\nu + 2t_{\nu;\rho}t_\mu = 0\,.$$

Multiplying by t^ρ and writing

$$2t_{\rho;\mu}t^\rho = (t_\rho t^\rho)_{;\mu}\ , \qquad 2t_{\nu;\rho}t^\rho = -2t_{\rho;\nu}t^\rho = -(t_\rho t^\rho)_{;\nu}\ ,$$

we thus obtain

$$(t_{\mu;\nu} - t_{\nu;\mu})t_\rho t^\rho + (t_\rho t^\rho)_{;\mu}t_\nu - (t_\rho t^\rho)_{;\nu}t_\mu = 0\,.$$

After division by $(t_\rho t^\rho)^2$, this can be written as

$$\left(\frac{t_\mu}{t_\rho t^\rho}\right)_{;\nu} - \left(\frac{t_\nu}{t_\rho t^\rho}\right)_{;\mu} = 0 \qquad \Longleftrightarrow \qquad \left(\frac{t_\mu}{t_\rho t^\rho}\right)_{,\nu} - \left(\frac{t_\nu}{t_\rho t^\rho}\right)_{,\mu} = 0\,,$$

which means that $\frac{t_\mu}{t_\rho t^\rho}$ has vanishing rotation. It thus must represent a gradient of some scalar (call it Φ),

$$\frac{t_\mu}{t_\rho t^\rho} = \Phi_{,\mu} \qquad \Longrightarrow \qquad t_\mu = (t_\rho t^\rho)\Phi_{,\mu}\ .$$

Plugging here $t_\rho t^\rho = g_{\rho\sigma}t^\rho t^\sigma = g_{tt}$ (and $t_\mu = g_{\mu t}$), we have

$$g_{\mu t} = g_{tt}\Phi_{,\mu} \quad \Longrightarrow \quad \text{i)}\ g_{it} = g_{tt}\Phi_{,i}\ , \quad \text{ii)}\ g_{tt} = g_{tt}\Phi_{,t} \ \Rightarrow\ \Phi_{,t}=1 \ \Rightarrow\ \Phi = t + f(x^i)\,.$$

Finally, perform the time transformation exactly given by $t' = \Phi = t + f(x^i)$ (and $x'^j = x^j$). This means $t = t' - f(x^i)$, so

$$\frac{\partial x^j}{\partial x'^i} = \delta^j_i\,, \qquad \frac{\partial t}{\partial x'^i} = -\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = -\frac{\partial f}{\partial x^i} = -\Phi_{,i}\,, \qquad \frac{\partial t}{\partial t'} = 1,$$

and thus the induced metric transformation reads

$$g'_{it} = \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial t'}\,g_{\alpha\beta} = \frac{\partial t}{\partial x'^i}\frac{\partial t}{\partial t'}\,g_{tt} + \frac{\partial x^j}{\partial x'^i}\frac{\partial t}{\partial t'}\,g_{jt} = -\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i}\,g_{tt} + \frac{\partial x^j}{\partial x'^i}\,g_{jt} = -\Phi_{,i}\,g_{tt} + g_{it} = 0\,.$$

One may also check that the time symmetry t^μ keeps its coordinate components,

$$t'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,t^\nu = \frac{\partial x'^\mu}{\partial t} = \frac{\partial x'^\mu}{\partial t'} = \delta^\mu_0\,,$$

so $g_{\mu\nu}$ remains independent of time (now of t').

24.5 Vector fields and adapted coordinates

In GR, the choice of coordinates seems to be irrelevant, but we know very well – from Schwarzschild already – that this is not the case actually, because i) usually just a few coordinate systems are practical in specific situations, and ii) different coordinates generally cover *different* parts of a given manifold, and they may offer its considerably different interpretation. Unfortunately, in a sufficiently generic case it may be very difficult to check whether some two metrics (written in two different coordinates) describe the same geometry.

For practical work, usually such coordinates are chosen which are **adapted** to some important vector fields. Specifically, it is natural to let one of the coordinates be given by parameter of the integral lines of the field. If the field is time-like (u^μ), that is the case of the time coordinate (t); in such an adapted system, often called **comoving**, the field has components $u^\mu = (u^t, 0, 0, 0)$.

If the field u^μ is hypersurface-orthogonal, it is natural, on the contrary, to put the corresponding *covector* u_α to a one-component form. It is sufficient to select, as the time coordinate, the scalar function Φ whose gradient is proportional to u_α: one thus has $u_\alpha = -f\Phi_{,\alpha} = -f t_{,\alpha}$, which in the (t, x^i) coordinates reduces to $u_\alpha = -f\delta^0_\alpha$, i.e. $u_\alpha = (u_t, 0, 0, 0)$ (with $u_t = -f$). For time we have used the same letter (t) as in the preceding paragraph, but note that the two choices need not be the same: the contravariant spatial components of the field $u^i = g^{i\alpha}u_\alpha = g^{it}u_t$ will only vanish in the Φ-adapted coordinates if $g^{it} = 0$ (which does *not* hold for the Kerr space-time, for example). Vice versa, if one succeeds in selecting the coordinates so that *both* u_α and u^μ have but the time components, it means that the metric lacks the mixed "time-space" terms,

$$\mathrm{d}s^2 = (-u_\mu u_\nu + h_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu = -(u_t)^2\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j\,.$$

We may recall the four-velocity field of the ZAMO congruence in circular space-times as an example of the hypersurface-orthogonal field. In asymptotically flat space-times, this field is orthogonal to isosurfaces of Killing time which at infinity corresponds to the proper time of observers standing there at rest. The contravariant spatial components of the ZAMO's four-velocity, u^i, only vanish if the space-time is **static**; in that case, the ZAMO congruence coincides with the static congruence, integral to the time Killing vector field $\partial/\partial t$.

Further simplification is possible in the case of a *geodesic* congruence, $a_\alpha = 0$. If such coordinates exist in which both $u^i = 0$ and $u_j = 0$, i.e. $g_{ti} = 0 = g^{ti}$, one has

$$0 = a_i = (\cancel{u_{i,\sigma}} - \Gamma^\rho{}_{i\sigma}u_\rho)\, u^\sigma = -\Gamma^t{}_{it}u_t u^t = \Gamma^t{}_{it} = \frac{1}{2}g^{tt}g_{tt,i} = \frac{g_{tt,i}}{2g_{tt}} \,,$$

hence g_{tt} can only depend on t. In such a case, however, it is possible to adjust the time (just by rescaling) in such a way that $g_{tt} = -1$. Then t stands for the proper time tied to the congruence u^μ. This is the situation in homogeneous and isotropic cosmological models. Their privileged field (geodesic and orthogonal to the hypersurfaces of homogeneity) is the four-velocity field of the cosmic fluid, and the corresponding metric is the FLRW metric (see Chapter 13).

CHAPTER 25

3+1 splitting of space-time

It may be heard, occasionally, that general relativity is "just kind of kinematics where everything is determined once for ever". Reality is just the opposite. Everything is only fixed in exact solutions (such as that of Kerr, or in a specific FLRW cosmological model, for example). These, however, stem from very special (symmetry) assumptions and thus can only describe the virtual world approximately. In general, GR is *the most dynamical (classical) theory*, because it even treats as dynamical (and having its own degrees of freedom) the space-time which hosts any physical process; in fact the space-time *actively participates in all processes*, if one does not resort to some kind of approximation. The related deep methodological issues will mainly emerge in chapter on Cauchy problem (Chapter 26), but already from Section 7.6 we know how inherently problematic the GR problems are due to the entanglement between the sources and the geometry, namely, due to the circumstance that one does not know the space-time *a priori*, so actually does not know terms in which the situation should be described. How problematic it is to say "now", for instance, if one does not know what was *before* and what will happen *after*!

Such aspects are seldom uncomfortable in the Lagrangian approach, because there one *assumes to know* the boundary conditions (and seeks how the physical configuration should look "in-between", relying on extremization of the postulated action). But the Hamiltonian approach (Chapter 27) is just the opposite – there, one assumes to know the configuration (plus possibly its time gradient) *at a certain moment*, and seeks how the given initial conditions evolve according to the prescribed law (Einstein equations in the GR case). For such an approach, it is necessary to understand and describe properly what it means to take the space-time "at a given time", to prescribe its momentary tendency for change, and to evolve such "initial conditions" to "the next instant".

25.1 Foliation of space-time by space-like hypersurfaces

It is by no means automatic that it is possible to "have a space-time at some instant of time". Namely, although there do exist time-like world-lines passing through any event, and to every such world-line one can in general assign a certain time parameter t, the local hyperplanes $t=\mathrm{const}$ need not be extendible (*integrable*) to hypersurfaces spanning the whole manifold. Actually, we saw in Chapter 24 (Section 24.4) that such a nice property only holds if the world-lines form a congruence with zero vorticity. In such a case, the tangent field of the congruence can be written as proportional to a gradient of a scalar function – and that function may be used to define time, i.e. a global function which monotonously increases along any time-like world-line. Note (again) that this function need not correspond to proper time of any physical observer. (Extremely nice is the situation in the FLRW cosmologies where hypersurfaces of homogeneity stand for isochrones of the time which even represents proper time of the cosmic fluid everywhere.)

So we will assume to have such space-time where it *is* possible to say, globally, "now and then": let a space-time (whether already known or yet virtual) be **foliated by a smooth family of space-like hypersurfaces** Σ_t, where t is some continuous parameter "numbering" the hypersurfaces (it will be identified as time later). More specifically, the notion *hypersurface* means a 3D smooth submanifold (a subset of space-time which has no cusps, edges or self-intersections), and the notion *foliation* means that each point of the space-time ($\equiv$ each event) is passed by exactly one of the hypersurfaces. The existence and description of submanifolds – and specifically hypersurfaces – is a classical part of differential geometry and has been exposed in many textbooks (though mostly for *Riemannian* manifolds, with the $+++\ldots$ metric signature).

Let us denote by n^μ the future-oriented (and necessarily time-like) unit normal to Σ_t. Since $t=\mathrm{const}$ on each of the hypersurfaces, the normal has to read

$$n_\alpha := -N\,\frac{\partial t}{\partial x^\alpha}\,, \tag{25.1}$$

where N is a normalization factor. By definition, this vector field is hypersurface-orthogonal, so it has zero vorticity.[1]

Let some congruence also exist defining what it means to "stay at rest" (it is thus usually assumed to be time-like, factually representing a family of observers). Along this congruence, we will identify the spatial coordinate positions on different hypersurfaces Σ_t: having some coordinates $y^m(t=t_0)$ on one of the hypersurfaces (Σ_{t_0}), on other hypersurfaces Σ_t we assign the same coordinate values $y^m(t)$ to the points which are threaded by the respective same "streamlines" of the congruence. Let t^μ stand for the tangent field to the latter congruence, thus indicating the "direction of time",

$$t^\mu := \frac{\partial x^\mu}{\partial t} = \frac{\partial x^\mu}{\partial y^m}\frac{\partial y^m}{\partial t}\,. \tag{25.2}$$

It follows immediately that

$$-n_\alpha t^\alpha = N\,\frac{\partial t}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial t} = N\,.$$

[1] It is also possible to start the whole picture from assuming one has a time-like congruence with unit tangent n^α which has zero vorticity. We know from Frobenius theorem that in such a case the orthogonal local spaces are integrable into a foliation of space-time by 3D space-like hypersurfaces (Σ_t). You may meet the notions **slicing / threading point of view** which indicate whether the space-like foliation or the time-like congruence is the primary ingredient of the splitting.

This function is called the **lapse**; we have already met it in Sections 16 and 19.4 (we will see the correspondence before long). It will also be useful to decompose t^μ, with respect to Σ_t, into the normal and tangent part,

$$t^\mu = \delta^\mu_\alpha t^\alpha = -n^\mu n_\alpha t^\alpha + (\delta^\mu_\alpha + n^\mu n_\alpha)\, t^\alpha = N n^\mu + N^\mu \,, \tag{25.3}$$

where the "pure spatial" vector

$$N^\mu := (\delta^\mu_\alpha + n^\mu n_\alpha)\, t^\alpha =: h^\mu_\alpha t^\alpha$$

representing projection of t^μ on Σ_t is called the **shift**. We have denoted by $h_{\mu\nu}$ the metric of Σ_t as usual, $h_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$. A scheme of the decomposition is given in Figure 25.1.

Note that t^μ is really *not* in general orthogonal to Σ_t (i.e. proportional to n^μ): in coordinates adapted to the decomposition, (t, x^i) (with x^i covering the hypersurfaces Σ_t), one has

$$t^\mu = \delta^\mu_0 \,, \qquad \text{whereas} \quad n^\mu = g^{\mu\alpha} n_\alpha = -N g^{\mu\alpha} \delta^0_\alpha = -N g^{\mu 0} \,.$$

See, for example, the stationary and axisymmetric (in fact circular) space-times where t^μ is the time Killing field, whereas $n^\mu = N^{-1}(t^\mu - N^\mu) = N^{-1}(t^\mu + \omega\phi^\mu)$ is the vector field with zero angular momentum (ϕ^μ is the axial Killing field). Only in the *static* case ($g_{0i} = 0 = g^{0i}$, i.e. $\omega = 0$) it holds $t^\mu = N n^\mu$, i.e. $N^\mu = 0$.

Denote, once more, by y^m some *intrinsic* coordinates in the 3D manifold Σ_t. The embedding of this manifold as a hypersurface in the 4D space-time is described by relations $x^\mu = x^\mu(t, y^m)$. By differentiation, we have

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial t}\,\mathrm{d}t + \frac{\partial x^\mu}{\partial y^m}\,\mathrm{d}y^m \,,$$

where $\frac{\partial x^\mu}{\partial t} \equiv t^\mu$ and $\frac{\partial x^\mu}{\partial y^m}$ are coordinate vectors tangent to Σ_t (more precisely, their components in the "space-time" coordinates x^μ). Substituting there for t^μ the decomposition (25.3), we obtain

$$\mathrm{d}x^\mu = N n^\mu \mathrm{d}t + N^\mu \mathrm{d}t + \frac{\partial x^\mu}{\partial y^m}\,\mathrm{d}y^m = N n^\mu \mathrm{d}t + \frac{\partial x^\mu}{\partial y^m}\,(N^m \mathrm{d}t + \mathrm{d}y^m) \,,$$

where $N^m \equiv N^\mu \frac{\partial y^m}{\partial x^\mu}$ represent components of N^μ in the intrinsic coordinates y^m (the vector N^μ is tangent to Σ_t, so its decomposition in the basis y^m is complete). The metric decomposition immediately follows

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\left[N n^\mu \mathrm{d}t + \frac{\partial x^\mu}{\partial y^m}\,(N^m \mathrm{d}t + \mathrm{d}y^m)\right]\left[N n^\nu \mathrm{d}t + \frac{\partial x^\nu}{\partial y^n}\,(N^n \mathrm{d}t + \mathrm{d}y^n)\right] = \\ &= -N^2 \mathrm{d}t^2 + h_{mn}(N^m \mathrm{d}t + \mathrm{d}y^m)(N^n \mathrm{d}t + \mathrm{d}y^n) \,, \end{aligned} \tag{25.4}$$

where just the definition properties have been employed

$$g_{\mu\nu} n^\mu n^\nu = -1 \,, \qquad g_{\mu\nu} n^\mu \frac{\partial x^\nu}{\partial y^n} = 0 \,,$$
$$g_{\mu\nu} \frac{\partial x^\mu}{\partial y^m}\frac{\partial x^\nu}{\partial y^n} = (h_{\mu\nu} - n_\mu n_\nu)\frac{\partial x^\mu}{\partial y^m}\frac{\partial x^\nu}{\partial y^n} = h_{\mu\nu}\frac{\partial x^\mu}{\partial y^m}\frac{\partial x^\nu}{\partial y^n} \equiv h_{mn} \,.$$

25.1.1 Normal, shift and metric in adapted coordinates

The decomposition holds for any space-time coordinates x^μ, but the "3+1" splitting naturally suggests to choose $x^0 \equiv t$ and x^i covering Σ_t, in which case one identifies $x^i \equiv \delta^i_m y^m$ and thus obtains

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(N^i\mathrm{d}t + \mathrm{d}x^i)(N^j\mathrm{d}t + \mathrm{d}x^j)\,. \tag{25.5}$$

Let us find the components of the above quantities in such adapted coordinates. First, we have $t^\mu = \delta^\mu_0$ and $n_\alpha = -N\delta^0_\alpha$ directly from definitions. The shift is purely spatial, so it reads $N^\mu = (0, N^i)$ in the adapted coordinates. From (25.3) we thus have

$$n^\mu = N^{-1}(t^\mu - N^\mu) = N^{-1}(\delta^\mu_0 - N^\mu) = N^{-1}(1, -N^i)\,.$$

Multiplying (25.3) by N_μ, it also follows $(t^\mu N_\mu =)\; N_0 = N^\mu N_\mu = N^j N_j$ and, from its covariant form,

$$t_\alpha = N n_\alpha + N_\alpha = -N^2\delta^0_\alpha + N_\alpha = (-N^2 + N_0, N_i) = (-N^2 + N_j N^j, N_i)\,.$$

Finally, the contravariant metric can be found by comparing the above components of n^μ with the expression $n^\mu = g^{\mu\alpha}n_\alpha = -Ng^{\mu 0}$, from where $g^{\mu 0} = \frac{1}{N^2}(-1, N^i)$, plus (spatial components) by substituting n^i into $g^{ik} = -n^i n^k + h^{ik} = -N^{-2}N^i N^k + h^{ik}$. Let us summarize the decompositions:

- Normal and the time vector:

$$n_\alpha = -N\delta^0_\alpha\,, \quad n^\mu = \frac{1}{N}(1, -N^i)\,, \qquad t^\mu = \delta^\mu_0\,, \quad t_\alpha = (-N^2 + N_j N^j, N_i)\,.$$

- Covariant-metric decomposition:

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu} = \begin{pmatrix} -N^2 + h_{00} = -N^2 + N_j N^j & h_{0k} = N_k \\ h_{i0} = N_i & h_{ik} \end{pmatrix}. \tag{25.6}$$

- Contravariant-metric decomposition:

$$g^{\mu\nu} = -n^\mu n^\nu + h^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} + h^{00} = -\frac{1}{N^2} & \frac{N^k}{N^2} + h^{0k} = \frac{N^k}{N^2} \\ \frac{N^i}{N^2} + h^{i0} = \frac{N^i}{N^2} & -\frac{N^i N^k}{N^2} + h^{ik} \end{pmatrix}. \tag{25.7}$$

  From the components of $g^{\mu\nu}$ it is seen that $h^{0\mu} = 0$, so h^{ik} is the inverse to the purely spatial metric h_{ik},

$$h^{ij}h_{jk} = h^{i\sigma}h_{\sigma k}(= h^{i\sigma}g_{\sigma k}) = h^i_k = \delta^i_k + n^i \cancel{n_k} = \delta^i_k\,.$$

- Mixed components h^μ_ν:

$$h^\mu_\nu = h^{\mu\iota}g_{\iota\nu} = h^{\mu\iota}h_{\iota\nu} = \cancel{h^{\mu 0}}h_{0\nu} + h^{\mu j}h_{j\nu}\,,$$

  from where we see, besides the already known $h^i_k = \delta^i_k$, that

$$h^0_\nu = 0\,, \qquad h^i_0 = h^{ij}N_j \left(= g^{i\iota}N_\iota\right) = N^i\,.$$

- Relation between the metric determinants is most easily obtained from the well known formula $g^{tt} = \frac{\mathrm{minor}(g_{tt})}{\det(g_{\mu\nu})} = \frac{h}{g}$:

$$-g = \frac{h}{-g^{tt}} = N^2 h\,. \tag{25.8}$$

25.1.2 Physical sense of lapse and shift

Besides their geometric meaning, the lapse and shift also have clear physical sense. The normal field n^μ obviously plays the role of four-velocity, so one can introduce a "proper time" accordingly, $n^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. Comparing this to the components of n^μ in the (t, x^i) coordinates, we have

$$n^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1, v^i\right) \quad \longleftrightarrow \quad \frac{1}{N}\left(1, -N^i\right). \tag{25.9}$$

Therefore, the lapse N represents the dilation factor between the proper time of observers with four-velocity n^μ and the global time t; in other words, N is the Lorentz factor between the observers n^μ and those static with respect to the coordinates (the latter having four-velocity proportional to t^μ); well, this was clear from the beginning – from the relation $N = -n_\sigma t^\sigma$. And the shift N^i represents spatial velocity with which the "static" observers move relative to the observers orthogonal to the hypersurface.

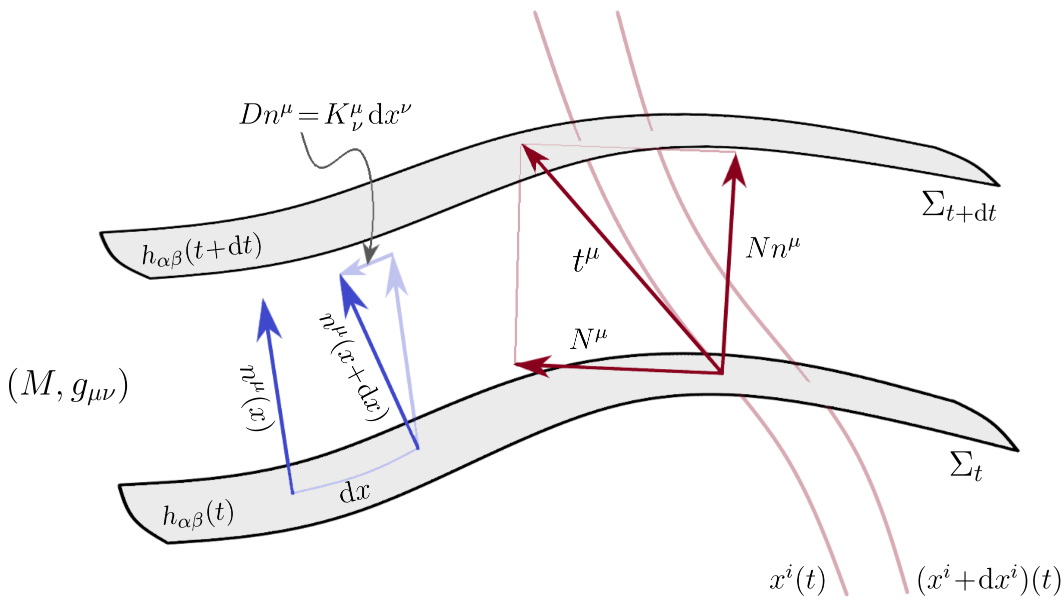

Figure 25.1 *A scheme of the 3+1 decomposition based on the foliation of the space-time manifold* $(M, g_{\mu\nu})$ *by a family of space-like hypersurfaces* Σ_t.

25.1.3 Acceleration of the hypersurface-orthogonal field

It will certainly be useful to know how the hypersurface-orthogonal field n^μ evolves along itself, i.e. to know its four-acceleration. Directly from the definition $n_\mu = -Nt_{,\mu}$ we obtain

$$n_{\mu;\nu} - n_{\nu;\mu} = n_{\mu,\nu} - n_{\nu,\mu} = -N_{,\nu}t_{,\mu} - \cancel{Nt_{,\mu\nu}} + N_{,\mu}t_{,\nu} + \cancel{Nt_{,\nu\mu}} = \frac{N_{,\nu}}{N}n_\mu - \frac{N_{,\mu}}{N}n_\nu\,. \tag{25.10}$$

Multiplying the above by n^ν and putting $n_{\nu;\mu}n^\nu = 0$ (from normalization), we further have $n_{\mu;\nu}n^\nu = (n_{\mu,\nu} - n_{\nu,\mu})n^\nu$, so

$$a_\mu \equiv n_{\mu;\nu}n^\nu = (n_{\mu,\nu} - n_{\nu,\mu})n^\nu = \left(\frac{N_{,\nu}}{N}\,n_\mu - \frac{N_{,\mu}}{N}\,n_\nu\right)n^\nu = \frac{N_{,\nu}}{N}\,h^\nu_\mu =: \frac{N_{|\mu}}{N}\,, \qquad (25.11)$$

where we have denoted by the vertical stroke the spatial gradient (the one which acts within Σ_t) – see Section 25.3 below. In the adapted coordinates, one thus obtains

$$a_\mu \equiv \frac{N_{,\nu}}{N}\,h^\nu_\mu = \frac{N_{,\nu}}{N}\,(N^\nu, h^\nu_i) = \frac{1}{N}\left(N_{,j}N^j, N_{,i}\right).$$

In the circular space-times, for example, $N^\nu = \omega\phi^\nu$, so $N_{,\nu}N^\nu = 0$ and one is left with the Kerr-like form of the acceleration of ZAMOs, $a_\mu = N_{,\mu}/N$.

25.2 Extrinsic curvature: the second fundamental form

We have been suggesting that the vector field t^μ represents time flow and hypersurfaces Σ_t represent "spatial geometry at given time t". However, if employing the 3+1 picture to treat the Cauchy initial problem, the bulk space-time is not known (it does not exist) at the moment when the problem is being formulated. In such a case, it is more natural to consider the hypersurfaces Σ_t as *identified* by the flow of t^μ, and to speak about the evolution of the spatial geometry h_{ik} on a *certain hypersurface* Σ, rather than about the evolution of a hypersurface Σ_t *in certain space-time*. The dynamical variable will be the intrinsic geometry of that hypersurface, as described by the metric $h_{ik}(t)$, also called the **first fundamental form** of the submanifold. In order to be able to compute the evolution $h_{ik}(t)$ (as given by Einstein equations which are second-order in its derivatives), the initial conditions have to also include the time derivative $h_{ik,t}$. Unfortunately, the latter is clearly *not* a tensorial quantity.

It turns out that a suitable tensorial quantity is the derivative of h_{ik} in the direction of spatial normal n^μ. We have already met this tensor in Chapter 24.1 (see remarx at its end), it was called the **expansion tensor** there, $\Theta_{\mu\nu}$ (24.8),

$$\Theta_{\mu\nu} = h^\alpha_\mu h^\beta_\nu n_{(\alpha;\beta)} = h^\alpha_{(\mu}h^\beta_{\nu)}n_{\alpha;\beta} = \delta^\alpha_{(\mu}h^\beta_{\nu)}n_{\alpha;\beta} = h^\beta_{(\nu}n_{\mu);\beta} = n_{(\mu;\nu)} + a_{(\mu}n_{\nu)} = \frac{1}{2}\pounds_{\mathbf{n}}h_{\mu\nu}\,,$$

where $a_\mu := n_{\mu;\beta}n^\beta$ is the acceleration of the field n^μ (please note that we denoted there the tangent field of the time-like congruence by u^μ, while in the present chapter it is usual to use n^μ). Expansion tensor was introduced as the symmetric part of the projected gradient of the tangent field; its skew-symmetric counterpart was called the vorticity (or twist) tensor ($\omega_{\mu\nu}$). Now, recall that if a field is hypersurface-orthogonal (which our $n_\mu = -Nt_{,\mu}$ is by definition), it is vorticity-free, $\omega_{\mu\nu} = 0$ (and vice versa) – in other words, its projected gradient $h^\alpha_\mu h^\beta_\nu n_{\alpha;\beta}$ is *symmetric*, just represented by the above expansion tensor. If you do not want to return to the Frobenius theorem, just show it easily from (25.10) and (25.11):

$$n_{\mu;\nu} - n_{\nu;\mu} + a_\mu n_\nu - n_\mu a_\nu = \frac{N_{,\nu}}{N}\,n_\mu - \frac{N_{,\mu}}{N}\,n_\nu + \frac{N_{,\sigma}}{N}\,h^\sigma_\mu n_\nu - \frac{N_{,\sigma}}{N}\,h^\sigma_\nu n_\mu = 0\,,$$

where you only rewrite, in pairs,

$$\frac{N_{,\nu}}{N}\,n_\mu - \frac{N_{,\sigma}}{N}\,h^\sigma_\nu n_\mu = \frac{n_\mu}{N}\left(\delta^\sigma_\nu - h^\sigma_\nu\right)N_{,\sigma} = -\frac{n_\mu n_\nu}{N}\,N_{,\sigma}n^\sigma\,,$$
$$-\frac{N_{,\mu}}{N}\,n_\nu + \frac{N_{,\sigma}}{N}\,h^\sigma_\mu n_\nu = -\frac{n_\nu}{N}\left(\delta^\sigma_\mu - h^\sigma_\mu\right)N_{,\sigma} = \frac{n_\nu n_\mu}{N}\,N_{,\sigma}n^\sigma\,.$$

Just to stress this simple result once more, *for a hypersurface-orthogonal field, the expansion tensor $\Theta_{\mu\nu}$ is symmetric*. In such a case, and especially when dealing with the 3+1 decomposition, this tensor is canonically denoted by $K_{\mu\nu}$,

$$K_{\mu\nu} := n_{\alpha;\beta}h^\alpha_\mu h^\beta_\nu = \left(n_{\mu;\beta} + \cancel{n_{\alpha;\beta}n^\alpha} n_\mu\right)h^\beta_\nu = n_{\mu;\beta}h^\beta_\nu = n_{\mu;\nu} + a_\mu n_\nu = \frac{1}{2}\pounds_{\mathbf{n}}h_{\mu\nu}\,, \quad (25.12)$$

and called **the extrinsic curvature** or **the second fundamental form** of the submanifold (its mixed components represent what is called **the shape operator** or **the Weingarten map**).[2] Needless to say, this tensor lives on Σ_t (as $\Theta_{\mu\nu}$ in general), so we may summarize

$$h_{\mu\nu}n^\nu = 0\,, \quad a_\mu n^\mu = 0\,, \quad K_{\mu\nu}n^\nu = 0\,; \quad K := h^{\mu\nu}K_{\mu\nu}(\equiv g^{\mu\nu}K_{\mu\nu}) = K^\mu_\mu = n^\mu{}_{;\mu}\,.$$

Why "extrinsic curvature" or "shape operator"? Because, while $h_{\mu\nu}$ describes the intrinsic geometry of the hypersurface, $K_{\mu\nu}$ says how the hypersurface is curved *as a submanifold of a given manifold* – what is its shape when viewed from the bulk. Indeed, consider that the equation $n_{\mu;\beta}V^\beta = 0$ generally means that the (co)vector n_μ is parallel transported in the direction V^α. On the other hand, $n_{\mu;\beta}h^\beta_\nu = 0$ means that n_μ is parallel transported in any direction orthogonal to n^α. Hence, $n_{\mu;\beta}h^\beta_\nu$ itself says how much the n_μ is *not* transported parallelly along Σ_t – that is, to what extent it "points in a different direction" at different locations of Σ_t (see Figure 25.1). This, however, tells about curvature of the manifold Σ_t as a hypersurface embedded in the 4D space-time. The trace K has a clear geometrical sense as well: as it is given by the expansion of n^μ (by Θ), it means that $K > 0$ (normal field is diverging) when Σ is convex, whereas $K < 0$ (normal field is converging) when Σ is concave. (The shape-operator trace K divided by dimension of Σ_t is being called the **mean curvature** of Σ_t.)

We already know from $\Theta_{\mu\nu}$ the elegant definition in terms of $\frac{1}{2}\pounds_{\mathbf{n}}h_{\mu\nu}$. Let us also check how it is related to the Lie derivative of $h_{\mu\nu}$ in the "purely temporal" direction $t^\alpha = Nn^\alpha + N^\alpha$. Since

$$t^\alpha{}_{;\mu} = N_{;\mu}n^\alpha + Nn^\alpha{}_{;\mu} + N^\alpha{}_{;\mu} \quad \Longrightarrow \quad t^\alpha{}_{;\mu}h_{\alpha\nu} = \left(Nn^\alpha{}_{;\mu} + N^\alpha{}_{;\mu}\right)h_{\alpha\nu}\,,$$

the Lie-derivative definition simply yields the "linear" relation

$$\pounds_{\mathbf{t}}h_{\mu\nu} = h_{\mu\nu;\alpha}t^\alpha + t^\alpha{}_{;\mu}h_{\alpha\nu} + t^\alpha{}_{;\nu}h_{\mu\alpha} =$$

[2] More rigorously: the extrinsic curvature = the second fundamental form is a symmetric (0,2) tensor field on Σ_t, so at some specific point it yields a real number for any two vectors tangent to Σ_t, while the shape operator = Weingarten map is a mapping from a tangent space of Σ_t (at some point) to itself, so for any vector (s^μ) tangent to Σ_t it gives the change of n^μ in that direction, $n^\mu{}_{;\nu}s^\nu = K^\mu{}_\nu s^\nu$.

$$
\begin{aligned}
&= h_{\mu\nu;\alpha}(Nn^\alpha + N^\alpha) + (Nn^\alpha{}_{;\mu} + N^\alpha{}_{;\mu})\, h_{\alpha\nu} + (Nn^\alpha{}_{;\nu} + N^\alpha{}_{;\nu})\, h_{\mu\alpha} = \\
&= Nh_{\mu\nu;\alpha}n^\alpha + Nn^\alpha{}_{;\mu}h_{\alpha\nu} + Nn^\alpha{}_{;\nu}\, h_{\mu\alpha} + h_{\mu\nu;\alpha}N^\alpha + N^\alpha{}_{;\mu}h_{\alpha\nu} + N^\alpha{}_{;\nu}h_{\mu\alpha} = \\
&= N\pounds_{\mathbf{n}}h_{\mu\nu} + \pounds_{\mathbf{N}}h_{\mu\nu}\,. \qquad (25.13)
\end{aligned}
$$

In short,

$$K_{\mu\nu} = \frac{1}{2}\pounds_{\mathbf{n}}h_{\mu\nu} = \frac{1}{2N}\left(\pounds_{\mathbf{t}}h_{\mu\nu} - \pounds_{\mathbf{N}}h_{\mu\nu}\right).$$

In the above relation, we surely know that $K_{\mu\nu}$ lives on Σ, whereas the terms on the right-hand side need not be such individually. If we preferred to strictly work on the hypersurface, we can project there both the terms – the relation will stay the same:

$$K_{\mu\nu} = \frac{1}{2}\pounds_{\mathbf{n}}h_{\mu\nu} = \frac{1}{2N}\, h^\alpha_\mu h^\beta_\nu \left(\pounds_{\mathbf{t}}h_{\alpha\beta} - \pounds_{\mathbf{N}}h_{\alpha\beta}\right). \qquad (25.14)$$

The second term can also be expressed as

$$
\begin{aligned}
&\pounds_{\mathbf{N}}h_{\alpha\beta} = h_{\alpha\beta;\sigma}N^\sigma + N^\sigma{}_{;\alpha}h_{\sigma\beta} + N^\sigma{}_{;\beta}h_{\alpha\sigma} = h_{\alpha\beta;\sigma}h^\sigma_\rho t^\rho + N_{\sigma;\alpha}h^\sigma_\beta + N_{\sigma;\beta}h^\sigma_\alpha \\
&\Longrightarrow \quad h^\alpha_\mu h^\beta_\nu \pounds_{\mathbf{N}}h_{\alpha\beta} = \cancel{h_{\mu\nu|\rho}}\, t^\rho + N_{\nu|\mu} + N_{\mu|\nu}\,, \qquad (25.15)
\end{aligned}
$$

so the relation becomes

$$K_{\mu\nu} = \frac{1}{2N}\, h^\alpha_\mu h^\beta_\nu \left(\pounds_{\mathbf{t}}h_{\alpha\beta} - \pounds_{\mathbf{N}}h_{\alpha\beta}\right) =: \frac{1}{2N}\left(\dot{h}_{\mu\nu} - N_{\mu|\nu} - N_{\nu|\mu}\right). \qquad (25.16)$$

25.3 Covariant derivative on a hypersurface

Projection by the h^μ_α tensor ensures a unique and geometric transition between the tensors living in space-time and those living/acting on the hypersurface Σ_t. Let us inquire now whether a similarly simple relation holds between the covariant derivatives acting in space-time and on the hypersurface. Sure, the latter will only have sense for tensors which themselves live/act on tangent and cotangent spaces of Σ_t. However, let us stress right away that even if acting on such "purely spatial" quantities, it would be incorrect to define it only by projecting the space-time gradient on Σ_t, i.e. in the $T^{\cdots}{}_{\ldots;\sigma}h^\sigma_\rho$ style, because the tensor field thus obtained need *not* be purely spatial any more. This point may not be totally intuitive, so let us illustrate it on the 3D metric $h_{\mu\nu}$ itself (so on a quantity which is tied to Σ_t more tightly than anything else): making its space-time gradient and projecting on the normal n_μ, one does *not* obtain zero,

$$(h^\mu_{\alpha;\sigma}h^\sigma_\rho)n_\mu = \cancel{n^\mu{}_{;\sigma}n_\mu}\, n_\alpha h^\sigma_\rho + n^\mu n_\mu n_{\alpha;\sigma}h^\sigma_\rho = -n_{\alpha;\sigma}h^\sigma_\rho = -K_{\alpha\rho}\,.$$

Indeed, in order that the derivative map *all* 3D tensors to 3D tensors again, the result of the 4D gradient has to be projected on Σ_t *in all its indices*.

Lemma: The operation defined by

$$T^{\mu\cdots}{}_{\alpha\ldots|\rho} := T^{\nu\cdots}{}_{\beta\ldots;\sigma}\, h^\mu_\nu \ldots h^\beta_\alpha \ldots h^\sigma_\rho \qquad (25.17)$$

is a covariant derivative corresponding to the Levi-Civita connection of the metric $h_{\mu\nu}$.
Proof: The projection modifies nothing on that the operation is linear and satisfies the Leibniz rule. Besides that, it corresponds to the Levi-Civita connection of the metric $h_{\mu\nu}$, because, first, $h_{\mu\nu}$ is constant with respect to it,

$$h_{\mu\nu|\rho} \equiv h_{\kappa\lambda;\sigma} h^\kappa_\mu h^\lambda_\nu h^\sigma_\rho = (n_{\kappa;\sigma} n_\lambda + n_\kappa n_{\lambda;\sigma})\, h^\kappa_\mu h^\lambda_\nu h^\sigma_\rho = 0\,,$$

and, second, it is torsion-free: for an arbitrary scalar field (f), we have

$$\begin{aligned} f_{|\mu\nu} &\equiv \left(f_{;\kappa} h^\kappa_\mu\right)_{|\nu} \equiv \left(f_{;\kappa} h^\kappa_\iota\right)_{;\lambda} h^\iota_\mu h^\lambda_\nu = \\ &= f_{;\kappa\lambda} h^\kappa_\mu h^\lambda_\nu + f_{;\kappa}(n^\kappa{}_{;\lambda} n_\iota + n^\kappa n_{\iota;\lambda}) h^\iota_\mu h^\lambda_\nu = (f_{,\kappa})_{;\lambda} h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa n_{\iota;\lambda} h^\iota_\mu h^\lambda_\nu = \\ &= (f_{,\kappa\lambda} - \Gamma^\alpha{}_{\kappa\lambda} f_{,\alpha})\, h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa K_{\mu\nu} = (f_{,\lambda\kappa} - \Gamma^\alpha{}_{\lambda\kappa} f_{,\alpha})\, h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa K_{\nu\mu} = \\ &= f_{;\lambda\kappa} h^\lambda_\nu h^\kappa_\mu + f_{,\kappa} n^\kappa K_{\nu\mu} \;=\; f_{|\nu\mu}\,. \end{aligned}$$

□

Remark (important actually): Offering itself was to use $h^\kappa_{\mu|\nu} = 0$ at the very beginning, and write $\left(f_{;\kappa} h^\kappa_\mu\right)_{|\nu} = (f_{;\kappa})_{|\nu} h^\kappa_\mu$, but the expression $(f_{;\kappa})_{|\nu}$ is not well defined, since the 3D derivative only acts on the hypersurface Σ_t – and the 4D gradient $f_{;\kappa}$ may stick out of it. Actually, one has to always be sure that the 3D derivative is only employed for 3D quantities. (Still important:) This does *not* mean that a generic 4D gradient could not be totally projected on Σ_t, but if the differentiated quantity is 4D (*not* tangent to Σ_t), one should not claim that the operation has the meaning of its 3D gradient. Nota bene, exactly such an operation appears in the definition of extrinsic curvature, doesn't it – and we did *not* call it $n_{\mu|\nu}$.

25.3.1 Relation between the 3D and 4D covariant derivatives

On tangent spaces of the Σ_t hypersurface, both the 3D and the 4D derivative can operate, naturally. We will compare their result on a vector V_μ and a covector V^γ (co)tangent to Σ_t:

$$V_{\mu|\nu} \equiv V_{\alpha;\beta} h^\alpha_\mu h^\beta_\nu = V_{\mu;\beta} h^\beta_\nu + V_{\alpha;\beta} n^\alpha n_\mu h^\beta_\nu = V_{\mu;\beta} h^\beta_\nu - V_\alpha n^\alpha{}_{;\beta} n_\mu h^\beta_\nu = V_{\mu;\beta} h^\beta_\nu - V_\alpha K^\alpha_\nu n_\mu\,, \tag{25.18}$$

$$V^{\gamma|\delta} = V_{\mu|\nu} h^{\mu\gamma} h^{\nu\delta} = V_{\mu;\beta} h^{\mu\gamma} h^{\beta\delta} = V^{\mu;\beta} h^\gamma_\mu h^\delta_\beta = \;\ldots\; = V^{\gamma;\beta} h^\delta_\beta - V^\mu K^\delta_\mu n^\gamma\,. \tag{25.19}$$

By contraction, we also find several expressions for the 3D divergence,

$$\begin{aligned} V^\nu{}_{|\nu} &= h^{\mu\nu} V_{\mu|\nu} = h^{\mu\nu} V_{\alpha;\beta} h^\alpha_\mu h^\beta_\nu = h^{\alpha\beta} V_{\alpha;\beta} = \\ &= V^\beta{}_{;\beta} + n^\alpha n^\beta V_{\alpha;\beta} = V^\beta{}_{;\beta} - n^\alpha{}_{;\beta} n^\beta V_\alpha = V^\beta{}_{;\beta} - a^\alpha V_\alpha\,. \end{aligned} \tag{25.20}$$

25.4 Decomposition of curvature: Gauss, Codazzi and Ricci equations

Knowing the covariant derivative acting on Σ_t, we can find the 3D Riemann tensor from commutator (6.3),

$$V_{\nu|\kappa\lambda} - V_{\nu|\lambda\kappa} = {}^{(3)}R^\mu{}_{\nu\kappa\lambda} V_\mu\,,$$

where V_ν is an arbitrary covector field on Σ_t (so remember that $V_\mu n^\mu = 0$). –Yes, we can:

$$\begin{aligned}
V_{\nu|\kappa\lambda} &\equiv (V_{\mu;\gamma}h^\mu_\nu h^\gamma_\kappa)_{|\lambda} \equiv \left(V_{\mu;\gamma}h^\mu_\alpha h^\gamma_\rho\right)_{;\sigma} h^\alpha_\nu h^\rho_\kappa h^\sigma_\lambda = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_{\alpha;\sigma}h^\gamma_\kappa h^\alpha_\nu h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_\nu h^\gamma_{\rho;\sigma}h^\rho_\kappa h^\sigma_\lambda = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}(n^\mu{}_{;\sigma}n_\alpha + n^\mu n_{\alpha;\sigma})h^\gamma_\kappa h^\alpha_\nu h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_\nu(n^\gamma{}_{;\sigma}n_\rho + n^\gamma n_{\rho;\sigma})h^\rho_\kappa h^\sigma_\lambda = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}n^\mu n_{\alpha;\sigma}h^\gamma_\kappa h^\alpha_\nu h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_\nu n^\gamma n_{\rho;\sigma}h^\rho_\kappa h^\sigma_\lambda = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}n^\mu h^\gamma_\kappa K_{\nu\lambda} + V_{\mu;\gamma}h^\mu_\nu n^\gamma K_{\kappa\lambda} = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda - V_\mu n^\mu{}_{;\gamma}h^\gamma_\kappa K_{\nu\lambda} + V_{\mu;\gamma}h^\mu_\nu n^\gamma K_{\kappa\lambda} = \\
&= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda - V_\mu K^\mu_\kappa K_{\nu\lambda} + V_{\mu;\gamma}h^\mu_\nu n^\gamma K_{\kappa\lambda}\ ,
\end{aligned}$$

from where

$$\begin{aligned}
{}^{(3)}R^\mu{}_{\nu\kappa\lambda}V_\mu := V_{\nu|\kappa\lambda} - V_{\nu|\lambda\kappa} &= V_{\mu;\gamma\sigma}h^\mu_\nu\left(h^\gamma_\kappa h^\sigma_\lambda - h^\gamma_\lambda h^\sigma_\kappa\right) - V_\mu\left(K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa}\right) = \\
&= \left(V_{\mu;\gamma\sigma} - V_{\mu;\sigma\gamma}\right)h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda - V_\mu\left(K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa}\right) = \\
&= R^\alpha{}_{\mu\gamma\sigma}V_\alpha h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda - V_\mu\left(K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa}\right) = \\
&= \left(R^\mu{}_{\beta\gamma\delta}h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda - K^\mu_\kappa K_{\nu\lambda} + K^\mu_\lambda K_{\nu\kappa}\right)V_\mu\ .
\end{aligned}$$

The covector V_μ is arbitrary yet tangent to Σ_t, so the relation holds irrespectively of it, one just has to project it on Σ_t (by h^μ_α):

$$\boxed{R^\alpha{}_{\beta\gamma\delta}h^\mu_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = {}^{(3)}R^\mu{}_{\nu\kappa\lambda} + K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa}}\ . \tag{25.21}$$

This equation relates the Riemann tensor of the hypersurface Σ_t to the "purely spatial" projection of the space-time Riemann tensor. It is known from geometry as the **Gauss equation**.

The 4D Riemann tensor can also be projected in two other ways, $(R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda)$ and $(R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu n^\gamma h^\delta_\lambda)$ (projections involving more than two n^μ are zero due to the Riemann's antisymmetries), so let us find how these are related to the quantities of the 3+1 decomposition. Both relations can be obtained from projection of the 4D Ricci identity $R^\alpha{}_{\beta\gamma\delta}n_\alpha = n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma}$. On the right-hand side, one may surmise the "commutator" of the swapped first derivatives of extrinsic curvature, so let us first compute

$$\begin{aligned}
K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma} &= (n_{\beta;\gamma} + a_\beta n_\gamma)_{;\delta} - (n_{\beta;\delta} + a_\beta n_\delta)_{;\gamma} = \\
&= n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma} + a_{\beta;\delta}n_\gamma - a_{\beta;\gamma}n_\delta + a_\beta(n_{\gamma;\delta} - n_{\delta;\gamma}) = \\
&= n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma} + a_{\beta;\delta}n_\gamma - a_{\beta;\gamma}n_\delta - a_\beta(a_\gamma n_\delta - a_\delta n_\gamma) = \\
&= n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma} + (a_{\beta;\delta} + a_\beta a_\delta)\,n_\gamma - (a_{\beta;\gamma} + a_\beta a_\gamma)\,n_\delta\ .
\end{aligned}$$

Substituting to the "hh-projected" Ricci identity for n_β, we thus have

$$R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\delta_\lambda = (n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma})h^\beta_\nu h^\delta_\lambda = (K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma})h^\beta_\nu h^\delta_\lambda - (a_{\nu|\lambda} + a_\nu a_\lambda)n_\gamma\ . \tag{25.22}$$

Projecting the latter by h^γ_κ, we arrive at the **Codazzi(-Mainardi) equation**

$$\boxed{R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = \left(K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma}\right)h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda \equiv K_{\nu\kappa|\lambda} - K_{\nu\lambda|\kappa}}\ , \tag{25.23}$$

while projecting the same relation on n^γ yields the **Ricci(-Kühne) equation**

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} n_\alpha h^\beta_\nu n^\gamma h^\delta_\lambda \; (= R_{\alpha\nu\gamma\lambda} n^\alpha n^\gamma) &= (K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma}) h^\beta_\nu n^\gamma h^\delta_\lambda + a_{\nu|\lambda} + a_\nu a_\lambda = \\ &= a_{\nu|\lambda} + a_\nu a_\lambda - K_{\beta\gamma} h^\beta_\nu n^\gamma{}_{;\delta} h^\delta_\lambda - K_{\beta\delta;\gamma} h^\beta_\nu n^\gamma h^\delta_\lambda = \\ &= a_{\nu|\lambda} + a_\nu a_\lambda - K_{\nu\gamma} K^\gamma_\lambda - h^\beta_\nu h^\delta_\lambda K_{\beta\delta;\gamma} n^\gamma = \qquad (25.24) \\ &= a_{\nu|\lambda} + a_\nu a_\lambda + K_{\nu\gamma} K^\gamma_\lambda - \pounds_{\mathbf{n}} K_{\nu\lambda} \, . \qquad (25.25) \end{aligned}$$

The last expression – involving the Lie derivative of $K_{\nu\lambda}$ – is obtained by

$$\begin{aligned} h^\beta_\nu h^\delta_\lambda K_{\beta\delta;\gamma} n^\gamma &= K_{\nu\lambda;\gamma} n^\gamma + n^\delta n_\lambda K_{\nu\delta;\gamma} n^\gamma + n^\beta n_\nu K_{\beta\lambda;\gamma} n^\gamma + n^\beta n^\delta K_{\beta\delta;\gamma} n^\gamma n_\nu n_\lambda = \\ &= K_{\nu\lambda;\gamma} n^\gamma - a^\delta n_\lambda K_{\nu\delta} - a^\beta n_\nu K_{\beta\lambda} = \\ &= K_{\nu\lambda;\gamma} n^\gamma - (K^\delta_\lambda - n^\delta{}_{;\lambda}) K_{\nu\delta} - (K^\beta_\nu - n^\beta{}_{;\nu}) K_{\beta\lambda} = \\ &= \pounds_{\mathbf{n}} K_{\nu\lambda} - 2 K_{\nu\gamma} K^\gamma_\lambda \, . \qquad (25.26) \end{aligned}$$

We actually know equation (25.24) from previous chapter already: it is the equation (24.33) for evolution of the expansion tensor $\Theta_{\mu\nu}$ along u^γ, one only has to realize that i) now u^γ is denoted by n^γ, ii) it has automatically zero vorticity $\omega_{\mu\nu}$ (since it is defined as hypersurface-orthogonal), iii) $\Theta_{\mu\nu}$ thus assumes the role of $K_{\mu\nu}$; and, finally, iv) the equation has now been projected by "hh", so the terms proportional to u_ν disappeared.

25.4.1 Decomposition of the Ricci tensor

... follows directly from the latter's definition,

$$R_{\beta\delta} := g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} = h^{\alpha\gamma} R_{\alpha\beta\gamma\delta} - n^\alpha n^\gamma R_{\alpha\beta\gamma\delta} = h^{\mu\kappa} R_{\alpha\beta\gamma\delta} h^\alpha_\mu h^\gamma_\kappa - n^\alpha n^\gamma R_{\alpha\beta\gamma\delta} \, . \qquad (25.27)$$

Three different projections are possible, apparently. For the first two we obtain, from above,

$$R_{\beta\delta} h^\beta_\nu h^\delta_\lambda = h^{\mu\kappa} R_{\alpha\beta\gamma\delta} h^\alpha_\mu h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda - R_{\alpha\nu\gamma\lambda} n^\alpha n^\gamma \, , \qquad (25.28)$$
$$R_{\beta\delta} n^\beta h^\delta_\lambda = h^{\mu\kappa} R_{\alpha\beta\gamma\delta} h^\alpha_\mu n^\beta h^\gamma_\kappa h^\delta_\lambda = -h^{\nu\kappa} R_{\alpha\beta\gamma\delta} n^\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda \, , \qquad (25.29)$$

which yields, after substituting from equations (25.21), (25.23) and (25.24),

$$R_{\beta\delta} h^\beta_\nu h^\delta_\lambda = {}^{(3)}R_{\nu\lambda} + K K_{\nu\lambda} - a_{\nu|\lambda} - a_\nu a_\lambda + h^\beta_\nu h^\delta_\lambda K_{\beta\delta;\gamma} n^\gamma \, , \qquad (25.30)$$
$$R_{\beta\delta} n^\beta h^\delta_\lambda = K^\kappa_{\lambda|\kappa} - K_{|\lambda} \, . \qquad (25.31)$$

The remaining, third projection is best derived by contraction of (25.24) or (25.25), i.e. by their multiplication by $h^{\nu\lambda}$,

$$\begin{aligned} R_{\alpha\gamma} n^\alpha n^\gamma = a^\lambda{}_{|\lambda} + a^\lambda a_\lambda - K^\lambda_\gamma K^\gamma_\lambda - h^{\beta\delta} K_{\beta\delta;\gamma} n^\gamma &= a^\delta{}_{;\delta} - K^\lambda_\gamma K^\gamma_\lambda - K_{,\gamma} n^\gamma = \qquad (25.32) \\ &= a^\delta{}_{;\delta} + K^\lambda_\gamma K^\gamma_\lambda - h^{\nu\lambda} \pounds_{\mathbf{n}} K_{\nu\lambda} \, . \qquad (25.33) \end{aligned}$$

Don't you recognize this equation? It is the Raychaudhuri equation (24.35), describing the evolution of expansion of the vector field n^μ – here without the vorticity term, however (bear in mind that n^μ is hypersurface-orthogonal). We have employed the relation (25.20) for $V^\mu \equiv a^\mu$, specifically,

$$a^\lambda{}_{|\lambda} = h^{\nu\lambda} a_{\nu|\lambda} = h^{\nu\lambda} a_{\beta;\delta} h^\beta_\nu h^\delta_\lambda = h^{\beta\delta} a_{\beta;\delta} = a^\delta{}_{;\delta} - a^\beta a_\beta \ , \tag{25.34}$$

and the very advantageous arrangement

$$\begin{aligned} h^{\nu\lambda} h^\beta_\nu h^\delta_\lambda K_{\beta\delta;\gamma} n^\gamma &= h^{\beta\delta} K_{\beta\delta;\gamma} n^\gamma = (g^{\beta\delta} K_{\beta\delta})_{;\gamma} n^\gamma + n^\beta n^\delta K_{\beta\delta;\gamma} n^\gamma = \\ &= K_{;\gamma} n^\gamma - a^\beta \cancel{n^\delta K_{\beta\delta}} - a^\delta \cancel{n^\beta K_{\beta\delta}} \ . \end{aligned} \tag{25.35}$$

25.4.2 Decomposition of the Ricci scalar

... follows directly from definition by substituting (25.28),

$$\begin{aligned} R \equiv g^{\beta\delta} R_{\beta\delta} &= h^{\beta\delta} R_{\beta\delta} - n^\beta n^\delta R_{\beta\delta} = h^{\nu\lambda} R_{\beta\delta} h^\beta_\nu h^\delta_\lambda - R_{\alpha\gamma} n^\alpha n^\gamma = \\ &= h^{\mu\kappa} h^{\nu\lambda} R_{\alpha\beta\gamma\delta} h^\alpha_\mu h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda - 2 R_{\alpha\gamma} n^\alpha n^\gamma \ . \end{aligned} \tag{25.36}$$

In the first row, use (25.30) and (25.32), while in the second row just rewrite the first term using (25.21),

$$R = {}^{(3)}R + K^2 + K_{\nu\gamma} K^{\nu\gamma} - 2 a^\delta{}_{;\delta} + 2 K_{,\gamma} n^\gamma = \tag{25.37}$$

$$\quad = {}^{(3)}R + K^2 - K_{\nu\kappa} K^{\nu\kappa} - 2 R_{\alpha\gamma} n^\alpha n^\gamma \ . \tag{25.38}$$

This equation generalizes the Gauss' famous result called **Theorema Egregium** (remarkable theorem). Gauss felt it remarkable since it links the *extrinsic* properties of Σ_t (characterised by $K_{\mu\nu}$) to its *intrinsic* properties (represented by the 3D Ricci scalar). Gauss actually obtained this result for a 2D surface in a Euclidean space, so his "bulk" Ricci was zero and the equation reduced to ${}^{(3)}R + K^2 - K_{ij} K^{ij} = 0$ (to be precise, we should have written opposite signs at the K-terms here, since $\mathbb{E}^3$ is a Riemannian rather than Lorentzian space).

25.4.3 Decomposition of the Einstein tensor

... comes out, consequently,

$$\begin{aligned} G_{\beta\delta} h^\beta_\nu h^\delta_\lambda = \ & R_{\beta\delta} h^\beta_\nu h^\delta_\lambda - \tfrac{R}{2} h_{\nu\lambda} = {}^{(3)}G_{\nu\lambda} + K K_{\nu\lambda} - a_{\nu|\lambda} - a_\nu a_\lambda + h^\beta_\nu h^\delta_\lambda K_{\beta\delta;\gamma} n^\gamma + \\ & - \frac{1}{2} \left(K^2 + K_{\beta\gamma} K^{\beta\gamma} - 2 a^\delta{}_{;\delta} + 2 K_{,\gamma} n^\gamma \right) h_{\nu\lambda} \ , \end{aligned} \tag{25.39}$$

$$G_{\beta\delta} n^\beta h^\delta_\lambda = R_{\beta\delta} n^\beta h^\delta_\lambda = K^\kappa_{\lambda|\kappa} - K_{|\lambda} \ , \tag{25.40}$$

$$G_{\alpha\gamma} n^\alpha n^\gamma = R_{\alpha\gamma} n^\alpha n^\gamma + \tfrac{R}{2} = \frac{1}{2} \left({}^{(3)}R + K^2 - K_{\nu\gamma} K^{\nu\gamma} \right) . \tag{25.41}$$

The last relation can be inferred best from the equality (25.36), if expressing from it

$$h^{\mu\kappa} h^{\nu\lambda} R_{\alpha\beta\gamma\delta} h^\alpha_\mu h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = 2 \left(R_{\alpha\gamma} n^\alpha n^\gamma + \frac{R}{2} \right) = 2 G_{\alpha\gamma} n^\alpha n^\gamma$$

and employing (25.21) on the left-hand side.

25.5 Decomposition of Einstein equations to constraints and evolution equations

Everything is thus ready for 3+1 decomposition of the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}$. Submitting them in the projections (25.40) and (25.41), we get the **momentum and Hamiltonian constraints**,

$$K^{\kappa}_{\lambda|\kappa} - K_{|\lambda} = 8\pi T_{\beta\delta} n^{\beta} h^{\delta}_{\lambda} \ (\equiv -8\pi j_{\lambda}) \,, \tag{25.42}$$

$$^{(3)}\!R + K^2 - K_{\nu\gamma} K^{\nu\gamma} = 16\pi T_{\alpha\gamma} n^{\alpha} n^{\gamma} + 2\Lambda \ (\equiv 16\pi\rho + 2\Lambda) \,. \tag{25.43}$$

All the quantities on the left-hand sides are bound to Σ_t, without involving any information about their *evolution*, so these are *constraints*, i.e. relations which have to be fulfilled *at a given moment* ($\equiv$ on Σ_t).

Evolution equations can most easily be reached by submitting the "dual" form of the field equations $R_{\beta\delta} = 8\pi \left(T_{\beta\delta} - \frac{1}{2} T g_{\beta\delta}\right) + \Lambda g_{\beta\delta}$ to the "hh-projection" of Ricci (25.30):

$$^{(3)}\!R_{\nu\lambda} + K K_{\nu\lambda} - a_{\nu|\lambda} - a_{\nu} a_{\lambda} + h^{\beta}_{\nu} h^{\delta}_{\lambda} K_{\beta\delta;\gamma} n^{\gamma} = 8\pi T_{\beta\delta} h^{\beta}_{\nu} h^{\delta}_{\lambda} + (\Lambda - 4\pi T) h_{\nu\lambda} \,. \tag{25.44}$$

"Cross-check" of the validity of $^{(3)}\!R_{\nu\lambda} h^{\nu\lambda} = {}^{(3)}\!R$ leads to a condition which is just ensured by equation (25.32), namely (after substitution for $R_{\alpha\gamma}$)

$$a^{\delta}{}_{;\delta} - K_{\nu\gamma} K^{\nu\gamma} - K_{,\gamma} n^{\gamma} = 8\pi\rho + 4\pi T - \Lambda \,. \tag{25.45}$$

Nevertheless, the equations (25.44) describe the change (of the extrinsic curvature) in the direction of n^{γ}, whereas by "evolution" one usually understands the change in the direction of $t^{\mu} \equiv \partial x^{\mu}/\partial t$. In the case of $h_{\mu\nu}$, we solved such a "transformation" by relation (25.13). But $K_{\mu\nu}$ is normal to n^{μ} as well, so it holds

$$t^{\alpha}{}_{;\mu} K_{\alpha\nu} = \left(N n^{\alpha}{}_{;\mu} + N^{\alpha}{}_{;\mu}\right) K_{\alpha\nu}$$

analogously as for $h_{\mu\nu}$, and hence one also ends up with the same relation as (25.13),

$$\pounds_{\mathbf{t}} K_{\nu\lambda} = \ \ldots \ = N \pounds_{\mathbf{n}} K_{\nu\lambda} + \pounds_{\mathbf{N}} K_{\nu\lambda} \,. \tag{25.46}$$

In order to write this quantity in terms of "spatial" quantities only, one substitutes for $\pounds_{\mathbf{n}} K_{\nu\lambda}$ the expression (25.26),

$$\pounds_{\mathbf{t}} K_{\nu\lambda} = N\, h^{\beta}_{\nu} h^{\delta}_{\lambda} K_{\beta\delta;\gamma} n^{\gamma} + 2N\, K_{\nu\gamma} K^{\gamma}_{\lambda} + \pounds_{\mathbf{N}} K_{\nu\lambda} \,,$$

and then plugs here $h^{\beta}_{\nu} h^{\delta}_{\lambda} K_{\beta\delta;\gamma} n^{\gamma}$ expressed from the hh-projection of the field equations (25.44):

$$\begin{aligned} \pounds_{\mathbf{t}} K_{\nu\lambda} = {} & N \left[a_{\nu|\lambda} + a_{\nu} a_{\lambda} - K K_{\nu\lambda} + 2 K_{\nu\gamma} K^{\gamma}_{\lambda} - {}^{(3)}\!R_{\nu\lambda} + 8\pi T_{\beta\delta} h^{\beta}_{\nu} h^{\delta}_{\lambda} + (\Lambda - 4\pi T) h_{\nu\lambda}\right] + \\ & + \pounds_{\mathbf{N}} K_{\nu\lambda} \,. \end{aligned} \tag{25.47}$$

Let us remark that (25.11) implies

$$a_{\mu|\nu} = \left(\frac{N_{|\mu}}{N}\right)_{|\nu} = \frac{N_{|\mu\nu}}{N} - \frac{N_{|\mu} N_{|\nu}}{N^2} = \frac{N_{|\mu\nu}}{N} - a_{\mu} a_{\nu} \,,$$

so one can everywhere abbreviate

$$N(a_{\mu|\nu} + a_{\mu} a_{\nu}) = N_{|\mu\nu} \,. \tag{25.48}$$

25.6 In adapted coordinates...

Although the background picture of the 3+1 splitting naturally is $\Sigma(t) \equiv \{t = \mathrm{const}\}$, we have not yet employed specific coordinates. Let us realize now how the important equations appear in the coordinates (t, x^i) adapted to the decomposition. The normal has components

$$n_\mu = \frac{-t_{,\mu}}{\sqrt{-g^{tt}}} \equiv -Nt_{,\mu} = -N\,\delta^0_\mu \qquad \Longrightarrow \qquad h_{\mu j} = g_{\mu j}\,, \;\; h^\mu_j = \delta^\mu_j\,, \;\; h^0_0 = 0\,, \;\; h^{0\mu} = 0$$

and, according to (25.18) and (25.19), the "spatial" derivative of "spatial" quantities satisfies $V_{j|k} = V_{j;k}$, yet in general $V^{j|k} \neq V^{j;k}$. From the Gauss-Codazzi equations it follows

$$R_{ijkl} = {}^{(3)}R_{ijkl} + K_{ik}K_{jl} - K_{jk}K_{il}\,, \tag{25.49}$$

$$R^0{}_{jkl} = -N^{-1}\left(K_{jk;l} - K_{jl;k}\right) \; = \; -N^{-1}\left(K_{jk|l} - K_{jl|k}\right). \tag{25.50}$$

It also holds $a^0 = 0$, $K^0_\mu = 0$, $K^{0\mu} = 0$, so the remaining projections of the Riemann tensor (25.24) have spatial components

$$R_{\alpha j\gamma l}n^\alpha n^\gamma = a_{j;l} + a_j a_l - K_{jk}K^k_l - K_{jl;\gamma}n^\gamma \tag{25.51}$$

and the Ricci-tensor projections (25.30)–(25.32) appear as

$$R_{jl} = {}^{(3)}R_{jl} + KK_{jl} - a_{j|l} - a_j a_l + K_{jl;\gamma}n^\gamma\,, \tag{25.52}$$

$$R_{\beta l}n^\beta = K^k_{l\,|k} - K_{|l} \;\; (= -R^0_l N)\,, \tag{25.53}$$

$$R_{\alpha\gamma}n^\alpha n^\gamma = a^l{}_{;l} - K_{jk}K^{jk} - K_{,\gamma}n^\gamma \;\; (= R^{00}N^2)\,. \tag{25.54}$$

Other equations from preceding paragraphs do not simplify significantly.

CHAPTER 26

Initial (Cauchy) problem

The space-times of isolated stationary black holes or homogeneous & isotropic cosmologies (Chapters 12–19) are very simplistic – assuming such high symmetries, they can only roughly approximate situations in the real Universe. There exist much more families of exact analytical solutions of Einstein equations, but also these rest on special assumptions. Most of them are even less realistic, if possessing interpretation at all. "Reality" can somewhat be approached by perturbation of known exact solutions. However, perturbation techniques can only render solutions which just slightly differ from the original ones, so their compass is rather limited. Most of the realistic situations thus remains a challenge either for *approximation techniques* such as the post-Newtonian or the post-Minkowskian expansions, the self-force approach or the "effective-one-body" formalism, or for purely *numerical solution of the Einstein equations*. This especially applies to the situations most interesting from the point of view of GR as well as astrophysics, i.e. to highly non-stationary processes involving very strong field, such as collisions of compact objects. In these cases, one can expect a complicated time evolution about which it is difficult to assume anything in advance. The problem is formulated then as **the initial-value problem (Cauchy problem)**: the system is supposed to be known *at a certain instant of time*, and its subsequent evolution is being sought.

In order that a given theory permit to solve evolutions from initial conditions, several features have to be fulfilled which we – vaguely, first – mention below:

- The theory has to permit the formulation of the problem, i.e. stating of the initial conditions which determine further evolution of the system **uniquely**. If the theory involves *constraints*, the initial conditions have to satisfy them. Needless to add, good if the solution of the problem at all **exists** :-).

- The initial-value problem has to be **well posed**, i.e. stable and causal. By **stability** we mean that the evolution must not depend on the initial conditions *sensitively*, at least for a certain early time – a small change of the initial conditions has to only invoke a small change of the system's "trajectory". One then speaks of a **continuous dependence of the solution on initial conditions**.

And by **causality** we mean that a change of the initial conditions within any spatial region must not invoke any change out of the causal future of that region, i.e. out of the union of future light cones of points of that region. (Otherwise the perturbation might propagate with a superluminal speed.)

In GR, initial conditions involve the metric and its first time derivative specified on some space-like hypersurface.[1] If it is not a *vacuum* problem, the initial data has to also be given for the present matter and non-gravitational fields; besides Einstein's equations, it is then necessary to simultaneously solve the coupled equations for the pertinent quantities. (In the following, we will mainly focus on the vacuum case.) The Cauchy problem is a notorious physics problem. In GR it has certain peculiarities:

- Standardly, the Cauchy problem is being solved in a *given* space-time. When, for example, we were discussing the motion of free *test* particles, we had to know the space-time (specifically, the affine connection in that case). The motion was found as the solution of the geodesic equation, from initial conditions including the initial location and three components of four-velocity (the fourth being fixed by normalization). Similarly one can seek the evolution of a test EM field in a *given* space-time by treating the Maxwell equations. When, however, the matter or fields present are *not* test, i.e., when they do themselves affect the space-time geometry, it is necessary to compute, together with their behaviour, the (coupled) evolution of the space-time geometry. Yet in GR *every* kind of energy-momentum represents a source, so an exact, consistent solution (of *every* Cauchy problem in fact) should be obtained solely from the metric and other quantities and their first derivatives only known on a certain hypersurface – the whole future of space-time has to be found, together with the evolution of "non-gravitational" dynamical variables.

- The above brings a number of "philosophical postscripts". The "world" does not at all exist at the beginning of the solution (one only assumes to know a certain "initial", "spatial" Riemannian manifold and its normal behaviour), so the usual picture is not actually at place: it is not clear, a priori, in which space-time the initial manifold should represent a Cauchy hypersurface; it is not clear in which region we seek the solution (because the region only arises by the very evolution of the initial 3D manifold); and it is of course not clear how to represent the initial hypersurface within the searched space-time (lapse, shift) and which coordinates might be convenient...

26.1 Cauchy problem in flat space-time

Actually, in classical mechanics first... Consider p mass points, mutually interacting, whose evolution is described by equations

$$\frac{\mathrm{d}^2\vec{q}_i}{\mathrm{d}t^2} = F_i\left(\vec{q}_1, ..., \vec{q}_p; \dot{\vec{q}}_1, ..., \dot{\vec{q}}_p\right) , \qquad i = 1, ..., p\,.$$

[1] It is also possible to fix the initial conditions on a *null* hypersurface or on a light cone, i.e. on characteristic surfaces of the gravitational equations. In such a case, one speaks of a **characteristic initial-value problem** – see Section 26.2.

The theory of (ordinary) differential equations says that such a system provides a unique solution for any set $\{\vec{q}_1, ..., \vec{q}_p\}_{\rm in}$, $\{\dot{\vec{q}}_1, ..., \dot{\vec{q}}_p\}_{\rm in}$ of initial conditions, at least for some time interval $\Delta t > 0$. However, this interval may be limited, even in very simple cases: consider, for example, just one particle ($p = 1$) in one spatial dimension, described by the equation $\dot{q} = q^\alpha$, with $q > 0$ and $\alpha \neq 1$ (here standing for *power*, not for an upper index). If α is a constant and time is assumed to increase from $t = 0$, one can separate variables as ($C > 0$, upper/lower sign applying for $\alpha < 1\,/\,\alpha > 1$)

$$\frac{dq}{q^\alpha} = {\rm d}t \quad \Longrightarrow \quad \frac{-q^{-(\alpha-1)}}{\alpha-1} = t \pm C \quad \Longrightarrow \quad \frac{1}{q^{\alpha-1}} = -(\alpha-1)(t \pm C)$$

$$\Longrightarrow \quad \underline{\alpha < 1}: \quad q = [(1-\alpha)(t+C)]^{\frac{1}{1-\alpha}}, \qquad \underline{\alpha > 1}: \quad q = \left[\frac{1}{(\alpha-1)(C-t)}\right]^{\frac{1}{\alpha-1}}.$$

For $\alpha < 1$, the solution is finite for any t, whereas for $\alpha > 1$ it diverges for $t = C$.

26.1.1 Maxwell in Minkowski

Maxwell equations include six evolution equations (with curls) for the components of the fields, and two constraints for their divergences. Imagine we know $\vec{E}$ and $\vec{B}$ at some initial instant of time $t = t_0$. They have to satisfy the constraints ${\rm div}\vec{E} = 0$, ${\rm div}\vec{B} = 0$ of course (in a source-free case). If they are really known "everywhere" at the initial time, one can compute all their spatial derivatives, in particular ${\rm rot}\vec{E}$ and ${\rm rot}\vec{B}$. Imagine, further, that one makes the Taylor expansion of the fields in time from the initial instant,

$$\vec{E}(t = t_0 + \delta t, \vec{x}) = \vec{E}(t_0) + \frac{\partial \vec{E}}{\partial t}\Big|_{t_0} \delta t + \frac{1}{2}\frac{\partial^2 \vec{E}}{\partial t^2}\Big|_{t_0} (\delta t)^2 + ..., \qquad \vec{B}(t = t_0 + \delta t, \vec{x}) = \text{likewise}\,.$$

With ${\rm rot}\vec{E}$ and ${\rm rot}\vec{B}$ at $t = t_0$ known, the evolution equations provide

$$\frac{\partial \vec{E}}{\partial t}\Big|_{t_0} = {\rm rot}\vec{B}(t_0)\,, \qquad \frac{\partial \vec{B}}{\partial t}\Big|_{t_0} = -{\rm rot}\vec{E}(t_0)$$

(we assume vacuum and source-free case, $J^\mu = 0$, and set $c = 1$). Second terms of the expansions can be found by time derivative of the Maxwell equations: we calculate ${\rm rot}\frac{\partial \vec{E}}{\partial t}$, ${\rm rot}\frac{\partial \vec{B}}{\partial t}$ at t_0 and thus find

$$\frac{\partial^2 \vec{E}}{\partial t^2}\Big|_{t_0} = {\rm rot}\,\frac{\partial \vec{B}}{\partial t}\Big|_{t_0}, \qquad \frac{\partial \vec{B}}{\partial t}\Big|_{t_0} = -{\rm rot}\frac{\partial \vec{E}}{\partial t}\Big|_{t_0}.$$

The procedure leading to determination of the whole expansion – and thus of the fields' evolution – is clear now.

Note that the constraints are automatically preserved by the evolution equations,

$$({\rm div}\vec{E})_{,t} = {\rm div}(\vec{E}_{,t}) = {\rm div}[{\rm rot}\vec{B}(t_0)] = 0\,, \quad ({\rm div}\vec{B})_{,t} = {\rm div}(\vec{B}_{,t}) = -{\rm div}[{\rm rot}\vec{E}(t_0)] = 0\,.$$

Formal proof of that the above works is based on the existence and uniqueness of Taylor expansion with finite radius of convergence. The only assumption is the **analyticity** of the initial data – the *viability* of their Taylor expansion. (It is a *very strong* assumption in fact: were the world analytic, it would not be possible to get up from bed and go to the lecture...)

26.1.2 Klein-Gordon in Minkowski

The Klein-Gordon equation

$$\Box\psi - m^2\psi = 0 \qquad \Longleftrightarrow \qquad \frac{\partial^2\psi}{\partial t^2} = \vec{\nabla}^2\psi - m^2\psi$$

describes the evolution of a massive (m) scalar field ψ in Minkowski. The initial data to be supplied are ψ and $\psi_{,t}$ at the instant t_0 of some inertial time. The field equation in fact determines $\psi_{,tt}(t_0)$ just from $\psi(t_0)$, *without* the need to also know $\psi_{,t}(t_0)$. However, $\psi_{,t}(t_0)$ has to be provided for the second step of the "Taylor ladder" in a second. (It may even be needed for the first step in the case of more general initial hypersurface, not given as $t_0 = \mathrm{const}$, or in a curved background.) The solution again relies on **analyticity of the initial data**, yet now analyticity in the *spatial* directions (i.e. over the initial hypersurface) is more important. First, if the initial data are analytic there, $\psi_{,tt}(t_0)$ is too, so one can also compute all its spatial derivatives $(\psi_{,tt})_{,i}$. Then, by time derivative of the KG equation, one finds $\psi_{,ttt}(t_0)$ from $\psi_{,t}(t_0)$ and $\vec{\nabla}^2\psi_{,t}(t_0)$, etc... Knowing, finally, all the time derivatives, one can write the Taylor series of $\psi(t_0 + \delta t, \vec{x})$, thus finding ψ after δt at any $\vec{x}$. Therefore, there exist as many analytic solutions of the KG equation as the number of pairs of (arbitrary) analytic functions $\psi(t_0)$, $\psi_{,t}(t_0)$ of the spatial variables $\vec{x}$.

26.1.3 The Cauchy-Kowalevski theorem

Let $(t, x^1, ..., x^m) =: (t, \vec{x})$ be coordinates in $\mathbb{R}^{1+m}$. Let a system of linear partial differential equations for n unknown functions ϕ_i $(i = 1, ..., n)$ defined on $\mathbb{R}^{1+m}$ have the form

$$\frac{\partial^2\phi_i}{\partial t^2} = F_i\left(t, \vec{x}; \phi_j, \frac{\partial\phi_j}{\partial t}, \frac{\partial\phi_j}{\partial x^a}, \frac{\partial^2\phi_j}{\partial t\partial x^a}, \frac{\partial^2\phi_j}{\partial x^a\partial x^b}\right), \tag{26.1}$$

where all the functions F_i are analytic in all variables. Let $f_i(\vec{x})$, $g_i(\vec{x})$ be analytic, too. Then there is a certain open neighbourhood of the hypersurface $t = t_0$ such that there exists a unique analytic solution $\{\phi_i(t)\}$, such that ϕ_i assume on $t = t_0$ the initial values $f_i(\vec{x})$ and $g_i(\vec{x})$,

$$\phi_i(t_0, \vec{x}) = f_i(\vec{x}), \qquad \frac{\partial\phi_i}{\partial t}(t_0, \vec{x}) = g_i(\vec{x}).$$

Instead of a proof, let us quote a nice, didactic Prague story from Wikipedia:

In August 1833 Cauchy left Turin for Prague to become the science tutor of the thirteen-year-old Duke of Bordeaux, Henri d'Artois (1820–1883), the exiled Crown Prince and grandson of Charles X. As a professor of the École Polytechnique, Cauchy had been a notoriously bad lecturer, assuming levels of understanding that only a few of his best students could reach, and cramming his allotted time with too much material. Henri d'Artois had neither taste nor talent for either mathematics or science. Although Cauchy took his mission very seriously, he did this with great clumsiness, and with surprising lack of authority over Henri d'Artois. [...] Cauchy's role as tutor lasted until Henri d'Artois became eighteen years old, in September 1838. Cauchy did hardly any research during those five years, while Henri d'Artois acquired a lifelong dislike of mathematics.

The theorem by Cauchy and Kowalevskaya (1842 and 1874) however does not address the **well-posedness** of the problem. Actually, in order to say something on **stability** of the solution, one would have to know how to assess "distance" between different sets of initial conditions and between the respective different solutions. This requires to know/introduce a certain norm in the space of functions acting on $\Sigma_0 \equiv \{t = t_0\}$, such as, for example (for a 3D space, with x, y, z),

$$\| f_1 - f_2 \|_{\Sigma_0}^{k \geqslant 0} := \sum_{k_1, k_2, k_3 \geqslant 0} \sup_{x \in \Sigma_0} \left| \frac{\partial^{k_1+k_2+k_3}(f_1 - f_2)}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}} \right|_{k_1+k_2+k_3 \leqslant k},$$

where "sup" stands for supremum (the least upper bound).

The theorem also does not address the **causality** of propagation of the fields. Actually, it even *cannot* do so, because it considers *analytic* functions. Namely, analytic functions are *uniquely* given by their value and all derivatives in an infinitesimal neighbourhood of a certain point, so to change them at one point necessarily means to change them *everywhere on* Σ_0 (and consequently within the whole solution region). In order to judge causality, one has to be able to follow the propagation of a certain "feature", i.e. of some non-analyticity in the initial data. For example, one may consider δ-function data, "rectangle" (compact-support) data, or – the "nicest" non-analytical option – the "bump"-type data (given by C^∞ yet non-analytical functions) for such a study.

26.1.4 Well posedness from energy estimates

In the modern treatments of stability of particular GR space-times (Minkowski, Kerr, etc.) against small perturbations, "energy estimates" are often applied, which we will try to at least shortly taste now. Consider, as an example, a scalar field in Minkowski. Minkowski space-time possesses 10 Killing vector fields. Let the initial and final hypersurfaces Σ_0 and Σ be chosen orthogonal to the time-translation symmetry $t^\mu = \partial x^\mu / \partial t$. We know from (11.29) that in GR $(T^{\alpha\beta}\xi_\alpha)_{;\beta} = 0$ for *any* Killing field ξ^α, hence

$$(T^{\alpha\beta}t_\alpha)_{;\beta} = 0 \qquad \text{... (in Minkowski coordinates:)} \;\; (T^{\alpha\beta}t_\alpha)_{,\beta} = 0\ .$$

Aiming at the Gauss theorem, let us fix the integration domain precisely. Denote

Σ_0 ... space-like hypersurface on which initial conditions are assigned $(t = t_0)$
Σ ... space-like hypersurface "shortly after Σ_0" $(t > t_0)$
S_0 ... 3D ball within Σ_0
$D^+(S_0)$ ... future domain of dependence of S_0
... $:= \{q \in \mathbb{M}$: every past causal curve from q intersects $S_0\}$
S ... 3D ball within Σ, given by $S = D^+(S_0) \cap \Sigma$
$J^-(p)$ ... causal past of point $p \in \mathbb{M}$
... $:= \{q \in \mathbb{M}$: there exists a causal curve connecting q with $p\}$
$J^-(S)$ ... causal past of S, $J^-(S) = \bigcup_{p \in S} J^-(p)$
Ω ... $\Omega := D^+(S_0) \cap J^-(\Sigma)$ ... our integration domain

H ... light-like "side" boundary of Ω

ℓ^α ... future-directed normal to H (i.e. "generator of H": it is also *tangent* to H) .

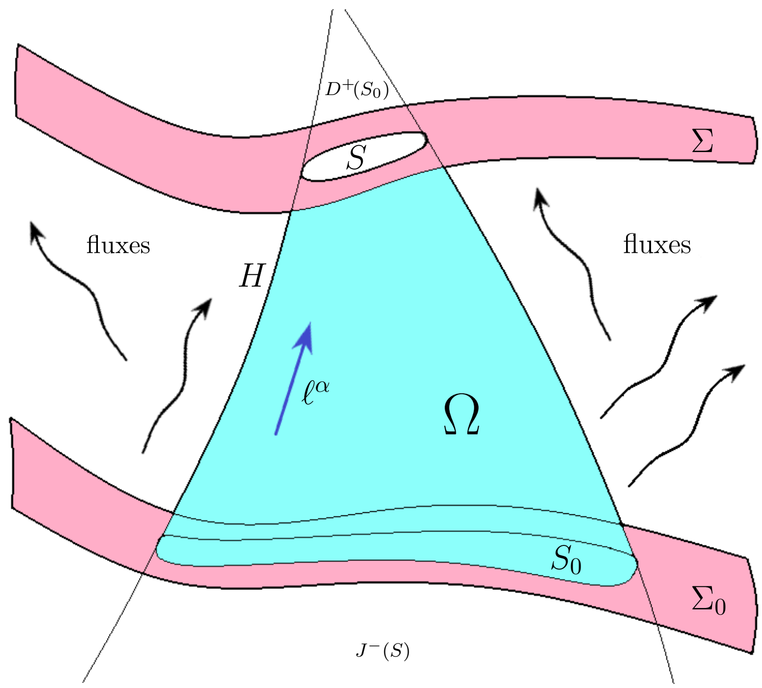

Figure 26.1 *Illustration of the above-defined regions. The integration domain Ω is defined as $D^+(S_0) \cap J^-(\Sigma)$, because $D^+(S_0)$ alone continues above Σ and $J^-(\Sigma)$ continues below Σ_0. Energy (and other quantities) may outflow from Ω across H, but cannot inflow across there, because H is a shrinking light-cone.*

By Gauss theorem, we now express the integral of $(T_{\alpha\beta}t^\alpha)_{,\beta}$ over Ω in terms of the integrals of the respective fluxes across the boundaries of Ω (bear in mind that t^α is unit in Minkowski, $\eta_{\alpha\beta}t^\alpha t^\beta = \eta_{tt} = -1$),

$$0 = \int_\Omega (T_{\alpha\beta}t^\alpha)_{,\beta}\,\mathrm{d}\Omega = \int_S T_{\alpha\beta}t^\alpha t^\beta\,\mathrm{d}V + \int_H T_{\alpha\beta}t^\alpha \ell^\beta\,\mathrm{d}\omega - \int_{S_0} T_{\alpha\beta}t^\alpha t^\beta\,\mathrm{d}V\,, \tag{26.2}$$

where the opposite sign of the last term reflects that t^μ is the *inward* normal to S_0 (whereas the outward one to S), and $\mathrm{d}\omega$ is the element on the null sides of Ω (called H).

Invariant 2-content on a null cone

The element $\mathrm{d}\omega$ needs more introduction. At some point $x^\mu = a^\mu$, consider a light cone as the limit of the hyperboloids representing constant time-like interval measured from a^μ,

$$\eta_{\rho\sigma}(x^\rho - a^\rho)(x^\sigma - a^\sigma) = -T^2$$

(the light cone is obtained for $T \to 0$). Let t^μ be a unit future-pointing time-like vector at $x^\mu = a^\mu$ (necessarily orthogonal to the constant-time hyperboloids), and let n^μ be an arbitrary other vector of that type, not parallel to t^μ; the two may be considered four-velocities of two different observers crossing the given point $x^\mu = a^\mu$. The trajectories of both t^μ and n^μ are orthogonal to all the hyperboloidal time contours (similarly as all radial straight lines outgoing from some point are orthogonal to all the spheres drawn about that point in $\mathbb{E}^3$). Take an infinitesimal element $\mathrm{d}S_{\mathrm{H}}$ on arbitrary nearby hyperboloid, at the point where n^μ intersects the latter. This element is orthogonal to n^μ, but *not* to t^μ. However, it may be *projected* on any of the planes orthogonal to t^μ easily, because the angle between the element $\mathrm{d}S_{\mathrm{H}}$ (orthogonal to n^μ) and the local plane orthogonal to t^μ is the same as the angle between n^μ and t^μ themselves, i.e. $n_\mu t^\mu$: the $\perp t^\mu$ -projection of the element $\mathrm{d}S_{\mathrm{H}}$ we denote by

$$\mathrm{d}S_\Sigma := |t_\mu n^\mu| \, \mathrm{d}S_{\mathrm{H}} \, .$$

With a^μ and T chosen, one has a specific hyperboloid, and with n^μ chosen one also has the point where the latter intersects the chosen hyperboloid – that is x^μ in the above equation. Since n^μ is unit, it holds $Tn^\mu = x^\mu - a^\mu$, hence the projected element can be written

$$\mathrm{d}S_\Sigma := \frac{|t_\mu(x^\mu - a^\mu)|}{T} \, \mathrm{d}S_{\mathrm{H}} \, .$$

Now the announced limit $T \to 0$ is performed and the 2D element $\mathrm{d}S_{\mathrm{H}}/T$ obtained in that limit is denoted by $\mathrm{d}\omega$,

$$\mathrm{d}\omega := \lim_{T\to 0} \frac{\mathrm{d}S_{\mathrm{H}}}{T} = \frac{\mathrm{d}S_\Sigma}{|t_\mu(x^\mu - a^\mu)|} \, . \tag{26.3}$$

The result is in fact independent of the choice of n^μ (as it should have been expected). It is being called the *invariant 2-content on a null cone (or on a null hypersurface in general).*

Dominant energy condition

In Section 7.7, the main *energy conditions* for $T_{\mu\nu}$ were listed, requiring/claiming/ensuring that "gravity is attractive and causal" in different ways. In the initial problem discussion, it is natural to demand the dominant energy condition which states that for any future-directed time-like vector (for any "observer") $\hat{u}^\mu$, the momentum density (energy-density flow) $-T^\alpha{}_\beta \hat{u}^\beta$ is a future-directed time-like or null vector,

$$g_{\alpha\beta} T^\alpha{}_\mu \hat{u}^\mu T^\beta{}_\nu \hat{u}^\nu \leqslant 0 \, ,$$

i.e. that the energy cannot propagate faster than light. On illustrations provided in Section 7.7 we also saw that from the dominant energy condition there also follows the *weak energy condition* stating that every observer $\hat{u}^\mu$ has to measure a non-negative energy density, $T_{\alpha\beta}\hat{u}^\alpha \hat{u}^\beta \geqslant 0$.

Now, if $-T^\alpha{}_\beta t^\beta$ is a future-directed time-like or null vector, then for ℓ^μ a future-directed null vector their scalar product must satisfy

$$\ell_\alpha(-T^\alpha{}_\beta t^\beta) \leqslant 0 \qquad \Longrightarrow \qquad T_{\alpha\beta}\ell^\alpha t^\beta \geqslant 0 \, .$$

Back to equation (26.2), finally

The last inequality means that the null-boundary term $\int_H T_{\alpha\beta}t^\alpha\ell^\beta\,\mathrm{d}\omega$ in equation (26.2) is certainly non-negative, hence the equation implies

$$\int\limits_S T_{\alpha\beta}t^\alpha t^\beta\,\mathrm{d}V - \int\limits_{S_0} T_{\alpha\beta}t^\alpha t^\beta\,\mathrm{d}V \leqslant 0\,,$$

where it has become obvious why such bounds are being called “energy estimates”.

Let us illustrate the importance of this result on scalar field, with

$$T_{\alpha\beta} = \psi_{,\alpha}\psi_{,\beta} - \frac{1}{2}\,\eta_{\alpha\beta}(\psi_{,\gamma}\psi^{,\gamma} + m^2\psi^2)\,. \tag{26.4}$$

In adapted coordinates in which $t^\mu = \delta^\mu_t$, we have

$$\text{integrand} = (\psi_{,t})^2 - \frac{1}{2}\,\eta_{tt}\left[\eta^{tt}(\psi_{,t})^2 + \psi_{,i}\psi^{,i} + m^2\psi^2\right] = \frac{1}{2}\left[(\psi_{,t})^2 + \psi_{,i}\psi^{,i} + m^2\psi^2\right],$$

so the inequality assumes the form

$$\int\limits_S \left[(\psi_{,t})^2 + \psi_{,i}\psi^{,i} + m^2\psi^2\right]\mathrm{d}V \leqslant \int\limits_{S_0} \left[(\psi_{,t})^2 + \psi_{,i}\psi^{,i} + m^2\psi^2\right]\mathrm{d}V\,. \tag{26.5}$$

Interpretation of the result is clear: energy contained in S is *smaller than or equal to* the energy contained in S_0. It is due to the “leakage of energy” across the null surface H (which has been shown to be positive). It is perfectly intuitive, because a region whose sides are null and “shrinking” (with the speed of light) clearly cannot be entered by anything propagating with at most the same speed. This ensures, at the same time, causality of the initial-value problem, because $D^+(S_0)$ cannot be influenced by anything outside S_0.

Uniqueness of the solution

Unique determination of the solution by the initial data can be proved by contradiction: assume ψ_1 and ψ_2 are two solutions (supposed to be C^2 at least, in order that the derivatives involved in the Klein-Gordon equation exist), both arising from the *same* initial conditions on S_0 (or possibly the whole Σ_0). Since the Klein-Gordon equation is linear, the difference $\Delta\psi := \psi_2 - \psi_1$ also satisfies it. But $\Delta\psi = 0$ on S_0 (Σ_0), so the initial-value integral in (26.5) is zero. The inequality (26.5) thus implies

$$\int\limits_S \left[(\Delta\psi_{,t})^2 + \Delta\psi_{,i}\Delta\psi^{,i} + m^2\Delta\psi^2\right]\mathrm{d}V \leqslant 0\,,$$

which can only be satisfied if the integral vanishes, because the integrand cannot be negative. And that can only be satisfied with $\Delta\psi$ also vanishing on any future hypersurface. (For $m \neq 0$ it is immediate. With $m = 0$, one has that $\Delta\psi_{,t} = 0$ and $\Delta\psi_{,i} = 0$ in $D^+(S_0)$, which, together with $\Delta\psi$ being zero on S_0, implies that $\Delta\psi$ has to vanish in $D^+(S_0)$.)

Continuous dependence on the initial data

Proof of the continuous dependence on initial conditions is more involved, yet let us at least give a glimpse. First, take the KG equation

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - m^2\psi$$

and differentiate it by x^μ. The equation is linear, and the differentiation is linear, so *all the derivatives of ψ satisfy the same KG equation as well*. Therefore, by the same procedure as above, one obtains for them similar "energy" inequalities as (26.5). In these inequalities, one can express – by the KG equation – all terms computed on S_0 with more than one time derivative in terms of the initial data ψ and $\psi_{,t}$ and their spatial derivatives along Σ_0. Putting all these terms (integrals over S_0) to the r.h. sides, their sum can schematically be written as

$$C^k \,\|\, \psi \,\|^k_{S_0} + C^k_t \,\|\, \psi_{,t} \,\|^{k-1}_{S_0} \,, \quad \text{where} \quad \|\, \psi \,\|^k_{S_0} := \left[\int_{S_0}\left(|\psi|^2 + ... + \sum_{\sum k_i \leqslant k} |\partial^{k_i}_{x^j}\psi|^2\right)\right]^{\frac{1}{2}} ,$$

where C^k and C^k_t are some constants (depending on the maximal differentiation order k) and the *Sobolev norm* of the k-th order $\|\, \psi \,\|^k_{S_0}$ only includes *spatial* derivatives $\partial^{k_i}_{x^j}\psi$ and $\psi_{,t}$ on the initial hypersurface. On the left-hand sides of the inequalities of the (26.5) type, one *cannot* express higher time derivatives of ψ in terms of the spatial-derivative terms (without having a particular solution at disposal), so the sum of the left-hand sides schematically appears as

$$\|\, \psi \,\|^k_S \; := \left[\int_S \left(|\psi|^2 + ... + \sum_{\sum k_i \leqslant k} |\partial^{k_i}_{x^\mu}\psi|^2\right)\right]^{\frac{1}{2}} ,$$

where now higher (than first) derivatives by t *are* present, so the partial derivatives $\partial^{k_i}_{x^\mu}$ do include those with respect to x^j *as well as* those with respect to t (as well as all kind of mixed ones). The "energy inequality" (26.5) thus assumes the form

$$\|\, \psi \,\|^k_S \; \leqslant \; C^k \,\|\, \psi \,\|^k_{S_0} + C^k_t \,\|\, \psi_{,t} \,\|^{k-1}_{S_0} \; .$$

Last point is to employ the following result, valid for domains in $\mathbb{R}^d$ satisfying the *uniform interior cone condition*, i.e. such that at every their point, including the boundary ones, a cone of certain (uniformly prescribed) height and vertex angle can be drawn *entirely lying within the given domain*.[2] The property is that for $k > d/2$ the $\|\, \psi \,\|$ norm bounds the numerical values which $|\psi|$ reaches in the domain. Connecting the results for $D^+(S_0)$ as the

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[2] Wald [50] states the result on the basis of this concept. It has in fact been shown (later) that it is equivalent to the requirement that the domain be a *Lipschitz domain*, i.e. a domain whose boundary is sufficiently regular (without too sharp spikes); more precisely, it has to be possible to represent the boundary in terms of a Lipschitz-continuous function (= function "with a limited slope").

domain and for $k = 3$ as the least k for which the property works (in the $d = 4$ domain), we have

$$\sup_{x \in D^+(S_0)} |\psi| \leqslant \| \psi \|_S^3 \leqslant C^3 \| \psi \|_{S_0}^3 + C_t^3 \| \psi_{,t} \|_{S_0}^2 .$$

Thanks to the KG equation, similar "bounds by initial data" can be claimed for the derivatives of ψ, so one confirms the *continuous dependence of ψ and its derivatives on initial conditions*.

Continuity is also the basis to prove the *existence of smooth solution for arbitrary smooth initial data*, which we however omit.

26.2 Generalizations

26.2.1 Characteristic initial-value problem

The initial-value problem is more problematic for hypersurfaces which are *characteristics* of the given field equations, and also for normals to these called *bicharacteristics*. Generally, the problem then has either no solution, or infinite number of solutions.

As a simple example, consider the (1+1)-dimensional wave equation $\psi_{,tt} - \psi_{,xx} = 0$. Characteristics correspond to the speed of propagation (involved in the equation), which has been chosen "unity" here ($c = 1$), so they are given by $t = \pm x + \mathrm{const}$. Let $\psi = 0$ and $\psi_{,t} = 0$ on some "initial" characteristic (e.g. $t = x$). Clearly $\psi = a(t-x)^2$ is a possible solution which satisfies such initial conditions. It is not unique, since the constant a is totally free.

Same equation, different example: in null coordinates $u = t - x$, $v = t + x$ (retarded and advanced time), the equation assumes the form $\psi_{,uv} = 0$. Indeed,

$$\begin{aligned}
&\psi_{,t} = \psi_{,u} u_{,t} + \psi_{,v} v_{,t} = \psi_{,u} + \psi_{,v} , \qquad \psi_{,x} = \psi_{,u} u_{,x} + \psi_{,v} v_{,x} = -\psi_{,u} + \psi_{,v} , \\
&\psi_{,tt} = \psi_{,uu} u_{,t} + \psi_{,uv} v_{,t} + \psi_{,vu} u_{,t} + \psi_{,vv} v_{,t} = \psi_{,uu} + \psi_{,uv} + \psi_{,vu} + \psi_{,vv} , \\
&\psi_{,xx} = -\psi_{,uu} u_{,x} - \psi_{,uv} v_{,x} + \psi_{,vu} u_{,x} + \psi_{,vv} v_{,x} = \psi_{,uu} - \psi_{,uv} - \psi_{,vu} + \psi_{,vv} \\
&\Longrightarrow \quad \psi_{,tt} - \psi_{,xx} = 4\psi_{,uv} .
\end{aligned}$$

Let the initial conditions be fixed on the $u = 0$ characteristic, $\psi(u = 0, v) = \psi_0(v)$, $\psi_{,u^+}(u = 0, v) = \psi_1(v)$. Now, in order that $\psi_{,uv} = 0$ be satisfied at $u = 0$, we have to satisfy there

$$(\psi_{,u^+})_{,v} = (\psi_1)_{,v} = 0 ,$$

which however implies $\psi_1(v) = \mathrm{const}$. This means the problem in general has no solution, because we are *not* allowed to choose the initial condition $\psi_1(v)$ freely. On the other hand, if we do choose $\psi_{,u^+}(u = 0, v) = \psi_1(v) = \mathrm{const} =: A$ (and $\psi_{,uv} = 0$ *is* thus satisfied at $u = 0$), one can integrate it to

$$\psi(u, v) = \psi_0(v) + Au + \chi(u) ,$$

where $\chi(u)$ is (only) constrained by the conditions $\chi(u = 0) = 0$ and $\chi_{,u}(u = 0) = 0$ required for the initial conditions to hold. Therefore, in this very special case when $\psi_{,u^+}(u = 0, v) = \mathrm{const}$, the solution contains *arbitrary function* $\chi(u)$ (satisfying the above initial behaviour), so it is far from being unique.

26.2.2 Cauchy-Kowalevski theorem for higher-derivative equations

The Cauchy-Kowalevski theorem has also been generalized to equations of higher derivative order. Let us represent such, as a generalization of (26.1), in the form (for just one "field" ϕ)

$$\frac{\partial^n \phi}{\partial t^n} = F\left(t, \vec{x}; \phi, ..., \frac{\partial^k \phi}{\partial t^{k_0} \partial x_1^{k_1} ... \partial x_m^{k_m}}, ...\right), \quad \text{where} \quad k_0 + k_1 + ... + k_m \leqslant n \ \text{ and } k_0 < n\,,$$

with the initial conditions

$$\phi(t=t_0, \vec{x}) = F_0(\vec{x})\,, \quad \phi_{,t}(t=t_0, \vec{x}) = F_1(\vec{x})\,, \quad ..., \quad \frac{\partial^{n-1}\phi}{\partial t^{n-1}}(t=t_0, \vec{x}) = F_{n-1}(\vec{x})\,.$$

The statement: if F on the right-hand side is analytic in all its variables in the neighbourhood of the initial state (at $t = t_0$), and all the initial-value functions F_k are also analytic in the neighbourhood of $\vec{x}(t = t_0)$, then there exists a neighbourhood of $(t_0, \vec{x}(t_0))$ (of the initial hypersurface) in which the solution exists, and it is analytic and unique.

26.2.3 Curved initial hypersurface

The initial (space-like) hypersurface from which the evolution of a given system starts may of course be different from flat (from a hyperplane). Specifically in GR, it is impossible typically to even consider such a case. However, if the initial hypersurface is sufficiently regular (which is standardly being assumed), then it is possible – at least locally – to transform to adapted coordinates in which the hypersurface corresponds to $t = \mathrm{const}$. Such a *generalized Cauchy problem* is relevant in GR.

26.2.4 Generalized Klein-Gordon in generic background

Let us consider a *generic* (i.e. non-flat) 4D space-time now, and on it a test scalar field described by the *generalized Klein-Gordon equation*

$$\Box\psi + A^\mu \psi_{;\mu} + B\psi + C = 0\,, \qquad \Box\psi := g^{\mu\nu}\psi_{;\mu\nu}\,,$$

where $g^{\mu\nu}$ is a *given* smooth Lorentzian metric and $A^\mu(x)$, $B(x)$ and $C(x)$ are some smooth functions of x^μ. It has been proved that, provided the background described by $g^{\mu\nu}$ is *globally hyperbolic* (see next section), the equation admits a well-posed initial-value problem based on giving, on a certain initial Cauchy hypersurface Σ_0, the initial functions $\psi|_{\Sigma_0}$ and $n^\alpha \psi_{;\alpha}|_{\Sigma_0}$ (with n^α the future-pointing normal to Σ_0).

The above has been shown to also hold in a still more general case of a *system* of n equations for n functions,

$$\Box\psi_{(i)} + \sum_{j=1}^{n} A^\mu_{(i)(j)} \nabla_\mu \psi_{(j)} + \sum_{j=1}^{n} B_{(i)(j)}\psi_{(j)} + C_{(i)} = 0\,, \qquad i = 1, ..., n\,,$$

where the requirements on functions involved are the same, and the initial conditions are fixed similarly as above.

However, although the equations mentioned in this section are more general than the problems considered before, they are still **linear** (unknowns $\psi_{(i)}$ and their derivatives only appear in the first power). With the Einstein equations, one has to consider a system of **non-linear** partial differential equations.

26.3 On Cauchy problem in general relativity

Einstein equations represent a quasi-linear system of partial differential equations (the highest derivatives of unknowns are linear, but not the lower ones). In order to reproduce and understand crucial theorems and results of this area, we first give definitions of several concepts from the theory of **causal structure** of a manifold (M).

- **Chronological future of a subset** Ω of $\mathcal{M}$ is the set $I^+(\Omega)$ of events that can be reached by some (at least one) future-directed time-like curve from some (at least one) point of Ω. It is a "future light cone of Ω" without its lateral surface (thus it is open).
- **Causal future of a subset** Ω of $\mathcal{M}$ is the set $J^+(\Omega)$ of events that can be reached by some (at least one) future-directed causal (time-like or light-like) curve from some (at least one) point of Ω. It is a "future light cone of Ω" including its lateral surface (thus it is closed, if there are no missing points etc.).
- **Achronal set** is such a subset Ω of $\mathcal{M}$ whose no points (events) can be connected by a time-like curve, i.e. which is completely disjunct from its chronological future, $I^+(\Omega) \cap \Omega = \emptyset$. Generally, this is the property of **space-like** or **light-like hypersurfaces**.
- **Acausal set** is such a subset Ω of $\mathcal{M}$ whose no points (events) can be connected by a causal curve, i.e. which is completely disjunct from its causal future, $J^+(\Omega) \cap \Omega = \emptyset$. Generally, this is the property of **space-like hypersurfaces**.
- **Past/future inextendible curve** is such a curve in $\mathcal{M}$ which has no past/future endpoint in $\mathcal{M}$. For an inextendible *causal* curve, there are three possibilities basically: i) the curve runs to infinity, ii) it is finally trapped in a certain finite region where it circulates for ever, iii) it hits a space-time singularity.
- **Future domain of dependence** of a subset Ω of $\mathcal{M}$ is the set $D^+(\Omega)$ of all points such that every past-oriented inextendible causal curve from them intersects Ω. It is the set of all events whose *whole* past light cones intersect Ω.
- **Past domain of dependence** of a subset Ω of $\mathcal{M}$ is the set $D^-(\Omega)$ of all points such that every future-oriented inextendible causal curve from them intersects Ω. It is the set of all events whose *whole* future light cones intersect Ω.
- **Cauchy hypersurface** Σ is such a hypersurface in $\mathcal{M}$ which is acausal and whose $D^+(\Sigma)$ and $D^-(\Sigma)$ together cover the whole $\mathcal{M}$, i.e. for which $D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}$. In other words, a Cauchy hypersurface *depends on everything what happened before, and influences everything that is to its future*, it cannot be bypassed from either direction. Since it is acausal, it is clear that every inextendible causal curve intersects it *exactly once*.

- **Globally hyperbolic space-time** is a space-time in which there exists a (global) Cauchy hypersurface. The point is that in a globally hyperbolic space-time, the entire future / the entire past of the universe can be predicted / retrodicted uniquely from conditions on the Cauchy hypersurface.

  Remark: We suggested three options for inextendible curves. In a globally hyperbolic space-time, it is clear that only the first can apply (the curve runs to infinity), because all causal curves have to intersect a certain (Cauchy) hypersurface. If some of them were trapped in a finite region or ended in a singularity, they might not satisfy that requirement. -Very intuitive it is in the case of a *naked* singularity (imagine it would occur in future of the "initial" hypersurface): its "future light cone" has *Cauchy horizon* as its boundary, as we e.g. know from the Reissner-Nordström space-time.

  On the other hand, if there is a black hole, it in general "eats information out of the space-time" (due to its "no-hair" properties), so it is not possible to *retrodict* then, from any "later" hypersurface, a complete information about what happened in the past. Yet it may still be possible to make unique *predictions* in the domain of outer communications:

- An *asymptotically flat* space-time is said to be **strongly asymptotically predictable** if the closure of the causal past of its future null infinity, $\overline{M \cap J^{-}(\mathscr{I}^{+})}$, is globally hyperbolic. (More precisely, the latter should be covered by some globally hyperbolic open region within the conformally extended manifold.) In such a space-time, the complement of $M \cap J^{-}(\mathscr{I}^{+})$ is called the **black hole** and its boundary is called the **(future) event horizon**.

Einstein equations may be written in many ways. When deriving the metric for stationary space-times, one usually chooses the parameter of that symmetry directly as the time coordinate ($t^{\mu} = \delta^{\mu}_{t}$), which means that the metric is independent of (that) time t. The field equations then lead to an *elliptic system*, describing spatial behaviour rather than time evolution. However, even stationary space-time can be understood as an *evolution* problem, if it is described in different coordinates (see e.g. how we discussed the *dynamics* of Schwarzschild in Kruskal coordinates). If studying the Cauchy problem, we naturally suppose to have brought the Einstein equations into such evolution-type, *hyperbolic system* of the following form:

$$g^{\alpha\beta}(x;\phi_{(j)},\nabla_{\gamma}\phi_{(j)})\,\nabla_{\alpha}\nabla_{\beta}\phi_{(i)} = F_{(i)}(x;\phi_{(j)},\nabla_{\gamma}\phi_{(j)})\,, \qquad i,j = 1,...,n\,, \tag{26.6}$$

where $g^{\alpha\beta}$ is a smooth Lorentzian matrix (Lorentzian $\equiv$ with "$-+++$" eigen-values, in order to ensure hyperbolicity of the system) which depends on position, but also on the unknown fields, yet only up to their first derivatives.[3] In the case of the (vacuum) Einstein equations, there is just one field ϕ – the metric $g_{\mu\nu}$. Yet still the left-hand side is far from trivial, because it may be non-linear in the first derivatives, and also the dependence of $g^{\alpha\beta}$ on $g_{\mu\nu}$ is itself non-linear ($g^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu}$, or realize how they are related through determinant). Notice that the highest (the second) derivatives of $\phi_{(i)}$ only appear linearly, so the system is *quasi-linear*, which is crucial in the results below.

[3] The derivative need not be the covariant one. Thinking of the GR case ($\phi_{(j)} \to g_{\mu\nu}$), it would actually be better to write partial derivatives, of course...

26.3.1 The Leray theorem

The fundamental theorem is from 1952, from the *unpublished notes* called *Hyperbolic differential equations*, written at Princeton Institute of Advanced Study by a French mathematician Jean Leray:

Be $\left\{\mathcal{M}, g^{\alpha\beta}(x;\phi_{(j)},\nabla_\gamma\phi_{(j)})\right\}_0$ (*some*) globally hyperbolic manifold, where $\left\{\phi_{(1)},...,\phi_{(n)}\right\}_0$ solve the system (26.6). Let Σ denote a smooth Cauchy hypersurface in $\left\{\mathcal{M}, g^{\alpha\beta}\right\}_0$, with some "initial values" on it, corresponding to the given solution. THEN the initial-value problem for the system (26.6) is well posed in the following sense: for every initial data sufficiently close to the data for the above reference solution, there exists a neighbourhood $\mathcal{O}$ of Σ (i.e. a certain "future of Σ") such that (26.6) has a solution $\{\phi_{(1)},...,\phi_{(n)}\}$ in $\mathcal{O}$, with the space-time region $\left\{\mathcal{O}, g^{\alpha\beta}(x;\phi_{(j)},\nabla_\gamma\phi_{(j)})\right\}$ evolved from perturbed data globally hyperbolic. Moreover, i) the solution is unique in $\mathcal{O}$; ii) it is also causal, namely, if some other data agree with those for the reference solution on some subset S of Σ, then the respective two solutions agree in $\mathcal{O}\cap D^+(S)$; and iii) the solution depends continuously on the initial data.

Remarks: $\{\mathcal{M}, g^{\alpha\beta}\}_0$ is really "*some* manifold": it satisfies the given equations automatically, serving as a unique reference solution. From a *perturbed* initial data, a piece of a *different* manifold is obtained. And, important again: "sufficient closeness" can only be assessed if an appropriate norm is known in the pertinent functional space, such as the Sobolev-type norm mentioned earlier in this chapter.

Intuitive grasp of a proof:
Assume we have a certain reference solution $\left\{\phi_{(1)},...,\phi_{(n)}\right\}_0$ and a certain Cauchy hypersurface on which the solution "records" certain initial conditions. We substitute the solution to $g^{\alpha\beta}$ on the left-hand side and to $F_{(i)}$ on the right-hand side of (26.6), and we solve the system, yet with *perturbed* initial conditions. Mathematically, we thus solve, in this first step, the *linear* problem (because all zeroth and first derivatives are known)

$$\left[\text{known functions } g^{\alpha\beta}(\ldots)\right] * \nabla_\alpha\nabla_\beta\phi_{(i)} = \left[\text{known functions } F_{(i)}(\ldots)\right] .$$

For a *linear* system, the Cauchy-Kowalevski theorem(s) ensure the solution exists, uniquely. One thus obtains the "first iteration" of $\phi_{(i)}$, symbolically $\left\{\phi_{(i)}\right\}_1 = \left\{\phi_{(i)}\right\}_0 + \delta\left\{\phi_{(i)}\right\}$. The second step is to substitute the first iteration $\left\{\phi_{(i)}\right\}_1$ back to $g^{\alpha\beta}$ and $F_{(i)}$ of (26.6), and solve the system – again a *linear one* – for $\phi_{(i)}$ again (with the perturbed initial conditions already kept from the first step, of course), to obtain the "second iteration" $\left\{\phi_{(i)}\right\}_2$. In such a way, one obtains a sequence of approximations, about which Leray showed that for a data sufficiently close to the original ones they converge to a solution of the full, non-linear system (26.6).

26.3.2 Einstein equations as a hyperbolic system

It is by no means automatic that the Einstein equations could be put into the form (26.6). And, as already mentioned, it clearly depends on the coordinates chosen. The first task will be to tackle these issues.

The "zeroth" task actually is to extract, from the left-hand side of Einstein's equations, the "genuine curvature" part = the one depending on the highest derivatives of the metric.

Recalling the fully covariant Riemann tensor (6.8),

$$R_{\mu\beta\kappa\delta} = \frac{1}{2}\left(g_{\mu\delta,\beta\kappa} + g_{\beta\kappa,\mu\delta} - g_{\mu\kappa,\beta\delta} - g_{\beta\delta,\mu\kappa}\right) + \left[\Gamma^2 \text{ terms}\right] ,$$

we obtain, from the second-derivative ("genuine curvature") terms solely,

$$\begin{aligned} R_{\nu\lambda} - \frac{1}{2}Rg_{\nu\lambda} &= \frac{1}{2}\,g^{\mu\kappa}\left(\delta^{\beta}_{\nu}\delta^{\delta}_{\lambda} - \frac{1}{2}\,g^{\beta\delta}g_{\nu\lambda}\right)\left(g_{\mu\delta,\beta\kappa} + g_{\beta\kappa,\mu\delta} - g_{\mu\kappa,\beta\delta} - g_{\beta\delta,\mu\kappa}\right) = \\ &= \frac{1}{2}\,g^{\mu\kappa}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) - \frac{1}{2}\,g_{\nu\lambda}g^{\mu\kappa}g^{\beta\delta}\left(g_{\mu\delta,\beta\kappa} - g_{\mu\kappa,\beta\delta}\right) . \end{aligned}$$

Still this does not correspond to the Leray system (26.6), the reason being that the Einstein equations contain **constraints**. To analogize it again to electromagnetism: there, the Maxwell equations involve 6 *evolution* equations with the first time derivatives, and 2 *constraints* which do not contain time derivatives at all. Similarly, the Einstein equations involve 6 *evolution* equations with the second time derivatives, and 4 *constraints* with only the first time derivatives.[4] Therefore, it is the constraint equations which are *not* in the form (26.6), because they do not contain the second-time-derivative term $g^{00}(...)\,\partial_0\partial_0 g_{\mu\nu}$, hence they are not at all of the hyperbolic type.

We met the constraints in Chapter 25 and will discuss them more in Chapter 27, yet it is anyway easy (albeit t&b) to show that they have the form $G_{\nu\lambda}n^{\lambda} = 0$, with $G_{\nu\lambda}$ the Einstein tensor and n^{λ} the future normal to some (practically: Cauchy) hypersurface, i.e., in adapted coordinates, the normal to the $t = \mathrm{const}$ hypersurfaces,[5]

$$n_{\alpha} = \frac{\partial t}{\partial x^{\alpha}} = \delta^{0}_{\alpha} \qquad \Longrightarrow \qquad n^{\lambda} = g^{\lambda\alpha}n_{\alpha} = g^{\lambda 0} \qquad \Longrightarrow \qquad G_{\nu\lambda}n^{\lambda} = G_{\nu\lambda}g^{\lambda 0} = G^{0}_{\nu} .$$

Actually, multiplying the curvature terms of $G_{\nu\lambda}$ by $g^{\lambda 0}$, one has (without 1/2)

$$g^{\mu\kappa}g^{\lambda 0}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) - g^{\mu\kappa}g^{\beta\delta}\delta^{0}_{\nu}\left(g_{\mu\delta,\beta\kappa} - g_{\mu\kappa,\beta\delta}\right) .$$

Collecting only the second-time-derivative terms, the first part provides

$$\begin{aligned} & g^{\mu\kappa}g^{\lambda 0}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) \\ & \longrightarrow \quad g^{\mu 0}g^{\lambda 0}g_{\mu\lambda,\nu 0} + \cancel{g^{0\kappa}g^{00}g_{\nu\kappa,00}} - g^{\mu\kappa}g^{00}g_{\mu\kappa,\nu 0} - \cancel{g^{00}g^{\lambda 0}g_{\nu\lambda,00}} \end{aligned}$$

and the second-term scalar part (the one which multiplies δ^{0}_{ν}) provides

$$\begin{aligned} & g^{\mu\kappa}g^{\beta\delta}\left(g_{\mu\kappa,\beta\delta} - g_{\mu\delta,\beta\kappa}\right) \\ & \longrightarrow \quad g^{\mu\kappa}g^{00}g_{\mu\kappa,00} - g^{\mu 0}g^{0\delta}g_{\mu\delta,00} . \end{aligned}$$

[4] The difference in the derivative order is no surprise, since the Einstein equations are for a quantity (the metric) on the level of potential, whereas the Maxwell equations are for *fields*. However, expressing the EM tensor $F_{\mu\nu}$ in terms of potentials, one *also* obtains equations with the second time derivatives (the wave equations).

[5] The *unit* normal would be divided by $g^{\mu\nu}n_{\mu}n_{\nu} = g^{00}$, but normalization is not important at this point.

Both parts only contribute to the $g_{..,00}$ terms for $\nu = 0$, and the contribution is anyway zero,

$$\cancel{g^{\mu 0} g^{\lambda 0} g_{\mu\lambda,00}} - \cancel{g^{\mu\kappa} g^{00} g_{\mu\kappa,00}} + \cancel{g^{\mu\kappa} g^{00} g_{\mu\kappa,00}} - \cancel{g^{\mu 0} g^{0\delta} g_{\mu\delta,00}} = 0\,.$$

Hence, the equations $G_{\nu\lambda} n^\lambda = 0$ indeed do not contain any second time derivatives.

However, the issue of constraints can be remedied thanks to the following

$\boxed{\text{Lemma}}$: Consider vacuum Einstein equations. Thanks to the Bianchi identities, the constraints $G^{\nu\lambda} n_\lambda = 0$ automatically stay satisfied (if they hold on some initial hypersurface), provided that the spatial dynamical equations $G_{\nu\lambda} h^\nu_\rho h^\lambda_\sigma = 0$ are satisfied, with $h^\nu_\rho := \delta^\nu_\rho + n^\nu n_\rho$ projector to the hypersurface(s) orthogonal to n^μ (now already taking *unit* n^μ).

<u>Proof:</u> In adapted coordinates, projection to h^α_λ means taking the respective spatial component, and projection to n_λ means taking the time component, $G^{\nu\lambda} h^\rho_\nu h^\sigma_\lambda \to G^{ij}$, $G^{\nu\lambda} n_\lambda \to G^{\nu 0}$. Also, $\partial/\partial x^i$ mean the derivatives along the initial hypersurface, while $\partial/\partial t$ means the derivative "away from it", to its future. *At the initial hypersurface* ($t = t_0$), $G^{\nu\lambda} = 0$ (because dynamical equations $G^{ij} = 0$ are supposed to hold everywhere, and there is the initial constraint $G^{\nu 0} = 0$ valid at $t = t_0$), hence also the spatial derivatives vanish, $G^{\nu\lambda}{}_{,i} = 0$. Now, we know that contraction of the Bianchi identities implies $G^{\nu\lambda}{}_{;\nu} = 0$. We thus have, at $t = t_0$,

$$0 = G^{\nu\lambda}{}_{;\nu} = G^{\nu\lambda}{}_{,\nu} + \Gamma^\nu{}_{\nu\sigma} G^{\sigma\lambda} + \Gamma^\lambda{}_{\nu\sigma} G^{\nu\sigma} \stackrel{t=t_0}{=} G^{0\lambda}{}_{,0} + G^{i\lambda}{}_{,i} = G^{0\lambda}{}_{,0}\,.$$

Hence, $G^{0\lambda}$ vanish on the initial hypersurface *together with their first time derivative*.

One can proceed similarly to higher time derivatives. Actually, by time derivative of the above Bianchi identities one obtains

$$0 = G^{\nu\lambda}{}_{,\nu 0} + \Gamma^\nu{}_{\nu\sigma,0} G^{\sigma\lambda} + \Gamma^\nu{}_{\nu\sigma} G^{\sigma\lambda}{}_{,0} + \Gamma^\lambda{}_{\nu\sigma,0} G^{\nu\sigma} + \Gamma^\lambda{}_{\nu\sigma} G^{\nu\sigma}{}_{,0}\,.$$

Again, at $t = t_0$, one has $G^{\sigma\lambda} = 0$, so the terms with derivatives of Gammas are out. $G^{\sigma\lambda}{}_{,0}$ vanish there too, because the dynamical equations are supposed to hold *everywhere* (hence $G^{ij}{}_{,0} = 0$) and $G^{0\lambda}{}_{,0} = 0$ has been found in the first step. So the equation reduces to

$$0 = G^{\nu\lambda}{}_{,\nu 0} = G^{0\lambda}{}_{,00} + G^{i\lambda}{}_{,i0}\,,$$

of which, however, the second term vanish for the same reasons: since $G^{ij} = 0$ everywhere, one has $G^{ij}{}_{,i0} = 0$, and since $G^{0\lambda}{}_{,0} = 0$ on the initial hypersurface (from the first step), one also has $G^{0\lambda}{}_{,i0} = G^{0\lambda}{}_{,0i} = 0$ there. Hence, $G^{0\lambda}$ vanish on the initial hypersurface *together with their first two time derivatives*. Etc... $\square$

It is "needless to say", but let us stress that the initial satisfaction of the constraints and ensuring that they stay valid along the whole evolution is crucial in numerical relativity. In a non-linear theory such as GR, the initial-constraint satisfaction itself is a tricky point, because exact analytical solution is typically not available, and neither any dynamical evolution (yet) which would yield the given situation as a consistent outcome. One thus often sees various "transition effects" at the beginnings of simulations, during which the non-satisfaction of the initial constraints is being smoothed out using iterations or dynamical equations.

As a more familiar example, we may once more recall electrodynamics (as in Section 26.1.1). In a source-free vacuum, the fulfilment of the constraints $\mathrm{div}\vec{E}=0$, $\mathrm{div}\vec{B}=0$ is automatically propagated along the evolution thanks to the dynamical equations $\frac{\partial \vec{E}}{\partial t}=\mathrm{rot}\vec{B}$, $\frac{\partial \vec{B}}{\partial t}=-\mathrm{rot}\vec{E}$ (which are supposed to hold everywhere),

$$\frac{\partial}{\partial t}(\mathrm{div}\vec{E}) \equiv \mathrm{div}\left(\frac{\partial \vec{E}}{\partial t}\right) = \mathrm{div}(\mathrm{rot}\vec{B}) = 0\,, \quad \frac{\partial}{\partial t}(\mathrm{div}\vec{B}) \equiv \mathrm{div}\left(\frac{\partial \vec{B}}{\partial t}\right) = -\mathrm{div}(\mathrm{rot}\vec{E}) = 0\,.$$

The above means that the Einstein equations represent an **underdetermined system**: since 4 of them are constraints, they provide, for 10 metric components $g_{\mu\nu}$, only 6 evolution equations. The underdetermination actually is *mandatory*[6] from the point of view of the covariance principle: one has to always have freedom to choose 4 coordinate functions.

26.3.3 Harmonic coordinates in Einstein equations

The diffeomorphism invariance implies that every attempt to prove uniqueness of a solution in GR has to be performed in certain specific coordinates (otherwise the coordinate freedom simply makes any result non-unique). Perhaps the most common choice are the **harmonic coordinates** given by $H^\mu := \Box x^\mu = 0$. We showed in Section 22.4.1 that such a condition is equivalent to $(\sqrt{-g}\,g^{\alpha\mu})_{,\alpha}=0$. In the linearized theory, it lead to $\gamma^{\alpha\mu}{}_{,\alpha}=0$ and was also referred to as the de Donder, Hilbert or Lorenz condition. The harmonic coordinates turned out to be mainly useful in asymptotically flat space-times, e.g. for studying isolated sources of gravitation (whereas they are not so suitable in cosmology). In the Cauchy problem, however, they are of crucial importance, because in them the Einstein equations *can* be written in the form (26.6) to which the Leray theorem applies, which we will now embark on.

Let us write out, explicitly,

$$0 = H^\mu = \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,g^{\alpha\mu})_{,\alpha} = g^{\alpha\mu}{}_{,\alpha} + \frac{1}{\sqrt{-g}}\,g^{\alpha\mu}(\sqrt{-g})_{,\alpha} = g^{\alpha\mu}{}_{,\alpha} + \frac{1}{2}\,g^{\alpha\mu}g^{\rho\sigma}g_{\rho\sigma,\alpha}\,,$$

where we have employed (A.21). In the following, it will be important that H^μ contains only first derivatives of the metric, so $H^\mu{}_{,\nu}$ also contain second derivatives. The trick with the Einstein equations (vacuum ones) is that one adds to their left-hand side a term proportional to derivatives of H^μ (thus *vanishing in the harmonic coordinates*). With that term, the equations do already have the Leray form.

Now details: take the vacuum field equations without the cosmological term, $R_{\nu\lambda}=0$, and modify the Ricci tensor according to

$$R^{\mathrm{H}}_{\nu\lambda} := R_{\nu\lambda} + \frac{1}{2}\,(g_{\iota\nu}H^\iota{}_{,\lambda} + g_{\iota\lambda}H^\iota{}_{,\nu})\,. \tag{26.7}$$

[6] Footnote on three English kind-of synonyms: “mandatory” means required by an official law or rule; “compulsory” means the same, perhaps slightly weaker, and only used by default in connection with school/education or some public (also military) service; and “obligatory” is the least official of the three, stemming not only from official laws/rules, but also from moral or social requirements (also “requirements”, meaning “stresses”) or just from other circumstances. (Refinements welcome!)

Substituting the explicit expressions, one finds

$$\begin{aligned}
R^{\rm H}_{\nu\lambda} &= \frac{1}{2}\, g^{\mu\kappa}(g_{\mu\lambda,\nu\kappa} + g_{\nu\kappa,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) + \Gamma^{\mu}{}_{\mu\iota}\Gamma^{\iota}{}_{\lambda\nu} - \Gamma^{\mu}{}_{\lambda\iota}\Gamma^{\iota}{}_{\mu\nu} + \\
&\quad + \frac{1}{2}\, g_{\iota\nu} g^{\alpha\iota}{}_{,\alpha\lambda} + \frac{1}{4}\, g_{\iota\nu}(g^{\alpha\iota}g^{\rho\sigma}g_{\rho\sigma,\alpha})_{,\lambda} + \frac{1}{2}\, g_{\iota\lambda} g^{\alpha\iota}{}_{,\alpha\nu} + \frac{1}{4}\, g_{\iota\lambda}(g^{\alpha\iota}g^{\rho\sigma}g_{\rho\sigma,\alpha})_{,\nu} = \\
&= \frac{1}{2}\, \cancel{g^{\mu\kappa}g_{\mu\lambda,\nu\kappa}} + \frac{1}{2}\, \cancel{g^{\mu\kappa}g_{\nu\kappa,\mu\lambda}} - \frac{1}{2}\, \cancel{g^{\mu\kappa}g_{\mu\kappa,\nu\lambda}} - \frac{1}{2}\, g^{\mu\kappa}g_{\nu\lambda,\mu\kappa} + \left\{(g_{\beta\gamma,\delta})^2 \text{ terms}\right\} - \\
&\quad - \frac{1}{2}\, \cancel{g_{\iota\nu,\alpha\lambda}g^{\alpha\iota}} - \frac{1}{2}\, \cancel{g_{\iota\lambda,\alpha\nu}g^{\alpha\iota}} + \left\{(g_{\beta\gamma,\delta})^2 \text{ terms}\right\} + \frac{1}{2}\, \cancel{g^{\rho\sigma}g_{\rho\sigma,\nu\lambda}} + \left\{(g_{\beta\gamma,\delta})^2 \text{ terms}\right\} \\
&= -\frac{1}{2}\, g^{\mu\kappa}g_{\nu\lambda,\mu\kappa} + \left\{(g_{\beta\gamma,\delta})^2 \text{ terms}\right\}\ ,
\end{aligned}$$

where we have used, for the $g_{\iota\nu}g^{\alpha\iota}{}_{,\alpha\lambda}$ and $g_{\iota\lambda}g^{\alpha\iota}{}_{,\alpha\nu}$ terms,

$$\delta^{\alpha}_{\nu} = g_{\iota\nu}g^{\alpha\iota} \quad\Rightarrow\quad 0 = g_{\iota\nu,\alpha}g^{\alpha\iota} + g_{\iota\nu}g^{\alpha\iota}{}_{,\alpha} \quad\Rightarrow\quad 0 = g_{\iota\nu,\alpha\lambda}g^{\alpha\iota} + g_{\iota\nu}g^{\alpha\iota}{}_{,\alpha\lambda} + \left\{(g_{\beta\gamma,\delta})^2 \text{ terms}\right\}.$$

Therefore, *in harmonic coordinates* (in which $H^{\mu}=0$, so the original and the modified Ricci tensors are identical), the vacuum Einstein equations can be written

$$\boxed{-\tfrac{1}{2}\, g^{\mu\kappa}g_{\nu\lambda,\mu\kappa} + F_{\nu\lambda}(g_{\beta\gamma}, g_{\beta\gamma,\delta}) = 0}\ , \tag{26.8}$$

which already *is* in the form to which the Leray theorem applies. It is known as the **reduced Einstein equations**. Hence, with the initial conditions for $g_{ij} \equiv h_{ij}$ and $h_{ij,t}$ given on the initial hypersurface $t = t_0$, the local (temporary) existence of solutions is guaranteed. Note that $g_{0\mu,t}$ are not at all constrained by the initial conditions, so, in order to ensure consistency, these should be chosen in such a way that $H^{\mu}=0$ really hold initially.

One naturally asks now: *how does the harmonic condition $H^{\mu} = 0$ propagate in time, according to the field equations?* To answer this, it is advantageous to also compute the H-modified Ricci scalar. From (26.7),

$$R^{\rm H} = R + \frac{1}{2}\, g^{\nu\lambda}(g_{\iota\nu}H^{\iota}{}_{,\lambda} + g_{\iota\lambda}H^{\iota}{}_{,\nu}) = R + H^{\iota}{}_{,\iota}\ .$$

The generic-coordinate field equations are thus related to the harmonic-coordinate ones by

$$0 = R_{\nu\lambda} - \frac{1}{2}\, Rg_{\nu\lambda} = R^{\rm H}_{\nu\lambda} - \frac{1}{2}\, R^{\rm H}g_{\nu\lambda} - \frac{1}{2}\, (g_{\iota\nu}H^{\iota}{}_{,\lambda} + g_{\iota\lambda}H^{\iota}{}_{,\nu} - g_{\nu\lambda}H^{\iota}{}_{,\iota})\,.$$

In the following, everything will be taken on the initial hypersurface: choosing, there, $H^{\mu}=0$ ensures that $R_{\nu\lambda} - \frac{1}{2}Rg_{\nu\lambda} = 0$ are satisfied there *as well as* $R^{\rm H}_{\nu\lambda} - \frac{1}{2}R^{\rm H}g_{\nu\lambda} = 0$. The last term (i.e. the parenthesis) thus has to vanish, so its contraction tells that $H^{\iota}{}_{,\iota} = 0$, which implies, when substituted back into the parenthesis, $g_{\iota\nu}H^{\iota}{}_{,\lambda} + g_{\iota\lambda}H^{\iota}{}_{,\nu} = 0$. Writing this in adapted coordinates where $H^{\mu}=0$ involves that the spatial derivatives of H^{μ} vanish initially as well, we have

$$\text{(e.g. for } \lambda = j,\ \nu = t\text{:)} \qquad g_{\iota t}\cancel{H^{\iota}{}_{,j}} + g_{\iota j}H^{\iota}{}_{,t} = 0 \quad \text{(for any } j)\,.$$

The harmonic condition is thus "propagated" along the time evolution.

One more nice property can be shown concerning the harmonic condition: that $H^\mu = 0$ can itself be written in the Leray form. Actually, write the contracted Bianchi identities again,

$$0 = G_{\nu\lambda}{}^{;\nu} = (G^{\mathrm{H}}_{\nu\lambda})^{;\nu} - \frac{1}{2}\,(g_{\iota\nu}H^\iota{}_{,\lambda} + g_{\iota\lambda}H^\iota{}_{,\nu} - g_{\nu\lambda}H^\iota{}_{,\iota})^{;\nu} \ .$$

The satisfaction of the harmonic condition $H^\mu = 0$ ensures $G_{\mu\nu} = G^{\mathrm{H}}_{\mu\nu}$, and that in turn means that the first term of the above vanishes, hence the same has to hold for the second. However, the latter can obviously be decomposed as

$$0 = -\frac{1}{2}\,(\cancel{H^\iota{}_{,\lambda\iota}} + g_{\iota\lambda}H^\iota{}_{,\nu}{}^{\nu} - \cancel{H^\iota{}_{,\iota\lambda}}) + \{\text{terms without } H^\iota{}_{,\kappa\lambda}\} \ .$$

Multiplying this by $g^{\lambda\alpha}$, we arrive at

$$g^{\mu\nu}H^\alpha{}_{,\mu\nu} = \{\text{terms without } H^\iota{}_{,\kappa\lambda}\} \ .$$

This is indeed the form (26.6) to which the Leray theorem can be applied, thus ensuring that $H^\mu = 0$ can really be found as a solution at $t \geqslant t_0$, provided that $H^\mu = 0$ and $H^\mu{}_{,t} = 0$ hold (which we have shown) at $t = t_0$.

Let us add that the well-posedness of the Einstein equations and the uniqueness of their solution were mostly proved by Yvonne Choquet-Bruhat in the 1950s, using the harmonic coordinates. At that time Choquet-Bruhat (then Fourès-Bruhat, actually) was a postdoc working with J. Leray at Princeton, also discussing certain points with late A. Einstein. (Being just 8 years younger then general relativity, she is exactly 100 today when the present Chapter is being typed, i.e. at the beginning of 2024.)

26.3.4 Maximal Cauchy development: a summary

Here we put together the results indicated above, following Wald [50], Theorem 10.2.2.:

- Let Σ be a 3D smooth manifold, let $h_{\mu\nu}$ be a smooth Riemannian metric on Σ, and let $K_{\mu\nu}$ (the extrinsic curvature of Σ) be a smooth symmetric tensor field on Σ which provides the information about the change of $h_{\mu\nu}$ in the direction orthogonal to Σ. Let $h_{\mu\nu}$ and $K_{\mu\nu}$ satisfy the constraint equations $G_{\mu\nu}n^\mu = 0$ for the Einstein tensor.

- Then there exists a unique solution of Einstein's equations $(\mathcal{M}, g_{\mu\nu})$ called the *maximal Cauchy development* of $(\Sigma, h_{\mu\nu}, K_{\mu\nu})$, satisfying the following properties:

  – The space-time $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic, with Σ its Cauchy hypersurface described by the induced metric $h_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$.

  – Uniqueness and "maximality": every other space-time satisfying the above can be mapped isometrically onto a subset of $(\mathcal{M}, g_{\mu\nu})$.

  – Let $(\Sigma, h_{\mu\nu}, K_{\mu\nu})$ and $(\Sigma', h'_{\mu\nu}, K'_{\mu\nu})$ be two sets of initial conditions with the respective maximal Cauchy developments $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}', g'_{\mu\nu})$. If there exists a diffeomorphism between certain $S \subset \Sigma$ and $S' \subset \Sigma'$ which relates $(\Sigma, h_{\mu\nu}, K_{\mu\nu})$ and $(\Sigma', h'_{\mu\nu}, K'_{\mu\nu})$, then the domain of dependence $D(S') \subset \mathcal{M}'$ is isometric to the domain of dependence $D(S) \subset \mathcal{M}$.

  – The solution $g_{\mu\nu}$ depends continuously on the initial data $h_{\mu\nu}$, $K_{\mu\nu}$ assigned on Σ.

26.3.5 Remarks

The existence of a well posed initial-value formulation is by no means an automatic feature of a theory. Besides obvious fundamental importance (evolution from "initial conditions" is *the* central problem of every quantitative science), the existence of such a formulation provides various practical advantages. Actually, however non-unique the correspondence between the bulk (globally hyperbolic) space-time and the initial conditions is (thanks to the freedom in choosing the initial hypersurface Σ), its existence can make various problems easier, because it is often simpler to translate them to the language of initial conditions and, subsequently, to investigate the initial-data constraints rather than the full 4D Einstein equations. A suitable choice of the initial hypersurface certainly is the crucial point of such an approach.

The Cauchy problem with sources ($T_{\mu\nu} \neq 0$)

can, in general, be formulated in a similar way as for the pure-vacuum GR, provided that the Einstein equations coupled with the equations governing the pertinent non-gravitational fields can be put into the Leray form (26.6), with the $T_{\mu\nu}$ solely depending on the fields and on the metric, together with their first derivatives. In particular, the problem of the coupled gravitational and Klein-Gordon fields, or the Einstein-Maxwell problem, can be formulated as well-posed initial-value problems. However, the Leray formulation is generally not a *necessary* condition for a system to possess a solvable Cauchy problem. For instance, a gravitating perfect fluid can be studied from a well-posed initial-value formulation, although the corresponding equations are *not* of the form (26.6).

CHAPTER 27

Hamiltonian formulation of the Einstein equations

The Lagrangian formulation of Einstein equations is "4D", whereas the Hamiltonian formulation has to be "3+1", since the Hamilton equations determine *time evolution* of the phase variables and thus necessarily require a specific choice of the time coordinate. The Hamiltonian formulation is thus suitable for evolution problems such as the Cauchy initial problem, while, on the other hand, it may also be useful in statements about "space at some given time" (in definitions of integral quantities, for example). The Hamiltonian view is also the starting point of the canonical quantization of the theory.

The approach is similar as in classical mechanics, only that continuous systems and fields have to be described in terms of proper spatial *densities* of the quantities (of the Lagrangian and Hamiltonian, in particular). Suppose we have decomposed the space-time to "time" t (parameter along the vector field t^μ) and "space" (hypersurfaces Σ_t) like in Chapter 25. Let the proper density of Lagrangian only depend on configuration variables ("fields") ψ and their first derivatives,[1] $\mathcal{L}(\psi, \dot{\psi}, \psi_{,i})$. The variables are tensorial in general (yet we will not indicate this by indices for clarity) and are evaluated on Σ_t; their gradient $\psi_{;\alpha}$ we decompose into the time derivative $\dot{\psi} := \pounds_{\mathbf{t}}\psi = \psi_{,\iota}t^\iota$ $(= \psi_{,\iota}\delta^\iota_t = \psi_{,t}$ in adapted coordinates) and the spatial derivatives $\psi_{,i}$ (along the coordinate directions of Σ_t). The canonical momenta Π associated with the fields ψ we define as the vector densities

$$\Pi := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\psi}}\,. \tag{27.1}$$

Therefrom we express the "velocities" $\dot{\psi} = \dot{\psi}(\psi, \Pi, \psi_{,i})$ and submit these to the definition of the Hamiltonian density (Legendre transformation)

$$\mathcal{H}(\psi, \Pi, \psi_{,i}) := \Pi\cdot\dot{\psi} - \sqrt{-g}\,\mathcal{L}\,. \tag{27.2}$$

[1] Good to recall Section 23.1 on functional derivative in order to understand how exactly arise, in performing the variation, the derivatives with respect to the field variables ψ and their derivatives (here only the first ones).

Due to the presence of $\sqrt{-g}$, this is *not* an invariant, but a scalar density of weight $+1$. We will thus integrate it over a *coordinate* volume, since the latter is a scalar density of weight -1, so the Hamiltonian will thus be invariant. (Though the spatial density of Hamiltonian is a scalar density, we will *not* write it in the gothic font – simply liking $\mathcal{H}$ more than $\mathfrak{H}$.)

The stationarity of action is equivalent to the **Hamilton canonical equations**. Indeed,

$$\delta S := \delta \int_{\Omega} \mathcal{L}\sqrt{-g}\,\mathrm{d}^4 x \xrightarrow{1+3} \delta \int_{t_1}^{t_2}\int_{\Sigma_t} \mathcal{L}\sqrt{-g}\,\mathrm{d}^3 x\,\mathrm{d}t = \delta \int_{t_1}^{t_2}\int_{\Sigma_t} (\Pi\cdot\dot{\psi} - \mathcal{H})\,\mathrm{d}^3 x\,\mathrm{d}t =$$

$$= \int_{t_1}^{t_2}\int_{\Sigma_t} \left(\delta\Pi\cdot\dot{\psi} + \Pi\cdot\delta\dot{\psi} - \frac{\partial\mathcal{H}}{\partial\psi}\cdot\delta\psi - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi - \frac{\partial\mathcal{H}}{\partial\psi_{,i}}\cdot\delta\psi_{,i}\right)\mathrm{d}^3 x\,\mathrm{d}t\,.$$

The second term we "per-partes", already dropping the boundary term as usual (better argument is that $\delta\psi$ is assumed to vanish at the marginal times t_1 and t_2),

$$\int_{t_1}^{t_2}\int_{\Sigma_t} \Pi\cdot\delta\dot{\psi}\,\mathrm{d}^3 x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\Sigma_t} \dot{\Pi}\cdot\delta\psi\,\mathrm{d}^3 x\,\mathrm{d}t\,.$$

Similarly we process the last term (time integration is not important in it),

$$-\int_{\Sigma_t} \frac{\partial\mathcal{H}}{\partial\psi_{,i}}\cdot\delta\psi_{,i}\,\mathrm{d}^3 x = \cancel{-\int_{\Sigma_t}\left(\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\cdot\delta\psi\right)_{,i}\mathrm{d}^3 x} + \int_{\Sigma_t}\left(\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\right)_{,i}\cdot\delta\psi\,\mathrm{d}^3 x\,;$$

here the first term has dropped out, since by Gauss law it can be rewritten as an integral from $\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\cdot\delta\psi$ over a 2D boundary $\partial\Sigma_t$ (which may possibly lie at infinity), where we assume $\delta\psi = 0$. (Note that the remaining term may look similar, but its integration goes over the whole Σ_t, not over any 2D boundary, so it does not vanish.)

To summarize,

$$\delta S = \int_{t_1}^{t_2}\int_{\Sigma_t} \left[\delta\Pi\cdot\dot{\psi} - \dot{\Pi}\cdot\delta\psi - \frac{\partial\mathcal{H}}{\partial\psi}\cdot\delta\psi - \frac{\partial\mathcal{H}}{\partial\Pi}\cdot\delta\Pi + \left(\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\right)_{,i}\cdot\delta\psi\right]\mathrm{d}^3 x\,\mathrm{d}t\,,$$

from where we see, by requiring zeros at both independent variations $\delta\Pi$ and $\delta\psi$, that

$$\delta S = 0 \qquad \Longleftrightarrow \qquad \dot{\psi} := \pounds_{\mathbf{t}}\psi = \frac{\partial\mathcal{H}}{\partial\Pi}\,, \quad \dot{\Pi} := \pounds_{\mathbf{t}}\Pi = -\frac{\partial\mathcal{H}}{\partial\psi} + \left(\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\right)_{,i}\,. \tag{27.3}$$

It is clear from the derivation that these Hamilton equations are equivalent to the Euler-Lagrange equations obtained from the Lagrangian approach. Actually, substituting $\sqrt{-g}\,\mathcal{L} = \Pi\cdot\dot{\psi} - \mathcal{H}$ and the definition of Π to the Euler-Lagrange equation, one has

$$\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi_{,t}}\right]_{,t} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\psi_{,i}}\right]_{,i} = -\frac{\partial\mathcal{H}}{\partial\psi} - \dot{\Pi} + \left[\frac{\partial\mathcal{H}}{\partial\psi_{,i}}\right]_{,i}\,,$$

which is zero due to the Hamilton equation for $\dot{\Pi}$.

27.1 Klein-Gordon field and EM field: a warm up

Before embarking on the Einstein equations, let us illustrate the Hamiltonian approach on the Klein-Gordon scalar field and on the electromagnetic field. In the latter case, we will meet the important circumstance which later will also occur in the gravitation problem – thanks to a gauge freedom in the field variables, some of the field equations become constraints.

Suppose, for simplicity, that we deal with a situation where $N^i = 0$, so, according to (25.7), $g^{\mu\nu} = \mathrm{diag}(-N^{-2}, h^{ik})$. The Lagrangian density of the Klein-Gordon scalar field ($\psi \equiv \psi$, now really without indices) then reads

$$\mathcal{L} = -\frac{1}{2}\left(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + m^2\psi^2\right) = -\frac{1}{2}\left(g^{tt}\dot{\psi}^2 + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right).$$

From it, we have

$$\Pi := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\psi}} = -\sqrt{-g}\,g^{tt}\dot{\psi} \qquad \Longrightarrow \qquad \dot{\psi} = -\frac{\Pi}{\sqrt{-g}\,g^{tt}}$$

and the Hamiltonian density

$$\mathcal{H} := \Pi\dot{\psi} - \sqrt{-g}\,\mathcal{L} = -\frac{\Pi^2}{\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(g^{tt}\frac{\Pi^2}{-g(g^{tt})^2} + h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right) =$$
$$= -\frac{\Pi^2}{2\sqrt{-g}\,g^{tt}} + \frac{\sqrt{-g}}{2}\left(h^{ik}\psi_{,i}\psi_{,k} + m^2\psi^2\right) ,$$

from which we finally find evolution equations

$$\dot{\psi} = \frac{\partial\mathcal{H}}{\partial\Pi} = -\frac{\Pi}{\sqrt{-g}\,g^{tt}} = \frac{\Pi N^2}{\sqrt{-g}} ,$$
$$\dot{\Pi} = -\frac{\partial\mathcal{H}}{\partial\psi} + \left(\frac{\partial\mathcal{H}}{\partial\psi_{,j}}\right)_{,j} = -\sqrt{-g}\,m^2\psi + \left(\sqrt{-g}\,h^{jk}\psi_{,k}\right)_{,j} .$$

This result really leads to the Klein-Gordon equation,

$$\Box\psi \equiv g^{\mu\nu}\psi_{;\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{\mu\nu}\psi_{,\mu}\right)_{,\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,g^{tt}\psi_{,t}\right)_{,t} + \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,h^{jk}\psi_{,j}\right)_{,k} =$$
$$= -\frac{\dot{\Pi}}{\sqrt{-g}} + \frac{1}{\sqrt{-g}}\left(\dot{\Pi} + \sqrt{-g}\,m^2\psi\right) = m^2\psi .$$

Second, let us test the Hamiltonian approach on a free EM field in the Minkowski space-time ($g_{\mu\nu} = \eta_{\mu\nu}$, $\sqrt{-g} = 1$). Suppose the configuration variable is the four-potential in this case, $\psi \equiv A_\mu$. We split it to time and spatial components with respect to Σ_t, i.e. to the "scalar" and "vector" potentials

$$\phi := -A_\mu n^\mu, \qquad \vec{A} := A_\mu h^\mu_\alpha ,$$

and write down the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi}\,F^{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(E^2 - B^2) = \frac{1}{8\pi}(\vec{\nabla}\phi + \dot{\vec{A}})\cdot(\vec{\nabla}\phi + \dot{\vec{A}}) - \frac{1}{8\pi}(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A}) ,$$

where

$$E_\mu \equiv F_{\mu\nu}n^\nu, \quad B_\mu \equiv -{}^*F_{\mu\nu}n^\nu \qquad (\iff F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \epsilon_{\mu\nu\rho\sigma}n^\rho B^\sigma)$$

are the electric and magnetic fields defined with respect to Σ_t. For quantities "living on Σ_t" we have employed the three-vector notation, in particular

$$\vec{E} := -\vec{\nabla}\phi - \dot{\vec{A}}, \qquad \vec{B} := \vec{\nabla} \times \vec{A}\,.$$

The momenta conjugated to the scalar and vector potentials come out

$$\Pi_t := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\phi}} = 0\,, \qquad \vec{\Pi} := \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\dot{\vec{A}}} = \frac{1}{4\pi}(\vec{\nabla}\phi + \dot{\vec{A}}) = -\frac{\vec{E}}{4\pi}\,.$$

Here comes *the issue*: the first of these relations cannot be inverted, namely, it is not possible to express from it $\dot{\phi}$, so it is also not possible to find the Hamiltonian density $\mathcal{H} = \Pi_t\dot{\phi} + \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L}$ (remember that $\sqrt{-g} = 1$). This "accident", related to the gauge freedom of the four-potential, is being remedied in a simple way: if Π_t vanishes identically, it is clearly not appropriate to consider ϕ a *dynamical* variable. Actually, if the Lagrangian does not depend on some of the "velocities", it means that the change of the respective quantity has *no effect in the theory*. So, if dropping ϕ and only leaving $\vec{A}$ as configuration variables, we may continue: we express $\dot{\vec{A}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi$ and submit it, together with $\mathcal{L} = 2\pi\vec{\Pi}\cdot\vec{\Pi} - \frac{B^2}{8\pi}$, to the "restricted" Hamiltonian-density prescription,

$$\mathcal{H} = \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L} = \vec{\Pi}\cdot(4\pi\vec{\Pi} - \vec{\nabla}\phi) - 2\pi\vec{\Pi}\cdot\vec{\Pi} + \frac{B^2}{8\pi} = 2\pi\Pi^2 - \vec{\Pi}\cdot\vec{\nabla}\phi + \frac{B^2}{8\pi}\,. \tag{27.4}$$

The Hamilton equations yield

$$\dot{\vec{A}} = \frac{\partial\mathcal{H}}{\partial\vec{\Pi}} = 4\pi\vec{\Pi} - \vec{\nabla}\phi = -\vec{E} - \vec{\nabla}\phi\,,$$

$$\dot{\vec{\Pi}}\left(= -\frac{\dot{\vec{E}}}{4\pi}\right) = -\cancel{\frac{\partial\mathcal{H}}{\partial\vec{A}}} + \left(\frac{\partial\mathcal{H}}{\partial\vec{A}_{,j}}\right)_{,j} = \frac{1}{8\pi}\left(\frac{\partial B^2}{\partial\vec{A}_{,j}}\right)_{,j} = -\frac{\vec{\nabla}\times\vec{B}}{4\pi}\,.$$

The last equality of the latter equation can best be computed "in components":

$$B^2 \equiv B_k B^k = \epsilon_{klm}A^{m,l}\epsilon^{kno}A_{o,n} = (\delta^n_l\delta^o_m - \delta^n_m\delta^o_l)A^{m,l}A_{o,n} = (\delta^{nl}\delta^{om} - \delta^{nm}\delta^{ol})A_{m,l}A_{o,n}\,,$$

so

$$\begin{aligned}\frac{\partial B^2}{\partial A_{i,j}} &= (\delta^{nl}\delta^{om} - \delta^{nm}\delta^{ol})(\delta^i_m\delta^j_l A_{o,n} + A_{m,l}\delta^i_o\delta^j_n) = \\ &= (\delta^{nj}\delta^{oi} - \delta^{ni}\delta^{oj})A_{o,n} + (\delta^{jl}\delta^{im} - \delta^{jm}\delta^{il})A_{m,l} = 2(A^{i,j} - A^{j,i}) \equiv 2F^{ji} \\ &\Longrightarrow \quad \left(\frac{\partial B^2}{\partial A_{i,j}}\right)_{,j} = 2F^{ji}{}_{,j} = 2\,\epsilon^{jik}B_{k,j} \equiv -2(\vec{\nabla}\times\vec{B})^i\,.\end{aligned}$$

The first equation thus reproduces the expression of $\vec{E}$ in terms of the potentials, thanks to which (plus $\vec{B}=\vec{\nabla}\times\vec{A}$) holds the second set of Maxwell equations, $\vec{\nabla}\times\vec{E}=-\vec{\nabla}\times\dot{\vec{A}}=-\dot{\vec{B}}$ and $\vec{\nabla}\cdot\vec{B}=0$. The second Hamilton equation yields the Maxwell equation $\vec{\nabla}\times\vec{B}=\dot{\vec{E}}$.

Finally, for consistence, it is necessary to add the equation for the derivative by non-dynamical variable ϕ,

$$0=\dot{\Pi}_t=-\cancel{\frac{\partial\mathcal{H}}{\partial\phi}}+\left(\frac{\partial\mathcal{H}}{\partial\phi_{,j}}\right)_{,j}=-\left[\frac{\partial(\vec{\Pi}\cdot\vec{\nabla}\phi)}{\partial\phi_{,j}}\right]_{,j}=-\left(\Pi^j\right)_{,j}=-\vec{\nabla}\cdot\vec{\Pi}=\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}\,.$$

This Maxwell equation represents a **constraint**. (Sure, it cannot be an evolution equation, because it does not contain the time derivative. The same also applies to the similar equation $\vec{\nabla}\cdot\vec{B}=0$.) Note that the equation can even simpler be obtained by rewriting, in the Hamiltonian (27.4),

$$-\vec{\Pi}\cdot\vec{\nabla}\phi=-\vec{\nabla}\cdot(\phi\,\vec{\Pi})+\phi\,\vec{\nabla}\cdot\vec{\Pi} \tag{27.5}$$

and omitting the first term (it is a divergence only contributing to the surface term): then

$$0=\dot{\Pi}_t=-\frac{\partial\mathcal{H}}{\partial\phi}+\cancel{\left(\frac{\partial\mathcal{H}}{\partial\phi_{,j}}\right)_{,j}}=-\vec{\nabla}\cdot\vec{\Pi}\,.$$

Knowing that the term $-\vec{\Pi}\cdot\vec{\nabla}\phi$ is thus irrelevant, the Hamiltonian (27.4) is seen to represent the energy density of the EM field,

$$\mathcal{H}=2\pi\Pi^2+\frac{B^2}{8\pi}=\frac{1}{2}\,\frac{E^2+B^2}{4\pi}\,.$$

The EM field exemplifies the **Hamiltonian system with a constraint**. Constraints occur in theories which possess a gauge or coordinate freedom. Their "gaugeable" variables are *not dynamical*, effectively playing the role of Lagrangian multipliers which enforce the fulfilment of certain **constraints**. Above, the constraint $\Pi_t=0$ is called a **primary constraint**; it follows from Lagrangian itself, without using any evolution equations. The constraint $\dot{\Pi}_t=0$ (implying the field equation $\vec{\nabla}\cdot\vec{E}=0$) is called a **secondary constraint** since it is required by consistency of the evolution equations with the respective primary constraint.

27.2 Gravitational field

We will start from the Lagrangian density $\mathcal{L}_{\rm g}=R-2\Lambda$ (will only consider vacuum), rewriting it to the "3+1" form. For simplicity, we will not take into account "boundary" (surface) terms – those given by divergence of some vector field (here we have covariant divergence in mind), because such can be expressed, thanks to the Gauss law, as integrals from the flows of the pertinent vector fields over the boundary of the integration region. Let us emphasize, however, that we are speaking now about the *Lagrangian itself* rather than about its *variation*, so we are *not* claiming that the "boundary" terms vanish as a consequence of vanishing of the variations on the boundary – actually, we saw in Section 23.4.6 that the boundary parts of the Lagrangian typically *aren't* zero and depend on the geometric properties of the boundary.

Will be suitable to slightly modify the 3+1 form of the scalar curvature (25.37). If writing (remember that $K = n^{\alpha}{}_{;\alpha}$)

$$K_{;\gamma} n^{\gamma} = (K n^{\gamma})_{;\gamma} - K n^{\gamma}{}_{;\gamma} = (K n^{\gamma})_{;\gamma} - K^2 \,,$$

the Ricci scalar can be cast into the form

$$\begin{aligned} R &= {}^{(3)}R + K^2 + K_{\nu\gamma} K^{\nu\gamma} - 2a^{\delta}{}_{;\delta} + 2K_{,\gamma} n^{\gamma} = \\ &= {}^{(3)}R - K^2 + K_{\nu\gamma} K^{\nu\gamma} - 2a^{\delta}{}_{;\delta} + 2(K n^{\gamma})_{;\gamma} \,. \end{aligned} \tag{27.6}$$

Therefore, if omitting the surface terms $-2a^{\delta}{}_{;\delta} + 2(Kn^{\gamma})_{;\gamma}$, given by divergences, the gravitational Lagrangian density reads

$$\mathcal{L}_{\mathrm{g}} = {}^{(3)}R - K^2 + K_{\nu\gamma} K^{\nu\gamma} - 2\Lambda = {}^{(3)}R + K_{\kappa\lambda} K_{\rho\sigma} (h^{\kappa\rho} h^{\lambda\sigma} - h^{\kappa\lambda} h^{\rho\sigma}) - 2\Lambda \,. \tag{27.7}$$

As configuration variables on a given Cauchy hypersurface Σ_t, we choose the latter's metric $h_{\mu\nu}$, and the lapse and shift functions N and N_{α}. Wishing to define the respective canonical-momentum densities as derivatives of the Lagrangian density by "velocities"

$$\dot{h}_{\mu\nu} := h^{\alpha}_{\mu} h^{\beta}_{\nu} \pounds_{\mathbf{t}} h_{\alpha\beta} \,, \qquad \dot{N} := \pounds_{\mathbf{t}} N = N_{,\sigma} t^{\sigma} \,, \qquad \dot{N}_{\alpha} := h^{\beta}_{\alpha} \pounds_{\mathbf{t}} N_{\beta} \,,$$

we must express the $\mathcal{L}_{\mathrm{g}}$ in terms of the latter. The time derivative of $h_{\mu\nu}$ occurs in the extrinsic curvature, as we know from equation (25.16), i.e. from the formula

$$K_{\mu\nu} = \frac{1}{2N} h^{\alpha}_{\mu} h^{\beta}_{\nu} \left(\pounds_{\mathbf{t}} h_{\alpha\beta} - \pounds_{\mathbf{N}} h_{\alpha\beta} \right) = \frac{1}{2N} \left(\dot{h}_{\mu\nu} - N_{\nu|\mu} - N_{\mu|\nu} \right) .$$

Now it is possible to define the "momenta" canonically conjugated to $h_{\mu\nu}$,

$$\Pi^{\mu\nu} := \frac{\partial(\sqrt{-g}\mathcal{L}_{\mathrm{g}})}{\partial \dot{h}_{\mu\nu}} = \frac{\partial(\sqrt{-g}\mathcal{L}_{\mathrm{g}})}{\partial K_{\alpha\beta}} \frac{\partial K_{\alpha\beta}}{\partial \dot{h}_{\mu\nu}} \,.$$

Substituting from (27.7)

$$\frac{\partial(\sqrt{-g}\mathcal{L}_{\mathrm{g}})}{\partial K_{\alpha\beta}} = \sqrt{-g} \left(\delta^{\alpha}_{\kappa} \delta^{\beta}_{\lambda} K_{\rho\sigma} + K_{\kappa\lambda} \delta^{\alpha}_{\rho} \delta^{\beta}_{\sigma} \right) \left(h^{\kappa\rho} h^{\lambda\sigma} - h^{\kappa\lambda} h^{\rho\sigma} \right) = 2\sqrt{-g} \left(K^{\alpha\beta} - K h^{\alpha\beta} \right)$$

and from (25.16) $\frac{\partial K_{\alpha\beta}}{\partial \dot{h}_{\mu\nu}} = \frac{1}{2N} \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta}$, we finally arrive at

$$\Pi^{\mu\nu} = \frac{\sqrt{-g}}{N} \left(K^{\mu\nu} - K h^{\mu\nu} \right) = \sqrt{h} \left(K^{\mu\nu} - K h^{\mu\nu} \right) . \tag{27.8}$$

The inverse relation follows by tracing,

$$\Pi := \Pi^{\mu}_{\mu} = -2\sqrt{h}\, K \quad \Longrightarrow \quad K^{\mu\nu} = \frac{1}{\sqrt{h}} \left(\Pi^{\mu\nu} - \frac{1}{2} \Pi h^{\mu\nu} \right) . \tag{27.9}$$

The Lagrangian does not contain $\dot{N}$ and $\dot{N}_\alpha$, so the momenta conjugated with N and N_α identically vanish, $\Pi_N = 0$ and $\Pi_{N_\alpha} = 0$. These are primary constraints of the theory. In analogy with the situation occurring in electrodynamics, we interpret it in such a way that N, N_α in fact *aren't dynamical variables*. Similarly as in the EM case, it is related to "gauge" freedom: N and N_α do not describe any *intrinsic* properties of space-time, they are *elective* components of the time vector t^μ. Specifically, N tells "how far" it is from Σ_{t_1} to Σ_{t_2}, so it scales the time coordinate, and N_α specifies the coordinates that cover the Σ_t hypersurfaces. One thus should not be looking for any dynamical equations for N and N_α, and so we will not employ them in designing the Hamiltonian.

Now we have everything to find the gravitational part of the Hamiltonian density, more accurately its "bulk" part (surface terms are not included, as already stressed above),

$$
\begin{aligned}
\mathcal{H}_{\rm g} &= \Pi^{\mu\nu}\dot{h}_{\mu\nu} - \sqrt{-g}\,\mathcal{L}_{\rm g} = \\
&= \sqrt{h}\,(K^{\mu\nu} - Kh^{\mu\nu})(2NK_{\mu\nu} + N_{\nu|\mu} + N_{\mu|\nu}) - N\sqrt{h}\left({}^{(3)}\!R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\Lambda\right) = \\
&= 2\sqrt{h}\,(K^{\mu\nu} - Kh^{\mu\nu})\,N_{(\mu|\nu)} + N\sqrt{h}\left(K_{\mu\nu}K^{\mu\nu} - K^2 - {}^{(3)}\!R + 2\Lambda\right).
\end{aligned}
$$

Employing the formula (27.8) for momenta $\Pi^{\mu\nu}$, one easily finds the relations

$$
\begin{aligned}
&\Pi_{\mu\nu}\Pi^{\mu\nu} = h\left(K_{\mu\nu}K^{\mu\nu} + K^2\right), \qquad \Pi^2 := (\Pi^\mu_\mu)^2 = (-2\sqrt{h}\,K)^2 = 4hK^2 \\
&\Longrightarrow \quad 2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = 2h\left(K_{\mu\nu}K^{\mu\nu} - K^2\right),
\end{aligned}
$$

thanks to which one finally arrives to

$$
\boxed{\mathcal{H}_{\rm g} = 2\Pi^{\mu\nu}N_{(\mu|\nu)} + \frac{N}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) - N\sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right)}\,. \tag{27.10}
$$

27.2.1 Hamilton equations: constraints

Let us first look at how *secondary constraints* arise, given by vanishing of the derivatives of $\mathcal{H}_{\rm g}$ with respect to the non-dynamical variables N and N_α (required by consistency with the identical vanishing of Π_N and Π_{N_α} found from the Lagrangian):[2]

$$
\begin{aligned}
&0\left(=\dot{\Pi}_N\right) = -\frac{\partial\mathcal{H}_{\rm g}}{\partial N} + \left(\cancel{\frac{\partial\mathcal{H}_{\rm g}}{\partial N_{|\iota}}}\right)_{|\iota} = -\frac{1}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + \sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right) \\
&\iff \quad 2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = 2h\left({}^{(3)}\!R - 2\Lambda\right), \qquad\qquad (27.11) \\
&0\left(=\dot{\Pi}_{N_\alpha}\right) = -\cancel{\frac{\partial\mathcal{H}_{\rm g}}{\partial N_\alpha}} + \left(\frac{\partial\mathcal{H}_{\rm g}}{\partial N_{\alpha|\iota}}\right)_{|\iota} = \left(2\Pi^{\mu\nu}\frac{\partial N_{\mu|\nu}}{\partial N_{\alpha|\iota}}\right)_{|\iota} = \left(2\Pi^{\mu\nu}\delta^\alpha_\mu\delta^\iota_\nu\right)_{|\iota} = 2\Pi^{\alpha\iota}{}_{|\iota} \\
&\iff \quad \Pi^{\alpha\iota}{}_{|\iota} = 0\,. \qquad\qquad (27.12)
\end{aligned}
$$

---

[2] In the following, it is indeed OK to apply the 3D divergence in the $\left(\frac{\partial\mathcal{H}_{\rm g}}{\partial N_{\alpha|\iota}}\right)_{|\iota}$ and similar terms, because the constraints should hold *on the hypersurfaces* Σ_t, they do not include any "normal" behaviour. (The latter will be described by evolution equations later.)

Yet simpler way how to get the second equation is to rewrite the first term of (27.10) in the same way as we did with the term $-\vec{\Pi}\cdot\vec{\nabla}\phi$ in electrodynamics (27.5), namely as

$$2\Pi^{\mu\nu}N_{\mu|\nu} = 2(\Pi^{\mu\nu}N_\mu)_{|\nu} - 2\Pi^{\mu\nu}{}_{|\nu}N_\mu$$

and discard the first, "boundary" part given by divergence. Then the last equation appears as

$$0\left(=\dot{\Pi}_{N_\alpha}\right) = -\frac{\partial\mathcal{H}_{\rm g}}{\partial N_\alpha} + \cancel{\left(\frac{\partial\mathcal{H}_{\rm g}}{\partial N_{\alpha|\iota}}\right)_{|\iota}} = 2\Pi^{\alpha\nu}{}_{|\nu}\,.$$

Submitting $2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2 = 2h(K_{\mu\nu}K^{\mu\nu}-K^2)$ and $\Pi^{\alpha\nu}=\sqrt{h}\,(K^{\alpha\nu}-Kh^{\alpha\nu})$ to the above equations reveals that they represent the **Hamiltonian and momentum constraints**,[3]

$${}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 2\Lambda \qquad \text{and} \qquad K^{\alpha\nu}{}_{|\nu} - K^{|\alpha} = 0$$

– see equations (25.42) and (25.43).

27.2.2 Hamiltonian as a combination of constraints

Let us point out that in order to infer dynamics from the Hamiltonian (see below), the constraints *must not* be plugged back into it – the Hamiltonian has to remain "off shell" (i.e. not evaluated along the actual evolution), otherwise it would not be possible to compute its derivatives "in arbitrary direction". Actually, it is easy to check that plugging the constraints back would even make the Hamiltonian *zero*. On the other hand, if computing global quantities by integrating the Hamiltonian over spatial slices (rather than differentiating it), the constraints *should* be satisfied, so its "dynamical part" really vanishes then. To such integral quantities thus only contribute the *boundary terms* (which we have been omitting yet).

The "on-shell" vanishing of $\mathcal{H}_{\rm g}$ is a symptom of a more general property: in theories invariant with respect to general diffeomorphisms, the Hamiltonian in fact turns out to be given by a linear combination of its constraints[4] (plus the boundary terms). Worth to realize again that the constraints have appeared as a consequence of independence of the theory (of the Lagrangian) of time derivatives of quantities which are subject to *gauge or coordinate freedom* and thus do not represent dynamical variables. (They stand as coefficients in the combination of constraints which represent the Hamiltonian.) Indeed, there is a one-to-one correspondence between local symmetries of a theory and its constraints. We will analyse it more in Chapter 28, specifically in sections on Noether theorems.

In designing a theory, the above correspondence can factually be employed in the opposite way: instead of starting from certain symmetry requirements, one may prescribe a certain set of constraints and design the Hamiltonian as their combination; the constraints then "generate" certain local symmetries.

---

[3] Sometimes they are called the scalar and vector constraints, with the latter (the momentum one) also being called the **diffeomorphism constraint**, because it is linked to the choice of coordinates on the Σ_t slices.

[4] The opposite need not be true: vanishing of the constrained Hamiltonian does not necessarily imply that the theory is diffeomorphism-invariant.

27.2.3 Hamilton equations: evolutions

The main work is still to be done. The **evolution equations** are given by Hamilton equations

$$\dot{h}_{\alpha\beta} := h^{\mu}_{\alpha} h^{\nu}_{\beta} \pounds_{\mathbf{t}} h_{\mu\nu} = \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial \Pi^{\alpha\beta}} \,,$$
$$\dot{\Pi}^{\alpha\beta} := h^{\alpha}_{\mu} h^{\beta}_{\nu} \pounds_{\mathbf{t}} \Pi^{\mu\nu} = -\frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial h_{\alpha\beta}} + \left(\frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} - \left(\text{possibly terms given by } \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial h_{\alpha\beta,\iota\kappa}}\right).$$

In the first equation, differentiation of (27.10) is clear, we only compute more carefully

$$\frac{\partial \Pi^2}{\partial \Pi^{\alpha\beta}} = 2\Pi \, \frac{\partial \Pi}{\partial \Pi^{\alpha\beta}} = 2\Pi \, \frac{\partial}{\partial \Pi^{\alpha\beta}} (\Pi^{\kappa\lambda} h_{\kappa\lambda}) = 2\Pi \, \delta^{\kappa}_{\alpha} \delta^{\lambda}_{\beta} h_{\kappa\lambda} = 2\Pi h_{\alpha\beta} \,,$$

thus arriving at

$$\dot{h}_{\alpha\beta} = \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial \Pi^{\alpha\beta}} = 2N_{(\alpha|\beta)} + \frac{N}{\sqrt{h}} \left(2\Pi_{\alpha\beta} - \Pi h_{\alpha\beta}\right) . \tag{27.13}$$

This exactly repeats the definition (27.8) of $\Pi_{\alpha\beta}$ and, recalling that $2\Pi_{\alpha\beta} - \Pi h_{\alpha\beta} = 2\sqrt{h}\, K_{\alpha\beta}$, the relation (25.16) between $K_{\alpha\beta}$ and $\dot{h}_{\mu\nu}$.

For the second equation, let us tackle each of the Hamiltonian (27.10) terms separately:

- In the last term, $-N\sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right)$, we use the knowledge from the variational derivation of Einstein equations. We learned there – see equation (23.15) – that if dropping the surface terms (given by behaviour of the metric *derivatives* on the integration-region boundary), then

$$\delta \left[\sqrt{-g}\,(R - 2\Lambda)\right] = \sqrt{-g} \, \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) \delta g^{\mu\nu}$$
$$\Longrightarrow \quad \frac{\partial}{\partial g^{\mu\nu}} \left[\sqrt{-g}\,(R - 2\Lambda)\right] = \sqrt{-g} \, \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right).$$

  However, we would prefer to know the derivative with respect to *covariant* metric, which reverses the sign, as we know from Section 23.4.1: specifically, we obtained there $\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}$, so

$$\delta \left[\sqrt{-g}\,(R - 2\Lambda)\right] = -\sqrt{-g} \, \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \Lambda g^{\alpha\beta}\right) \delta g_{\alpha\beta} \,,$$
$$\Longrightarrow \quad \frac{\partial}{\partial g_{\alpha\beta}} \left[\sqrt{-g}\,(R - 2\Lambda)\right] = -\sqrt{-g} \, \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \Lambda g^{\alpha\beta}\right).$$

  In the 3D analogy to this result, we may claim that

$$\frac{\partial}{\partial h_{\alpha\beta}} \left[\sqrt{h}\left({}^{(3)}\!R - 2\Lambda\right)\right] = -\sqrt{h} \, \left({}^{(3)}\!R^{\alpha\beta} - \frac{1}{2}{}^{(3)}\!R h^{\alpha\beta} + \Lambda h^{\alpha\beta}\right). \tag{27.14}$$

- In the middle term of (27.10), $\frac{N}{2\sqrt{h}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2\right)$, we rewrite

$$2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2=\Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2h_{\kappa\mu}h_{\lambda\nu}-h_{\kappa\lambda}h_{\mu\nu}\right)$$

in order to differentiate it by $h_{\mu\nu}$,

$$\begin{aligned}\frac{\partial}{\partial h_{\alpha\beta}}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2\right)&=\Pi^{\kappa\lambda}\Pi^{\mu\nu}\left(2\delta^\alpha_\kappa\delta^\beta_\mu h_{\lambda\nu}+2h_{\kappa\mu}\delta^\alpha_\lambda\delta^\beta_\nu-\delta^\alpha_\kappa\delta^\beta_\lambda h_{\mu\nu}-h_{\kappa\lambda}\delta^\alpha_\mu\delta^\beta_\nu\right)=\\&=4\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu}-2\Pi\Pi^{\alpha\beta}\ .\end{aligned}$$

Recalling Section 23.4.1 once again, specifically equation (23.10) for the derivative of the metric determinant, $\frac{\partial(-g)}{\partial g_{\mu\nu}}=(-g)g^{\mu\nu}$, we analogously take $\frac{\partial h}{\partial h_{\alpha\beta}}=hh^{\alpha\beta}$ here on Σ_t, so

$$\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{1}{\sqrt{h}}\right)=-\frac{1}{2h^{3/2}}\frac{\partial h}{\partial h_{\alpha\beta}}=-\frac{1}{2h^{3/2}}\,hh^{\alpha\beta}=-\frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\ .$$

Hence, in total,

$$\begin{aligned}&\frac{\partial}{\partial h_{\alpha\beta}}\left(\frac{2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2}{\sqrt{h}}\right)=\\&=\frac{2}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu}-\Pi\Pi^{\alpha\beta}\right)-\frac{1}{2\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu}-\Pi^2\right)\ .\end{aligned}\tag{27.15}$$

- Finally, in the first term of (27.10), $2\Pi^{\mu\nu}N_{(\mu|\nu)}$, the momenta $\Pi^{\mu\nu}$ are taken as independent of $h_{\alpha\beta}$, and we rewrite

$$\begin{aligned}N_{\mu|\nu}&=h_{\mu\kappa}N^\kappa{}_{|\nu}=h_{\mu\kappa}\left(N^\kappa{}_{,\nu}+{}^{(3)}\Gamma^\kappa{}_{\nu\lambda}N^\lambda\right)=h_{\mu\kappa}N^\kappa{}_{,\nu}+{}^{(3)}\Gamma_{\mu\nu\lambda}N^\lambda=\\&=h_{\mu\kappa}N^\kappa{}_{,\nu}+\frac{1}{2}\left(h_{\mu\nu,\lambda}+h_{\lambda\mu,\nu}-h_{\nu\lambda,\mu}\right)N^\lambda\\\Longrightarrow\quad 2N_{(\mu|\nu)}&=h_{\mu\kappa}N^\kappa{}_{,\nu}+h_{\nu\kappa}N^\kappa{}_{,\mu}+h_{\mu\nu,\lambda}N^\lambda\end{aligned}$$

(the last two terms in the parenthesis together are anti-symmetric in μ,ν), so

$$\begin{aligned}\frac{\partial(2\Pi^{\mu\nu}N_{(\mu|\nu)})}{\partial h_{\alpha\beta}}&=\Pi^{\mu\nu}\delta^{(\alpha}_\mu\delta^{\beta)}_\kappa N^\kappa{}_{,\nu}+\Pi^{\mu\nu}\delta^{(\alpha}_\nu\delta^{\beta)}_\kappa N^\kappa{}_{,\mu}=\Pi^{\nu(\alpha}N^{\beta)}{}_{,\nu}+\Pi^{\mu(\alpha}N^{\beta)}{}_{,\mu}=\\&=2\Pi^{\mu(\alpha}N^{\beta)}{}_{,\mu}=\Pi^{\mu\alpha}N^\beta{}_{,\mu}+\Pi^{\mu\beta}N^\alpha{}_{,\mu}\ ,\\\frac{\partial(2\Pi^{\mu\nu}N_{(\mu|\nu)})}{\partial h_{\alpha\beta,\iota}}&=\Pi^{\mu\nu}\delta^\alpha_\mu\delta^\beta_\nu\delta^\iota_\lambda N^\lambda=\Pi^{\alpha\beta}N^\iota\ ,\end{aligned}$$

where the symmetrization in the former term is dictated by the symmetry of $h_{\alpha\beta}$. Putting the two terms together and rewriting them in terms of the 3D covariant derivative, we have

$$\begin{aligned}&\left[\frac{\partial(2\Pi^{\mu\nu}N_{(\mu|\nu)})}{\partial h_{\alpha\beta,\iota}}\right]_{,\iota}-\frac{\partial(2\Pi^{\mu\nu}N_{(\mu|\nu)})}{\partial h_{\alpha\beta}}=\\&\quad=\Pi^{\alpha\beta}{}_{,\iota}N^\iota+\Pi^{\alpha\beta}N^\iota{}_{,\iota}-\Pi^{\mu\alpha}N^\beta{}_{,\mu}-\Pi^{\mu\beta}N^\alpha{}_{,\mu}=\end{aligned}$$

$$
\begin{aligned}
&= \left(\Pi^{\alpha\beta}{}_{|\iota} - \cancel{{}^{(3)}\Gamma^{\alpha}{}_{\iota\sigma}\Pi^{\sigma\beta}} - \cancel{{}^{(3)}\Gamma^{\beta}{}_{\iota\sigma}\Pi^{\alpha\sigma}}\right) N^{\iota} + \Pi^{\alpha\beta} N^{\iota}{}_{,\iota} - \\
&\quad - \Pi^{\mu\alpha}\left(N^{\beta}{}_{|\mu} - \cancel{{}^{(3)}\Gamma^{\beta}{}_{\mu\sigma}N^{\sigma}}\right) - \Pi^{\mu\beta}\left(N^{\alpha}{}_{|\mu} - \cancel{{}^{(3)}\Gamma^{\alpha}{}_{\mu\sigma}N^{\sigma}}\right) = \\
&= \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}N^{\iota}{}_{,\iota} - \Pi^{\mu\alpha}N^{\beta}{}_{|\mu} - \Pi^{\mu\beta}N^{\alpha}{}_{|\mu} = \\
&= \Pi^{\alpha\beta}{}_{|\iota}N^{\iota} + \Pi^{\alpha\beta}\sqrt{h}\left(\frac{N^{\iota}}{\sqrt{h}}\right)_{|\iota} - \Pi^{\mu\alpha}N^{\beta}{}_{|\mu} - \Pi^{\mu\beta}N^{\alpha}{}_{|\mu} = \\
&= (\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\mu\alpha}N^{\beta}{}_{|\mu} - \Pi^{\mu\beta}N^{\alpha}{}_{|\mu} \,.
\end{aligned}
$$

Therefore, collecting the above results for the three Hamiltonian terms, we conclude that

$$
\begin{aligned}
\dot{\Pi}^{\alpha\beta} &:= h^{\alpha}_{\mu}h^{\beta}_{\nu}\pounds_{\mathbf{t}}\Pi^{\mu\nu} = -\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial h_{\alpha\beta}} + \left(\frac{\partial\mathcal{H}_{\mathrm{g}}}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} = \\
&= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right) - \frac{N}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) + \\
&\quad + \frac{N}{4\sqrt{h}}h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + (\Pi^{\alpha\beta}N^{\iota})_{|\iota} - \Pi^{\mu\alpha}N^{\beta}{}_{|\mu} - \Pi^{\mu\beta}N^{\alpha}{}_{|\mu} \,.
\end{aligned}
\tag{27.16}
$$

27.2.4 Boundary contribution to the field equations

In Chapter 23 on Lagrangian approach, a specific attention had to be devoted to the behaviour of the metric (in fact of its derivatives) on the boundary of the chosen space-time region. Actually, we showed in Section 23.4.6 that if only $\delta g_{\mu\nu}$ are assumed to vanish there, whereas the variation of the metric *derivatives* is left free, a boundary term appears which then has to be subtracted from the action in order to obtain correct field equations. However, the boundary term could be expressed concisely as an integral of the shape-operator trace K (called mean curvature) over the boundary.

Here, one dimension lower, the situation is more tricky. Let us primarily tackle the same issue as recalled above: what kind of contribution to the Hamiltonian arises if we release the assumption about vanishing of $\delta g_{\mu\nu,\alpha}$ on the boundary $\delta\Omega$ of the 4D region Ω? What does it imply for the situation on the 3D space-like hypersurface Σ_t where we formulate the Hamiltonian picture?

It is natural to restrict to Σ_t by considering the space-time region Ω to be a "cylinder" with space-like bases and time-like cylindrical hypersurface, and to obtain the 3D picture by shrinking the height of the cylinder to zero. In such a limit, the conditions on the (now identified) bases should go over to the conditions on the hypersurface Σ_t itself, thus also *inside* the relevant region (S_t) within that hypersurface, so one may expect that if the variations $\delta g_{\mu\nu,\alpha}$ do not vanish there, it might in fact contribute to the "bulk" ("volume") behaviour. Below, we demonstrate that the extra term thus originating can be expressed as proportional to $\delta h_{\alpha\beta}$, so it indeed contributes to the variation with respect to $h_{\alpha\beta}$, and thus to the derivative of the Hamiltonian with respect to it. The behaviour on the now 2D, spatial boundary ∂S_t which arose by shrinking the originally 3D cylindrical surface, apparently should be queried as well, but we will *not* address that point in this section, effectively assuming that *there* the variation of $h_{\alpha\beta,\iota}$ does vanish.

Following again the analogy with the 4D case, we start from the 4D variation (23.13) and notice the surface term (the second one), specifically how we fixed it in (23.14). For convenience, we repeat the whole expression:

$$\delta\int(R-2\Lambda)\sqrt{-g}\,\mathrm{d}^4x=\int\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x+\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x\,,$$

where the second term we arranged into the form

$$\int(g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\nu\mu}-g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\nu\mu})_{;\lambda}\sqrt{-g}\,\mathrm{d}^4x\,,$$

rewrote its integrand as a partial divergence of the vector $\sqrt{-g}\left(g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\nu\mu}-g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\nu\mu}\right)$ and thus got rid of it (with a little help from our friend C.F.G.).

In analogy, we thus have, one dimension lower, the integrand

$$\begin{aligned}&\delta\left[N\sqrt{h}\left({}^{(3)}R-2\Lambda\right)\right]=\\&=-N\sqrt{h}\left({}^{(3)}R^{\alpha\beta}-\frac{1}{2}{}^{(3)}Rh^{\alpha\beta}+\Lambda h^{\alpha\beta}\right)\delta h_{\alpha\beta}+N\sqrt{h}\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}-h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right)_{|\delta}\,,\end{aligned}$$

where the opposite sign at the first term is because we work in terms of $\delta h_{\alpha\beta}$ rather than $\delta h^{\alpha\beta}$, and N stays "intact" under the variation with respect to $h_{\alpha\beta}$. The first part we have already covered – see (27.14) – while the second term we are newly questioning now.

- First, write the new term as a "divergence of everything minus the extra term thus added",

$$\begin{aligned}&N\sqrt{h}\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}-h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right)_{|\delta}=\\&=\sqrt{h}\left(Nh^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}-Nh^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right)_{|\delta}-\sqrt{h}\,N_{|\delta}\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}-h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right),\end{aligned}\tag{27.17}$$

  of which the first term can be written in terms of partial divergence, so it does not contribute, *provided that* $\delta^{(3)}\Gamma$s vanish on the 2D boundary ∂S_t of our region S_t within Σ_t.

- Second, in perfect analogy with the 4D-case formulae (23.21) and (23.22) for the variations of Christoffel symbols, the variations of 3D Gammas read

$$\delta^{(3)}\Gamma_{\iota\beta\alpha}=\frac{1}{2}\left[(\delta h_{\iota\beta})_{|\alpha}+(\delta h_{\alpha\iota})_{|\beta}-(\delta h_{\beta\alpha})_{|\iota}\right]+{}^{(3)}\Gamma^{\sigma}{}_{\beta\alpha}\delta h_{\sigma\iota}\,,\tag{27.18}$$

$$\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}=\frac{1}{2}h^{\delta\iota}\left[(\delta h_{\iota\beta})_{|\alpha}+(\delta h_{\alpha\iota})_{|\beta}-(\delta h_{\beta\alpha})_{|\iota}\right].\tag{27.19}$$

- As in the 4D case again, we can thus simplify

$$\begin{aligned}&\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha}-h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right)=\\&=\frac{1}{2}(h^{\alpha\beta}h^{\delta\iota}-h^{\alpha\delta}h^{\beta\iota})\left[(\delta h_{\iota\beta})_{|\alpha}+(\delta h_{\alpha\iota})_{|\beta}-(\delta h_{\beta\alpha})_{|\iota}\right]=\\&=\frac{1}{2}(h^{\alpha\beta}h^{\delta\iota}-h^{\alpha\delta}h^{\beta\iota})\left[(\delta h_{\iota\beta})_{|\alpha}-(\delta h_{\beta\alpha})_{|\iota}\right]=(h^{\alpha\beta}h^{\delta\iota}-h^{\alpha\delta}h^{\beta\iota})\,(\delta h_{\iota\beta})_{|\alpha}\end{aligned}$$

(because both the parenthesis and the bracket are anti-symmetric in $[\alpha, \iota]$), and so the remaining term from the first step (the second one there) can be written as

$$\begin{aligned} &-\sqrt{h}\,N_{|\delta}\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha} - h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right) = -\sqrt{h}\,N_{|\delta}\left(h^{\alpha\beta}h^{\delta\iota} - h^{\alpha\delta}h^{\beta\iota}\right)(\delta h_{\iota\beta})_{|\alpha} = \\ &= \sqrt{h}\,\left(N^{|\alpha}h^{\beta\iota} - N^{|\iota}h^{\alpha\beta}\right)(\delta h_{\beta\iota})_{|\alpha}\,. \end{aligned}$$

- Now the usual trick, finally: the above term we rewrite as

$$\begin{aligned} &\cancel{\sqrt{h}\,\left[\left(N^{|\alpha}h^{\beta\iota} - N^{|\iota}h^{\alpha\beta}\right)\delta h_{\beta\iota}\right]_{|\alpha}} - \sqrt{h}\left(N^{|\alpha}h^{\beta\iota} - N^{|\iota}h^{\alpha\beta}\right)_{|\alpha}\delta h_{\beta\iota} = \\ &= -\sqrt{h}\left(N^{|\alpha}{}_{\alpha}h^{\beta\iota} - N^{|\beta\iota}\right)\delta h_{\beta\iota}\,, \end{aligned}$$

  where the first term represents partial divergence of $\left[\sqrt{h}\left(N^{|\alpha}h^{\beta\iota} - N^{|\iota}h^{\alpha\beta}\right)\delta h_{\beta\iota}\right]$, which, due to the Gauss law, equals the flux of this vector over the 2D boundary ∂S_t – and that is zero due to the vanishing of $\delta h_{\beta\iota}$ there.

Thereby, we have derived that the correct result for the variation of the last term of the Hamiltonian (27.10) reads

$$\begin{aligned} &\delta\left[N\sqrt{h}\left({}^{(3)}R - 2\Lambda\right)\right] = \\ &= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right)\delta h_{\alpha\beta} + N\sqrt{h}\left(h^{\alpha\beta}\delta^{(3)}\Gamma^{\delta}{}_{\beta\alpha} - h^{\alpha\delta}\delta^{(3)}\Gamma^{\beta}{}_{\beta\alpha}\right)_{|\delta} = \\ &= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right)\delta h_{\alpha\beta} - \sqrt{h}\left(N^{|\delta}{}_{\delta}h^{\alpha\beta} - N^{|\alpha\beta}\right)\delta h_{\alpha\beta}\,, \end{aligned}$$

and hence the overall result (27.16) has to be supplemented by the above second term,

$$\boxed{\begin{aligned} \dot{\Pi}^{\alpha\beta} &:= h^{\alpha}_{\mu}h^{\beta}_{\nu}\pounds_{\mathbf{t}}\Pi^{\mu\nu} = -\frac{\partial\mathcal{H}_{\rm g}}{\partial h_{\alpha\beta}} + \left(\frac{\partial\mathcal{H}_{\rm g}}{\partial h_{\alpha\beta,\iota}}\right)_{,\iota} = \\ &= -N\sqrt{h}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}{}^{(3)}Rh^{\alpha\beta} + \Lambda h^{\alpha\beta}\right) - \sqrt{h}\left(N^{|\delta}{}_{\delta}h^{\alpha\beta} - N^{|\alpha\beta}\right) - \\ &\quad - \frac{N}{\sqrt{h}}\left(2\Pi^{\alpha\lambda}\Pi^{\beta\nu}h_{\lambda\nu} - \Pi\Pi^{\alpha\beta}\right) + \frac{N}{4\sqrt{h}}\,h^{\alpha\beta}\left(2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2\right) + \\ &\quad + \left(\Pi^{\alpha\beta}N^{\iota}\right)_{|\iota} - \Pi^{\mu\alpha}N^{\beta}{}_{|\mu} - \Pi^{\mu\beta}N^{\alpha}{}_{|\mu}\,. \end{aligned}} \tag{27.20}$$

27.2.5 Boundary part of the Hamiltonian

Although we have supplemented the boundary contribution *to the field equations*, the boundary part *of the Hamiltonian* has to be completed more carefully. We will include it via Legendre transformation, so we need to first evaluate, in the 3+1 manner, the Lagrangian of the action (23.25), i.e. of

$$S = \int\limits_{\Omega}(R - 2\Lambda + 16\pi\mathcal{L}_{\rm ng})\sqrt{-g}\;{\rm d}^4x + 2\epsilon\oint\limits_{\partial\Omega}(K - K_{\rm flat})\,\sqrt{h}\;{\rm d}^3y\,,$$

where $\epsilon := n_\iota n^\iota$ with n^α the outward normal to $\partial\Omega$. (We will focus on the gravitational part, thus not taking $\mathcal{L}_{\rm ng}$ into account.) Two contributions have to be added to (27.10): one arising from the 3+1 evaluation of the above boundary term $2\epsilon \oint_{\partial\Omega}(K - K_{\rm flat})\sqrt{h}\,\mathrm{d}^3y$, and the other arising from the divergence terms present in the 3+1 decomposition of the Ricci scalar R (which we omitted in the derivation of (27.10)).

- The evaluation of the boundary term rests in proper grasp of the boundary $\partial\Omega$. In the 3+1 picture, $\partial\Omega$ is composed of two space-like regions S_1 and S_2 selected within certain two (partial) Cauchy hypersurfaces Σ_{t_1} and (later) Σ_{t_2}, and of a cylindrical-type, smooth time-like "sleeve" $\mathcal{B}$ which is supposed to connect them *in an orthogonal manner*.[5] On $\mathcal{B}$, the relevant quantities are defined by the unit *space-like* outer normal of $\mathcal{B}$ (denote it by r^μ) rather than by n^μ: the extrinsic curvature reads there $\mathcal{K}_{ab} = r_{\alpha;\beta}\frac{\partial x^\alpha}{\partial z^a}\frac{\partial x^\beta}{\partial z^b}$, with z^i some intrinsic coordinates on $\mathcal{B}$, and the corresponding mean curvature is $\mathcal{K} = \gamma^{ab}\mathcal{K}_{ab}$, where γ^{ab} is an inverse to the metric induced on $\mathcal{B}$, i.e. to $\gamma_{ab} := g_{\mu\nu}\frac{\partial x^\mu}{\partial z^a}\frac{\partial x^\nu}{\partial z^b}$. Clearly γ^{ab} is the space-time metric "without the r^μ direction" and it is $(-{+}{+})$, namely $\gamma^{ab}\frac{\partial x^\alpha}{\partial z^a}\frac{\partial x^\beta}{\partial z^b} = g^{\alpha\beta} - r^\alpha r^\beta$.
  Hence, the boundary term (without normalization $K_{\rm flat}$ for now) decomposes as
  $$2\epsilon \oint_{\partial\Omega} K\sqrt{\text{3D metric on } \partial\Omega}\,\mathrm{d}^3y = 2\int_{S_1} K\sqrt{h}\,\mathrm{d}^3y - 2\int_{S_2} K\sqrt{h}\,\mathrm{d}^3y + 2\int_{\mathcal{B}} \mathcal{K}\sqrt{-\gamma}\,\mathrm{d}^3z\,,$$
  because ϵ is -1 for the bases S_1 and S_2 whereas it is $+1$ for $\mathcal{B}$, while, moreover, the future-pointing normal is an *inward* one for S_1, thus the "opposite" sign of the first term.

- In derivation of the gravitational-Hamiltonian density (27.10), we were omitting the boundary terms, so we took as the Lagrangian density just
  $$\mathcal{L}_{\rm g} = R - 2\Lambda = {}^{(3)}\!R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2\Lambda\,.$$
  If *not* dropping the divergence terms, we would have had
  $$\mathcal{L}_{\rm g} = {}^{(3)}\!R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2\Lambda - 2a^\delta{}_{;\delta} + 2(Kn^\gamma)_{;\gamma}\,,$$
  as given by the full Ricci-tensor decomposition (27.6). Using the Gauss theorem, the omitted terms contribute by $-2\oint_{\partial\Omega}(a^\delta - Kn^\delta)\,\mathrm{d}\Sigma_\delta$. Within the S_1 and S_2 regions, $a^\delta n_\delta = 0$ (since $n^\delta n_\delta \equiv \epsilon = \text{const}$) and $\mathrm{d}\Sigma_\delta = n_\delta\sqrt{h}\,\mathrm{d}^3y$, so one obtains there
  $$-2\int_{S_{1,2}} (a^\delta - Kn^\delta)\,n_\delta\sqrt{h}\,\mathrm{d}^3y = 2\epsilon\int_{S_{1,2}} K\sqrt{h}\,\mathrm{d}^3y\,,$$
  which exactly compensates the boundary term of the first item. On the side hypersurface $\mathcal{B}$, however, $a^\delta n_\delta \neq 0$ since a^μ is the four-acceleration of n^μ and n^μ is *not* normal to $\mathcal{B}$. Actually, the normal of $\mathcal{B}$ is r^μ, so on $\mathcal{B}$ the term $-2\oint_{\partial\Omega}(a^\delta - Kn^\delta)\,\mathrm{d}\Sigma_\delta$ yields
  $$-2\int_{\mathcal{B}}(n^\delta{}_{;\iota}n^\iota - \mathcal{K}n^\delta)\,r_\delta\sqrt{-\gamma}\,\mathrm{d}^3z = -2\int_{\mathcal{B}} n^\delta{}_{;\iota}n^\iota r_\delta\sqrt{-\gamma}\,\mathrm{d}^3z = 2\int_{\mathcal{B}} r_{\delta;\iota}n^\delta n^\iota\sqrt{-\gamma}\,\mathrm{d}^3z\,.$$

[5] That is, $\mathcal{B}$ is supposed to be orthogonal to all the Cauchy hypersurfaces Σ_t which it intersects, thus the field n^μ is everywhere tangent to it. Therefore, the normal to $\mathcal{B}$ which we are about to call r^μ fulfils $g_{\mu\nu}n^\mu r^\nu = 0$.

Summarizing from above, we have found that the gravitational part of the action (23.25) yields, if boundary contributions are included,

$$S_{\mathrm{g}} = \int\limits_{\Omega} ({}^{(3)}R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2\Lambda)\sqrt{-g}\,\mathrm{d}^4x + 2\int\limits_{\mathcal{B}} (\mathcal{K} + r_{\delta;\iota}n^\delta n^\iota)\sqrt{-\gamma}\,\mathrm{d}^3z\,. \tag{27.21}$$

The boundary term can be worked out further. Writing the mean curvature of $\mathcal{B}$ as

$$\mathcal{K} \equiv \gamma^{ab}\mathcal{K}_{ab} \equiv \gamma^{ab}\left(r_{\alpha;\beta}\frac{\partial x^\alpha}{\partial z^a}\frac{\partial x^\beta}{\partial z^b}\right) = r_{\alpha;\beta}\left(\gamma^{ab}\frac{\partial x^\alpha}{\partial z^a}\frac{\partial x^\beta}{\partial z^b}\right) = r_{\alpha;\beta}(g^{\alpha\beta} - r^\alpha r^\beta)\,,$$

we obtain

$$\mathcal{K} + r_{\delta;\iota}n^\delta n^\iota = r_{\alpha;\beta}(g^{\alpha\beta} - r^\alpha r^\beta + n^\alpha n^\beta) =: \hbar\,.$$

This means the projection of $r_{\alpha;\beta}$ on the (spheroidal) 2D surface ∂S_t given by intersection of $\mathcal{B}$ with Σ_t, thus standing for the extrinsic curvature of that surface *as a submanifold of* Σ_t (because r^μ itself is tangent to Σ_t). Denoting such a quantity by $\hbar$ as indicated, we thus arrive at the boundary result

$$2\int_{\mathcal{B}} (\hbar - \hbar_{\mathrm{flat}})\sqrt{-\gamma}\,\mathrm{d}^3z\,, \tag{27.22}$$

where we have restored the appropriate normalization term $\hbar_{\mathrm{flat}}$ which ensures that the boundary contribution does not diverge in the limit when the 2D surface $\partial S_t \equiv \Sigma_t \cap \mathcal{B}$ is taken at spatial infinity. (Geometrically, $\hbar_{\mathrm{flat}}$ represents the extrinsic curvature of that surface as embedded in flat 3D space.)

In order to clearly identify the full Lagrangian and then the Hamiltonian (rather than their spatial densities), let us finally decompose the integration over $\sqrt{-g}\,\mathrm{d}^4x$ into the integration over S_t (i.e. $\sqrt{h}\,\mathrm{d}^3y$) and the time integration ($N\mathrm{d}t$). The integration on $\mathcal{B}$ can also be decomposed accordingly, if choosing one of the z^a coordinates as t. One thus obtains the complete gravitational action (27.21) in the form

$$S = \int\limits_{t_1}^{t_2}\left[\int_{S_t}\left({}^{(3)}R - K^2 + K_{\nu\gamma}K^{\nu\gamma} - 2\Lambda\right)N\sqrt{h}\,\mathrm{d}^3y + 2\oint\limits_{\partial S_t}(\hbar - \hbar_{\mathrm{flat}})N\sqrt{\sigma}\,\mathrm{d}^2\theta\right]\mathrm{d}t\,,$$

where $\sqrt{\sigma}\,\mathrm{d}^2\theta$ has been used for the space-like area element on ∂S_t.

Finally, taking the Legendre transformation

$$\mathcal{H}_{\mathrm{g}} = \frac{\partial(\sqrt{-g}\mathcal{L}_{\mathrm{g}})}{\partial\dot{h}_{\mu\nu}}\,\dot{h}_{\mu\nu} - \sqrt{-g}\,\mathcal{L}_{\mathrm{g}} = \frac{\partial(\sqrt{-g}\mathcal{L}_{\mathrm{g}})}{\partial K_{\alpha\beta}}\frac{\partial K_{\alpha\beta}}{\partial\dot{h}_{\mu\nu}}\,\dot{h}_{\mu\nu} - \sqrt{-g}\,\mathcal{L}_{\mathrm{g}}$$

and realizing that the newly added boundary term (27.22) does not depend on $\dot{h}_{\mu\nu}$ (on $K_{\alpha\beta}$), one sees that it is sufficient, in order to get the *full* gravitational Hamiltonian, to subtract from the "dynamical" part of its density (27.10) the boundary part of $\sqrt{-g}\,\mathcal{L}_{\mathrm{g}}$, and integrate it over S_t and ∂S_t, respectively,

$$
\begin{aligned}
H_{\mathrm{g}} &= \int\limits_{S_t} \mathcal{H}_{\mathrm{g}}\,\mathrm{d}^3 y - 2 \oint\limits_{\partial S_t} (\cancel{k} - \cancel{k}_{\mathrm{flat}}) N \sqrt{\sigma}\,\mathrm{d}^2\theta = \\
&= \int\limits_{S_t} \left[2\Pi^{\mu\nu} N_{(\mu|\nu)} + \frac{N}{2\sqrt{h}} (2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2) - N\sqrt{h}\,({}^{(3)}R - 2\Lambda) \right] \mathrm{d}^3 y - \\
&\quad - 2 \oint\limits_{\partial S_t} (\cancel{k} - \cancel{k}_{\mathrm{flat}}) N \sqrt{\sigma}\,\mathrm{d}^2\theta\,.
\end{aligned}
$$

For the last point, it is more convenient to return to the original, "Lagrangian" variables by

$$
\Pi^{\mu\nu} = \sqrt{h}\,(K^{\mu\nu} - K h^{\mu\nu})\,, \qquad 2\Pi_{\mu\nu}\Pi^{\mu\nu} - \Pi^2 = 2h\left(K_{\mu\nu}K^{\mu\nu} - K^2\right)
$$

and to write

$$
\begin{aligned}
H_{\mathrm{g}} &= \int\limits_{S_t} \left[2(K^{\mu\nu} - K h^{\mu\nu}) N_{(\mu|\nu)} + N\left(K_{\mu\nu}K^{\mu\nu} - K^2 - {}^{(3)}R + 2\Lambda\right)\right] \sqrt{h}\,\mathrm{d}^3 y - \\
&\quad - 2 \oint\limits_{\partial S_t} (\cancel{k} - \cancel{k}_{\mathrm{flat}}) N \sqrt{\sigma}\,\mathrm{d}^2\theta\,.
\end{aligned}
$$

Namely, the last point is to rewrite the first term as

$$
2(K^{\mu\nu} - K h^{\mu\nu}) N_{(\mu|\nu)} = 2\left[(K^{\mu\nu} - K h^{\mu\nu}) N_\mu\right]_{|\nu} - 2(K^{\mu\nu} - K h^{\mu\nu})_{|\nu} N_\mu
$$

and see that this term consists of the "bulk" part $-2(K^{\mu\nu} - K h^{\mu\nu})_{|\nu} N_\mu$ and the 3-divergence part which can be shifted to the boundary ∂S_t,

$$
2 \int\limits_{S_t} \left[(K^{\mu\nu} - K h^{\mu\nu}) N_\mu\right]_{|\nu} \sqrt{h}\,\mathrm{d}^3 y = 2 \oint\limits_{\partial S_t} (K^{\mu\nu} - K h^{\mu\nu}) N_\mu r_\nu \sqrt{\sigma}\,\mathrm{d}^2\theta\,.
$$

The complete Hamiltonian thus acquires the form

$$
\boxed{
\begin{aligned}
H_{\mathrm{g}} &= \int\limits_{S_t} \left[-2(K^{\mu\nu} - K h^{\mu\nu})_{|\nu} N_\mu + N\left(K_{\mu\nu}K^{\mu\nu} - K^2 - {}^{(3)}R + 2\Lambda\right)\right] \sqrt{h}\,\mathrm{d}^3 y + \\
&\quad + 2 \oint\limits_{\partial S_t} \left[(K^{\mu\nu} - K h^{\mu\nu}) N_\mu r_\nu - N(\cancel{k} - \cancel{k}_{\mathrm{flat}})\right] \sqrt{\sigma}\,\mathrm{d}^2\theta\,.
\end{aligned}
}
\tag{27.23}
$$

The second, boundary part does not contribute to the variation and thus to the field equations, but one has to take it into account when attempting to find integral quantities from the Hamiltonian (from the Hamiltonian *itself*, not from its variation). In fact *only* the boundary terms contribute "on-shell" because, as we have learnt, the "volume" terms of the Hamiltonian vanish if constraint equations are satisfied.

27.2.6 Mass and angular momentum of asymptotically flat space-times

Since the value of the Hamiltonian represents mass-energy, one may obtain this global quantity by “on-shell” evaluation of the Hamiltonian. In such a case, only its boundary part is left. This part is *also* zero for compact space-times, whereas for non-compact ones its value depends on asymptotic behaviour at spatial infinity. We will compute the mass-energy for asymptotically flat space-times.

Recall the basic relation (25.3) of the 3+1 decomposition, $t^\mu := \frac{\partial x^\mu}{\partial t} = Nn^\mu + N^\mu$. How can this be expected to behave asymptotically? In the asymptotically flat case, it is natural to require that, far away, t goes over to the asymptotic inertial time and t^μ goes over to n^μ which at infinity becomes the four-velocity of inertial observers. This corresponds to the asymptotic values $N^\mu = 0$, $N = 1$. Under such a behaviour, the boundary Hamiltonian reduces to

$$M_{\rm ADM} = -\frac{1}{8\pi} \lim_{\partial S_t \to \infty} \oint\limits_{\partial S_t} (\not k - \not k_{\rm flat})\sqrt{\sigma}\, {\rm d}^2\theta\,, \tag{27.24}$$

where we have added $1/(16\pi)$ in correspondence with the $1/(16\pi)$-factor difference between the gravitational and non-gravitational parts of the action (23.25). The quantity is called the **ADM mass**, according to R. Arnowitt, S. Deser and C. W. Misner. Since the vector field t^μ generates time flow, the ADM mass represents the value of the Hamiltonian connected with asymptotic time translation.

With the above hint, one could also judge that the total angular momentum of space-time should similarly be connected with asymptotic rotation, i.e. with the vector field $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$, where ϕ is the asymptotic rotation angle. Such a choice of the flow would correspond to $N = 0$, $N^\mu = \phi^\mu$, under which the Hamiltonian yields

$$J_{\rm ADM} = -\frac{1}{8\pi} \lim_{\partial S_t \to \infty} \oint\limits_{\partial S_t} \left(K^{\mu\nu} - Kh^{\mu\nu}\right) \phi_\mu r_\nu \sqrt{\sigma}\, {\rm d}^2\theta \tag{27.25}$$

(the minus sign just ensures the standard convention for the angular momentum orientation – to be tested below).

Asymptotically flat circular space-times

Let us check what the ADM formulae lead to for the “Kerr-type”, asymptotically flat circular space-times. We know from Section 22.4.3, equation (22.27), that at $r \gg M$ the Kerr-type metric reads

$${\rm d}s^2 = -\left(1 - \frac{2M}{r}\right){\rm d}t^2 - \frac{4J}{r}\sin^2\theta\, {\rm d}t\, {\rm d}\phi + \left(1 + \frac{2M}{r}\right)({\rm d}r^2 + r^2 {\rm d}\Omega^2)\,.$$

From (17.21), it is also easy to infer the inverse far-field metric, in particular,

$$g^{tt} = -\frac{\Sigma\Delta + (2Mr - Q^2)(r^2 + a^2)}{\Sigma\Delta} = -1 - \frac{(2Mr - Q^2)(r^2 + a^2)}{\Sigma\Delta} \to -1 - \frac{2M}{r}\,,$$

$$g^{t\phi} = -\frac{2Mr - Q^2}{\Sigma\Delta}\, a \to -\frac{2J}{r^3}\,.$$

The hypersurfaces Σ_t are naturally chosen as given by the constant Killing time t, and the boundary ∂S_t of a region S_t within Σ_t is chosen to be the coordinate sphere $r = \mathrm{const}$. The respective induced 3D and 2D metrics read

$$h_{ij}\mathrm{d}y^i\mathrm{d}y^j = \left(1+\frac{2M}{r}\right)(\mathrm{d}r^2+r^2\mathrm{d}\Omega^2), \qquad \sigma_{AB}\mathrm{d}\theta^A\mathrm{d}\theta^B = \left(1+\frac{2M}{r}\right)r^2\mathrm{d}\Omega^2 .$$

The future unit normal n_μ of Σ_t is proportional to the gradient $t_{,\mu}$, with the normalization $g^{\mu\nu}n_\mu n_\nu = -1$. Similarly, the unit outer normal r_μ of ∂S_t surfaces, tangent to Σ_t, is proportional to the gradient $r_{,\mu}$, with the normalization $g^{\mu\nu}r_\mu r_\nu = 1$. In the coordinates (t, r, θ, ϕ), this implies $t_{,\mu} = \delta^t_\mu$ and $r_{,\mu} = \delta^r_\mu$, with

$$n_t = \frac{-1}{\sqrt{-g^{tt}}} = \frac{-1}{\sqrt{1+\frac{2M}{r}}} \doteq -1+\frac{M}{r}, \quad r_r = \frac{1}{\sqrt{g^{rr}}} = \sqrt{g_{rr}} = \sqrt{1+\frac{2M}{r}} \doteq 1+\frac{M}{r} .$$

The 2D extrinsic curvature of ∂S_t (as a submanifold of Σ_t) thus amounts to

$$\Bbbk \equiv r_{\alpha;\beta}(g^{\alpha\beta}-r^\alpha r^\beta+n^\alpha n^\beta) = r^\beta{}_{;\beta}+r_{\alpha;\beta}n^\alpha n^\beta = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,r^r\right)_{,r} - \Gamma_{rtt}r^r(n^t)^2 .$$

Substituting there $\Gamma_{rtt} = -\frac{1}{2}\,g_{tt,r} = \frac{M}{r^2}$, $\sqrt{-g} = \Sigma\sin\theta \to r^2\sin\theta$ and

$$r^r = \frac{1}{\sqrt{g_{rr}}} \to 1-\frac{M}{r}, \quad n^t = g^{tt}n_t \to 1+\frac{M}{r}, \quad n^\phi = g^{\phi t}n_t \to \frac{2J}{r^3}\left(1-\frac{M}{r}\right),$$

we find

$$\Bbbk = \frac{1}{r^2}(2r-M) - \frac{M}{r^2}\left(1-\frac{M}{r}\right)\left(1+\frac{M}{r}\right)^2 = \frac{2}{r^2}(r-M)+O(r^{-3}) .$$

The mean curvature of a sphere $r = \mathrm{const}$ in a 3D Euclidean space can be computed by the same formula, where however $g_{tt} = -1 \Rightarrow \Gamma_{rtt} = 0$ now, so only left is

$$\Bbbk_{\mathrm{flat}} = r^\beta{}_{;\beta} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,r^r\right)_{,r} = \frac{2}{r} .$$

Hence, $\Bbbk - \Bbbk_{\mathrm{flat}} = -2M/r^2$, which, together with $\sqrt{\sigma}\,\mathrm{d}^2\theta = \left(1+\frac{2M}{r}\right)r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$, yields

$$M_{\mathrm{ADM}} = -\frac{1}{8\pi}\lim_{r\to\infty}\int_0^{2\pi}\!\!\int_0^{\pi}(\Bbbk-\Bbbk_{\mathrm{flat}})\left(1+\frac{2M}{r}\right)r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = \frac{1}{8\pi}\lim_{r\to\infty}\left(\frac{2M}{r^2}\,4\pi r^2\right) = M .$$

Now to the angular momentum (27.25). First, $h^{\mu\nu}\phi_\mu r_\nu = g^{\mu\nu}\phi_\mu r_\nu = 0$, so we only need to compute $K_{\mu\nu}\phi^\mu r^\nu = K_{\phi r}r^r = K_{\phi r}\left(1-\frac{M}{r}\right)$. By the definition $K_{\mu\nu} = n_{\mu;\beta}h^\beta_\nu$, we find

$$K_{\phi r} = n_{\phi;\beta}(\delta^\beta_r + n^\beta \cancel{n_r}) = n_{\phi;r} = -\Gamma^\iota{}_{r\phi}n_\iota = -\Gamma_{\iota r\phi}n^\iota = -\frac{1}{2}\,g_{\phi t,r}n^t - \frac{1}{2}\,g_{\phi\phi,r}n^\phi =$$

$$= -\frac{J}{r^2}\sin^2\theta\left(1+\frac{M}{r}\right) - (r+M)\sin^2\theta\,\frac{2J}{r^3}\left(1-\frac{M}{r}\right) = -\frac{3J}{r^2}\sin^2\theta + O(r^{-3})\,.$$

Therefore,

$$J_{\rm ADM} = -\frac{1}{8\pi}\lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi} K_{\mu\nu}\phi^\mu r^\nu\left(1+\frac{2M}{r}\right) r^2\sin\theta\,{\rm d}\theta{\rm d}\phi =$$
$$= \frac{3J}{4}\lim_{r\to\infty}\left[\left(1-\frac{M}{r}\right)\left(1+\frac{2M}{r}\right)\int_0^{\pi}\sin^3\theta\,{\rm d}\theta\right] = \frac{3J}{4}\int_0^{\pi}\sin^3\theta\,{\rm d}\theta = J\,.$$

Hence, for asymptotically flat space-times, the boundary part of the Hamiltonian likely provides a reasonable way how to compute the total amount of a quantity, as connected with a certain asymptotic vector-field flow. Some alternative options of how to find space-time's global parameters will be suggested in Section 28.10.

27.2.7 Checking the evolution equation for K-mu-nu

Hamiltonian equation (27.20) for $\dot{\Pi}^{\alpha\beta}$ should be equivalent to the evolution equation (25.47) for $K_{\mu\nu}$, thus, without sources ($T_{\beta\delta}=0$), to

$$\pounds_{\mathbf t}K_{\nu\lambda} = N\left(a_{\nu|\lambda} + a_\nu a_\lambda - KK_{\nu\lambda} + 2K_{\nu\gamma}K^\gamma_\lambda - {}^{(3)}R_{\nu\lambda} + \Lambda h_{\nu\lambda}\right) + \pounds_{\mathbf N}K_{\nu\lambda}\,. \tag{27.26}$$

The relation between the two quantities reads $\Pi^{\mu\nu}=\sqrt{h}\,(K^{\mu\nu}-Kh^{\mu\nu})$ (27.8), so one might like to check whether submission of the respective evolution equations for $K^{\mu\nu}$ and $h^{\mu\nu}$ into the Lie derivative of this relation really yields the same equation as (27.20).[6]

Regarding that

$$\frac{\partial h}{\partial h_{\rho\sigma}} = hh^{\rho\sigma} \quad\Longrightarrow\quad \frac{\partial\sqrt{h}}{\partial h_{\rho\sigma}} = \frac{1}{2\sqrt{h}}\,\frac{\partial h}{\partial h_{\rho\sigma}} = \frac{1}{2\sqrt{h}}\,hh^{\rho\sigma} = \frac{\sqrt{h}}{2}\,h^{\rho\sigma}\,,$$

the Lie derivative of $\Pi^{\mu\nu}=\sqrt{h}\,(K^{\mu\nu}-Kh^{\mu\nu})$ gives

$$\dot{\Pi}^{\mu\nu} = (\sqrt{h})^{\dot{}}\,(K^{\mu\nu}-Kh^{\mu\nu}) + \sqrt{h}\,(K^{\mu\nu}-Kh^{\mu\nu})^{\dot{}} =$$
$$= \frac{\partial\sqrt{h}}{\partial h_{\rho\sigma}}\,\dot{h}_{\rho\sigma}(K^{\mu\nu}-Kh^{\mu\nu}) + \sqrt{h}\,(\dot{K}^{\mu\nu}-\dot{K}h^{\mu\nu}-K\dot{h}^{\mu\nu}) =$$
$$= \frac{1}{2}\,h^{\rho\sigma}\dot{h}_{\rho\sigma}\Pi^{\mu\nu} + \sqrt{h}\,(\dot{K}^{\mu\nu}-\dot{K}h^{\mu\nu}-K\dot{h}^{\mu\nu})\,, \tag{27.27}$$

where we remind the notation $\dot{h}^{\mu\nu} := h^\mu_\alpha h^\nu_\beta \pounds_t h^{\alpha\beta}$ (and the like). The first useful term thus is $\dot{h}_{\rho\sigma}$ which we express from (25.16) and (27.9),

$$\dot{h}_{\rho\sigma} = 2NK_{\rho\sigma} + N_{\rho|\sigma} + N_{\sigma|\rho} = \frac{N}{\sqrt{h}}\,(2\Pi_{\rho\sigma} - \Pi h_{\rho\sigma}) + N_{\rho|\sigma} + N_{\sigma|\rho} \tag{27.28}$$

[6] Exclamation mark is rarely used in scientific literature, and it is even strictly excluded in some journals. Still E. Poisson uses one in [35] when warning the reader that it is tedious to derive equation (25.47). Yet much more it applies to our current whim... So, will you please consider this part an optional exercise.

$$\Longrightarrow \quad h^{\iota\sigma}\dot{h}_{\rho\sigma} = \frac{N}{\sqrt{h}}\left(2\Pi^{\iota}_{\rho} - \Pi h^{\iota}_{\rho}\right) + h^{\iota\sigma}(N_{\rho|\sigma} + N_{\sigma|\rho}) \tag{27.29}$$

$$\Longrightarrow \quad h^{\rho\sigma}\dot{h}_{\rho\sigma} = 2NK + 2N^{\sigma}{}_{|\sigma} = -\frac{N\Pi}{\sqrt{h}} + 2N^{\sigma}{}_{|\sigma}\ . \tag{27.30}$$

In the terms $\dot{h}^{\mu\nu}$ and $\dot{K}^{\mu\nu}$, mind *not* to naively rise indices of $\dot{h}_{\mu\nu}$, $\dot{K}_{\mu\nu}$, because the Lie derivative does not commute with the metric (Lie derivative of the metric is not zero in general). It is safer to compute the contravariant counter-parts of $\pounds_{\mathbf{n}} h_{\mu\nu} \equiv 2K_{\mu\nu}$ (25.14), $\pounds_{\mathbf{t}} h_{\mu\nu}$ (25.13) and $\pounds_{\mathbf{N}} h_{\mu\nu}$ (25.15) "from definitions":

$$\begin{aligned}
\pounds_{\mathbf{n}} h^{\alpha\beta} &= h^{\alpha\beta}{}_{;\iota} n^{\iota} - n^{\alpha}{}_{;\iota} h^{\iota\beta} - n^{\beta}{}_{;\iota} h^{\alpha\iota} = \\
&= \cancel{a^{\alpha} n^{\beta}} + \cancel{n^{\alpha} a^{\beta}} - n^{\alpha;\beta} - \cancel{a^{\alpha} n^{\beta}} - n^{\beta;\alpha} - \cancel{a^{\beta} n^{\alpha}} = -n^{\alpha;\beta} - n^{\beta;\alpha}\ , \qquad (27.31)\\
\pounds_{\mathbf{N}} h^{\alpha\beta} &= h^{\alpha\beta}{}_{;\iota} N^{\iota} - N^{\alpha}{}_{;\iota} h^{\iota\beta} - N^{\beta}{}_{;\iota} h^{\alpha\iota} = h^{\alpha\beta;\iota} h^{\rho}_{\iota} t_{\rho} - N^{\alpha;\iota} h^{\beta}_{\iota} - N^{\beta;\iota} h^{\alpha}_{\iota}\\
\Longrightarrow \quad h^{\mu}_{\alpha} h^{\nu}_{\beta} \pounds_{\mathbf{N}} h^{\alpha\beta} &= \cancel{h^{\mu\nu|\rho}}\, t_{\rho} - N^{\mu|\nu} - N^{\nu|\mu}\ , \qquad (27.32)\\
\pounds_{\mathbf{t}} h^{\alpha\beta} &= h^{\alpha\beta}{}_{;\iota} t^{\iota} - t^{\alpha}{}_{;\iota} h^{\iota\beta} - t^{\beta}{}_{;\iota} h^{\alpha\iota} = \\
&= h^{\alpha\beta}{}_{;\iota}(N n^{\iota} + N^{\iota}) - (N n^{\alpha} + N^{\alpha})_{;\iota} h^{\iota\beta} - (N n^{\beta} + N^{\beta})_{;\iota} h^{\alpha\iota} = \\
&= N \pounds_{\mathbf{n}} h^{\alpha\beta} - N^{|\beta} n^{\alpha} - N^{|\alpha} n^{\beta} + \pounds_{\mathbf{N}} h^{\alpha\beta} = \\
&= N \pounds_{\mathbf{n}} h^{\alpha\beta} + \pounds_{\mathbf{N}} h^{\alpha\beta} - N a^{\beta} n^{\alpha} - N a^{\alpha} n^{\beta}\ . \qquad (27.33)
\end{aligned}$$

By substitution of (27.31) and (27.32) to (27.33), we also reach the contravariant counter-parts of (25.14) and (25.16), respectively (note the opposite signs of the Lie-derivative terms!):

$$\begin{aligned}
&\pounds_{\mathbf{t}} h^{\alpha\beta} = -N(n^{\alpha;\beta} + n^{\beta;\alpha} + a^{\alpha} n^{\beta} + a^{\beta} n^{\alpha}) + \pounds_{\mathbf{N}} h^{\alpha\beta} = -2NK^{\alpha\beta} + \pounds_{\mathbf{N}} h^{\alpha\beta}\\
&\Longrightarrow \quad \dot{h}^{\mu\nu} \equiv h^{\mu}_{\alpha} h^{\nu}_{\beta} \pounds_{\mathbf{t}} h^{\alpha\beta} = h^{\mu}_{\alpha} h^{\nu}_{\beta}\left(-2NK^{\alpha\beta} + \pounds_{\mathbf{N}} h^{\alpha\beta}\right) = \\
&\qquad = -2NK^{\mu\nu} - N^{\mu|\nu} - N^{\nu|\mu} = -\frac{N}{\sqrt{h}}\left(2\Pi^{\mu\nu} - \Pi h^{\mu\nu}\right) - N^{\mu|\nu} - N^{\nu|\mu}\ , \qquad (27.34)
\end{aligned}$$

where (27.9) has been employed in the last equality.

Further, we express $\dot{K}^{\mu\nu}$ in terms of $\pounds_{\mathbf{t}} K_{\kappa\lambda}$:

$$\begin{aligned}
&h^{\kappa}_{\alpha} h^{\lambda}_{\beta} \pounds_{\mathbf{t}} K_{\kappa\lambda} \equiv \dot{K}_{\alpha\beta} = (h_{\alpha\rho} h_{\beta\sigma} K^{\rho\sigma})^{\cdot} = \dot{h}_{\alpha\rho} h_{\beta\sigma} K^{\rho\sigma} + h_{\alpha\rho} \dot{h}_{\beta\sigma} K^{\rho\sigma} + h_{\alpha\rho} h_{\beta\sigma} \dot{K}^{\rho\sigma} = \\
&\qquad\qquad = \dot{h}_{\alpha\rho} K^{\rho}_{\beta} + \dot{h}_{\beta\sigma} K^{\sigma}_{\alpha} + h_{\alpha\rho} h_{\beta\sigma} \dot{K}^{\rho\sigma} \qquad\qquad \Big|\ \cdot h^{\mu\alpha} h^{\nu\beta}\\
&\Longrightarrow \quad h^{\mu\alpha} h^{\nu\beta} \dot{K}_{\alpha\beta} = h^{\mu\alpha} \dot{h}_{\alpha\rho} K^{\rho\nu} + h^{\nu\beta} \dot{h}_{\beta\sigma} K^{\mu\sigma} + \dot{K}^{\mu\nu}\\
&\Longrightarrow \quad \dot{K}^{\mu\nu} = h^{\mu\alpha} h^{\nu\beta} \dot{K}_{\alpha\beta} - h^{\mu\alpha} \dot{h}_{\alpha\rho} K^{\rho\nu} - h^{\nu\beta} \dot{h}_{\beta\sigma} K^{\mu\sigma} \qquad (27.35)\\
&\Longrightarrow \quad h_{\mu\nu} \dot{K}^{\mu\nu} = h^{\alpha\beta} \dot{K}_{\alpha\beta} - h^{\alpha}_{\nu} \dot{h}_{\alpha\rho} K^{\rho\nu} - h^{\beta}_{\mu} \dot{h}_{\beta\sigma} K^{\mu\sigma} = h^{\alpha\beta} \dot{K}_{\alpha\beta} - 2\dot{h}_{\alpha\rho} K^{\alpha\rho}\\
&\Longrightarrow \quad \dot{K} = (h_{\mu\nu} K^{\mu\nu})^{\cdot} = \dot{h}_{\mu\nu} K^{\mu\nu} + h_{\mu\nu} \dot{K}^{\mu\nu} = h^{\alpha\beta} \dot{K}_{\alpha\beta} - \dot{h}_{\alpha\rho} K^{\alpha\rho}\ , \qquad (27.36)
\end{aligned}$$

where we have abbreviated $h^{\mu}_{\rho} h^{\nu}_{\sigma} \dot{K}^{\rho\sigma} \equiv h^{\mu}_{\rho} h^{\nu}_{\sigma} h^{\rho}_{\tau} h^{\sigma}_{\omega} \pounds_{\mathbf{t}} K^{\tau\omega} = h^{\mu}_{\tau} h^{\nu}_{\omega} \pounds_{\mathbf{t}} K^{\tau\omega} \equiv \dot{K}^{\mu\nu}$. The final preparatory point is to use the expressions (27.26) and (25.48) in the Lie-derivative terms of the above results for $\dot{K}^{\mu\nu}$ and $\dot{K}$. Firstly, expressing the $K_{\mu\nu}$ terms of (27.26) from (27.9),

$$-KK_{\nu\lambda} + 2K_{\nu\gamma}K^{\gamma}_{\lambda} = \frac{\Pi}{4h}\left(2\Pi_{\nu\lambda} - \Pi h_{\nu\lambda}\right) + \frac{1}{2h}\left(4\Pi_{\nu\gamma}\Pi^{\gamma}_{\lambda} - 4\Pi\Pi_{\nu\lambda} + \Pi^2 h_{\nu\lambda}\right) =$$

$$= \frac{1}{4h}\left(8\Pi_{\nu\gamma}\Pi^{\gamma}_{\lambda} - 6\Pi\Pi_{\nu\lambda} + \Pi^2 h_{\nu\lambda}\right) ,$$

equation (27.26) assumes the form

$$\begin{aligned} \pounds_{\mathbf{t}} K_{\nu\lambda} &= N_{|\nu\lambda} + \frac{N}{4h}\left(8\Pi_{\nu\gamma}\Pi^{\gamma}_{\lambda} - 6\Pi\Pi_{\nu\lambda} + \Pi^2 h_{\nu\lambda}\right) - N\left({}^{(3)}R_{\nu\lambda} - \Lambda h_{\nu\lambda}\right) + \pounds_{\mathbf{N}} K_{\nu\lambda} \\ \Longrightarrow \quad h^{\mu\alpha}h^{\nu\beta}\dot{K}_{\alpha\beta} &= N^{|\mu\nu} + \frac{N}{4h}\left(8\Pi^{\mu\gamma}\Pi^{\nu}_{\gamma} - 6\Pi\Pi^{\mu\nu} + \Pi^2 h^{\mu\nu}\right) - \\ &\quad - N\left({}^{(3)}R^{\mu\nu} - \Lambda h^{\mu\nu}\right) + h^{\mu\alpha}h^{\nu\beta}\pounds_{\mathbf{N}} K_{\alpha\beta} \,. \end{aligned} \tag{27.37}$$

The last ingredient needed is

$$\begin{aligned} &h^{\mu\alpha}h^{\nu\beta}\pounds_{\mathbf{N}} K_{\alpha\beta} = h^{\mu\alpha}h^{\nu\beta}\left(K_{\alpha\beta;\iota}N^{\iota} + N^{\iota}{}_{;\alpha}K_{\iota\beta} + N^{\iota}{}_{;\beta}K_{\alpha\iota}\right) = \\ &\quad = h^{\mu\alpha}h^{\nu\beta}\left(K_{\alpha\beta;\iota}h^{\iota}_{\kappa}N^{\kappa} + N_{\iota;\alpha}h^{\iota\kappa}K_{\kappa\beta} + N_{\iota;\beta}h^{\iota\kappa}K_{\kappa\alpha}\right) \equiv K^{\mu\nu}{}_{|\kappa}N^{\kappa} + N^{\kappa|\mu}K^{\nu}_{\kappa} + N^{\kappa|\nu}K^{\mu}_{\kappa} = \\ &\quad = \frac{1}{2\sqrt{h}}\left[2\Pi^{\mu\nu}{}_{|\kappa}N^{\kappa} - \Pi_{|\kappa}N^{\kappa}h^{\mu\nu} + N^{\kappa|\mu}(2\Pi^{\nu}_{\kappa} - \Pi h^{\nu}_{\kappa}) + N^{\kappa|\nu}(2\Pi^{\mu}_{\kappa} - \Pi h^{\mu}_{\kappa})\right] . \end{aligned}$$

Using the latter, (27.37) and (27.28) in (27.35) and in (27.36), we obtain

$$\begin{aligned} \dot{K}^{\mu\nu} &= N^{|\mu\nu} + \frac{N}{4h}\left(8\Pi^{\mu\gamma}\Pi^{\nu}_{\gamma} - 6\Pi\Pi^{\mu\nu} + \Pi^2 h^{\mu\nu}\right) - N\left({}^{(3)}R^{\mu\nu} - \Lambda h^{\mu\nu}\right) + \\ &\quad + \frac{1}{2\sqrt{h}}\left[2\Pi^{\mu\nu}{}_{|\kappa}N^{\kappa} - \Pi_{|\kappa}N^{\kappa}h^{\mu\nu} + \cancel{N^{\kappa|\mu}(2\Pi^{\nu}_{\kappa} - \Pi h^{\nu}_{\kappa})} + \cancel{N^{\kappa|\nu}(2\Pi^{\mu}_{\kappa} - \Pi h^{\mu}_{\kappa})}\right] - \\ &\quad - \frac{N}{2h}(2\Pi^{\mu}_{\rho} - \Pi h^{\mu}_{\rho})(2\Pi^{\rho\nu} - \Pi h^{\rho\nu}) - \frac{1}{2\sqrt{h}}(N^{\mu|\rho} + \cancel{N^{\rho|\mu}})(2\Pi^{\nu}_{\rho} - \Pi h^{\nu}_{\rho}) - \\ &\quad - \frac{N}{2h}(2\Pi^{\nu}_{\sigma} - \Pi h^{\nu}_{\sigma})(2\Pi^{\mu\sigma} - \Pi h^{\mu\sigma}) - \frac{1}{2\sqrt{h}}(N^{\nu|\sigma} + \cancel{N^{\sigma|\nu}})(2\Pi^{\mu}_{\sigma} - \Pi h^{\mu}_{\sigma}) = \\ &= N^{|\mu\nu} - \frac{N}{4h}\left(8\Pi^{\mu\gamma}\Pi^{\nu}_{\gamma} - 10\Pi\Pi^{\mu\nu} + 3\Pi^2 h^{\mu\nu}\right) - N\left({}^{(3)}R^{\mu\nu} - \Lambda h^{\mu\nu}\right) + \\ &\quad + \frac{1}{2\sqrt{h}}\left[2\Pi^{\mu\nu}{}_{|\kappa}N^{\kappa} - \Pi_{|\kappa}N^{\kappa}h^{\mu\nu} - 2\Pi^{\nu}_{\rho}N^{\mu|\rho} - 2\Pi^{\mu}_{\rho}N^{\nu|\rho} + \Pi(N^{\mu|\nu} + N^{\nu|\mu})\right] , \\ \dot{K} &= N^{|\sigma}{}_{\sigma} + \frac{N}{4h}\left(8\Pi^{\mu\gamma}\Pi_{\mu\gamma} - 3\Pi^2\right) - N({}^{(3)}R - 3\Lambda) + \\ &\quad + \frac{1}{2\sqrt{h}}\left[2\Pi_{|\kappa}N^{\kappa} - 3\Pi_{|\kappa}N^{\kappa} + \cancel{2N^{\kappa|\mu}(2\Pi_{\kappa\mu} - \Pi h_{\kappa\mu})}\right] - \\ &\quad - \frac{N}{2h}(2\Pi_{\alpha\rho} - \Pi h_{\alpha\rho})(2\Pi^{\alpha\rho} - \Pi h^{\alpha\rho}) - \cancel{\frac{1}{2\sqrt{h}}(N_{\alpha|\rho} + N_{\rho|\alpha})(2\Pi^{\alpha\rho} - \Pi h^{\alpha\rho})} = \\ &= N^{|\sigma}{}_{\sigma} - \frac{N\Pi^2}{4h} - N({}^{(3)}R - 3\Lambda) - \frac{1}{2\sqrt{h}}\Pi_{|\kappa}N^{\kappa} \,. \end{aligned}$$

Finally, we substitute these last relations for $\dot{K}^{\mu\nu}$ and $\dot{K}$, together with (27.30) and (27.34), to the formula (27.27), $\dot{\Pi}^{\mu\nu} = \frac{1}{2}\,h^{\rho\sigma}\dot{h}_{\rho\sigma}\Pi^{\mu\nu} + \sqrt{h}\,\dot{K}^{\mu\nu} - \sqrt{h}\,\dot{K}h^{\mu\nu} - \sqrt{h}\,K\dot{h}^{\mu\nu}$:

$$\dot{\Pi}^{\mu\nu} = -\frac{N\Pi}{2\sqrt{h}}\,\Pi^{\mu\nu} + N^{\sigma}{}_{|\sigma}\Pi^{\mu\nu} + \sqrt{h}\,N^{|\mu\nu} - \frac{N}{4\sqrt{h}}\left(8\Pi^{\mu\gamma}\Pi^{\nu}_{\gamma} - 10\Pi\Pi^{\mu\nu} + \cancel{3\Pi^2 h^{\mu\nu}}\right) -$$

$$
\begin{aligned}
& -N\sqrt{h}\left({}^{(3)}R^{\mu\nu} - \Lambda h^{\mu\nu}\right) + \Pi^{\mu\nu}{}_{|\kappa}N^\kappa - \cancel{\frac{1}{2}\Pi_{|\kappa}N^\kappa} h^{\mu\nu} - \Pi^\nu_\rho N^{\mu|\rho} - \Pi^\mu_\rho N^{\nu|\rho} + \\
& + \cancel{\frac{\Pi}{2}(N^{\mu|\nu} + N^{\nu|\mu})} - \sqrt{h}\,N^{|\sigma}{}_\sigma h^{\mu\nu} + \cancel{\frac{N\Pi^2}{4\sqrt{h}}} h^{\mu\nu} + N\sqrt{h}\,({}^{(3)}R - 3\Lambda)h^{\mu\nu} + \\
& + \cancel{\frac{1}{2}\Pi_{|\kappa}N^\kappa} h^{\mu\nu} - \frac{N\Pi}{2\sqrt{h}}\left(2\Pi^{\mu\nu} - \cancel{\Pi h^{\mu\nu}}\right) - \cancel{\frac{\Pi}{2}(N^{\mu|\nu} + N^{\nu|\mu})} = \\
= & -\frac{N}{\sqrt{h}}\left(2\Pi^{\mu\gamma}\Pi^\nu_\gamma - \Pi\Pi^{\mu\nu}\right) + \left(\Pi^{\mu\nu}N^\kappa\right)_{|\kappa} + \sqrt{h}\left(N^{|\mu\nu} - N^{|\sigma}{}_\sigma h^{\mu\nu}\right) - \\
& - \Pi^\nu_\rho N^{\mu|\rho} - \Pi^\mu_\rho N^{\nu|\rho} - N\sqrt{h}\left({}^{(3)}R^{\mu\nu} - {}^{(3)}Rh^{\mu\nu} + 2\Lambda h^{\mu\nu}\right).
\end{aligned}
$$

This indeed reproduces equation (27.20) we obtained in the Hamiltonian way, *including the boundary term*, provided that one employs the Hamiltonian constraint (27.11) and expresses

$$
N\sqrt{h}\left(\frac{1}{2}{}^{(3)}R - \Lambda\right)h^{\mu\nu} = \frac{N}{4\sqrt{h}}\left(2\Pi_{\alpha\beta}\Pi^{\alpha\beta} - \Pi^2\right)h^{\mu\nu}.
$$

Namely, it brings the Ricci-tensor parenthesis to the Einsteinian form,

$$
\begin{aligned}
& -N\sqrt{h}\left({}^{(3)}R^{\mu\nu} - {}^{(3)}Rh^{\mu\nu} + 2\Lambda h^{\mu\nu}\right) = \\
& \quad = -N\sqrt{h}\left({}^{(3)}R^{\mu\nu} - \frac{1}{2}{}^{(3)}Rh^{\mu\nu} + \Lambda h^{\mu\nu}\right) + N\sqrt{h}\left(\frac{1}{2}{}^{(3)}R - \Lambda\right)h^{\mu\nu} = \\
& \quad = -N\sqrt{h}\left({}^{(3)}R^{\mu\nu} - \frac{1}{2}{}^{(3)}Rh^{\mu\nu} + \Lambda h^{\mu\nu}\right) + \frac{N}{4\sqrt{h}}\left(2\Pi_{\alpha\beta}\Pi^{\alpha\beta} - \Pi^2\right)h^{\mu\nu}.
\end{aligned}
$$

$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota.$

CHAPTER 28

Conservation laws

It is extremely useful if a certain quantity is *conserved* along the evolution of a system. Such a circumstance represents a constraint on the degrees of freedom and serves as a "boundary condition" of a solution. Expressions for the conserved quantities ("integrals of the motion") typically contain one-degree-lower derivatives than the equations of motion, which itself enables to solve certain aspects of the problem in an easier way. A simple example is a (one-dimensional) motion of a particle in the potential $V(x)$: the corresponding equation of motion $m\ddot{x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$ can be rewritten, if multiplied by $\dot{x}$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\,m\dot{x}^2 + V(x)\right] = 0 \qquad \text{i.e.} \quad \frac{1}{2}\,m\dot{x}^2 + V(x) = \mathrm{const}\,.$$

The conservations are usually associated with certain *symmetries* of the system. Here above, it is the independence of the potential V of time t which brings the conservation of energy.

Or, in chapter 11 on Lie derivative, we observed that if the metric does not change under the flow of a certain vector field ξ^μ (a "space-time symmetry" exists), the four-momentum projection $p_\mu\xi^\mu$ stays constant along any geodesic. Here we will consider behaviour *of the whole theory* under transformations of variables (not just coordinates), in order to see whether a possible invariance again implies any conservation laws.

We will start from the Lagrangian formulation, Chapter 23. Let an action is given by a Lagrangian density ($\mathcal{L}$) depending on location x^μ and on the pertinent field(s) $\psi^\mu(x)$ and its (their) first and possibly second partial derivatives. The requirement that the action not be altered by change of the field(s), i.e. that its variation with respect to the field(s) vanish, one obtains the Euler-Lagrange equations (23.3),

$$[\mathrm{EL}(\mathfrak{L})]^\mu := \frac{\partial\mathfrak{L}}{\partial\psi^\mu} - \left[\frac{\partial\mathfrak{L}}{\partial{\psi^\mu}_{,\alpha}}\right]_{,\alpha} + \left[\frac{\partial\mathfrak{L}}{\partial{\psi^\mu}_{,\alpha\beta}}\right]_{,\alpha\beta} = 0\,, \tag{28.1}$$

where $[\mathrm{EL}]$ (with appropriate index) is called the **Euler operator** and where, for brevity, we employ the usual gothic notation $\mathfrak{L}(x,\psi(x)) := \sqrt{-g(x)}\;\mathcal{L}(x,\psi(x))$ for the Lagrangian scalar density (it has weight $w=+1$). Note that we will only sometimes list explicitly the dependences of $\mathfrak{L}$ on all the variables, i.e. on x (*four* x^μ actually), on field(s) $\psi(x)$ and on their derivatives up to a certain order (usually 1st or 2nd). If writing just $\mathfrak{L}$ or $\mathfrak{L}(x, \psi(x))$, we tacitly admit a possible dependence on derivatives of ψ as well.

28.1 Mystery of symmetry, mystery of action

What underlies the theory of relativity is a fundamental surprise: although the bodies and fields of the physical world are (of course) *not* invariant under generic coordinate transformations, still the *rules which govern their behaviour* do have such a *symmetry*. And it has become clear that there even exist continuous symmetries which do *not* concern spatio-temporal relations and which hide deeper in physical theories. They are called **gauge symmetries**, because they typically apply to quantities which represent more degrees of freedom than how many are possessed by actual physical system (and thus they can partially be "gauged" according to what problem one needs to address). In modern era, it is not even so that theorists would ask which symmetry the given theory has, but rather they require, from the very beginning, the symmetry of the theory (of the action) with respect to a certain Lie group of continuous ("gauge") transformations. To each generator of the Lie algebra associated with the given Lie group, there corresponds a gauge field which has to be properly inserted in the action in order to ensure the demanded invariance of the latter. (By quantization of this field, one obtains "gauge bosons" as carriers of the given interaction. In the gauge theories of fundamental interactions, these are photons, W and Z bosons, and gluons. In gravitation, a similar role should be played by the yet hypothetical graviton.)

The **symmetry transformations** of a given gauge theory can be **global or local**, the former acting everywhere in the same manner (having *constant* parameters), whereas the latter acting in dependence on location (their parameters are functions of coordinates). It is one of the "mantras of theoretical physics", as sometimes aptly referred to, that the symmetry of a (non-dissipative) system with respect to global continuous transformations implies the respective number of physical conservation laws, whereas the symmetry with respect to local continuous transformations implies mathematical identities which represent constraints for non-physical (gauge) degrees of freedom. Also common is to speak of **dynamical vs. non-dynamical symmetries**, because the former imply conservation laws for dynamical degrees of freedom, whereas the latter imply constraints for dynamical equations which effectively fix the gauge arbitrariness. The first/second case is addressed by the first/second **Noether's theorems**. They both belong to the most important results in theoretical physics.

The one-to-one link between symmetries and conservation laws naturally appears within the Lagrangian formalism, which itself builds on another mysterious feature of the physical Universe: although processes often seem to occur and run rather casually, they in fact follow very special routes along which certain quantities (action functionals) are extremized. This is *not* to claim that one could not study the symmetries directly on the equations of motion; after all, it is not that *every* valuable differential equation, possibly possessing interesting symmetries, follows by variation of any action. However, i) the Hamilton principle being so effective and generic, it also seems to be suitable for the discussion of symmetries; and, ii) the symmetries which appear on the level of action/Lagrangian are stronger in that they are "of shell", not necessarily depending on the equations of motion.

The most important message of this chapter will be that if the action is invariant under a certain (finite) Lie group of global transformations, then to every generator of the associated Lie algebra there corresponds a conserved **Noether current** (it satisfies a continuity equation), which in turn can be integrated to obtain a conserved **Noether charge**.

28.2 Action, Lagrangian, field equations, and transformations

Lagrangian depends on the position x^μ and on the fields ψ which themselves depend on x^μ. One can change both: the coordinates, which necessarily induces a certain transformation of the fields, and the fields themselves (without necessarily changing the coordinates); the latter are usually called *intrinsic*, *internal*, *geometrical*, or *gauge transformations*.[1]

- The field equations (or "equations of motion" in mechanical parallel) select, from within the "kinematically possible" configurations (or trajectories) of a system, those which the system could *really* assume (or follow). In the Lagrangian approach, one looks for them by varying the action with respect to the respective field(s) *at a given location and in a given coordinate system*. The basic variations (of the fields) are arbitrary yet assumed to vanish on a boundary of the region in question.

- When asking about conservation laws or constraints of a theory, one looks for the invariance of the action with respect to variations of *anything on which it depends* in principle, *irrespectively of any dynamical equations*. Usually it means variation of the fields *as well as of the coordinates* (thus including the coordinate location and the integration domain). Here the variations are *not* required to vanish on any boundary, yet it is clear they must be much more special to generate symmetry.

Physically, one asks when the variation with respect to a given variable "does not change physics". In GR, "physics" should primarily be invariant under general diffeomorphisms, so the Lagrangian $\mathcal{L}$ (and thus the action) has to be an invariant, hence $\mathfrak{L} \equiv \sqrt{-g}\,\mathcal{L}$ has to be a scalar density (of weight $+1$). However trivial such a demand might look, remember that we were able, in Section 23.4.5, to derive from it the conservation of the energy-momentum tensor. Still more interesting is to inspect the behaviour of the action under the change of the *fields* ψ (such as the metric in the GR vacuum case). The main question will be *what are the implications of the invariance with respect to various types of variations*.

"Not to change physics" actually involves a number of subtle issues. One generally has four levels of description: the action, the Lagrangian (density), the equations of motion (called field equations in a field theory), and specific solutions of the latter. These need not share the same symmetries. Most important is *the invariance of the field equations*, because *these determine the physical prediction*. In the Lagrangian formulation, it means to leave unchanged the Euler-Lagrange equations (28.1). First, though it is not necessary for the

[1] In reference to the fibre-bundle picture, the coordinate transformations are sometimes called "horizontal", while the intrinsic transformations are called "vertical". We don't consider transformations of parameters (e.g. coupling constants) on which the theory may also depend.

discussion of the conservation laws, it is natural in GR to assume that the fields ψ are tensorial and that the action and the Lagrangian are invariant with respect to coordinate transformations $(x^\mu \to x'^\mu)$. This ensures that the EL equations are tensorial (one then usually writes them in terms of covariant derivatives). Still there remains the question of what happens if, in addition, *the fields* change by some gauge transformation. How to *then* ensure that the EL equations keep their form? Since all their terms are given by the first derivatives of $\mathfrak{L}$ with respect to the fields and their derivatives, it is clearly *sufficient* (not [necessarily] necessary)[2] if the **Lagrangian density is form-invariant**, i.e. if it keeps the same functional dependence on its variables, "$\mathfrak{L}' = \mathfrak{L}$".

What about the action? The following computation is crucial for everything below, so let us perform it carefully, from both the active and the passive point of view:

- "Active" point of view: the coordinate part of the transformation shifts points (thus also the integration region), while the coordinate mesh "stays still". One proceeds like

$$\begin{aligned}
S'[\psi'] &\equiv \int_{\Omega'} \mathfrak{L}'(x', \psi'(x'))\, \mathrm{d}^4x' \overset{1)}{=} \int_{\Omega'} \mathfrak{L}(x', \psi'(x'))\, \mathrm{d}^4x' \overset{2)}{=} \int_{\Omega'} \mathfrak{L}(x, \psi'(x))\, \mathrm{d}^4x \overset{3)}{=} \\
&\overset{3)}{=} \int_{\Omega} \mathfrak{L}(x, \psi'(x))\, \mathrm{d}^4x + \int_{\partial\Omega} \mathfrak{L}(x, \psi'(x))\, \delta x^\alpha n_\alpha \mathrm{d}^3x \overset{4)}{=} \\
&\overset{4)}{=} \int_{\Omega} \mathfrak{L}(x, \psi'(x))\, \mathrm{d}^4x + \int_{\Omega} \left[\mathfrak{L}(x, \psi(x))\, \delta x^\alpha\right]_{,\alpha} \mathrm{d}^4x\,.
\end{aligned} \tag{28.2}$$

  Steps: 1) Form invariance of Lagrangian: $\mathfrak{L}' = \mathfrak{L}$. 2) The coordinates are "fixed", and x is just a dummy integration variable, so there is no difference to *denote* back $x' \to x$. 3) With the coordinate transformation, the integration region Ω changed infinitesimally, to Ω'. The corresponding change of the action is given by an integral of $\mathfrak{L}(x, \psi'(x))$ over the volume difference between Ω and Ω', i.e. over the boundary $\partial\Omega$ multiplied by projection of the shift δx^α to the outward normal of $\partial\Omega$ (we called it n_α). 4) This "boundary" term was finally expressed as a divergence integrated over the original domain, with ψ already used instead of ψ', because the term is already $O(\delta)$.

- "Passive" point of view: the integration domain Ω stays fixed, while the coordinates shift. This case is more delicate since the Jacobian of the transformation $\left|\frac{\partial x'}{\partial x}\right|$ enters the integral. It is worth computing its expansion to linear order in δ right now:

$$M^\mu_\alpha := \frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + (\delta x^\mu)_{,\alpha} \quad \Rightarrow \quad \mathrm{Tr}\, M = 4 + (\delta x^\mu)_{,\mu}\,, \quad M^n = \delta^\mu_\alpha + n(\delta x^\mu)_{,\alpha} + O(\delta^2)\,,$$

  so, substituting to the relation (A.10), i.e.

$$\det M^\mu_\alpha = \frac{1}{4!}\left[(\mathrm{Tr}\, M)^4 - 6\,(\mathrm{Tr}\, M)^2\, \mathrm{Tr}\, M^2 + 8\, \mathrm{Tr}\, M\ \mathrm{Tr}\, M^3 - 6\, \mathrm{Tr}\, M^4 + 3\,(\mathrm{Tr}\, M^2)^2\right],$$

---

[2] This note is not *necessarily* mysterious: multiplication of a Lagrangian density by a constant "trivially" does not change the EL equations, yet it is *not* covered by the option below. And note in passing that although it does not "change physics", such a scaling does change the action.

we have, to linear order,

$$
\begin{aligned}
&(\operatorname{Tr} M)^4 = 4^4\,[1 + (\delta x^\mu)_{,\mu}]\,, \quad (\operatorname{Tr} M)^2 = 8\,[2 + (\delta x^\mu)_{,\mu}]\,, \quad \operatorname{Tr} M^n = 4 + n(\delta x^\mu)_{,\mu} \\
&\implies \det\left[\delta^\mu_\alpha + (\delta x^\mu)_{,\alpha}\right] = 1 + (\delta x^\mu)_{,\mu}\,.
\end{aligned}
$$

So, let us compute the action integral in the "passive" point of view,

$$
\begin{aligned}
S'[\psi'] &\equiv \int_\Omega \mathfrak{L}'(x', \psi'(x'))\,\mathrm{d}^4x' \overset{1)}{=} \int_\Omega \mathfrak{L}(x', \psi'(x'))\,\mathrm{d}^4x' \overset{2)}{\equiv} \\
&\overset{2)}{\equiv} \int_\Omega \mathfrak{L}(x + \delta x, \psi'(x + \delta x)) \left|\frac{\partial x'}{\partial x}\right| \mathrm{d}^4x \overset{3)}{=} \\
&\overset{3)}{=} \int_\Omega \Big\{\mathfrak{L}(x, \psi'(x)) + [\mathfrak{L}(x, \psi'(x))]_{,\alpha}\,\delta x^\alpha\Big\}\,[1 + (\delta x^\mu)_{,\mu}]\ \mathrm{d}^4x \overset{4)}{=} \\
&\overset{4)}{=} \int_\Omega \mathfrak{L}(x, \psi'(x))\,\mathrm{d}^4x + \int_\Omega [\mathfrak{L}(x, \psi(x))\,\delta x^\alpha]_{,\alpha}\,\mathrm{d}^4x\,.
\end{aligned}
\tag{28.3}
$$

Steps: 1) Same as above. 2) Just a substitution. 3) Expansion of $\mathfrak{L}$ and of the Jacobian in δx to linear order. 4) Restricting to linear δ order in the result of the multiplication.

The last step is to evaluate the $\int_\Omega \mathfrak{L}(x, \psi'(x))\,\mathrm{d}^4x$ term:

$$
\begin{aligned}
&\int_\Omega \mathfrak{L}(x, \psi'(x))\,\mathrm{d}^4x \equiv \int_\Omega \mathfrak{L}(x, \psi + \bar\delta\psi)\,\mathrm{d}^4x = \int_\Omega \left[\mathfrak{L}(x, \psi(x)) + \frac{\delta\mathfrak{L}}{\delta\psi}\,\bar\delta\psi\right] \mathrm{d}^4x = \\
&\equiv S[\psi] + \int_\Omega \cancel{\frac{\delta\mathfrak{L}}{\delta\psi}}\,\bar\delta\psi\,\mathrm{d}^4x\,,
\end{aligned}
\tag{28.4}
$$

where the second term is zero *if the (original) Euler-Lagrange equations are valid*.

Hence, the EL equations keep their form if the Lagrangian density is form-invariant. In such a case, the action only (possibly) changes by a term which is given by a divergence. Is not it so that it actually makes no harm to add to the Lagrangian density *any* term given by total partial divergence? Yes, but, more precisely, if one considers a problem with $\mathfrak{L}$ depending on ψ up to the k-th derivatives, the divergence term can at most depend on the $(k-1)$-th derivatives. For a Lagrangian depending on the fields ψ and their 1st and 2nd derivatives, for example, this means to take four scalars $Q^\alpha(x^\mu, \psi(x), \psi_{,\rho}(x))$ such that

$$
S'[\psi'] = S[\psi] - \int_\Omega Q^\alpha{}_{,\alpha}(x^\mu, \psi, \psi_{,\rho})\,\mathrm{d}^4x \ \equiv\ S - \int_\Omega Q^\alpha{}_{,\alpha}\,\mathrm{d}^4x\,. \tag{28.5}
$$

Then the variation of S' with respect to ψ' really equals that of S with respect to ψ, because

$$
\begin{aligned}
&\int\limits_\Omega Q^\alpha{}_{,\alpha}\,\mathrm{d}^4x = \int\limits_{\partial\Omega} Q^\alpha\,n_\alpha\,\mathrm{d}^3x \\
&\implies \quad \bar\delta \int\limits_\Omega Q^\alpha{}_{,\alpha}\,\mathrm{d}^4x = \int\limits_{\partial\Omega} \bar\delta Q^\alpha\,n_\alpha\,\mathrm{d}^3x = \int\limits_{\partial\Omega} \frac{\partial Q^\alpha}{\partial\psi}\,\bar\delta\psi\,n_\alpha\,\mathrm{d}^3x + \int\limits_{\partial\Omega} \frac{\partial Q^\alpha}{\partial\psi_{,\rho}}\,\bar\delta\psi_{,\rho}\,n_\alpha\,\mathrm{d}^3x = 0
\end{aligned}
$$

(since both $\delta\psi$ and $\delta\psi_{,\rho}$ are supposed to vanish on the boundary $\partial\Omega$). As restated in a different way: under the standard assumptions on the boundary (vanishing of the variations), the Lagrangian density is not unique – for a given problem, there is the whole class of them mutually differing by divergence terms. Note, however, that the *value* of the action *is* modified by the divergence term, at least if the latter does not vanish on the boundary.

To summarize, combining the evaluation of $S'[\psi']$ with the finding that the EL equations are not altered by adding any divergence term to the Lagrangian density, we obtain

$$\int\limits_{\Omega} \mathfrak{L}(x,\psi'(x))\,\mathrm{d}^4x + \int\limits_{\Omega} \left[\mathfrak{L}(x,\psi(x))\,\delta x^{\alpha}\right]_{,\alpha} \mathrm{d}^4x = \int\limits_{\Omega} \mathfrak{L}(x,\psi(x))\,\mathrm{d}^4x - \int\limits_{\Omega} Q^{\alpha}{}_{,\alpha}\,\mathrm{d}^4x\,. \tag{28.6}$$

This relation will be crucial later. Remember that it follows from the form-invariance of the Lagrangian. The latter ensured the form-invariance of the EL equations, yet the relation does not require that they are *satisfied* (i.e., it is “off-shell”).

Remarks

- Irrelevance of the divergence terms applies to the Lagrangian (spatial) *densities*, employed to describe *continuous* problems, such as those of physical fields. (The densities have to be integrated over 4D regions to get actions.) *Discrete* problems, such as those of particles, are described by Lagrangians (which are only being integrated over time). In the Lagrangian, correspondingly, irrelevant are *time-derivative* terms:

  $$\bar{\delta}S' = \bar{\delta}S - \bar{\delta}\int\limits_{\tau_1}^{\tau_2} \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau}\,\mathrm{d}\tau = \bar{\delta}S - \bar{\delta}[\mathcal{F}]_{\tau_1}^{\tau_2} = \bar{\delta}S - \cancel{\bar{\delta}\mathcal{F}(\tau_2)} + \cancel{\bar{\delta}\mathcal{F}(\tau_1)}\,,$$

  because the variation vanishes at the end points, $\bar{\delta}\mathcal{F}(\tau_1)=0$, $\bar{\delta}\mathcal{F}(\tau_2)=0$.

- Some authors denote the “symmetry modulo the divergence term” as *quasi-symmetry*. Clearly the quasi-symmetry is designed so that it is sufficient for the Euler-Lagrange equations to stay *exactly* the same. It holds that every symmetry of the Lagrangian is inherited by the Euler-Lagrange equations, but the converse is not true – the equations may have richer symmetries than the Lagrangian. (We stressed that we speak of *sufficient* conditions for the EL-equations covariance.)

- (Quasi-)symmetries of the Lagrangian density may not map to the same (quasi-)symmetries of the action. One of the reasons is the dependence of S on the region Ω: if the region is more-than-1D, it need not satisfy the given symmetry. It may also hold in the opposite direction: the integration may “rectify” certain asymmetries of the Lagrangian. In other words, one has to also take into account how Ω changes under the (coordinate) transformation. On the other hand, it is clear from the notion of *quasi*-symmetry that some transformations may induce the extra divergence (or total derivative) terms in the Lagrangian, which however may not contribute to the action (in such a case, one would say that the quasi-symmetry of the Lagrangian translates to the full symmetry of the action).

- The last item in the chain are specific solutions of the field equations. These need not inherit the symmetries of the field equations – one then speaks of the *spontaneous symmetry breaking*. This phenomenon is extremely important in many areas (particle physics, condensed matter, cosmology, etc.), but it will not be in our focus here.

- If treating the variational problem *covariantly* (see section 23.2.1), one proceeds analogously, just using the bare Lagrangian density ($\mathcal{L}$, without $\sqrt{-g}$) and covariant derivatives of the fields instead of partial ones. In particular, the equivalent actions then differ by terms given by integrals of $Q^{\alpha}{}_{;\alpha}(x,\psi,\psi_{;\rho})$ over an *invariant* volume. (Needless to say, in order to keep the action invariant, Q^α has to be a four-vector then.)

28.2.1 Trivial (or null) Lagrangians

So, Lagrangian densities fully given by partial divergences of some quadruples of functions are trivial in the sense that, under proper boundary conditions, their Euler-Lagrange equations are *satisfied trivially* (by *any* function). Is a converse also true? -Yes:

Lemma: A **Lagrangian is trivial** (or **null**) (i.e., its EL equations are satisfied by *any* function $\psi(x)$ at every x) if and only if it is given by a total divergence $Q^{\alpha}{}_{,\alpha}$ of some four functions $Q^\alpha(x^\mu,\psi(x^\mu),\psi_{,\iota}(x^\mu),\psi_{,\iota\kappa}(x^\mu))$. (We consider such a specific setting for the demonstration, yet the statement readily generalizes to *more* fields ψ (or more components of them), to *any* dimension of the manifold and to the dependence of Q^α on *any* derivatives of ψ). In other words, the space of total divergences is the kernel of the Euler operator.

To prove it in the opposite direction than above, consider, for λ a real parameter, $\mathfrak{L}=\mathfrak{L}(x,\lambda\psi)$ and its derivative

$$\frac{\mathrm{d}\mathfrak{L}(x,\lambda\psi)}{\mathrm{d}\lambda}=\frac{\partial\mathfrak{L}}{\partial(\lambda\psi)}\,\psi+\frac{\partial\mathfrak{L}}{\partial(\lambda\psi_{,\iota})}\,\psi_{,\iota}+\frac{\partial\mathfrak{L}}{\partial(\lambda\psi_{,\iota\kappa})}\,\psi_{,\iota\kappa}=\frac{1}{\lambda}\left(\frac{\partial\mathfrak{L}}{\partial\psi}\,\psi+\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\,\psi_{,\iota}+\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\,\psi_{,\iota\kappa}\right).$$

The parenthesis can be expressed as

$$\begin{aligned}(\ldots)&=\frac{\partial\mathfrak{L}}{\partial\psi}\,\psi+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\,\psi\right)_{,\iota}-\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\right)_{,\iota}\psi+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\,\psi_{,\iota}\right)_{,\kappa}-\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\right)_{,\kappa}\psi_{,\iota}=\\&=\frac{\partial\mathfrak{L}}{\partial\psi}\,\psi+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\,\psi\right)_{,\iota}-\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\right)_{,\iota}\psi+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\,\psi_{,\iota}\right)_{,\kappa}-\left[\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\right)_{,\kappa}\psi\right]_{,\iota}+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\right)_{,\kappa\iota}\psi=\\&=[\mathrm{EL}(\lambda\psi)]\,\lambda\psi+\text{divergences}[\mathfrak{L}(x,\lambda\psi)]\,,\end{aligned}$$

where $[\mathrm{EL}(\lambda\psi)]$ is the left-hand side of the Euler-Lagrange equations (28.1) for $\lambda\psi$ and

$$\begin{aligned}\text{divergences}[\mathfrak{L}(x,\lambda\psi)]&:=\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota}}\,\psi\right)_{,\iota}+\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\,\psi_{,\iota}\right)_{,\kappa}-\left[\left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\iota\kappa}}\right)_{,\kappa}\psi\right]_{,\iota}=\\&=\left[\frac{\partial\mathfrak{L}(x,\lambda\psi)}{\partial\psi_{,\iota}}\,\psi+\frac{\partial\mathfrak{L}(x,\lambda\psi)}{\partial\psi_{,\kappa\iota}}\,\psi_{,\kappa}-\left(\frac{\partial\mathfrak{L}(x,\lambda\psi)}{\partial\psi_{,\iota\kappa}}\right)_{,\kappa}\psi\right]_{,\iota}=:[P^\iota(x,\psi;\lambda)]_{,\iota}\,.\end{aligned}$$

Now, if $[\mathrm{EL}] \equiv 0$ *trivially*, for *any* function, then we are left with

$$\frac{\mathrm{d}\mathfrak{L}(x, \lambda\psi)}{\mathrm{d}\lambda} = \frac{[P^\iota(x, \psi; \lambda)]_{,\iota}}{\lambda}$$

which, by integration from $\lambda = 0$ to $\lambda = 1$, yields

$$\mathfrak{L}(x, \psi) - \mathfrak{L}(x, 0) = \int_0^1 \frac{[P^\iota(x, \psi; \lambda)]_{,\iota}}{\lambda}\,\mathrm{d}\lambda = \left[\int_0^1 \frac{P^\iota(x, \psi; \lambda)}{\lambda}\,\mathrm{d}\lambda\right]_{,\iota} =: [\tilde{Q}^\iota(x, \psi)]_{,\iota}\ .$$

Finally, it is always possible to find such a set of functions $\hat{Q}^\iota(x)$ that $\mathfrak{L}(x^\mu, 0) = [\hat{Q}^\iota(x^\mu)]_{,\iota}$ within the (compact) domain Ω.[3] Hence, $\mathfrak{L}(x, \psi) = [\hat{Q}^\iota(x) + \tilde{Q}^\iota(x, \psi)]_{,\iota}$. □

28.2.2 "Off-shell" vs. "on-shell" variations

In the realm of variations and symmetries, often used is the attribute "on shell" / "off shell" which specifies whether the fields involved are assumed/required to satisfy the respective field equations or not. Correspondingly, symmetries (in fact *any* statements) are often called *weak* if they depend on specific field equations (or even on their specific solutions), vs. *strong* when they apply irrespectively of the latter. The subtle point is that certain results which are valid "on-shell" must be derived from "off-shell" action, because otherwise the argument would be trivial or circular. As an example, recall the Lagrangian derivation of the equation of motion for a massive particle in special relativity. When proceeding in a simpler way (without varying $\mathrm{d}\tau$), from the Lagrangian $\mathcal{L} = \frac{1}{2}\, m_0 \eta_{\mu\nu} u^\mu u^\nu$, one must not employ the $\eta_{\mu\nu} u^\mu u^\nu = -1$ normalization *before performing the variation*, because the normalization only holds along the actual trajectory, which however is just being searched for by the variation.

Indeed, it is very important to distinguish between the "off-shell" and "on shell" variations. They are quite opposite in fact: i) the **(quasi-)symmetries of the action** are meant to be off-shell, "**virtual**" features – they represent *conditions on the variation*, while the fields stay arbitrary (not constrained by any Euler-Lagrange equations), whereas ii) the **on-shell variations** are performed along the **actual** evolution of a system – they are arbitrary, but with the fields constrained to satisfy the pertinent EL equations. Similarly as with the above Lagrangian of a relativistic particle, it is worth to realize that the concept of an "on-shell (quasi-)symmetry" would have no sense, because if taking the variation of (28.5) with the corresponding EL equations *already satisfied*, $\delta S'$ as well as δS would vanish trivially (*by assumption*), and the boundary term would *of course* make the same (as it is *designed* so).

Example: charged particle in Kerr-Newman

A simple example offers itself: in treating the motion of charged particles in the Kerr-Newman space-time (Section 17.3), we employed the Lagrangian $\mathcal{L} = \frac{1}{2}\, m g_{\mu\nu} u^\mu u^\nu + q A_\mu u^\mu$ (Section 17.3.3). Imagine one did *not* know how the field A_μ behaves under the "gauge"

---

[3] $\mathfrak{L}(x^\mu, \psi)$ must be integrable in Ω, so one may e.g. use, for some $a^\mu \in \Omega$, $\hat{Q}^0(x^\mu) = \frac{1}{4}\int_{a^0}^{x^0} \mathfrak{L}(u, x^1, x^2, x^3, 0)\,\mathrm{d}u$, $\hat{Q}^1(x^\mu) = \frac{1}{4}\int_{a^1}^{x^1} \mathfrak{L}(x^0, u, x^2, x^3, 0)\,\mathrm{d}u$, $\hat{Q}^2(x^\mu) = \frac{1}{4}\int_{a^2}^{x^2} \mathfrak{L}(x^0, x^1, u, x^3, 0)\,\mathrm{d}u$, $\hat{Q}^3(x^\mu) = \frac{1}{4}\int_{a^3}^{x^3} \mathfrak{L}(x^0, x^1, x^2, u, 0)\,\mathrm{d}u$.

transformation. However, one requires that any possible transformation has to leave physics untouched, which means that it is allowed to change the Lagrangian by a total time derivative of some function $\mathcal{F}$, but nothing else. Admitting a very general freedom $\mathcal{F}(x^\mu, u^\mu, g_{\mu\nu}, A_\mu)$ and considering that x^μ is just a set of functions (for which covariant derivative is the same as ordinary one), one would have

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau} \equiv \frac{\mathrm{D}\mathcal{F}}{\mathrm{d}\tau} = \mathcal{F}_{,\mu} u^\mu + \frac{\partial \mathcal{F}}{\partial u^\mu} a^\mu + \frac{\partial \mathcal{F}}{\partial g_{\mu\nu}} \cancel{g_{\mu\nu;\alpha}} u^\alpha + \frac{\partial \mathcal{F}}{\partial A_\mu} A_{\mu;\alpha} u^\alpha \,.$$

Should the Lagrangian keep the same form, only the first term may actually be present, contributing as

$$\mathcal{L} = \frac{1}{2}\, m g_{\mu\nu} u^\mu u^\nu + q(A_\mu + \chi_{,\mu}) u^\mu \,, \qquad \chi := -\frac{\mathcal{F}}{q} \,,$$

so A_μ has to transform as $A_\mu \to A_\mu + \chi_{,\mu}$. The invariance of the Lagrangian thus enforces the properties of the gauge field A_μ. And, this in turn implies that the gauge field *has to be present* in general: even if not included from the beginning, it would appear due to the gauge transformation.

Example: scalar field

Have a free massless scalar field $\psi(x^\mu)$, described by $\mathcal{L} = -\frac{1}{2}\, g^{\alpha\beta} \psi_{;\alpha} \psi_{;\beta}$, and thus EL equations

$$0 = \frac{\cancel{\partial \mathcal{L}}}{\cancel{\partial \psi}} - \left[\frac{\partial \mathcal{L}}{\partial \psi_{;\mu}}\right]_{;\mu} = \frac{1}{2}\,(g^{\alpha\beta} \delta^\mu_\alpha \psi_{;\beta} + g^{\alpha\beta} \psi_{;\alpha} \delta^\mu_\beta)_{;\mu} = \psi^{;\mu}{}_{\mu} \equiv \Box\psi \,.$$

Consider the gauge transformation $\bar{\psi} = \psi + k_\iota x^\iota$, with the vector field k^μ null ($k_\mu k^\mu = 0$) and divergence-free ($k^\mu{}_{;\mu} = 0$). Under such a transformation, $\bar{\psi}_{;\alpha} = \psi_{;\alpha} + (k_\iota x^\iota)_{;\alpha} = \psi_{;\alpha} + k_\iota \delta^\iota_\alpha = \psi_{;\alpha} + k_\alpha$, so, assuming that $\bar{\mathcal{L}}$ depends on $\bar{\psi}$ in the same way as $\mathcal{L}$ depends on ψ,

$$\bar{\mathcal{L}}(\bar{\psi}) = -\frac{1}{2}\, g^{\alpha\beta} \bar{\psi}_{;\alpha} \bar{\psi}_{;\beta} = -\frac{1}{2}\, g^{\alpha\beta} (\psi_{;\alpha} + k_\alpha)(\psi_{;\beta} + k_\beta) = \mathcal{L}(\psi) - \psi_{;\alpha} k^\alpha = \mathcal{L}(\psi) - (\psi k^\alpha)_{;\alpha} \,,$$

that is, $\bar{\mathcal{L}}(\bar{\psi})$ differs from $\mathcal{L}(\psi)$ by a divergence term. The "primed" field equations really do not differ from the original ones,

$$0 = \frac{\cancel{\partial \bar{\mathcal{L}}}}{\cancel{\partial \bar{\psi}}} - \left[\frac{\partial \bar{\mathcal{L}}}{\partial \bar{\psi}_{;\mu}}\right]_{;\mu} = \ldots = \Box\bar{\psi} = (\psi^{;\mu}{}_{\mu} + \cancel{k^\mu{}_{;\mu}}) = \Box\psi \,.$$

Were the field not free, but rather "burdened" with some potential $V(\psi)$, one would have $\mathcal{L} = -\frac{1}{2}\, g^{\alpha\beta} \psi_{;\alpha} \psi_{;\beta} + V(\psi)$ and the Euler-Lagrange equations would yield

$$0 = \frac{\partial \mathcal{L}}{\partial \psi} - \left[\frac{\partial \mathcal{L}}{\partial \psi_{;\mu}}\right]_{;\mu} = \frac{\mathrm{d}V}{\mathrm{d}\psi} + \Box\psi \,.$$

In particular, the potential of the form $V(\psi) = -\frac{1}{2} m^2 \psi^2$ (with m constant) leads to the Klein-Gordon equation $\Box\psi - m^2\psi = 0$.

Example: electromagnetic field

We have $F_{\mu\nu}$, A_μ and J^μ. The "kinetic" invariant clearly is $F_{\mu\nu}F^{\mu\nu}$, and a plausible potential invariant should be $J^\mu A_\mu$ (the hint is that it contains charge density times the electrostatic potential in the first term). Another invariant is $J_\mu J^\mu$, but that does not depend on the field at all. Then there is $A_\mu A^\mu$, which however is *not* invariant under the gauge transformation $A_\mu \to A_\mu + \chi_{,\mu}$. And the latter is also true for the last tip $F^{\mu\nu}J_\mu A_\nu$. So let us take[4]

$$\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu$$

(the constant in front will be convenient later). Well, neither the $J^\mu A_\mu$ is automatically gauge invariant! Imagine how it behaves in the action subject to the gauge transformation:

$$\int_\Omega J^\mu(A_\mu+\chi_{,\mu})\,\sqrt{-g}\,\mathrm{d}^4x = \int_\Omega J^\mu A_\mu\,\sqrt{-g}\,\mathrm{d}^4x + \int_\Omega \left[(J^\mu\chi)_{;\mu} - J^\mu{}_{;\mu}\chi\right]\,\sqrt{-g}\,\mathrm{d}^4x\,.$$

By the divergence rule and by Gauss, the middle term is translated to the boundary as

$$\int_\Omega (J^\mu\chi)_{;\mu}\,\sqrt{-g}\,\mathrm{d}^4x = \int_\Omega (\sqrt{-g}\,J^\mu\chi)_{,\mu}\,\mathrm{d}^4x = \int_{\partial\Omega} \sqrt{-g}\,J^\mu\chi\,n_\mu\,\mathrm{d}^3x\,,$$

so it vanishes if either J^μ or χ vanish on the boundary $\partial\Omega$ (yet χ should *not* in general be required to vanish there, should it be *generic*). Then the potential term does not harm the gauge invariance of action if the last term vanishes as well, i.e. if $J^\mu{}_{;\mu}=0$. The J^μ is thus a simple example of Noether's current: the requirement of gauge invariance implies $J^\mu{}_{;\mu}=0$, which ensures the conservation of electric charge. More insight will arise from the second Noether theorem.

The Euler-Lagrange equations (23.4) read $\frac{\partial\mathcal{L}}{\partial A_\alpha} - \left[\frac{\partial\mathcal{L}}{\partial A_{\alpha;\beta}}\right]_{;\beta} = 0$, with $\frac{\partial\mathcal{L}}{\partial A_\alpha} = J^\mu\delta^\alpha_\mu = J^\alpha$. For the second term, we write

$$F_{\mu\nu}F^{\mu\nu} = (A_{\nu;\mu}-A_{\mu;\nu})(A^{\nu;\mu}-A^{\mu;\nu}) = 2A_{\nu;\mu}(A^{\nu;\mu}-A^{\mu;\nu}) = 2A_{\nu;\mu}(A_{\sigma;\rho}-A_{\rho;\sigma})\,g^{\sigma\nu}g^{\rho\mu}$$

and compute

$$-16\pi\,\frac{\partial\mathcal{L}}{\partial A_{\alpha;\beta}} = 2\delta^\alpha_\nu\delta^\beta_\mu(A^{\nu;\mu}-A^{\mu;\nu}) + 2A^{\sigma;\rho}(\delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\rho\delta^\beta_\sigma) = 4(A^{\alpha;\beta}-A^{\beta;\alpha}) = 4F^{\beta\alpha}\,.$$

The EL equations thus provide the first Maxwell series $F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$. The second series, $F_{\{\mu\nu;\rho\}}=0$, is well known to follow from the very definition of $F_{\mu\nu}$.

Appendix: It may seem we threw out the $A^\mu A_\mu$ term too easily. Actually, in correspondence

[4] This is the right place to mention the independent term ${}^*F^{\mu\nu}F_{\mu\nu}$ which would come to mind: for the action this is irrelevant, because it can be written as a divergence,

$$\frac{1}{2}\,\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}F_{\mu\nu} = 2\epsilon^{\mu\nu\kappa\lambda}A_{\kappa,\lambda}A_{\mu,\nu} = -\frac{2}{\sqrt{-g}}[\mu\nu\kappa\lambda]A_{\lambda,\kappa}A_{\mu,\nu} = -\frac{2}{\sqrt{-g}}\left([\mu\nu\kappa\lambda]A_{\lambda,\kappa}A_\mu\right)_{,\nu}\,.$$

to the massive scalar field, why not to add the term $m^2 A^\mu A_\mu$ (with mass m constant)? Look at how we treated the gauge invariance of the $J^\mu A_\mu$ term: we found that J^μ should vanish on the boundary and satisfy the continuity equation $J^\mu{}_{;\mu} = 0$. Analogously, for the $m^2 A^\mu A_\mu$ term, we would find that A^μ should vanish on the boundary and satisfy the Lorenz condition $A^\mu{}_{;\mu} = 0$. So, clearly, $m^2 A^\mu A_\mu$ is *not* gauge invariant. One may interpret this as the requirement for zero rest mass of photons, $m = 0$.

28.2.3 Diffeomorphisms vs. gauge transformations

Is the diffeomorphism invariance of GR a special kind of gauge symmetry? Opposite opinions can be heard, also depending on what level of the problem one wants to address. The word "gauge" mostly is used rather generically ("TT gauge", "harmonic gauge", etc.), yet let us shortly look at it.

- At first sight, the general diffeomorphism group of GR (smooth coordinate transformations), similarly as the groups of gauge transformations in the gauge theories, both represent certain invariances of physical laws ("symmetries"), so they may be understood on similar footing.

- At second sight, however, one realizes that the diffeomorphisms of GR transform the space-time *coordinates*, not the fields (the metric in particular) – the latter only change "secondarily" (according to their mathematical nature). Gauge transformations, on the other hand, directly transform the (gauge) fields, without changing the space-time coordinates. This difference could *technically* be ignored if any permitted "gauge" of the fields could alternatively be achieved by a pure coordinate transformation – if the scopes of changing the given field(s) by the gauge transformation and by the coordinate change were in some sense equivalent. But this is not the case: in electrodynamics, for example, gauging the four-potential by $A_\mu + \chi_{,\mu}$ cannot *always* be achieved by a coordinate transformation, i.e. as

$$\frac{\partial x^\nu}{\partial x'^\mu} A_\nu = A_\mu + \chi_{,\mu} \, .$$

  Trivial examples are $A_\mu = 0$ and $\chi_{,\mu} \neq 0$, or $\chi_{,\mu} = -A_\mu \, (\neq 0)$ ... these cannot be achieved by a coordinate change; more generally, the coordinate transformation does not change $g^{\mu\nu} A_\mu A_\nu$, whereas the gauge transformation does, in general.

- There is more to that when one considers what is the role, in the respective theories, of quantities which are subject to the transformations. The gauge fields are being "gauged" by the pertinent gauge group at each space-time point (the group lives on a "fibre existing above each such point" in the fibre-bundle picture), while themselves factually representing connections acting "across" the fibres, namely selecting, by their spatio-temporal behaviour, a specific spatio-temporal arrangement of the gauge fixing ($\equiv$ the "phase"). If, for a given such connection, the "phase" changes when going across a closed space-time loop (the connection has non-trivial holonomy), it means the connection gives rise to some curvature which in this picture corresponds to a given physical field (e.g. the electromagnetic field, $F_{\mu\nu}$). The gauge invariance means that the physical field (which is *the* measurable quantity) is independent of the choice of the gauge fixing, i.e. of the connection.

Analogous in GR *is* that the field – the "decent one", which cannot be transformed out, which is the curvature – is obtained from the holonomy of the connection. However, the gauge fields of gauge theories are *fundamental* fields, whereas in GR the (Levi-Civita) connection is obtained from the metric as the fundamental tensor. And, as already pointed out, what is "gauged" in GR rather are coordinates, not directly the connection; in the geometrical language, the diffeomorphism group of GR does not live "above" each manifold point to shift the local gauge-field value, it rather acts "horizontally", within the manifold, changing its points themselves. And, strictly speaking, as opposed to the gauge transformations, the diffeomorphisms do also change the field (the metric, the connection, the curvature) secondarily, with only certain scalars staying invariant.

In the quantum view, what is being quantized in qauge theories is the gauge field (the one being gauged), which in GR would correspond to the coordinates – and that apparently is not the case. One would rather guess that connection should be quantized, which however is represented by Christoffel symbols in GR and that are not tensors. (It is in fact not yet clear what should primarily be quantized in gravity.)

A more philosophical remark is at place. Under the diffeomorphism invariance, it actually has little sense to speak of the "field at a certain point", because the "manifold points" do not have any clear identity; they can only acquire one thanks to the *metric* (namely, the metric permits to say *when* and *where*, though it itself also changes under diffeomorphisms). Hence, as opposed to the standard view in gauge field theories ("a field in some space-time", not even speaking of the "field at a given point"), in GR *the field itself is the space-time*. As already quoted in Chapter 1, Einstein articulated that as follows: *"Space-time does not claim existence on its own, but only as a structural quality of the field."*

- With self-consistent physical fields dynamically tied to the space-time, the distinction is less pronounced, because they are subject to both (possible) gauge and diffeomorphism invariance. A propos, exactly in such a situation it is worth to clearly distinguish between the two types of transformations, and to elucidate the role of symmetries with respect to them in the structure of the theory, especially in the conservation laws.

- In the case of *infinitesimal* transformations, there exists a simple relation between the behaviour of tensorial quantities under the gauge ("intrinsic") and coordinate transformations, given by the Lie derivative (see the following section). Then, one *can* treat the two operations as *technically* akin.

28.3 Infinitesimal symmetry transformations

In looking for field equations (either in the Lagrangian or Hamiltonian manner), we were varying the Lagrangian/Hamiltonian at fixed coordinate position, i.e. all the change was due to the change of the fields. If asking about conservation laws, it is important to extend the picture and consider variations as caused by change of the fields themselves *as well as* by the change of coordinates. (In principle, one may ask about the behaviour of action under a change of *anything* on which it depends.) Actually, remember the derivation of $T^{\mu\nu}{}_{;\nu}=0$ from invariance of the "non-gravitational" action with respect to infinitesimal diffeomorphisms, where the variation was *entirely* induced by a coordinate transformation (Section 23.4.5).

Below, we assume both types of transformations form a *continuous (Lie) group*, so that any finite transformation can be composed of infinitesimal ones. Although in GR we mainly concern about tensors, the "fields" may be rather generic objects (their behaviour under a coordinate change is left unspecified yet). Let us look into how both types of (infinitesimal) transformations work and, mainly, what is their relation.

The **infinitesimal coordinate change** we write as[5]

$$x'^{\mu} = x^{\mu} - \epsilon\,\xi^{\mu}(x) = x^{\mu} + \delta x^{\mu}\,, \qquad \delta x^{\mu} := -\epsilon\,\xi^{\mu}(x)\,.$$

In addition to that, let the fields ψ undergo a certain infinitesimal *intrinsic* change independent of the coordinate shift, i.e. an **infinitesimal gauge transformation** (which would be there even if coordinates stayed unchanged). Denote the new field(s) in new values of the coordinates (the result of *both transformations performed simultaneously*) by $\psi'(x')$, with

$$\delta\psi(x) := \psi'(x') - \psi(x) \qquad \text{... } \textbf{total change of the field value}\,.$$

Note that this is an extension of the notation we use everywhere else: $\psi'(x')$ etc. we generally denote the result of a coordinate transformation "alone", whereas here it stands for the result of both the coordinate and the gauge shifts. Besides the total variation $\delta\psi(x)$, it will be useful to also define

$$\bar{\delta}\psi(x) := \psi'(x) - \psi(x) \qquad \text{... field-value change due to the } \textbf{change of the field form}\,.$$

This is *also* caused by *both* shifts, yet it is determined *at the same coordinate values*. Worth to realize that such a definition does *not* represent the change of the field *at a given point* (had it any clear meaning), but rather the difference between the original components of ψ at x and the transformed components ψ' at the point that is mapped onto x by the coordinate shift.

If there is no gauge change of ψ, the whole variation is solely induced by the coordinate transformation, with $\psi'(x')$ and $\delta\psi(x)$ meaning what we are used to (for instance, if ψ is a tensor, it is given by tensor transformation rule) and $\bar{\delta}\psi(x)$ representing the infinitesimal change of the field *along the flow* of ξ^{μ}. If, on the other hand, there is only a gauge change, without any coordinate shift, then $\delta\psi(x) = \bar{\delta}\psi(x)$ (one would tend then to call the altered field $\bar{\psi}(x)$) and they really represent a change of the field value "at the same point" (as fully caused by the change of the field form).[6]

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[5] Compare the following with the Lie-derivative section 11.3.1: there, we considered an infinitesimal shift along the flow of a vector field ξ^{μ}, writing that as $x^{\mu} = x_0^{\mu} + \epsilon\,\xi^{\mu}(x_0^{\alpha})$. Here we only use the opposite sign, in order to indicate that now we have a *coordinate* change in mind, rather than a "point-shifting" transformation. (These two options are usually called the "active" vs. the "passive" version of a transformation.)

[6] We adopt a *passive* view of the coordinate change here, so, "philosophically", $\delta\psi(x)$ means an **Eulerian variation** (at the same point yet with its coordinates changed), while $\bar{\delta}\psi(x)$ means a **Lagrangian variation** (at fixed coordinate values, which however represent different points before and after the transformation). In an *active* view, when "the point" is defined to be keeping its coordinates, it would be vice versa.

Note that if both the coordinate and gauge transformations are infinitesimal, their order is irrelevant, since the results only differ in $O(\delta^2)$. Independently of how transform the fields, the two variations are related by shift between the points in which the final field is evaluated,

$$\begin{aligned}\bar{\delta}\psi \;&\equiv\; \psi'(x)-\psi(x)=\psi'(x'-\delta x)-\psi(x)\doteq\psi'(x')-\psi'_{,\alpha}(x')\,\delta x^\alpha-\psi(x)\equiv\\ &\equiv\; \delta\psi-\psi'_{,\alpha}(x')\,\delta x^\alpha\doteq\delta\psi-\psi_{,\alpha}(x)\,\delta x^\alpha\;\equiv\;\delta\psi+\epsilon\,\psi_{,\alpha}\xi^\alpha \qquad (28.7)\end{aligned}$$

This is the well known relation between the Lagrangian and Eulerian variations ($\bar{\delta}\psi$ and $\delta\psi$). The existence of such a local relation between $\delta\psi$ and $\bar{\delta}\psi$ (and the coordinate shift) allows some authors to treat the variations "concisely", solely in terms of the "intrinsic" changes of the field. (Also take into account that the coordinate transformation is only a *shift*, the *type* of the coordinates is not altered, so the "primed-unprimed" notation is really just a notation.) Anyway, we will keep the δ-$\bar{\delta}$ distinction, for didactic reasons :-).

The Lagrangian variation $\bar{\delta}\psi$ commutes with partial derivative, because it is defined *at the same coordinate location*; the Eulerian variation, on the other hand, does not:

$$\begin{aligned}\bar{\delta}(\psi_{,\alpha})\equiv\bar{\delta}\left(\frac{\partial\psi}{\partial x^\alpha}\right)&=\frac{\partial\psi'(x)}{\partial x^\alpha}-\frac{\partial\psi(x)}{\partial x^\alpha}=\frac{\partial}{\partial x^\alpha}(\psi'(x)-\psi(x))\equiv(\bar{\delta}\psi)_{,\alpha}\,,\\ \delta(\psi_{,\alpha})\equiv\delta\left(\frac{\partial\psi}{\partial x^\alpha}\right)&=\frac{\partial\psi'(x')}{\partial x'^\alpha}-\frac{\partial\psi(x)}{\partial x^\alpha}=\frac{\partial\psi'(x'(x))}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\alpha}-\frac{\partial\psi(x)}{\partial x^\alpha}\neq\\ &\neq\frac{\partial}{\partial x^\alpha}(\psi'(x')-\psi(x))\equiv(\delta\psi)_{,\alpha}\,.\end{aligned}$$

If having just one field ψ of tensorial nature, and if performing only the infinitesimal coordinate shift, then

$$\begin{aligned}\delta\psi&\equiv\delta\psi^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}\equiv\psi'^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}(x')-\psi^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}(x)=\\ &=\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\ldots\frac{\partial x^\tau}{\partial x'^\kappa}\frac{\partial x^\omega}{\partial x'^\lambda}\ldots\psi^{\rho\sigma\ldots}{}_{\tau\omega\ldots}-\psi^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}=\\ &\doteq(\delta^\mu_\rho-\epsilon\,\xi^\mu{}_{,\rho})(\delta^\nu_\sigma-\epsilon\,\xi^\nu{}_{,\sigma})\ldots(\delta^\tau_\kappa+\epsilon\,\xi^\tau{}_{,\kappa})(\delta^\omega_\lambda+\epsilon\,\xi^\omega{}_{,\lambda})\ldots\psi^{\rho\sigma\ldots}{}_{\tau\omega\ldots}-\psi^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}=\\ &\doteq\epsilon\,(-\xi^\mu{}_{,\rho}\,\psi^{\rho\nu\ldots}{}_{\kappa\lambda\ldots}-\xi^\nu{}_{,\sigma}\,\psi^{\mu\sigma\ldots}{}_{\kappa\lambda\ldots}-\ldots+\xi^\tau{}_{,\kappa}\,\psi^{\mu\nu\ldots}{}_{\tau\lambda\ldots}+\xi^\omega{}_{,\lambda}\,\psi^{\mu\nu\ldots}{}_{\kappa\omega\ldots}+\ldots)=\\ &=\epsilon\,(\pounds_\xi\psi)^{\mu\nu\ldots}{}_{\kappa\lambda\ldots}-\epsilon\,\psi^{\mu\nu\ldots}{}_{\kappa\lambda\ldots,\alpha}\,\xi^\alpha\equiv\epsilon\,(\pounds_\xi\psi-\psi_{,\alpha}\xi^\alpha)\,. \qquad (28.8)\end{aligned}$$

In such a case, as expected (it is factually a *definition* of the Lie derivative, as telling the change of the field ψ under the flow of $\epsilon\xi^\mu$),

$$\bar{\delta}\psi=\delta\psi+\epsilon\,\psi_{,\alpha}\xi^\alpha=\epsilon\,\pounds_\xi\psi\;\;(\equiv\pounds_{\epsilon\xi}\psi)\,. \qquad (28.9)$$

We seek conditions under which the infinitesimal transformation is a **(quasi-)symmetry of an action**. Recall that the basic condition – ensuring the covariance of the EL equations – is that $S'[\psi']$ at most differs from $S[\psi]$ by a divergence term integrated over the (original) region Ω, with the Lagrangian keeping the same functional form. By combination of these two requirements, we obtained the crucial relation (28.6), i.e.[7]

$$\int_\Omega\mathfrak{L}(x,\psi'(x))\,\mathrm{d}^4x+\int_\Omega\left[\mathfrak{L}(x,\psi(x))\,\delta x^\alpha\right]_{,\alpha}\mathrm{d}^4x=\int_\Omega\mathfrak{L}(x,\psi(x))\,\mathrm{d}^4x-\int_\Omega Q^\alpha{}_{,\alpha}\,\mathrm{d}^4x\,.$$

[7] Very important remarque: in (28.4), we saw that $\int_\Omega\mathfrak{L}(x,\psi'(x))\,\mathrm{d}^4x$ equals $S[\psi]$ plus a term which vanishes

Let us denote, naturally, $\mathfrak{L}(x,\psi'(x))-\mathfrak{L}(x,\psi(x))=:\bar{\delta}\mathfrak{L}$, and let us consider that the Q^α term has to be $O(\bar{\delta})$ as well, so we denote it by $\bar{\delta}Q^\alpha$ (Q^α itself is just a symbol, after all). In such a way, we arrive at the symmetry condition

$$\int_\Omega \Big[\bar{\delta}\mathfrak{L} + \big(\mathfrak{L}(x,\psi)\,\delta x^\alpha + \bar{\delta}Q^\alpha(x,\psi)\big)_{,\alpha}\Big]\,\mathrm{d}^4x = 0\,. \tag{28.10}$$

Should this hold for any region Ω, we have the requirement

$$\boxed{\bar{\delta}\mathfrak{L} + (\mathfrak{L}\,\delta x^\alpha + \bar{\delta}Q^\alpha)_{,\alpha} = 0}\,. \tag{28.11}$$

28.4 Infinitesimal symmetry and weak conservation laws

Firstly, note that everything has yet been generic, "off shell". Let us now evaluate $\bar{\delta}\mathfrak{L}$ in (28.11) explicitly, assuming that $\mathfrak{L}$ at most depends on the second derivatives of ψ,

$$\bar{\delta}\mathfrak{L} \equiv \mathfrak{L}(x,\psi+\bar{\delta}\psi) - \mathfrak{L}(x,\psi) = \cancel{\mathfrak{L}(x,\psi)} + \frac{\partial\mathfrak{L}}{\partial\psi}\,\bar{\delta}\psi + \frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi_{,\alpha} + \frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}\,\bar{\delta}\psi_{,\alpha\beta} - \cancel{\mathfrak{L}(x,\psi)} =$$

$$= \left[\frac{\partial\mathfrak{L}}{\partial\psi} - \left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha}}\right)_{,\alpha} + \left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}\right)_{,\alpha\beta}\right]\bar{\delta}\psi + \left[\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi - \left(\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}\right)_{,\beta}\bar{\delta}\psi + \frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}\,\bar{\delta}\psi_{,\beta}\right]_{,\alpha} \tag{28.12}$$

(it has been important that the $\bar{\delta}$ variation commutes with coordinate partial derivative!). Look, the first part are just the Euler-Lagrange equations, and the second part is a total divergence!

This is the right place for an important supplement: for the sake of brevity, we have been denoting just by ψ all the fields possibly present. However, if performing the variation of any quantity depending on more-than-one fields (including the case of just one field which however is a vector or a tensor one, thus having more components), one obtains terms which have to be summed over all the fields (or their components). Indeed, this applies to *all* the terms which have appeared in the above variation $\bar{\delta}\mathfrak{L}$. Taking this into account, and using "A" for the corresponding index (numbering the fields or their components), the infinitesimal-symmetry condition (28.11) assumes the form

$$\boxed{[\mathrm{EL}]^A\,\bar{\delta}\psi_A + \bar{\delta}\,\mathfrak{J}^\alpha{}_{,\alpha} = 0}\,, \tag{28.13}$$

where $[\mathrm{EL}]^A$ denotes the left-hand sides of the Euler-Lagrange equations (28.1) pertaining to the A-th field (or the field's A-th component) ψ_A, and $\bar{\delta}\,\mathfrak{J}^\alpha$ denote the "infinitesimal currents",

$$[\mathrm{EL}]^A := \frac{\partial\mathfrak{L}}{\partial\psi_A} - \left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\alpha}}\right)_{,\alpha} + \left(\frac{\partial\mathfrak{L}}{\partial\psi_{A,\alpha\beta}}\right)_{,\alpha\beta}, \tag{28.14}$$

if the EL equations hold. This result was important there for showing that if the Lagrangian is form-invariant (which ensured the covariance of the EL equations), then, under a generic infinitesimal transformation, the action changes at most by a boundary term. At the *present* moment, on the other hand, we *must not* use the EL equations, because the (quasi-)symmetries of the action are to be treated *irrespectively of them*, "off-shell". It is thus necessary to use the result (28.2) *without* the step (28.4).

$$\bar{\delta}\,\mathfrak{J}^\alpha := \frac{\partial \mathfrak{L}}{\partial \psi_{A,\alpha}}\,\bar{\delta}\psi_A - \left(\frac{\partial \mathfrak{L}}{\partial \psi_{A,\alpha\beta}}\right)_{,\beta}\bar{\delta}\psi_A + \frac{\partial \mathfrak{L}}{\partial \psi_{A,\alpha\beta}}\,\bar{\delta}\psi_{A,\beta} + \mathfrak{L}\,\delta x^\alpha + \bar{\delta}Q^\alpha\,. \tag{28.15}$$

Equation (28.13) is called **the main Noether's identity**. It puts together what we mentioned as **on-shell variations** (the first term: it vanishes if the fields satisfy their EL equations, while the variation stays arbitrary) and **symmetries of the action** (the second term: it vanishes if the variation satisfies a certain constraint, while the fields are left free), announcing that *whenever the action is off-shell (quasi-)invariant with respect to some continuous global transformation, then there exists a certain current whose divergence vanishes on-shell.*[8]

If the Lagrangian depends on more fields, the first term of (28.13) appears for each of them. *When all these fields are dynamical* (dynamically interconnected with the rest of the theory), the corresponding Euler-Lagrange equations *are* satisfied and the symmetry condition thus implies the conservation law

$$\boxed{\bar{\delta}\,\mathfrak{J}^\alpha{}_{,\alpha} = 0}\,. \tag{28.16}$$

The law is "weak" in the sense of being off-shell, i.e. *only holding if the pertinent EL equations are satisfied.*

28.5 1st Noether's theorem: weak conservation

Let us consider the case of *global* transformations: let the infinitesimal coordinate transformations $\delta x^\alpha = -\epsilon^{(i)}\xi^\alpha_{(i)}(x)$ form an r-dimensional Lie group (r finite), with $\xi^\alpha_{(i)}(x)$ the corresponding Lie-algebra generators and $\epsilon^{(i)}$ *constant* coefficients, $i = 1, 2, ..., r$.[9] Denote the intrinsic field change(s) by $\bar{\delta}\psi_A$, as either induced by the coordinate shift or also otherwise, and assume that δx^α and $\bar{\delta}\psi_A$ together form a symmetry transformation, with the respective conserved infinitesimal current $\bar{\delta}\,\mathfrak{J}^\alpha$. Now, decompose $\bar{\delta}\psi_A$ and $\bar{\delta}\,\mathfrak{J}^\alpha$ into such bases (denoted $\eta_{A(i)}$ and $\mathfrak{t}^\alpha_{(i)}$, respectively) with respect to which the coefficients of both decompositions are exactly the $\epsilon^{(i)}$ constants, i.e., let $\eta_{A(i)}$ and $\mathfrak{t}^\alpha_{(i)}$ be *defined* so that

$$\bar{\delta}\psi_A = \epsilon^{(i)}\eta_{A(i)}(x, \psi, \psi_{,\rho}) \;\; (A = ...)\,, \qquad \bar{\delta}\,\mathfrak{J}^\alpha = -\epsilon^{(i)}\mathfrak{t}^\alpha_{(i)}(x, \psi, \psi_{,\rho}, \psi_{,\rho\sigma})\,. \tag{28.17}$$

Then (28.16) implies r (weak) conservation laws, one for each independent symmetry generator,

$$\boxed{\partial_\alpha \mathfrak{t}^\alpha_{(i)} = 0\,, \qquad i = 1, 2, ..., r}\,. \tag{28.18}$$

Note that the currents $\mathfrak{t}^\alpha_{(i)}$ are vector densities of weight $+1$. One can also define the corresponding vectors (dividing $\mathfrak{t}^\alpha_{(i)}$ by $\sqrt{-g}$) and write the divergence as a covariant one using the usual formula.

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[8] In Section 28.2, we remarked that the invariance of the action (up to a boundary term) is *sufficient, yet not necessary* for the symmetry of the field equations. Therefore, it is good to have in mind that the Noether identity (and thus the consequent conservation laws) do not *solely* follow from the symmetry of the EL equations, but *also* from the assumption/requirement that the action is diffeomorphism invariant.

[9] Poincaré transform is a simple example: $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ contains 4 (independent) parameters a^μ and 16 parameters $\Lambda^\mu{}_\nu$ of which 6 are independent (due to the 10 orthogonality constraints $\eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu}$).

Example: scalar field in Minkowski

Consider again the scalar field in flat space-time, now with generic mass m, described by the Klein-Gordon Lagrangian density $\mathfrak{L} = \mathcal{L} = -\frac{1}{2}\left(\eta^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + m^2\psi^2\right)$. Consider a pure *coordinate* shift $x'^{\mu} = x^{\mu} + \delta x^{\mu}$, $\delta x^{\mu} = -\epsilon^{(i)}\xi^{\mu}_{(i)}$, with $\epsilon^{(i)}$ constants and $\xi^{\mu}_{(i)}$ generators. Since ψ is invariant, one has $\delta\psi = 0$ and, from (28.9),

$$\bar{\delta}\psi = \epsilon^{(i)}\pounds_{\xi_{(i)}}\psi = \epsilon^{(i)}\psi_{,\mu}\xi^{\mu}_{(i)} \,.$$

From (28.15), it thus follows

$$\begin{aligned}\bar{\delta}\,\mathfrak{J}^{\alpha} &= \frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi - \left(\cancel{\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}}\right)_{,\beta}\bar{\delta}\psi + \cancel{\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha\beta}}}\,\bar{\delta}\psi_{,\beta} + \mathfrak{L}\,\delta x^{\alpha} + \bar{\delta}Q^{\alpha} = \\ &= -\psi^{,\alpha}\epsilon^{(i)}\psi_{,\mu}\xi^{\mu}_{(i)} - \mathfrak{L}\,\epsilon^{(i)}\xi^{\alpha}_{(i)} + \bar{\delta}Q^{\alpha} = -(\psi^{,\alpha}\psi_{,\mu} + \mathfrak{L}\,\delta^{\alpha}_{\mu})\,\epsilon^{(i)}\xi^{\mu}_{(i)} + \bar{\delta}Q^{\alpha} \,.\end{aligned}$$

Let us choose $\bar{\delta}Q^{\alpha} = 0$ and $\xi^{\mu}_{(i)} = \delta^{\mu}_{i}$ (i.e. generators of the four translational symmetries). With such a choice,

$$\bar{\delta}\,\mathfrak{J}^{\alpha} = -(\psi^{,\alpha}\psi_{,\mu} + \mathfrak{L}\,\delta^{\alpha}_{\mu})\,\epsilon^{(i)}\delta^{\mu}_{i} = -(\psi^{,\alpha}\psi_{,i} + \mathfrak{L}\,\delta^{\alpha}_{i})\,\epsilon^{(i)} \,.$$

Comparison with (28.17) yields (index A is useless, because there is just one field)

$$\eta_{(i)} = \psi_{,\mu}\xi^{\mu}_{(i)} \,, \qquad \mathfrak{t}^{\alpha}_{(i)} = \psi^{,\alpha}\psi_{,i} + \mathfrak{L}\,\delta^{\alpha}_{i} \qquad (i = 1,2,3,4 \;\; \ldots \;\; \text{or} \;\; i = 0,1,2,3) \,, \quad (28.19)$$

the latter standing for the energy-momentum tensor of the Klein-Gordon field. Returning to the original form $\mathfrak{t}^{\alpha}_{(i)} = -\frac{\partial\mathfrak{L}}{\partial\psi_{,\alpha}}\,\psi_{,i} + \mathfrak{L}\,\delta^{\alpha}_{i}$, it represents the so-called *canonical energy-momentum tensor*. (The latter is generally associated with the invariance of a system under constant translations. We will see later that it does not in general provide a satisfactory energy-momentum tensor, yet for the scalar field it works.)

28.6 2nd Noether's theorem: generalized Bianchi identities

Besides dynamical fields, theories may also contain *non-dynamical quantities* – those not constrained by any "equations of motion". Most notably, these include functions appearing in gauge transformations (such as $\chi(x)$ in the $A_{\mu} \to A_{\mu} + \chi_{,\mu}$ transformation), or – in GR, in particular – the coordinates (= four free functions). These are *general functions*, so they cannot be expressed in terms of any finite set of constant parameters $\epsilon^{(i)}$, or, more accurately, not even in terms of any *countable* set of constants, because the transformation freedom is *continuous* – one can independently change things at every point. Hence, transformations involving such general functions form a continuous *infinite*-dimensional (pseudo-)group.

The *local symmetries* should not constrain physical degrees of freedom, yet still they imply certain constraints on the field equations governing the physics. The number of these constraints is the same as the number of free functions in the gauge group. Namely, in order to somehow fix the originally free gauge functions, one has to impose certain gauge conditions. Although these are largely a matter of choice, once fixed, they have to be added to the original field equations, which means that the dynamical degrees of freedom finally have to satisfy more equations than is their number. The problem thus would be overdetermined and may not have any solution. The only non-trivial way out is that the dynamical equations are not independent – that they are bound by the pertinent number of constraints.

The transformation (pseudo-)group is usually taken not just continuous, but even smooth, so one in fact considers an "infinite Lie group" (nomenclature is not unique, because sometimes the Lie groups are automatically taken as finite-dimensional). In our description, the transition from a finite to an infinite group can be performed by replacing the *constant* parameters $\epsilon^{(i)}$ with s (say) *generic functions* $\epsilon^{(\omega)}(x)$ ($\omega = 1, ..., s$); for example, the covariance of four-dimensional GR corresponds to $s = 4$ – the number of free coordinate functions.[10] (Such a transition is often called the *localization of the group*.) As a generalization of the above, we thus assume the transformation of x^μ and of each of the fields to have the form

$$\begin{aligned} \delta x^\alpha &= -\epsilon^{(\omega)}(x)\,\xi^\alpha_{(\omega)}(x) \qquad (\omega = 1, ..., s)\,, \\ \bar\delta\psi_A &= \epsilon^{(\omega)}(x)\,\eta_{A(\omega)}(x,\psi,\psi_{,\rho},\psi_{,\rho\sigma}) + \epsilon^{(\omega)}{}_{,\alpha}(x)\,\eta^\alpha_{A(\omega)}(x,\psi,\psi_{,\rho},\psi_{,\rho\sigma})\,, \end{aligned} \tag{28.20}$$

where $\epsilon^{(\omega)}(x)$ are generic functions now, supposed to vanish on the boundary $\partial\Omega$ together with their derivatives $\epsilon^{(\omega)}{}_{,\alpha}(x)$. In general, one might consider a more general $\bar\delta\psi_A$ involving higher derivatives of $\epsilon^{(\omega)}(x)$, but we will only comment on this finally.

Abbreviating again by [EL] the left-hand side of the Euler-Lagrange equations, we insert the calculation (28.12) of $\bar\delta\mathfrak{L}$ into the basic equation (28.10), while assuming that the divergence terms vanish due to the vanishing of the variations (of $\epsilon^{(\omega)}$ and $\epsilon^{(\omega)}_{,\alpha}$, actually) on the boundary,

$$\begin{aligned} 0 &= \int_\Omega \bar\delta\mathfrak{L}\,\mathrm{d}^4x + \cancel{\int_\Omega \big(\mathfrak{L}(x,\psi)\,\delta x^\alpha + \bar\delta Q^\alpha(x,\psi)\big)_{,\alpha}\,\mathrm{d}^4x} = \int_\Omega [\mathrm{EL}]^A\,\bar\delta\psi_A\,\mathrm{d}^4x + \cancel{\int_\Omega \bar\delta\mathfrak{J}^\alpha{}_{,\alpha}}\,\mathrm{d}^4x = \\ &= \int_\Omega [\mathrm{EL}]^A\left(\epsilon^{(\omega)}\eta_{A(\omega)} + \epsilon^{(\omega)}{}_{,\alpha}\,\eta^\alpha_{A(\omega)}\right)\mathrm{d}^4x = \int_\Omega \epsilon^{(\omega)}\left\{[\mathrm{EL}]^A\,\eta_{A(\omega)} - \partial_\alpha\big([\mathrm{EL}]^A\,\eta^\alpha_{A(\omega)}\big)\right\}\mathrm{d}^4x\,, \end{aligned}$$

where the last term has been obtained by "per partes",

$$\int_\Omega [\mathrm{EL}]^A\,\epsilon^{(\omega)}{}_{,\alpha}\,\eta^\alpha_{A(\omega)}\,\mathrm{d}^4x = \cancel{\int_\Omega \partial_\alpha\big([\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^\alpha_{A(\omega)}\big)}\,\mathrm{d}^4x - \int_\Omega \epsilon^{(\omega)}\partial_\alpha\big([\mathrm{EL}]^A\,\eta^\alpha_{A(\omega)}\big)\,\mathrm{d}^4x\,.$$

We thus arrive at s "generalized Bianchi identities"

$$\boxed{[\mathrm{EL}]^A\,\eta_{A(\omega)} - \partial_\alpha\Big([\mathrm{EL}]^A\,\eta^\alpha_{A(\omega)}\Big) = 0}\,, \tag{28.21}$$

since $\epsilon^{(\omega)}(x)$ have been arbitrary. Obviously, these identities impose certain *constraints* on the field equations, namely on $[\mathrm{EL}]^A$. Clearly they are taken "off-shell", otherwise they would be

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[10] The infinite-dimensionality of the problem is thus *not* brought by the fact that ω would be infinite (it may well be $s = 1$ as in Maxwell's electromagnetism), but by the continuous dependence on position x^μ as the independent variable.

trivial. Note that if the arbitrary functions $\epsilon^{(\omega)}(x)$ did *not* vanish on the boundary, one would obtain the boundary terms on the r.h. side, instead of the zero.

Imagine now, instead of (28.20), a transformation containing higher derivatives of the $\epsilon^{(\omega)}(x)$ functions – up to the order k, say – and follow the above derivation once more: clearly, after rewriting, similarly as in deriving the [EL] equations themselves, the "expansion" of $\bar{\delta}\psi$ in terms of as many divergence terms as possible (given by divergence of expressions containing $\epsilon^{(\omega)}$ and its derivatives up to the order $k-1$ ... which all vanish on the boundary), one is left with k terms containing the [EL] expression with its derivatives up to the k-th order. For instance, were the field variation given by

$$\bar{\delta}\psi_A = \epsilon^{(\omega)}\eta_{A(\omega)} + \epsilon^{(\omega)}{}_{,\alpha}\,\eta^{\alpha}_{A(\omega)} + \epsilon^{(\omega)}{}_{,\alpha\beta}\,\eta^{\alpha\beta}_{A(\omega)}\,,$$

the generalized Bianchi identities would read

$$[\mathrm{EL}]^A\,\eta_{A(\omega)} - \partial_\alpha\big([\mathrm{EL}]\,\eta^{\alpha}_{A(\omega)}\big) + \partial_\alpha\partial_\beta\Big([\mathrm{EL}]\,\eta^{\alpha\beta}_{A(\omega)}\Big) = 0\,.$$

28.6.1 Implication for the initial problem

The occurrence of Bianchi identities makes the initial-value problems for the theories possessing local symmetries more complicated, because the constraints *have to be already satisfied by initial conditions*. Let us go into it a bit.

The Bianchi identities bring an important message about the dependence of $[\mathrm{EL}]^A$ on the highest derivatives of ψ_A. Let us denote these derivatives by $\psi^{(n)}_A$. If the EL equations only depend on the highest derivatives linearly (as in the case of GR), we can write

$$[\mathrm{EL}]^A = \kappa^{AB}\psi^{(n)}_B + \mathcal{I}^A\,,$$

where κ^{AB} and $\mathcal{I}^A$ only depend on lower-than-(n)th derivatives of ψ_A. Inserting such an ansatz into the Bianchi identities (28.21), we obtain

$$0 = \partial_\alpha\big([\mathrm{EL}]^A\,\eta^{\alpha}_{A(\omega)}\big) + ... = \partial_\alpha\left(\kappa^{AB}\psi^{(n)}_B\eta^{\alpha}_{A(\omega)}\right) + ... = \kappa^{AB}\eta^{\alpha}_{A(\omega)}\partial_\alpha\psi^{(n)}_B + ...\,,$$

where we have gradually isolated the only term containing the now highest, $(n+1)$th derivatives of ψ_A. Imagine, in particular, that the above applies to the x^0 coordinate. Then the highest-derivative term (in x^0) specifically is the $\alpha = 0$ one,

$$\kappa^{AB}\eta^{0}_{A(\omega)}\partial_0\psi^{(n)}_B = \kappa^{AB}\eta^{0}_{A(\omega)}\psi^{(n+1)}_B \equiv \kappa^{AB}\eta^{0}_{A(\omega)}\,\frac{\partial^{n+1}\psi_B}{\partial(x^0)^{n+1}}\,.$$

Now, since we are off-shell, the ψ_A and its/their derivatives are not determined by the EL equations, they may be rather arbitrary. In order to satisfy the Bianchi identities for all such virtual field configurations, it is necessary that $\kappa^{AB}\eta^{0}_{A(\omega)} = 0$. The latter involves the action of a matrix κ^{AB} on the vectors $\eta^{0}_{A(\omega)}$ ($\omega = 1, ..., s$), specifically, it says that the matrix κ^{AB} has s independent eigen-vectors with zero eigen-values, hence the determinant of κ^{AB} is zero (it is a singular matrix). This means, however, that it is not possible to express, from the relations

$[\mathrm{EL}]^A = \kappa^{AB}\psi_B^{(n)} + \mathcal{I}^A$, all the highest derivatives (by x^0) $\psi_A^{(n)}$ in terms of lower derivatives. For theories possessing a local gauge symmetry, the field equations are thus not "naturally" of the Leray type (26.6). However, in most cases it is possible to reach the suitable form by a coordinate or gauge transformation (remember the harmonic gauge which "remedies" the Einstein equations, Section 26.3).

Example: EM field

The well known gauge freedom of electrodynamics with respect to $\bar{\delta}A_\kappa = \chi_{,\kappa}$ corresponds, in our present description, to $\psi_A \to A_\kappa$ and $s = 1$ (one free gauge function χ), so A is the usual space-time index ($A = 0,1,2,3 =: \kappa$) and ω can be omitted (it only has one value),

$$\bar{\delta}A_\kappa = \chi_{,\kappa} \quad \Longrightarrow \quad \epsilon^{(\omega)}(x) \to \epsilon(x) = \chi(x)\,, \quad \eta_{A(\omega)} \to \eta_\kappa = 0\,, \quad \eta^\alpha_{A(\omega)} \to \eta^\alpha_\kappa = \delta^\alpha_\kappa\,.$$

We know from the EM example before Section 28.2.3 that $\mathfrak{L} = \sqrt{-g}\left(-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right)$ yields

$$[\mathrm{EL}]^\kappa = \frac{\partial\mathfrak{L}}{\partial A_\kappa} - \left[\frac{\partial\mathfrak{L}}{\partial A_{\kappa;\beta}}\right]_{;\beta} = \sqrt{-g}\left(J^\kappa - \frac{1}{4\pi}F^{\kappa\beta}{}_{;\beta}\right) = \sqrt{-g}\,J^\kappa - \frac{1}{4\pi}\left(\sqrt{-g}\,F^{\kappa\beta}\right)_{,\beta}\,.$$

Hence, in the generalized Bianchi identities (28.21), the first term vanishes due to $\eta_\kappa = 0$, and the second term reads

$$\partial_\alpha\left([\mathrm{EL}]^\kappa\eta^\alpha_\kappa\right) = \partial_\alpha\left([\mathrm{EL}]^\alpha\right) = \left(\sqrt{-g}\,J^\alpha\right)_{,\alpha} - \cancel{\frac{1}{4\pi}\left(\sqrt{-g}\,F^{\alpha\beta}\right)_{,\beta\alpha}} = \sqrt{-g}\,J^\alpha{}_{;\alpha}\,.$$

Recall now how the continuity equation for the electric current has been appearing in the relativity course: first it was presented as following from the first set of Maxwell equations, so as an "on-shell" result. In the examples before Section 28.2.3, we presented it differently – as an *off-shell* condition for symmetry of the potential term in the Lagrangian with respect to the gauge transformation $A_\kappa \to A_\kappa + \chi_{,\kappa}$. Now we see that it really follows as an *off-shell* Bianchi-type constraint on the field equations (it is not necessary to satisfy $[\mathrm{EL}] = 0$ for it, since we have $\eta_{A(\omega)} = 0$).

28.7 3rd Noether's theorem: strong conservation and superpotentials

Finally, let us consider a combination of the above symmetry settings: let the symmetry transformations form an infinite Lie group, but let the latter contain an r-dimensional Lie subgroup (r finite) as in Section 28.5. We will again start from the basic Noether's symmetry condition (28.13), i.e. $[\mathrm{EL}]^A\,\bar{\delta}\psi_A + \bar{\delta}\,\mathfrak{J}^\alpha{}_{,\alpha} = 0$. The point is that when the symmetry group *only* consisted of the r global shifts, we were done: the condition implied r *weak* conservations (only valid if $[\mathrm{EL}]^A = 0$). Here, if there exists a larger, infinite symmetry group (containing the finite one as a subgroup), one also has the pertinent Bianchi identities (28.21) from where it is possible to express $[\mathrm{EL}]^A\,\bar{\delta}\psi_A$ in terms of the divergence terms, and thus to reach a set of *strong* conservations by adding them in that form to $\bar{\delta}\,\mathfrak{J}^\alpha{}_{,\alpha}$.

Now in more detail: plugging to the Noether's identity, as above, the decomposition $\bar{\delta}\psi_A = \epsilon^{(\omega)}\eta_{A(\omega)} + \epsilon^{(\omega)}{}_{,\alpha}\,\eta^{\alpha}_{A(\omega)}$ (dependences on position and field not indicated any more), and substituting $[\mathrm{EL}]^A\,\eta_{A(\omega)} = \partial_\alpha\big([\mathrm{EL}]^A\,\eta^{\alpha}_{A(\omega)}\big)$ from the Bianchi identities (28.21),[11] we have

$$[\mathrm{EL}]^A\,\bar{\delta}\psi_A = [\mathrm{EL}]^A\left(\epsilon^{(\omega)}\eta_{A(\omega)}+\epsilon^{(\omega)}{}_{,\alpha}\,\eta^{\alpha}_{A(\omega)}\right) = \epsilon^{(\omega)}\,\partial_\alpha\big([\mathrm{EL}]^A\,\eta^{\alpha}_{A(\omega)}\big) + [\mathrm{EL}]^A\,\epsilon^{(\omega)}{}_{,\alpha}\,\eta^{\alpha}_{A(\omega)} =$$
$$= \partial_\alpha\big([\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^{\alpha}_{A(\omega)}\big) \qquad \omega = 1,...,s\,.$$

The term $\bar{\delta}\,\mathfrak{J}^{\alpha}{}_{,\alpha}$ can further be "decomposed into the existing r constant parameters" $\epsilon^{(i)}$ as in Section 28.5,

$$\bar{\delta}\,\mathfrak{J}^{\alpha} = -\epsilon^{(i)}\mathfrak{t}^{\alpha}_{(i)}\,, \qquad i = 1,...,r\,,$$

so, altogether, the symmetry condition assumes the conservation-law form

$$\boxed{\partial_\alpha\Big([\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^{\alpha}_{A(\omega)} - \epsilon^{(i)}\mathfrak{t}^{\alpha}_{(i)}\Big) = 0}\,. \tag{28.22}$$

As announced already, these conservations are *strong*, i.e. valid irrespectively of whether the $[\mathrm{EL}]^A$ term(s) vanish or not. To summarize: if the finite Lie group of "global" symmetries is contained within the infinite-dimensional group of (generally) local ones, the basic symmetry condition (28.13), i.e. $[\mathrm{EL}]^A\,\bar{\delta}\psi_A+\bar{\delta}\,\mathfrak{J}^{\alpha}{}_{,\alpha} = 0$, can be rewritten as a total divergence (namely, the first term can also be cast into such a form).

The finding can be expressed in terms of the existence of a certain **superpotential**, namely such an antisymmetric matrix $U^{\alpha\beta}$ that

$$[\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^{\alpha}_{A(\omega)} - \epsilon^{(i)}\mathfrak{t}^{\alpha}_{(i)} = \partial_\beta U^{\alpha\beta}\,.$$

The antisymmetry of $U^{\alpha\beta}$ automatically ensures that

$$\partial_\alpha\left([\mathrm{EL}]\,\epsilon^{(\omega)}\eta^{\alpha}_{(\omega)} - \epsilon^{(i)}\mathfrak{t}^{\alpha}_{(i)}\right) \equiv \partial_\alpha\partial_\beta U^{\alpha\beta} = 0\,.$$

Apparently, the above strong conservation laws convert into weak conservation laws for $\mathfrak{t}^{\alpha}_{(i)}$ as in Section 28.5, whenever the field equations are satisfied:

$$\text{IF}\ \ [\mathrm{EL}]^A = 0\,, \quad \text{THEN}\ \ \partial_\alpha\mathfrak{t}^{\alpha}_{(i)} = 0\,. \tag{28.23}$$

28.8 Summary on Noether's theorems and conservation laws

- 1st Noether's theorem: invariance of the action S with respect to an r-dimensional Lie group of global transformations (involving r generators and r constant coefficients) implies r independent relations between the Lagrangian variation [EL] $\bar{\delta}\psi$ and divergences of certain currents $\mathfrak{t}^{\alpha}_{(i)}$. If $[\mathrm{EL}]=0$, these imply r ("weak") conservation laws for the currents.

[11] As it is standard, we do not include their divergence terms, possibly occurring if we had not required that the variations (namely the gauge functions $\epsilon^{(\omega)}$ and $\epsilon^{(\omega)}{}_{,\alpha}$) vanish on the boundary $\partial\Omega$.

- 2nd Noether's theorem: invariance of S with respect to an ∞-dimensional (pseudo-)group of local transformations, involving s free functions of x^μ and their derivatives up to the k-th order, leads to s differential identities (generalized Bianchi identities) which are linear in the Euler-Lagrange expression $[\mathrm{EL}]$ and in the latter's derivatives up to the k-th order.

- If S is invariant with respect to an ∞-dimensional (pseudo-)group of local transformations which, in addition, contains a certain r-dimensional Lie (sub)group, the corresponding r weak conservations from the 1st Noether's theorem can be converted into strong ones.

28.8.1 "Proper", "improper" and "trivial" conservation laws

What today is being called "weak" vs. "strong" laws, E. Noether called "proper" vs. "improper" laws, respectively. Noether's original language is not being much used today, only that "in the meantime" the weak conservation laws have occasionally been divided into **proper/improper** according to whether the finite-dimensional symmetry group of a given theory *cannot/can* be extended to an infinite-dimensional symmetry group *without introduction of auxiliary, non-dynamical fields*.
[Example: the invariance of special-relativity with respect to the 10-parametric Poincaré group can be extended to invariance with respect to the infinite-dimensional group of local diffeomorphisms by extension from Minkowskian to general coordinates. However, the field $g_{\mu\nu}$ which appears in it is not dynamical – there are no field equations for it.]

A notable contribution to the classification was provided by P. J. Olver [32]: he says that, for a certain set of differential equations for $\psi(x)$, a conservation law is any equation $(\mathrm{expression})^\mu{}_{,\mu} = 0$ satisfied for all solutions of the system (with "expression" being a tensor density depending on x^μ and on $\psi(x)$ including its derivatives up to a certain order). In our terminology, such laws would be "weak" (satisfied for *solutions* of the equations). Quite interesting to quote his very simple example: the Laplace or d'Alembert equation in *itself* represents a conservation law, $\Box\psi = (\psi^{;\beta})_{;\beta} = 0$, with further conservations e.g. following by its multiplication by $\psi^{;\alpha}$:

$$0 = \psi^{;\alpha}\psi^{;\beta}{}_{;\beta} = \left(\psi^{;\alpha}\psi^{;\beta} - \frac{1}{2}\, g^{\alpha\beta}\psi_{;\gamma}\psi^{;\gamma}\right)_{;\beta} .$$

On the other hand, instead of "improper" laws, he speaks of *trivial conservation laws*, recognizing two classes of them: the first class is "trivial" because not only $(\mathrm{expression})^\mu{}_{,\mu}$ vanishes, but even $(\mathrm{expression})^\mu$ itself (for all solutions of the given system). The second class is "trivial" because it holds *strongly* (in our language), for *all functions* $\psi(x)$, not just for those which solve the system. Interestingly, the conservation equation (28.22) obtained for the combined-symmetry situation, i.e. $\partial_\alpha\Big([\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^\alpha_{A(\omega)} - \epsilon^{(i)}\mathfrak{t}^\alpha_{(i)}\Big) = 0$, involves *all three* Olver's cases: the whole equation is "strong", that means "of the second trivial class"; the first term $[\mathrm{EL}]^A\,\epsilon^{(\omega)}\eta^\alpha_{A(\omega)}$ has zero divergence, but it is also itself zero for solutions of the EL equations, so it is "of the first trivial class"; and the remaining term $\mathfrak{t}^\alpha_{(i)}$ is non-zero while divergence-free only if the EL equations are satisfied, so it is a non-trivial conservation law.

28.9 Conservation laws for test fields, and canonical energy-momentum tensor

The variational problem gets different if the gravitational field $g_{\mu\nu}$ is *given as a fixed background*. In such a case, it is *not dynamical* (not reciprocally interconnected with the rest of the problem), which means that the variation of the Lagrangian density with respect to it is not required to vanish. We will vary, on such a fixed background, the "non-gravitational" Lagrangian density, assuming the latter depends on *test* fields ψ and their first derivatives, and also on the metric, but not on the metric derivatives,

$$\mathfrak{L}_{\rm ng}(g_{\mu\nu};\psi,\psi_{,\alpha}) \equiv \sqrt{-g}\,\mathcal{L}_{\rm ng}(g_{\mu\nu};\psi,\psi_{,\alpha})\,.$$

Specifically, we will consider a variation solely induced by an infinitesimal coordinate shift and will express it in terms of the field variations *at a given location* x^μ, i.e. in terms of $\bar{\delta}\psi(x)$. From (28.9) we know that $\bar{\delta}\psi$ induced by an infinitesimal diffeomorphism $x^\mu \to x^\mu - \epsilon\xi^\mu$ is determined by $\bar{\delta}\psi = \epsilon\,\pounds_\xi\psi$. We will arrive at certain conservations (if there are some spacetime symmetries), yet the main goal will be to introduce a **canonical energy-momentum tensor** and to find its relation to the "standard", **symmetric energy-momentum tensor**.

In the above setting, we have

$$\bar{\delta}\mathfrak{L}_{\rm ng} = \frac{\partial\mathfrak{L}_{\rm ng}}{\partial g_{\mu\nu}}\,\bar{\delta}g_{\mu\nu} + \frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi}\,\bar{\delta}\psi + \frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi_{,\alpha}\,. \tag{28.24}$$

From (23.6) we know that if the non-gravitational Lagrangian density only depends on $g_{\mu\nu}$ and not on its derivatives, one defines the energy-momentum tensor $T_{\alpha\beta}$ and its associated tensor density $\mathfrak{T}_{\alpha\beta}$ (of weight $w=1$) by

$$\frac{1}{2}\,\mathfrak{T}_{\alpha\beta} := \frac{1}{2}\sqrt{-g}\,T_{\alpha\beta} := -\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm ng})}{\partial g^{\alpha\beta}} \equiv -\frac{\partial\mathfrak{L}_{\rm ng}}{\partial g^{\alpha\beta}}\,.$$

We also know that $\frac{\partial\mathfrak{L}_{\rm ng}}{\partial g_{\mu\nu}}\,\bar{\delta}g_{\mu\nu} = -\frac{\partial\mathfrak{L}_{\rm ng}}{\partial g^{\mu\nu}}\,\bar{\delta}g^{\mu\nu}$, so the first term of (28.24) reads $\frac{1}{2}\,\mathfrak{T}^{\mu\nu}\bar{\delta}g_{\mu\nu}$. The remaining terms can be rearranged as usual,

$$\begin{aligned}\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi}\,\bar{\delta}\psi + \frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi_{,\alpha} &= \frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi}\,\bar{\delta}\psi + \left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi\right)_{,\alpha} - \left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\right)_{,\alpha}\bar{\delta}\psi = \\ &= [\mathrm{EL}]\,\bar{\delta}\psi + \partial_\alpha\!\left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi\right),\end{aligned}$$

where $[\mathrm{EL}(\mathfrak{L}_{\rm ng})]$ denotes the Euler operator arising from variation of $\mathfrak{L}_{\rm ng}$ with respect to ψ,

$$[\mathrm{EL}] := [\mathrm{EL}(\mathfrak{L}_{\rm ng})] \equiv \frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi} - \partial_\alpha\!\left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\right)$$

(and commutation of the $\bar{\delta}$ variation with the partial derivative was employed for $\bar{\delta}\psi_{,\alpha}$). Hence, the variation (28.24) assumes the form

$$\bar{\delta}\mathfrak{L}_{\rm ng} = \frac{1}{2}\,\mathfrak{T}^{\mu\nu}\bar{\delta}g_{\mu\nu} + [\mathrm{EL}]\,\bar{\delta}\psi + \partial_\alpha\!\left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\,\bar{\delta}\psi\right). \tag{28.25}$$

Recall now the coordinate-shift origin of the variations. In such a case, we know that invariants (in fact all tensors) behave as $\bar{\delta}T = \epsilon\, \pounds_\xi T = \epsilon\, T_{;\iota}\xi^\iota$. The same also applies to the associated scalar densities ($\mathfrak{T} \equiv \sqrt{-g}\, T$), because $\bar{\delta}\mathfrak{T} = T\, \bar{\delta}\sqrt{-g} + \sqrt{-g}\, \bar{\delta}T$, where

$$\bar{\delta}\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\bar{\delta}g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\epsilon\, \pounds_\xi g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\epsilon\, (\xi_{\mu;\nu} + \xi_{\nu;\mu}) = \epsilon\, \sqrt{-g}\, \xi^\mu{}_{;\mu}\ ,$$
$$\Longrightarrow \quad \bar{\delta}\mathfrak{T} = \epsilon\, \sqrt{-g}\, T\xi^\mu{}_{;\mu} + \epsilon\, \sqrt{-g}\, T_{;\iota}\xi^\iota = \epsilon\, (\sqrt{-g}\, T\xi^\iota)_{;\iota} \equiv \epsilon\, \pounds_\xi \mathfrak{T}\, .$$

Details are given in Section A.1.2. In particular, we show there that the above divergence can equally well be written as a partial one. Hence, to summarize, we substitute, to equation (28.25), the expressions $\bar{\delta}\psi = \epsilon\, \pounds_\xi\psi$ and

$$\bar{\delta}\mathfrak{L}_{\rm ng} = \epsilon\, \pounds_\xi \mathfrak{L}_{\rm ng} = \epsilon\, (\mathfrak{L}_{\rm ng}\, \xi^\iota)_{;\iota} = \epsilon\, (\mathfrak{L}_{\rm ng}\, \xi^\iota)_{,\iota}\ ,$$
$$\bar{\delta}g_{\mu\nu} = \epsilon\, \pounds_\xi g_{\mu\nu} = \epsilon\, (\xi_{\mu;\nu} + \xi_{\nu;\mu}) \quad \Longrightarrow \quad \frac{1}{2}\, \mathfrak{T}^{\mu\nu}\bar{\delta}g_{\mu\nu} = \frac{1}{2}\, \mathfrak{T}^{\mu\nu}\epsilon\, (\xi_{\mu;\nu} + \xi_{\nu;\mu}) = \epsilon\, \mathfrak{T}^{\mu\nu}\xi_{\mu;\nu}\ ,$$

to obtain (cancelling ϵ from the whole equation)

$$(\mathfrak{L}_{\rm ng}\, \xi^\iota)_{,\iota} = \mathfrak{T}^{\mu\nu}\xi_{\mu;\nu} + [\mathrm{EL}]\, \pounds_\xi\psi + \partial_\alpha\left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\, \pounds_\xi\psi\right).$$

Finally, shifting the $(\mathfrak{L}_{\rm ng}\, \xi^\iota)_{,\iota}$ term to the divergence on the r.h. side, we arrive at

$$\boxed{\mathfrak{T}^{\mu\nu}\xi_{\mu;\nu} + [\mathrm{EL}]\, \pounds_\xi\psi + \partial_\alpha\left(\frac{\partial\mathfrak{L}_{\rm ng}}{\partial\psi_{,\alpha}}\, \pounds_\xi\psi - \mathfrak{L}_{\rm ng}\, \xi^\alpha\right) = 0}\, . \tag{28.26}$$

It is clear now that in the special case when ξ^μ is a Killing field, one obtains weak conservation(s) for the divergence term, because then $\mathfrak{T}^{\mu\nu}\xi_{\mu;\nu} = \mathfrak{T}^{(\mu\nu)}\xi_{[\mu;\nu]} = 0$. Still more special is the case when the test field ψ follows the space-time symmetry, $\pounds_\xi\psi = 0$: then one is left just with $(\nabla_\alpha\mathfrak{L}_{\rm ng})\xi^\alpha = \sqrt{-g}(\partial_\alpha\mathcal{L}_{\rm ng})\xi^\alpha = 0$, because $\mathfrak{L}_{\rm ng}\xi^\alpha$ is a vector density for which partial and covariant derivatives coincide, $\xi^\alpha{}_{;\alpha} = 0$ for a Killing field, and $\nabla_\alpha\mathcal{L}_{\rm ng} = \partial_\alpha\mathcal{L}_{\rm ng}$.

Recall now that (28.26) is an *identity*, so it has to be satisfied for *arbitrary* ξ^μ, including its arbitrary derivatives. Therefore, not only that (28.26) has to vanish as a whole – vanish must *individually* its parts proportional to ξ^μ and to its derivatives. In order to check this, let us write the equation more explicitely. First of all, realize once again that ψ in general represents *more than one* fields, or a field with more components, so the terms such as $[\mathrm{EL}]\, \pounds_\xi\psi$ should in fact be written as $[\mathrm{EL}]^A\, \pounds_\xi\psi_A$ (sum over A). In order to extract the gradient of ξ^μ from the $\pounds_\xi\psi$ term, let us consider just a *single* field ψ yet of a *generic* type (ψ^K, say, with K a multi-index in general). Then, generally,

$$\pounds_\xi\psi^K = \psi^K{}_{;\alpha}\xi^\alpha - F^K{}_{\alpha L}{}^\beta\psi^L\xi^\alpha{}_{;\beta}\ , \tag{28.27}$$

where the coefficients $F^K{}_{\alpha L}{}^\beta$ (and ψ^L, of course) are independent of the properties of ξ^μ. In order to illustrate such a formula, take $K \equiv \mu\nu$, for example: one has

$$\pounds_\xi\psi^{\mu\nu} = \psi^{\mu\nu}{}_{;\alpha}\xi^\alpha - \xi^\mu{}_{;\beta}\psi^{\beta\nu} - \xi^\nu{}_{;\beta}\psi^{\mu\beta}\ ,$$

so should the corresponding formula

$$\pounds_\xi \psi^{\mu\nu} \big(\equiv \pounds_\xi \psi^K = \psi^K{}_{;\alpha}\xi^\alpha - F^K{}_{\alpha L}{}^\beta \psi^L \xi^\alpha{}_{;\beta} \stackrel{!}{=} \big)\ \psi^{\mu\nu}{}_{;\alpha}\xi^\alpha - F^{\mu\nu}{}_{\alpha\rho\sigma}{}^\beta \psi^{\rho\sigma}\xi^\alpha{}_{;\beta}$$

work, the coefficients have to read $F^{\mu\nu}{}_{\alpha\rho\sigma}{}^\beta = \delta^\mu_\alpha \delta^\beta_\rho \delta^\nu_\sigma + \delta^\nu_\alpha \delta^\mu_\rho \delta^\beta_\sigma$.

Let us now focus on the last term of (28.26). It is a divergence of a vector density, so one can apply on it covariant divergence equally well,

$$\begin{aligned}
\partial_\alpha \left(\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} \pounds_\xi \psi^K - \mathfrak{L}_{\rm ng}\, \xi^\alpha \right) &= \nabla_\alpha \left(\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} \pounds_\xi \psi^K - \mathfrak{L}_{\rm ng}\, \xi^\alpha \right) = \\
&= \nabla_\alpha \left(\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} \psi^K{}_{;\rho}\xi^\rho - \frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} F^K{}_{\rho L}{}^\beta \psi^L \xi^\rho{}_{;\beta} - \mathfrak{L}_{\rm ng}\, \xi^\alpha \right) = \\
&= \nabla_\alpha \Bigg[\underbrace{\left(\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} \psi^K{}_{;\rho} - \mathfrak{L}_{\rm ng}\, \delta^\alpha_\rho \right)}_{=:\, -\mathfrak{t}_\rho{}^\alpha} \xi^\rho - \underbrace{\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi^K{}_{,\alpha}} F^K{}_{\rho L}{}^\beta \psi^L}_{=:\, -\mathfrak{S}_\rho{}^{\beta\alpha}} \xi^\rho{}_{;\beta} \Bigg] = \\
&= -\mathfrak{t}_\rho{}^\alpha{}_{;\alpha}\, \xi^\rho - \mathfrak{t}_\rho{}^\alpha\, \xi^\rho{}_{;\alpha} + \mathfrak{S}_\rho{}^{\beta\alpha}{}_{;\alpha}\, \xi^\rho{}_{;\beta} + \mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;\beta\alpha}\,,
\end{aligned} \tag{28.28}$$

where $\mathfrak{t}_\rho{}^\alpha$ is known, from the field theory, as the **canonical energy-momentum tensor**,[12] and $\mathfrak{S}_\rho{}^{\beta\alpha}$ is know as the **(canonical) spin tensor**. The whole identity (28.26) now appears as

$$\begin{aligned}
&\mathfrak{T}^{\mu\nu}\xi_{\mu;\nu} + [\mathrm{EL}]_K \left(\psi^K{}_{;\alpha}\xi^\alpha - F^K{}_{\alpha L}{}^\beta \psi^L \xi^\alpha{}_{;\beta} \right) + \\
&\qquad - \mathfrak{t}_\rho{}^\alpha{}_{;\alpha}\, \xi^\rho - \mathfrak{t}_\rho{}^\alpha\, \xi^\rho{}_{;\alpha} + \mathfrak{S}_\rho{}^{\beta\alpha}{}_{;\alpha}\, \xi^\rho{}_{;\beta} + \mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;\beta\alpha} = 0
\end{aligned} \tag{28.29}$$

([EL] now clearly has the multi-index K at the bottom).

In order to fulfil this for arbitrary ξ^μ in all the derivative orders, note first that $F^K{}_{\alpha L}{}^\beta$, $\mathfrak{t}_\rho{}^\alpha$ and $\mathfrak{S}_\rho{}^{\beta\alpha}$ do not depend on ξ^μ at all. Second, the second derivatives of ξ^μ only appear in the very last term $\mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;\beta\alpha}$. For the partial-derivative part of this term, $\mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{,\beta\alpha}$, to vanish, $\mathfrak{S}_\rho{}^{\beta\alpha}$ has to be anti-symmetric in $[\alpha, \beta]$. Then, however, one has

$$\mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;\beta\alpha} = \mathfrak{S}_\rho{}^{[\beta\alpha]}\, \xi^\rho{}_{;\beta\alpha} = \mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;[\beta\alpha]} = -\mathfrak{S}_\rho{}^{\beta\alpha}\, R^\rho{}_{\mu\beta\alpha}\xi^\mu\,,$$

thanks to which we conclude that the term $\mathfrak{S}_\rho{}^{\beta\alpha}\, \xi^\rho{}_{;\beta\alpha}$ actually does not involve *any* derivatives of ξ^μ, and that there are *no* terms at all proportional to the second derivatives of ξ^μ in the equation.

Proceed now to the crucial part of (28.29), proportional to $\xi^\alpha{}_{,\beta}$: this will provide the relation between $\mathfrak{T}^\beta_\alpha$ and $\mathfrak{t}_\alpha{}^\beta$. From the first term, we extract the partial-derivative part as

$$\mathfrak{T}^{\mu\nu}\xi_{\mu;\nu} \;\equiv\; \mathfrak{T}^{\alpha\beta}\xi_{\alpha;\beta} \;\equiv\; \mathfrak{T}^\beta_\alpha \xi^\alpha{}_{;\beta} \;\longrightarrow\; \mathfrak{T}^\beta_\alpha \xi^\alpha{}_{,\beta}\,.$$

From the second term, $[\mathrm{EL}]\, \pounds_\xi\psi$, the [EL] part does not depend on ξ^μ, so only contributing is the $-F^K{}_{\alpha L}{}^\beta \psi^L \xi^\alpha{}_{;\beta}$ part, hence

$$[\mathrm{EL}]\, \pounds_\xi \psi \;\longrightarrow\; -[\mathrm{EL}]_K\, F^K{}_{\alpha L}{}^\beta \psi^L \xi^\alpha{}_{,\beta}\,.$$

[12] This tensor is more often being introduced with the opposite sign in the literature. Our choice leads, for a scalar field (e.g.), to $\mathfrak{t}_\rho{}^\alpha = -\frac{\partial \mathfrak{L}_{\rm ng}}{\partial \psi_{,\alpha}}\psi_{,\rho} + \mathfrak{L}_{\rm ng}\,\delta^\alpha_\rho = \psi^{,\alpha}\psi_{,\rho} + \mathfrak{L}_{\rm ng}\,\delta^\alpha_\rho$, which agrees with the result obtained from the conserved Noether current in (28.19), and also with that obtained from the formula (23.17) in (23.18).

Finally, from the rest of (28.29) we have (just renaming indices first...)

$$-\mathfrak{t}_\rho{}^\alpha\,\xi^\rho{}_{;\alpha}+\mathfrak{S}_\rho{}^{\beta\alpha}{}_{;\alpha}\,\xi^\rho{}_{;\beta}\;\equiv\;-\mathfrak{t}_\alpha{}^\beta\,\xi^\alpha{}_{;\beta}+\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;\rho}\,\xi^\alpha{}_{;\beta}\;\longrightarrow\;\left(-\mathfrak{t}_\alpha{}^\beta+\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;\rho}\right)\xi^\alpha{}_{,\beta}\,.$$

Therefore, in total we have reached, for the $\xi^\alpha{}_{,\beta}$-part of (28.29), the requirement

$$\boxed{\left(\mathfrak{T}^\beta_\alpha-[\mathrm{EL}]_K\,F^K{}_{\alpha L}{}^\beta\psi^L-\mathfrak{t}_\alpha{}^\beta+\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;\rho}\right)}\;\xi^\alpha{}_{,\beta}=0\,. \tag{28.30}$$

It has to be satisfied for *arbitrary* $\xi^\alpha{}_{,\beta}$, so vanish must the parenthesis. The formula represents the relation between the "symmetric" (or "metric") energy-momentum tensor $\mathfrak{T}^\beta_\alpha$ and the canonical one, $\mathfrak{t}_\alpha{}^\beta$. (They both are tensor *densities* actually, sure.)

Remark: in specific theories, the definition of the energy-momentum tensors is adjusted, by including constants, according to how exactly the pertinent field equations relate the fields to the sources. See the next section for the GR case where the formulas (28.28) in fact define $16\pi\mathfrak{t}_\rho{}^\alpha$, etc.

If the Euler-Lagrange equations are satisfied, $[\mathrm{EL}]_K=0$, the energy-momentum tensors just differ by the spin term. Diverging the thus reduced ("weak") condition while supposing conservation of the symmetric energy-momentum tensor, $\mathfrak{T}^\beta_{\alpha;\beta}=0$, we obtain

$$\mathfrak{t}_\alpha{}^\beta{}_{;\beta}=\cancel{\mathfrak{T}^\beta_{\alpha;\beta}}+\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;\rho\beta}\,.$$

Regarding, in addition, the skew-symmetry of $\mathfrak{S}_\alpha{}^{\beta\rho}$ in the upper indices, we have

$$\begin{aligned}\mathfrak{t}_\alpha{}^\beta{}_{;\beta}&=\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;\rho\beta}=\mathfrak{S}_\alpha{}^{[\beta\rho]}{}_{;\rho\beta}=\mathfrak{S}_\alpha{}^{\beta\rho}{}_{;[\rho\beta]}=R^\iota{}_{\alpha\rho\beta}\mathfrak{S}_\iota{}^{\beta\rho}+R_\iota{}^\beta{}_{\rho\beta}\mathfrak{S}_\alpha{}^{\iota\rho}+R_\iota{}^\rho{}_{\rho\beta}\mathfrak{S}_\alpha{}^{\beta\iota}=\\&=R^\iota{}_{\alpha\rho\beta}\mathfrak{S}_\iota{}^{\beta\rho}+\cancel{R_{\iota\rho}\mathfrak{S}_\alpha{}^{\iota\rho}}-\cancel{R_{\iota\beta}\mathfrak{S}_\alpha{}^{\beta\iota}}=-R^\iota{}_{\alpha\beta\rho}\mathfrak{S}_\iota{}^{\beta\rho}\,.\end{aligned}$$

So, the canonical tensor is *not* in general conserved in a curved space-time.

Example: EM field in Minkowski

Let us check how the currents and the canonical energy-momentum tensor look for the free EM field in Minkowski. For $\mathfrak{L}\equiv\mathcal{L}=-\frac{1}{16\pi}\,F_{\mu\nu}F^{\mu\nu}$, the diverged part of the main Noether's identity (28.13), i.e. the infinitesimal currents (28.15), read

$$\bar\delta\,\mathfrak{J}^\alpha=\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}}\,\bar\delta A_\kappa-\left(\cancel{\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha\beta}}}\right)_{,\beta}\bar\delta A_\kappa+\cancel{\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha\beta}}}\,\bar\delta A_{\kappa,\beta}+\mathfrak{L}\,\delta x^\alpha+\bar\delta Q^\alpha\,.$$

If the change of A_μ is solely induced by the infinitesimal translation $x'^\alpha=x^\alpha-\epsilon\,\xi^\alpha(x)$, we have, from the general relation (28.9), $\bar\delta A_\kappa=\epsilon\,\pounds_\xi A_\kappa=\epsilon\,(A_{\kappa,\iota}\xi^\iota+\xi^\iota{}_{,\kappa}A_\iota)$, hence

$$\bar\delta\,\mathfrak{J}^\alpha=\epsilon\left(\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}}\,A_{\kappa,\iota}-\mathfrak{L}\,\delta^\alpha_\iota\right)\xi^\iota+\epsilon\,\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}}\,A_\iota\xi^\iota{}_{,\kappa}+\bar\delta Q^\alpha\,.$$

Comparing this with the divergence term (28.28) of the test-field equation (28.26), we recognize the canonical energy-momentum tensor and the canonical spin tensor,

$$\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}}\,A_{\kappa,\iota}-\mathfrak{L}\,\delta^\alpha_\iota\equiv-\mathfrak{t}_\iota{}^\alpha\,,\qquad\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}}\,A_\iota\equiv-\mathfrak{S}_\iota{}^{\kappa\alpha}\,.$$

Substituting for $\frac{\partial\mathfrak{L}}{\partial A_{\kappa,\alpha}} = -\frac{1}{4\pi}\,F^{\alpha\kappa}$, this means

$$\mathfrak{t}_\iota{}^\alpha = \frac{1}{4\pi}\left(F^{\alpha\kappa}A_{\kappa,\iota} - \frac{1}{4}\,\delta^\alpha_\iota F_{\mu\nu}F^{\mu\nu}\right), \qquad \mathfrak{S}_\iota{}^{\kappa\alpha} = \frac{1}{4\pi}\,F^{\alpha\kappa}A_\iota\,.$$

It is not ideal: $\mathfrak{t}_{\iota\alpha}$ is neither symmetric nor trace-free in general, and it is even gauge-dependent (a change of A_κ would affect it).

However, we can employ the freedom in $\bar\delta Q^\alpha$ to rectify the result: choosing

$$\bar\delta Q^\alpha = \frac{\epsilon}{4\pi}\,(F^{\alpha\kappa}\xi^\iota A_\iota)_{,\kappa} = \frac{\epsilon}{4\pi}\,\cancel{F^{\alpha\kappa}{}_{,\kappa}}\,\xi^\iota A_\iota + \frac{\epsilon}{4\pi}\,F^{\alpha\kappa}\xi^\iota{}_{,\kappa}A_\iota + \frac{\epsilon}{4\pi}\,F^{\alpha\kappa}A_{\iota,\kappa}\xi^\iota$$

(thus “on-shell”), the first term exactly cancels the spin-tensor term and the second term complements the gauge-dependent term of $\mathfrak{t}_\iota{}^\alpha$ to the standard “correct” one,

$$\begin{aligned}\bar\delta\,\mathfrak{J}^\alpha &= -\epsilon\left(\mathfrak{t}_\iota{}^\alpha - \frac{1}{4\pi}\,F^{\alpha\kappa}A_{\iota,\kappa}\right)\xi^\iota - \epsilon\left(\cancel{\mathfrak{S}_\iota{}^{\kappa\alpha} - \frac{1}{4\pi}\,F^{\alpha\kappa}A_\iota}\right)\xi^\iota{}_{,\kappa} = \\ &= -\frac{1}{4\pi}\left(F^{\alpha\kappa}F_{\iota\kappa} - \frac{1}{4}\,\delta^\alpha_\iota F_{\mu\nu}F^{\mu\nu}\right)\epsilon\,\xi^\iota\,.\end{aligned}$$

Note that the above choice of $\bar\delta Q^\alpha$ is totally “harmless”, because not only that it leads to a divergence term, but even this term is zero, $(\bar\delta Q^\alpha)_{,\alpha}=0$, thanks to the antisymmetry of $F^{\alpha\kappa}$.

The above modification of the canonical energy-momentum tensor to a symmetric and gauge-invariant form is called the **Belinfante-Rosenfeld procedure**. Note once more that the correction term has been taken “on-shell”, so in general the favourable properties of the Belinfante-Rosenfeld tensor only apply if the field equations are satisfied.

28.10 Energy (and other “source” features) of the gravitational field

In Maxwell’s electromagnetism (in flat space-time), the currents J^α generate the field $F^{\alpha\beta}$, with the field bearing no characteristics of the source (i.e. of the current): there is no current in the field itself, which makes the notion of the current clear. Associated with the field is the energy-momentum tensor $T_{\mu\nu}$ (see Section 7.3), from which the energy of the field at some moment (on some space-like hypersurface Σ) can be computed as $\int_\Sigma T^\mu_\nu t^\nu n_\mu\,\mathrm{d}^3x$, with t^μ the time-translation Killing vector and n^μ the unit normal to Σ. Equations $T^\mu_{\nu,\mu}=0$ then represent the conservation laws which guarantee that the energy and momentum of the system are independent of time (independent of the choice of Σ) – see below. Besides that, it is always possible to say how much energy (and momentum) there is in a certain spatial region. The field can *transport* energy from one place to the other, as it is clear from what people are constantly doing (look around).

In GR, it is the energy-momentum $T_{\mu\nu}$ what represent sources, yet the field *itself bears the energy and momentum* and thus itself behaves as a source (“the field generates field”). And the field is capable of transporting the energy and momentum at distance: otherwise detection of gravitational waves would not be feasible. In fact, the field has its own degrees of freedom, independent of any "non-gravitational source" – even with $T_{\mu\nu}=0$, one has plethora of space-times which do contain non-trivial field, including those containing gravitational radiation. It is thus inherently problematic to say how much energy or momentum is here and how much there, because "the source is everywhere" ... the gravitational energy, as well as other source characteristics, are *non-localisable*. The situation is only clear for *test* bodies and fields which have no effect on the gravitational *background* (which are not *dynamically coupled* to the latter).

One may also comment on the above issue from the principle of equivalence: since the *gravitation is universal*, it is always possible to go over to LIFE... There, however, gravitational acceleration locally vanishes, so the situation is – *locally* – indistinguishable from flat space-time, i.e., as if it were *no gravitation at all*. In mathematical words, first derivatives of the metric (Christoffel symbols) themselves do not in general give rise to any tensorial quantity from which "strength of gravity" could be inferred and use to define the energetics.[13] You may ask, *why just metric and its first derivatives?* –It is because of the field equations: these contain second derivatives, so in order to have something conserved, it should be of one derivative order less.

Remark: Equations containing higher than second derivatives tend to predict unphysical properties, such as superluminal speed of propagation (acausality) or instability with respect to perturbations (modes with negative energy can occur). At a quantum level, this reveals as the occurrence of particles with imaginary mass/energy (*tachyons*) or with negative mass/energy (*ghosts*), possibly having negative or greater-than-one probability of being found in a certain state (thus breaking the unitarity). In the role of equations of motion and field equations, physics thus very much prefers the second-order differential equations.

Now the same in more technical words. In Einstein equations, $T_{\mu\nu}$ represents all non-gravitational sources. It *locally* conserves covariantly, $T^{\mu\nu}{}_{;\nu}=0$, yet still this does not generally imply a global "Gauss law" which would say that a *total* energy and momentum present in a certain region were conserved. Actually, the covariant divergence of a *symmetric* tensor cannot be rewritten solely in terms of partial divergence – see (5.17),

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\mu\nu}\right)_{,\nu} + \Gamma^{\mu}{}_{\nu\iota}T^{\iota\nu} .$$

To be more precise, this formula does provide conservation, but only in the stationary case. Indeed, writing it for a *mixed* tensor T^{ν}_{α}, we have

$$\begin{aligned} T^{\nu}_{\alpha;\nu} &\equiv T^{\nu}_{\alpha,\nu} + \Gamma^{\nu}{}_{\nu\iota}T^{\iota}_{\alpha} - \Gamma^{\iota}{}_{\nu\alpha}T^{\nu}_{\iota} = T^{\nu}_{\alpha,\nu} + \frac{(\sqrt{-g})_{,\iota}}{\sqrt{-g}}\,T^{\iota}_{\alpha} - \Gamma_{\iota\nu\alpha}T^{\iota\nu} = \\ &= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\nu}_{\alpha}\right)_{,\nu} - \frac{1}{2}\left(g_{\iota\nu,\alpha} + g_{\alpha\iota,\nu} - g_{\nu\alpha,\iota}\right)T^{\iota\nu} = \\ &= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\nu}_{\alpha}\right)_{,\nu} - \frac{1}{2}\,g_{\iota\nu,\alpha}\,T^{\iota\nu}. \end{aligned} \tag{28.31}$$

[13] In this sentence, "in general" means "in general space-time". In Kerr space-time, for example, there *are* invariant quantities (lapse N, dragging angular velocity ω, and circumferential radius given by $g_{\phi\phi}$) from which one can define vectors by gradients. In particular, $g^{\mu\nu}N_{,\mu}N_{,\nu}$ is a very good measure of the gravitational acceleration (squared). But all this is based on particular symmetries – the time and azimuthal Killing fields.

Something similar to what we know from geodesic equation (3.6) arises: if the metric does not depend on the α-th coordinate, the respective (α-th) row/column of the energy-momentum tensor is conserved, because then $0 = T^{\nu}_{\alpha;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\nu}_{\alpha}\right)_{,\nu}$. In particular, for a *stationary space-time*, one has $g_{\iota\nu,0} = 0$ (in terms of the Killing time), hence

$$0 = \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,T^{\nu}_{0})_{,\nu} \quad\Longrightarrow\quad 0 = \int_{\Omega} \frac{1}{\sqrt{-g}}\,(\sqrt{-g}\,T^{\nu}_{0})_{,\nu}\,\sqrt{-g}\;\mathrm{d}^4x = \oint_{\partial\Omega} \sqrt{-g}\,T^{\nu}_{0}\,n_{\nu}\,\mathrm{d}^3x\ .$$

Imagine the domain Ω as representing a history of some spatial region, i.e. as a 3D cylinder with its bases at $x^0 \equiv t = t_{\rm in}$ and at $x^0 \equiv t = t_{\rm fin}$. Assuming there are no energy fluxes across the walls of that cylinder, we have the common conservation of energy: $\int_{t_{\rm in}} \sqrt{-g}\,T^0_0\,\mathrm{d}^3x = \int_{t_{\rm fin}} \sqrt{-g}\,T^0_0\,\mathrm{d}^3x$ (because the normal points in an opposite sense on the bases).

However, in general no formula for energy follows straightforwardly which would be naturally conserved. This was one of the reasons why also the notion of gravitational radiation was for decades a matter of debate. In particular, the Einstein's "quadrupole formula" from 1918, determining the power emitted by a gravitational source, was a subject to non-negligible scepticism which marginally survived until 21st century. It was one of Einstein's many triumphs when it was shown, in 1980s, that the orbital period of the Hulse-Taylor binary pulsar (1974) decays in incredible agreement with the formula (down to 0.2% by 2005), not speaking about direct detection of the waves in 2015.

Today, it is better understood that the lack of the energy-density prescription is a *feature* rather than a bug of GR. Below, we mention a few ways how to deal with this issue.

28.10.1 Energy-momentum pseudo-tensors (complexes)

In Section 23.3, we showed how Einstein gravitational equations follow by a suitable variational method. Let us recall the procedure, yet in a more general setting: this time we will only specify that the Lagrangian density should depend on the metric and on its 1st and 2nd derivatives. So let

$$S = S_{\rm g} + S_{\rm ng} = \int_{\Omega} \mathcal{L}_{\rm g}(g_{\mu\nu}, g_{\mu\nu,\alpha}, g_{\mu\nu,\alpha\beta})\,\sqrt{-g}\;\mathrm{d}^4x + 16\pi \int_{\Omega} \mathcal{L}_{\rm ng}(\psi_{\rm ng}; g_{\mu\nu}, g_{\mu\nu,\alpha})\,\sqrt{-g}\;\mathrm{d}^4x\,,$$

where the 16π has been fixed according to how we did it specifically in GR in (23.5). The Lagrangians certainly also depends on x^μ, but we assume it does so only "implicitly" through the dependence on the field $g_{\mu\nu}$ and its derivatives. We know from Section 23.3 that if $\bar{\delta}g^{\mu\nu}$ as well as $\bar{\delta}g^{\mu\nu}{}_{,\iota}$ vanish on the boundary $\partial\Omega$, the variational derivatives of $S_{\rm g}$ and $S_{\rm ng}$ with respect to $g^{\mu\nu}$ yield, respectively,

$$\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm g})}{\partial g^{\mu\nu}} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm g})}{\partial g^{\mu\nu}{}_{,\alpha}}\right]_{,\alpha} + \left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm g})}{\partial g^{\mu\nu}{}_{,\alpha\beta}}\right]_{,\alpha\beta} =: [\mathrm{EL}(\mathfrak{L}_{\rm g})]_{\mu\nu}\ ,$$

$$\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm ng})}{\partial g^{\mu\nu}} - \left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm ng})}{\partial g^{\mu\nu}{}_{,\alpha}}\right]_{,\alpha} =: [\mathrm{EL}(\mathfrak{L}_{\rm ng})]_{\mu\nu}\ .$$

Specifically in GR, we know that these come out as

$$[\mathrm{EL}(\mathfrak{L}_{\rm g})]_{\mu\nu} = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}\,Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right), \qquad [\mathrm{EL}(\mathfrak{L}_{\rm ng})]_{\mu\nu} =: -\frac{\sqrt{-g}}{2}\,T_{\mu\nu}\ ,$$

and that the field equations relate them as $[\mathrm{EL}(\mathfrak{L}_\mathrm{g})]_{\mu\nu} + 16\pi\,[\mathrm{EL}(\mathfrak{L}_\mathrm{ng})]_{\mu\nu} = 0$.

Now, however, consider a generic situation (no specific field equations), and focus on the *gravitational* part S_g. Consider again its behaviour induced by an infinitesimal shift $x^\mu \to x^\mu - \epsilon\xi^\mu$. Yes, the steps will seem to be very similar to what we did above, yet the aim is very different. We will try to find a *conservation law which would include the gravitational field*, in order to obtain then, on its basis, some conserved global quantities also incorporating the pure-gravitational contributions. Such contributions have to be searched for *elsewhere* than in the standard energy-momentum tensor $T^{\mu\nu}$, because the latter only contains non-gravitational sources. And the GR experience says that the gravitational part of the field equations hardly provides any covariant information in this respect (in GR, the covariant divergence of the field-equations l.h. side vanishes automatically due to the Bianchi identities). The only hope is to find the conservation law in terms of *partial* divergence, for some tensor *density* which would contain the standard, non-gravitational part $\mathfrak{T}^{\mu\nu}$ plus a certain contribution due to the gravitational field. The latter cannot be expected to be of tensorial nature, yet it might still be useful, at least in problems with clearly privileged coordinates.

So, let us express the "barred" variation of $\mathfrak{L}_\mathrm{g}$ (the change *at a given point*) in the way we have already followed several times,

$$
\begin{aligned}
\bar\delta\mathfrak{L}_\mathrm{g} &= \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu}}\,\bar\delta g_{\mu\nu} + \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\,\bar\delta g_{\mu\nu,\alpha} + \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\,\bar\delta g_{\mu\nu,\alpha\beta} = \\
&= \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu}}\,\bar\delta g_{\mu\nu} + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\,\bar\delta g_{\mu\nu}\right)_{,\alpha} - \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\right)_{,\alpha}\bar\delta g_{\mu\nu} + \\
&\quad + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\,\bar\delta g_{\mu\nu,\alpha}\right)_{,\beta} - \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\beta}\bar\delta g_{\mu\nu,\alpha} = \\
&= \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu}}\,\bar\delta g_{\mu\nu} + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\,\bar\delta g_{\mu\nu}\right)_{,\alpha} - \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\right)_{,\alpha}\bar\delta g_{\mu\nu} + \\
&\quad + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\,\bar\delta g_{\mu\nu}\right)_{,\alpha\beta} - 2\left[\left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\alpha}\bar\delta g_{\mu\nu}\right]_{,\beta} + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\alpha\beta}\bar\delta g_{\mu\nu} = \\
&= [\mathrm{EL}(\mathfrak{L}_\mathrm{g})]^{\mu\nu}\bar\delta g_{\mu\nu} + \partial_\alpha\left[\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\,\bar\delta g_{\mu\nu} + \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\,\bar\delta g_{\mu\nu}\right)_{,\beta} - 2\left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\beta}\bar\delta g_{\mu\nu}\right] = \\
&= [\mathrm{EL}(\mathfrak{L}_\mathrm{g})]^{\mu\nu}\bar\delta g_{\mu\nu} + \partial_\alpha\left[\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha}}\,\bar\delta g_{\mu\nu} - \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\beta}\bar\delta g_{\mu\nu} + \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\mu\nu,\alpha\beta}}\,\left(\bar\delta g_{\mu\nu}\right)_{,\beta}\right],
\end{aligned}
$$

where we have again used the commutation of $\bar\delta$ with coordinate partial derivatives.

If the variation is completely induced by a coordinate shift $x^\mu \to x^\mu - \epsilon\xi^\mu(x)$, we know the "barred" variation of the quantities is given by their Lie derivative with respect to $\epsilon\xi^\mu$. The variation of metric thus reads $\pounds_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$. And, being $\mathfrak{L}_\mathrm{g} \equiv \sqrt{-g}\,\mathcal{L}_\mathrm{g}$ a scalar density of weight $+1$, we also know the Lie derivative acts on it according to

$$\bar\delta\mathfrak{L}_\mathrm{g} = \pounds_\xi\mathfrak{L}_\mathrm{g} = (\mathfrak{L}_\mathrm{g}\xi^\alpha)_{;\alpha} = (\mathfrak{L}_\mathrm{g}\xi^\alpha)_{,\alpha}\,. \tag{28.32}$$

Putting all together, we have the equation

$$(\mathfrak{L}_{\rm g}\xi^\alpha)_{,\alpha} = [\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\,\pounds_\xi g_{\mu\nu} + \partial_\alpha\left[\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha}}\,\pounds_\xi g_{\mu\nu} - \left(\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\beta}\pounds_\xi g_{\mu\nu} + \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha\beta}}\,(\pounds_\xi g_{\mu\nu})_{,\beta}\right],$$

that is,

$$0 = [\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\,\pounds_\xi g_{\mu\nu} + \big[...\big]^\alpha{}_{,\alpha} - (\mathfrak{L}_{\rm g}\xi^\alpha)_{,\alpha} = [\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\,\pounds_\xi g_{\mu\nu} + 16\pi\,\mathfrak{J}^\alpha{}_{,\alpha}\ , \qquad (28.33)$$

where the "gravitational flux" $\mathfrak{J}^\alpha$ has been introduced, similarly as in equation (28.15), by

$$16\pi\,\mathfrak{J}^\alpha := \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha}}\,\pounds_\xi g_{\mu\nu} - \left(\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha\beta}}\right)_{,\beta}\pounds_\xi g_{\mu\nu} + \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\mu\nu,\alpha\beta}}\,\left(\pounds_\xi g_{\mu\nu}\right)_{,\beta} - \mathfrak{L}_{\rm g}\xi^\alpha\ . \qquad (28.34)$$

Integrating this over Ω while assuming that ξ^μ vanishes at $\partial\Omega$ with sufficient number of derivatives to make $\bar\delta g_{\mu\nu}\equiv\pounds_\xi g_{\mu\nu}=0$, $\bar\delta g_{\mu\nu,\beta}=(\bar\delta g_{\mu\nu})_{,\beta}\equiv(\pounds_\xi g_{\mu\nu})_{,\beta}=0$ there, the divergence term $\mathfrak{J}^\alpha{}_{,\alpha}$ vanishes due to Gauss, so we are left with

$$\begin{aligned}0 &= \int_\Omega[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}(\xi_{\mu;\nu}+\xi_{\nu;\mu})\,\mathrm{d}^4x = 2\int_\Omega[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_{\mu;\nu}\,\mathrm{d}^4x = \\ &= \cancel{2\int_\Omega\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu\right)_{;\nu}\,\mathrm{d}^4x} - 2\int_\Omega\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\right)_{;\nu}\xi_\mu\,\mathrm{d}^4x\ ,\end{aligned}$$

where we have right cancelled the first term, because $[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu$ is a vector density of weight +1, so $\big([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu\big)_{;\nu} = \big([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu\big)_{,\nu}$ and its integral again vanishes due to Gauss under the given boundary assumptions ($\xi^\mu=0$ on $\partial\Omega$). Hence, since ξ_μ is generic, similarly as we obtained the GR conservation laws $T^{\mu\nu}{}_{;\nu}=0$ in (23.20), we have derived the generic contracted Bianchi identities

$$\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\right)_{;\nu} = 0\,.$$

By using the above Bianchi identities back in (28.33), it is possible to express the whole this identity as a divergence. Actually, after plugging there $\pounds_\xi g_{\mu\nu}=\xi_{\mu;\nu}+\xi_{\nu;\mu}$, rewrite again

$$\begin{aligned}[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\,\pounds_\xi g_{\mu\nu} + 16\pi\,\mathfrak{J}^\alpha{}_{,\alpha} &= 2[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_{\mu;\nu} + 16\pi\,\mathfrak{J}^\alpha{}_{,\alpha} = \\ &= 2\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu\right)_{;\nu} - 2\cancel{\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\right)_{;\nu}}\xi_\mu + 16\pi\,\mathfrak{J}^\alpha{}_{,\alpha}\ ,\end{aligned}$$

where the cancellation is exactly due to the Bianchi identities. Realizing finally that both $[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu$ and $\mathfrak{J}^\alpha$ are vector densities (of the same weight +1), we can write the result equally well in terms of covariant or partial divergence,

$$\boxed{\left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu + 8\pi\,\mathfrak{J}^\nu\right)_{;\nu} = \left([\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu}\xi_\mu + 8\pi\,\mathfrak{J}^\nu\right)_{,\nu} = 0}\ . \qquad (28.35)$$

Apparently, this is a *strong*-type conservation law (not necessarily requiring satisfaction of any EL equations). However, if one does substitute certain particular field equations, as e.g. those of the Einstein type, $[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\mu\nu} = 8\pi\,\mathfrak{T}^{\mu\nu} := 8\pi\sqrt{-g}\,T^{\mu\nu}$, it yields

$$\left(\mathfrak{T}^{\mu\nu}\xi_\mu + \mathfrak{J}^\nu\right)_{,\nu} = 0\,. \qquad (28.36)$$

In the history (cf. e.g. H. A. Lorentz or W. Pauli), $\mathfrak{T}^{\mu\nu}\xi_\mu$ and $\mathfrak{J}^\nu$ were respectively interpreted as the "matter energy-momentum flux" and the "energy-momentum flux due to the gravitational field", with the conservation being understood so that they just cancel out each other ($\mathfrak{T}^{\mu\nu}\xi_\mu + \mathfrak{J}^\nu$ itself vanishes). However, today's viewpoint is different:

While $\mathfrak{T}^{\mu\nu}$ is determined by the field equations, the "gravitational flux" $\mathfrak{J}^\nu$ is not unique. Actually, take a certain skew-symmetric matrix $V^{\mu\nu}(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, \xi^\iota)$, called **superpotential** (it need not be a tensor, and not even a tensor density, as already suggested by its variables not including $g_{\alpha\beta,\gamma\delta}$), and change the gravitational flux to

$$\mathfrak{J}'^\nu = \mathfrak{J}^\nu + V^{\nu\lambda}{}_{,\lambda} \, .$$

Even if the total flux $\mathfrak{T}^{\mu\nu}\xi_\mu + \mathfrak{J}^\nu$ were zero originally, now it need not be, while clearly $V^{\nu\lambda}{}_{,\lambda\nu} = 0$, so the conservation $\left(\mathfrak{T}^{\mu\nu}\xi_\mu + \mathfrak{J}'^\nu\right)_{,\nu} = 0$ still holds. In order to restrict the freedom in $\mathfrak{J}^\nu$, one e.g. requires that the integration of $\mathfrak{J}^0$ over the whole space (over its proper volume) yields the correct mass, such as M in the case of Schwarzschild.

Recall now that $\mathfrak{J}^\nu$ depends on the choice of ξ^μ – see the definition (28.34). A "canonical" option is to take *four* independent translations which in the adapted coordinates reduce to $\xi^\mu_{(\kappa)} = \delta^\mu_\kappa$, where $\kappa = 0,1,2,3$ numbers the vectors. The matrix of the gravitational fluxes $\mathfrak{J}^\nu$ obtained for these four $\xi^\mu_{(\kappa)}$ is called then the **energy-momentum pseudotensor** or the **energy-momentum complex**,

$$\mathsf{t}_\kappa{}^\nu := \mathfrak{J}^\nu(\xi^\mu = \xi^\mu_{(\kappa)} = \delta^\mu_\kappa) \, .$$

We know from (11.12) that $\xi^\mu \to \xi^\mu_{(\kappa)} = \delta^\mu_\kappa$ yields $\pounds_\xi g_{\alpha\beta} = g_{\alpha\beta,\kappa}$, hence the general form of $\mathfrak{J}^\nu$, (28.34), becomes

$$16\pi\,\mathfrak{J}^\nu \to 16\pi\,\mathsf{t}_\kappa{}^\nu = \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\alpha\beta,\nu}}\,g_{\alpha\beta,\kappa} - \left(\frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\alpha\beta,\nu\lambda}}\right)_{,\lambda} g_{\alpha\beta,\kappa} + \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\alpha\beta,\nu\lambda}}\,\left(g_{\alpha\beta,\kappa}\right)_{,\lambda} - \mathfrak{L}_\mathrm{g}\delta^\nu_\kappa \, . \tag{28.37}$$

In the special case when the Lagrangian density does not depend on the second derivatives of the metric, we see the prescription reduces to the "canonical"-type expression (28.28),[14]

$$16\pi\,\mathsf{t}_\kappa{}^\nu = \frac{\partial\mathfrak{L}_\mathrm{g}}{\partial g_{\alpha\beta,\nu}}\,g_{\alpha\beta,\kappa} - \mathfrak{L}_\mathrm{g}\delta^\nu_\kappa \, . \tag{28.38}$$

Before proceeding to specific examples, let us stress the pseudotensors/complexes are *not* tensorial quantities, so they cannot represent the energy-momentum fluxes in an invariant and unique way. In particular, they typically can locally be made vanish, which exactly indicates that it is impossible to *localize* the gravitational energy/momentum. This does not mean they are not useful, but they have to be employed in coordinates suitable for a given situation (in asymptotically flat space-times, for example, it has to be coordinates which

[14] Yes, there is the sign difference and there appear 16π here. However, remember that (28.28) was computed from $\mathfrak{L}_\mathrm{ng}$, whereas here we take it from $\mathfrak{L}_\mathrm{g}$. And, in equation $(\mathfrak{T}^{\mu\nu}\xi_\mu + \mathfrak{J}^\nu)_{,\nu} = 0$, we sum the non-gravitational and gravitational contributions ($\mathfrak{T}^{\mu\nu}\xi_\mu$ and $\mathfrak{J}^\nu$, respectively) with the *same* signs, while they stand on the opposite sides of the field equations.

asymptotically become Minkowskian and which do so sufficiently quickly, namely faster than $g_{\mu\nu} \to \eta_{\mu\nu} + O(1/\sqrt{r})$, with r asymptotically representing proper radial distance[15]).

The gravitational complexes should depend on the metric and its first derivatives (yet there are exceptions also involving the second derivatives, e.g. the suggestion by C. Møller[16]). They preferably should be symmetric, and in asymptotically flat space-times their integral over the whole "space" (over a properly chosen space-like hypersurface) should provide integral quantities (energy, angular momentum) of plausible properties and independent of the spatial coordinates. There exist two important examples of the energy-momentum complex:

Einstein's energy-momentum complex

Einstein himself suggested (right in 1916) a complex which can be obtained from the reduced formula (28.38), because he only isolated from the gravitational Lagrangian the part independent of the second metric derivatives. Actually, the Hilbert-Lagrangian scalar density can be decomposed as

$$\sqrt{-g}\,R = \sqrt{-g}\,g^{\beta\gamma}\left(\Gamma^{\lambda}{}_{\rho\lambda}\Gamma^{\rho}{}_{\beta\gamma} - \Gamma^{\lambda}{}_{\rho\gamma}\Gamma^{\rho}{}_{\beta\lambda}\right) + \left[\sqrt{-g}\,(g^{\beta\gamma}\Gamma^{\iota}{}_{\beta\gamma} - g^{\iota\beta}\Gamma^{\lambda}{}_{\beta\lambda})\right]_{,\iota}\ .$$

Importantly, the second part (the one depending on the second metric derivatives) is given by partial divergence, so it does *not* contribute to the field equations at all. Hence the first part of the expression *alone* represents a valid Lagrangian density for GR (only that it is not invariant). Indeed, with $\mathfrak{L}_{\rm g} \equiv \sqrt{-g}\,\mathcal{L}_{\rm g} := \sqrt{-g}\,g^{\beta\gamma}\left(\Gamma^{\lambda}{}_{\rho\lambda}\Gamma^{\rho}{}_{\beta\gamma} - \Gamma^{\lambda}{}_{\rho\gamma}\Gamma^{\rho}{}_{\beta\lambda}\right)$, the field equations appear as

$$[\mathrm{EL}(\mathfrak{L}_{\rm g})]^{\alpha\beta} \equiv \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta}} - \left(\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta,\iota}}\right)_{,\iota} = -16\pi\,[\mathrm{EL}(\mathfrak{L}_{\rm ng})]^{\alpha\beta} = 8\pi\,\mathfrak{T}^{\alpha\beta}\,. \qquad (28.39)$$

It is thus natural to introduce, by formula (28.38),

$$16\pi\,\mathsf{t}_{\kappa}{}^{\nu} := \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta,\nu}}\,g_{\alpha\beta,\kappa} - \mathfrak{L}_{\rm g}\delta^{\nu}_{\kappa}\,, \qquad (28.40)$$

yielding the explicit expression

$$\frac{16\pi\,\mathsf{t}_{\kappa}{}^{\nu}}{\sqrt{-g}} = g^{\rho\sigma}\left(\Gamma^{\iota}{}_{\iota\rho}\Gamma^{\nu}{}_{\kappa\sigma} + \Gamma^{\iota}{}_{\iota\kappa}\Gamma^{\nu}{}_{\rho\sigma} - 2\Gamma^{\nu}{}_{\rho\iota}\Gamma^{\iota}{}_{\kappa\sigma}\right) - \delta^{\nu}_{\kappa}\,g^{\rho\sigma}\left(\Gamma^{\tau}{}_{\rho\sigma}\Gamma^{\iota}{}_{\iota\tau} - \Gamma^{\tau}{}_{\rho\iota}\Gamma^{\iota}{}_{\sigma\tau}\right) + \\ + g^{\nu\iota}\left(\Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\iota\kappa} - \Gamma^{\sigma}{}_{\sigma\iota}\Gamma^{\rho}{}_{\rho\kappa}\right).$$

Taking a divergence of the definition (28.40), we have (recall that $\mathfrak{L}_{\rm g} = \mathfrak{L}_{\rm g}(g_{\alpha\beta}, g_{\alpha\beta,\iota})$)

$$16\pi\,\mathsf{t}_{\kappa}{}^{\nu}{}_{,\nu} = \left(\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu} g_{\alpha\beta,\kappa} + \cancel{\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta,\nu}}\,g_{\alpha\beta,\kappa\nu}} - \frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta}}\,g_{\alpha\beta,\kappa} - \cancel{\frac{\partial\mathfrak{L}_{\rm g}}{\partial g_{\alpha\beta,\iota}}\,g_{\alpha\beta,\iota\kappa}} =$$

[15] This type of subtlety is actually involved in the notion of asymptotic flatness as such: it is not sufficient that "the space is flat very far away", the metric has to approach the flat one in a specific manner.

[16] It is *the* Christian Møller who, besides writing the known GR textbook [30], in 1938 advised to O. Frisch to study nuclear fission as a possible source of vast energy. Also, in the 1930s when A. S. Eddington was giving a hard time to S. Chandrasekhar's theory of degenerate fermion gas in final stages of stellar-core evolution, Møller clearly supported Chandrasekhar and wrote a paper *Relativistic Degeneracy* with him.

$$= -[\mathrm{EL}(\mathfrak{L}_{\mathrm{g}})]^{\alpha\beta} g_{\alpha\beta,\kappa} = -8\pi\, \mathfrak{T}^{\alpha\beta} g_{\alpha\beta,\kappa} \,. \tag{28.41}$$

However, since we know, from (28.31), that the conservation of T_κ^ν reads, explicitely,

$$0 = \sqrt{-g}\, T^\nu_{\kappa;\nu} = (\sqrt{-g}\, T^\nu_\kappa)_{,\nu} - \frac{\sqrt{-g}}{2}\, g_{\iota\nu,\kappa}\, T^{\iota\nu} \equiv \mathfrak{T}^\nu_{\kappa,\nu} - \frac{1}{2}\, g_{\iota\nu,\kappa}\, \mathfrak{T}^{\iota\nu} \,,$$

we can express $\frac{1}{2}\, g_{\iota\nu,\kappa}\, \mathfrak{T}^{\iota\nu} = \mathfrak{T}^\nu_{\kappa,\nu}$, or, $8\pi\, \mathfrak{T}^{\alpha\beta} g_{\alpha\beta,\kappa} = 16\pi\, \mathfrak{T}^\beta_{\kappa,\beta}$, to obtain, from (28.41),

$$(\mathfrak{T}^\nu_\kappa + \mathfrak{t}_\kappa{}^\nu)_{,\nu} = 0 \,,$$

which is exactly the desired property.

It has been found that Einstein complex is generated by a superpotential $V_\kappa{}^{\nu\sigma}$ given by

$$16\pi\, V_\kappa{}^{\nu\sigma} = \frac{1}{\sqrt{-g}}\, g_{\kappa\lambda} \left[g\, (g^{\lambda\sigma} g^{\nu\iota} - g^{\lambda\nu} g^{\sigma\iota}) \right]_{,\iota} \;:\quad V_\kappa{}^{\nu\sigma}{}_{,\sigma} = \mathfrak{t}_\kappa{}^\nu \;\; (+\, \mathfrak{T}^\nu_\kappa \;\text{ if there is matter}) \,.$$

Note that since $V_\kappa{}^{\nu\sigma}$ is anti-symmetric in $[\nu\sigma]$, the conservation law $V_\kappa{}^{\nu\sigma}{}_{,\sigma\nu} = 0$ follows without any requirement on $g_{\mu\nu}$ (it need not satisfy Einstein equations), i.e. "strongly", so F. Klein and E. Noether would call it *improper*, while P. J. Olver would call it *trivial of the second class*.

Clearly the mixed Einstein complex transforms, under affine (linear, but not necessarily Lorentz) transformations, as a tensor density of weight $+1$, which is a good feature because thanks to it the integral

$$p_\kappa = \int_V \mathfrak{t}_\kappa{}^0 \, \mathrm{d}^3x$$

provides a quantity which – under the affine transformations – behaves as a *(co)vector* and which can thus represent the momentum (of the gravitational field). The total four-momentum of course is then introduced as

$$p^\mu = \int_\Sigma (\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu})\, \mathrm{d}\Sigma_\nu \,.$$

However, the standard definition $J^{\mu\nu} = \int_\Sigma \left[x^\mu (\mathfrak{T}^{\nu\kappa} + \mathfrak{t}^{\nu\kappa}) - x^\nu (\mathfrak{T}^{\mu\kappa} + \mathfrak{t}^{\mu\kappa}) \right] \mathrm{d}\Sigma_\kappa$ of total angular momentum is useless, because the non-mixed forms of the Einstein complex, $\mathfrak{t}_{\kappa\lambda}$ and $\mathfrak{t}^{\mu\nu}$, are not symmetric. This is the main drawback of Einstein's proposal.

Landau-Lifshitz energy-momentum complex

Landau & Lifshitz chose an opposite strategy in a sense ([25], section 96): they identified, in a freely falling frame, the "genuine-curvature" part of the Einstein tensor (i.e. the part solely given by the second derivatives of the metric), defined $\mathfrak{T}^{\mu\nu}$ from it, and then added a complex $\mathfrak{t}^{\mu\nu}$ in such a way that it correspond to the rest of the Einstein tensor within the conservation laws $(\sqrt{-g}\, \mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu})_{,\nu} = 0$.

Actually, in a local geodesic system, the first derivatives of the metric vanish, so the fully covariant Riemann (6.8) reduces to

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}\right) + [\Gamma^2 \text{ terms}]\ .$$

Plugging the curvature part into the Einstein tensor,

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} &= g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} - \frac{1}{2}\,g^{\beta\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta}g^{\mu\nu} = \\ &= \frac{1}{2}\,g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) - \frac{1}{2}\,g^{\beta\delta}g^{\alpha\gamma}g^{\mu\nu}(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta})\,, \end{aligned}$$

one can define the corresponding $T^{\mu\nu}$ from the field equations,

$$\begin{aligned} &16\pi\,T^{\mu\nu}\,(-2\Lambda g^{\mu\nu}) = 2R^{\mu\nu} - Rg^{\mu\nu} = \\ &\quad = g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) - g^{\beta\delta}g^{\alpha\gamma}g^{\mu\nu}(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta})\,, \end{aligned}$$

which – when neglecting the first-metric-derivative terms again – turns out to equal

$$16\pi\,T^{\mu\nu}\,(-2\Lambda g^{\mu\nu}) = \frac{1}{-g}\left[g\left(g^{\mu\sigma}g^{\nu\iota} - g^{\mu\nu}g^{\sigma\iota}\right)\right]_{,\iota\sigma} =: \frac{16\pi}{-g}\,V^{\mu\nu\sigma}{}_{,\sigma}\,,$$

i.e. (we omit the cosmological term already)

$$\sqrt{-g}\,\mathfrak{T}^{\mu\nu} \equiv (-g)\,T^{\mu\nu} = V^{\mu\nu\sigma}{}_{,\sigma}\,, \qquad \text{with} \quad 16\pi\,V^{\mu\nu\sigma} := \left[g\left(g^{\mu\sigma}g^{\nu\iota} - g^{\mu\nu}g^{\sigma\iota}\right)\right]_{,\iota}\ .$$

In a *generic* coordinate system, there also appear the $g_{\mu\nu,\iota}$ terms (the Christoffel symbols), so $\sqrt{-g}\,\mathfrak{T}^{\mu\nu} = V^{\mu\nu\sigma}{}_{,\sigma}$ no longer holds. Let us denote the difference as $\mathfrak{t}^{\mu\nu}$,

$$\sqrt{-g}\,\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu} = V^{\mu\nu\sigma}{}_{,\sigma} \qquad \Longrightarrow \qquad (\sqrt{-g}\,\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu})_{,\nu} = 0\ .$$

Obviously, the above superpotential can be obtained from that of the Einstein complex simply by $(V_{\rm LL})^{\mu\nu\sigma} = \sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}$. Since the matter contribution $\mathfrak{T}^{\mu\nu}$ means the same in both cases, one can thus relate the two gravitational complexes,

$$\begin{aligned} \sqrt{-g}\,\mathfrak{T}^{\mu\nu} + (\mathfrak{t}_{\rm LL})^{\mu\nu} \equiv (V_{\rm LL})^{\mu\nu\sigma}{}_{,\sigma} &= [\sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}]_{,\sigma} = \\ &= (\sqrt{-g}\,g^{\mu\kappa})_{,\sigma}(V_{\rm E})_\kappa{}^{\nu\sigma} + \sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\sigma} \equiv \\ &\equiv (\sqrt{-g}\,g^{\mu\kappa})_{,\sigma}(V_{\rm E})_\kappa{}^{\nu\sigma} + \sqrt{-g}\,g^{\mu\kappa}\left[\mathfrak{T}^\nu_\kappa + (\mathfrak{t}_{\rm E})_\kappa{}^\nu\right] \end{aligned}$$

$$\Longrightarrow \qquad (\mathfrak{t}_{\rm LL})^{\mu\nu} = \sqrt{-g}\,g^{\mu\kappa}(\mathfrak{t}_{\rm E})_\kappa{}^\nu + (\sqrt{-g}\,g^{\mu\kappa})_{,\sigma}(V_{\rm E})_\kappa{}^{\nu\sigma}\ .$$

And it also reveals how both complexes can be conserved simultaneously,

$$\begin{aligned} (V_{\rm LL})^{\mu\nu\sigma}{}_{,\sigma\nu} &= [\sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}]_{,\sigma\nu} = [(\sqrt{-g}\,g^{\mu\kappa})_{,\sigma}(V_{\rm E})_\kappa{}^{\nu\sigma} + \sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\sigma}]_{,\nu} = \\ &= \cancel{(\sqrt{-g}\,g^{\mu\kappa})_{,\sigma\nu}(V_{\rm E})_\kappa{}^{\nu\sigma}} + (\sqrt{-g}\,g^{\mu\kappa})_{,\sigma}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\nu} + \\ &\quad + (\sqrt{-g}\,g^{\mu\kappa})_{,\nu}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\sigma} + \sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\sigma\nu} = \\ &= \cancel{2(\sqrt{-g}\,g^{\mu\kappa})_{,(\sigma}\partial_{\nu)}(V_{\rm E})_\kappa{}^{\nu\sigma}} + \sqrt{-g}\,g^{\mu\kappa}(V_{\rm E})_\kappa{}^{\nu\sigma}{}_{,\sigma\nu}\ . \end{aligned}$$

To derive an explicit formula for the LL $\mathfrak{t}^{\mu\nu}$ is tedious, it reads

$$\begin{aligned}
\frac{16\pi\,\mathfrak{t}^{\mu\nu}}{-g} &= \left(2\Gamma^{\sigma}{}_{\kappa\lambda}\Gamma^{\rho}{}_{\rho\sigma} - \Gamma^{\sigma}{}_{\kappa\rho}\Gamma^{\rho}{}_{\lambda\sigma} - \Gamma^{\sigma}{}_{\kappa\sigma}\Gamma^{\rho}{}_{\lambda\rho}\right)\left(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\nu}g^{\kappa\lambda}\right) + \\
&+ \left(\Gamma^{\nu}{}_{\kappa\rho}\Gamma^{\rho}{}_{\lambda\sigma} + \Gamma^{\nu}{}_{\lambda\sigma}\Gamma^{\rho}{}_{\kappa\rho} - \Gamma^{\nu}{}_{\rho\sigma}\Gamma^{\rho}{}_{\kappa\lambda} - \Gamma^{\nu}{}_{\kappa\lambda}\Gamma^{\rho}{}_{\rho\sigma}\right) g^{\mu\kappa}g^{\lambda\sigma} + \\
&+ \left(\Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\rho}{}_{\lambda\sigma} + \Gamma^{\mu}{}_{\lambda\sigma}\Gamma^{\rho}{}_{\kappa\rho} - \Gamma^{\mu}{}_{\rho\sigma}\Gamma^{\rho}{}_{\kappa\lambda} - \Gamma^{\mu}{}_{\kappa\lambda}\Gamma^{\rho}{}_{\rho\sigma}\right) g^{\nu\kappa}g^{\lambda\sigma} + \\
&+ \left(\Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\nu}{}_{\lambda\rho} - \Gamma^{\mu}{}_{\kappa\lambda}\Gamma^{\nu}{}_{\rho\sigma}\right) g^{\kappa\lambda}g^{\rho\sigma}\ .
\end{aligned}$$

The LL complex is a tensor density of weight +2, not +1 as the Einstein complex (more accurately, they behave so under linear transformations). This is not ideal for the definition of the momentum (the respective integral over coordinate volume yields a vector density of weight +1 rather than a vector), though suggestions exist in the literature how to work around this issue. On the other hand, a clear advantage over the Einstein proposal is the LL-complex symmetry.

An example, finally: consider the natural definition of four-momentum,

$$\begin{aligned}
p^{\alpha} &= \int_{\Sigma}\left(\sqrt{-g}\;\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu}\right)\mathrm{d}\Sigma_{\nu} \equiv \int_{V}\left(\sqrt{-g}\;\mathfrak{T}^{\mu0} + \mathfrak{t}^{\mu0}\right)\mathrm{d}V = \\
&= \int_{V} V^{\mu0\sigma}{}_{,\sigma}\,\mathrm{d}V = \int_{V} V^{\mu0k}{}_{,k}\,\mathrm{d}V = \int_{S} V^{\mu0k}\,\mathrm{d}S_k\,,
\end{aligned}$$

with the last but one equality holding in a stationary situation, and with $S := \partial V$. Substituting

$$V^{\mu0k} = \frac{1}{16\pi}\left[g\left(g^{\mu k}g^{0i} - g^{\mu0}g^{ki}\right)\right]_{,i}\ ,$$

we see immediately that $V^{i0k} = \frac{1}{16\pi}\left[g\left(g^{ik}g^{0i} - g^{i0}g^{ki}\right)\right]_{,i} = 0$, so $p^i = 0$. For p^0, consider an asymptotically flat space-time and recall the far-field metric (22.24), (22.27) we derived for a stationary quasi-Newtonian source in the centre-of-mass coordinates (those in which $p^i = 0$). We found there

$$g_{00} = -1 - 2\Phi\,, \quad g_{0i} = O(r^{-2})\,, \quad g_{ik} = (1 - 2\Phi)\delta_{ik}$$

(whether indices are down or up is irrelevant since the coordinates are of linear type and far away they are raised/lowered by a flat metric). Substituting the leading terms

$$\begin{aligned}
&\cancel{g^{0k}g^{0i}} - g^{00}g^{ki} \simeq (1 + 2\Phi)(1 - 2\Phi)\delta^{ik} = (1 - 4\Phi^2)\delta^{ik}\,, \\
&g \simeq -(1 + 2\Phi)(1 - 2\Phi)^3 \simeq -1 + 4\Phi\,,
\end{aligned}$$

we have

$$\begin{aligned}
&g\left(\cancel{g^{\mu k}g^{0i}} - g^{\mu0}g^{ki}\right) \simeq (-1 + 4\Phi)(1 - 4\Phi^2)\delta^{ik} \simeq (-1 + 4\Phi)\delta^{ik} \simeq \left(-1 - \frac{4M}{r}\right)\delta^{ik} \\
&\Longrightarrow \quad \left[g\left(g^{\mu k}g^{0i} - g^{\mu0}g^{ki}\right)\right]_{,i} \simeq \left(-1 - \frac{4M}{r}\right)_{,r} = \frac{4M}{r^2}\ .
\end{aligned}$$

Hence, by integrating over a sphere $r = \mathrm{const}$, we find

$$p^0 = \frac{1}{16\pi}\int_{S}\frac{4M}{r^2}\,4\pi r^2\,\mathrm{d}r = M\ .$$

More than enough complexes

In order to indicate the freedom one has in the design of a complex, consider an arbitrary tensor density $\mathfrak{t}_\kappa{}^\nu$ which is conserved, i.e. which is constrained by the continuity equation $\mathfrak{t}_\kappa{}^\nu{}_{;\nu}=0$. The corresponding superpotential $V_\kappa{}^{[\nu\sigma]}$ is required to fulfill $V_\kappa{}^{\nu\sigma}{}_{,\sigma}=\mathfrak{t}_\kappa{}^\nu$. However, the conservation will also apply to any quantity $\mathfrak{t}'^{\,\nu}_{\kappa}:=(\varphi\,V_\kappa{}^{\nu\sigma})_{,\sigma}$, with $\varphi(x^\alpha)$ arbitrary function, thanks to the anti-symmetry of $V_\kappa{}^{\nu\sigma}$.

Clearly, another way to alternative complexes is to add two superpotentials (which both generate a certain complex), $V_\kappa{}^{[\nu\sigma]}+W_\kappa{}^{[\nu\sigma]}$.

28.10.2 Komar integrals

We learned in (9.2) and in (11.27) that the EM four-potential A^μ and any Killing vector ξ^μ are coupled to curvature through equations

$$\Box A^\alpha - R^\alpha{}_\mu A^\mu = -4\pi J^\alpha\,, \qquad \Box\xi^\alpha + R^\alpha{}_\mu \xi^\mu = 0\,,$$

respectively. In a vacuum ($R^\alpha{}_\mu=0$), these are very similar (the source term J^α is not a big distinction actually, because the electromagnetism is considered *test* here). Generally, without the Lorenz condition $A^\alpha{}_{;\alpha}=0$, the EM wave equation reads

$$(A^{\beta;\alpha}-A^{\alpha;\beta})_{;\beta} = 4\pi J^\alpha\,.$$

One can define the total charge from its density standing in J^0 by

$$Q=\int\limits_{x^0=\mathrm{const}} J^0\sqrt{-g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\,.$$

In this analogy, it is natural to introduce the "mass-current ($\equiv$ mometum) density" as

$$p^\alpha := (\xi^{\beta;\alpha}-\xi^{\alpha;\beta})_{;\beta} = -2\xi^{\alpha;\beta}{}_\beta$$

and from it to define "mass" (with obvious idea and notation)

$$\begin{aligned} M &= -\frac{1}{8\pi}\int\limits_{x^0=\mathrm{const}} p^0\sqrt{-g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \equiv \frac{1}{4\pi}\int\limits_{x^0=\mathrm{const}} \xi^{0;\beta}{}_\beta\sqrt{-g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \\ &= \frac{1}{4\pi}\int\limits_{x^0=\mathrm{const}} (\sqrt{-g}\,\xi^{0;\beta})_{,\beta}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \frac{1}{4\pi}\int\limits_{x^0=\mathrm{const}} (\sqrt{-g}\,\xi^{0;i})_{,i}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \\ &= \frac{1}{4\pi}\oint\limits_S \xi^{0;i}\,\mathrm{d}S_i \end{aligned}$$

(remember that $\xi^{\mu;\nu}$ is anti-symmetric, so $\xi^{0;0}=0$), where S is some 2-surface enclosing the source and $\mathrm{d}S_i$ is its vector proper-area element.

Let us check whether this might work on the Schwarzschild space-time where the time-like Killing field writes, in the Schwarzschild coordinates, $\xi^\mu\equiv t^\mu=\delta^\mu_0$. Choosing naturally

$\{t=\mathrm{const}, r=\mathrm{const}\}$ as the integration surface S, i.e. $\mathrm{d}S_i=\delta^r_i\sqrt{g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}\theta\,\mathrm{d}\phi$, it is sufficient to know the r-component of the gradient,

$$\begin{aligned} t^{0;r} &= g^{r\sigma}t^0{}_{;\sigma} = g^{rr}(\cancel{t^0{}_{,r}} + \Gamma^0{}_{r0}) = \frac{1}{2}\,g^{rr}g^{0\rho}(\cancel{g_{\rho r,0}} + g_{0\rho,r} - \cancel{g_{r0,\rho}}) = \\ &= \frac{1}{2}\,g^{rr}g^{00}g_{00,r} = -\frac{1}{2}\,g_{00,r} = \frac{M}{r^2}\,. \end{aligned}$$

The above prescription really yields the correct Schwarzschild value,

$$\frac{1}{4\pi}\oint_S t^{0;i}\,\mathrm{d}S_i = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} t^{0;r}\sqrt{g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi}\frac{M}{r^2}\,r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = M\,. \tag{28.42}$$

Besides stationarity, Schwarzschild (but e.g. Kerr-Newman as well) also possesses axial symmetry, so there also exists a space-like Killing vector field with circular orbits, $\phi^\mu=\delta^\mu_3$ (if assigning $\phi\equiv x^3$). Similarly as the mass, here it is the angular momentum what can be defined accordingly,

$$J = -\frac{1}{8\pi}\oint_S \phi^{0;i}\,\mathrm{d}S_i\,.$$

Notice the factor $-1/8\pi$ instead of $1/4\pi$ (which is an "anomaly", sometimes ascribed to the universally attractive and spin-2 character of gravity, while more often just considered a weak point of this type of definition).

After this brief intro, let us motivate the Komar integrals more thoroughly. Consider a stationary, asymptotically flat space-time which is vacuum at least "far away". A family of observers "at rest with respect to infinity", i.e. those with four-velocity proportional to the time Killing field $t^\mu\equiv\frac{\partial x^\mu}{\partial t}$, $u^\mu=\frac{t^\mu}{\sqrt{-g_{\rho\sigma}t^\rho t^\sigma}}\equiv\frac{t^\mu}{\sqrt{-g_{tt}}}$, have four-acceleration

$$a^\mu = u^\mu{}_{;\nu}u^\nu = \left(\frac{t^\mu}{\sqrt{-g_{tt}}}\right)_{;\nu}\frac{t^\nu}{\sqrt{-g_{tt}}} = \frac{t^\mu{}_{;\nu}t^\nu}{-g_{tt}} + \frac{1}{2}\,\frac{t^\mu\,\cancel{t^\nu g_{tt;\nu}}}{(-g_{tt})^2} = \frac{t^\mu{}_{;\nu}t^\nu}{-g_{tt}}\,, \tag{28.43}$$

because

$$t^\nu g_{tt;\nu} \equiv t^\nu(g_{\alpha\beta}t^\alpha t^\beta)_{;\nu} = t^\nu(g^{\alpha\beta}t_\alpha t_\beta)_{;\nu} = 2t^\nu g^{\alpha\beta}t_{\alpha;\nu}t_\beta = 2t^\nu t_{[\alpha;\nu]}t^\alpha = 0$$

(t^μ being Killing, $t_{\alpha;\nu}$ is anti-symmetric). "Renormalizing" now the four-acceleration "with respect to the asymptotic inertial frame" (similarly as we did when introducing the surface gravity in Section 19.4.1), i.e. multiplying it by $\sqrt{-g_{tt}}$, we finally get the expression $\frac{t^\mu{}_{;\nu}t^\nu}{\sqrt{-g_{tt}}}$ for the force necessary to keep a unit-mass particle at rest at a given location (as taken with respect to infinity, e.g. as if hanging from there on a massless string). The force necessary to support the whole spheroidal *shell* S of unit-density mass is thus given by

$$F = \oint_S \frac{t^\mu{}_{;\nu}t^\nu}{\sqrt{-g_{tt}}}\,r_\mu\,\mathrm{d}S\,, \tag{28.44}$$

where r^μ is the unit outward normal to S, orthogonal to t^μ, i.e. $r_\mu r^\mu = 1$, $r_\mu t^\mu = 0$, $g_{tt} \equiv t_\mu t^\mu$. It is easy to check – on Schwarzschild again – that the integral agrees with what we suggested above: writing

$$r_\mu = g_{\mu\nu} r^\nu = g_{\mu\nu} \frac{\frac{\partial x^\nu}{\partial r}}{\sqrt{g_{\kappa\lambda} \frac{\partial x^\kappa}{\partial r} \frac{\partial x^\lambda}{\partial r}}} = \frac{g_{\mu\nu}}{\sqrt{g_{rr}}} \delta^\nu_r = \sqrt{g_{rr}}\, \delta^r_\mu\,, \quad r_\mu \mathrm{d}S = \sqrt{g_{rr}}\, \delta^r_\mu \sqrt{g_{\theta\theta} g_{\phi\phi}}\, \mathrm{d}\theta\, \mathrm{d}\phi\,,$$

we have

$$\frac{t^\mu{}_{;\nu} t^\nu}{\sqrt{-g_{tt}}}\, r_\mu\, \mathrm{d}S = \frac{t^{\mu;0} t_0}{\sqrt{-g_{tt}}}\, r_\mu\, \mathrm{d}S = \frac{t^{\mu;0} g_{tt}}{\sqrt{-g_{tt}}}\, r_\mu\, \mathrm{d}S = -\frac{t^{0;\mu} g_{tt}}{\sqrt{-g_{tt}}}\, r_\mu\, \mathrm{d}S = t^{0;\mu} \sqrt{-g_{tt}}\, r_\mu\, \mathrm{d}S =$$
$$= t^{0;r} \sqrt{-g_{tt} g_{rr}}\, \sqrt{g_{\theta\theta} g_{\phi\phi}}\, \mathrm{d}\theta\, \mathrm{d}\phi = t^{0;r} \sqrt{g_{\theta\theta} g_{\phi\phi}}\, \mathrm{d}\theta\, \mathrm{d}\phi\,,$$

which really is the same as what we integrated in (28.42).

Back to a *generic* stationary space-time. The vectors t^μ and r^μ form the (non-unit) time-like and (unit) space-like normals to the surface S, so the metric can be decomposed as

$$g_{\mu\nu} = -\frac{t_\mu t_\nu}{-g_{tt}} + r_\mu r_\nu + \sigma_{\mu\nu} = \frac{1}{-g_{tt}} (t_\mu r^\iota - r_\mu t^\iota)(t_\iota r_\nu - r_\iota t_\nu) + \sigma_{\mu\nu} = N_\mu{}^\iota N_{\iota\nu} + \sigma_{\mu\nu}$$

($\sigma_{\mu\nu}$ being the induced metric on S), where a bivector normal to S has been introduced,

$$N_{\mu\nu} := \frac{1}{\sqrt{-g_{tt}}} (t_\mu r_\nu - r_\mu t_\nu) = N_{[\mu\nu]}\,.$$

In terms of the latter, one can write the integrand of (28.44) in terms of the tensor area element on S,

$$\frac{t^{\mu;\nu} t_\nu}{\sqrt{-g_{tt}}}\, r_\mu = \frac{t^{[\mu;\nu]} t_\nu}{\sqrt{-g_{tt}}}\, r_\mu = \frac{1}{2}\, t^{\mu;\nu} N_{\nu\mu}\,, \qquad \mathrm{d}S_{\nu\mu} = -N_{\nu\mu} \mathrm{d}S = N_{\mu\nu} \mathrm{d}S = N_{\mu\nu} \sqrt{\sigma}\, \mathrm{d}^2 x\,.$$

Using the Stokes theorem for the volume V between two spheroidal surfaces S and S' $(> S)$, one has (schematic notation for integral "limits")

$$F' - F = \oint_S^{S'} \frac{t^\mu{}_{;\nu} t^\nu}{\sqrt{-g_{tt}}}\, r_\mu\, \mathrm{d}S = \frac{1}{2} \oint_S^{S'} t^{\mu;\nu} N_{\nu\mu}\, \mathrm{d}S = -\frac{1}{2} \oint_S^{S'} t^{\mu;\nu}\, \mathrm{d}S_{\nu\mu} = \frac{1}{2} \oint_S^{S'} t^{\mu;\nu}\, \mathrm{d}S_{\mu\nu} =$$
$$= \int_V t^{\mu;\nu}{}_\nu\, \mathrm{d}\Sigma_\mu = -\int_V R^\mu{}_\nu t^\nu\, \mathrm{d}\Sigma_\mu = \int_V R^\mu{}_\nu t^\nu\, n_\mu\, \mathrm{d}V\,, \tag{28.45}$$

where the Killing property of t^μ has been employed, $\Box t^\mu = -R^\mu{}_\nu t^\nu$, and $\mathrm{d}\Sigma_\mu = -n_\mu\, \mathrm{d}V$ is the vector proper-volume element on V (with n^μ the future-directed unit normal to V; for example, if V was within $t = \mathrm{const}$, then $n_\mu = -t_{,\mu}$ and $n^\mu = -g^{\mu\nu} t_{,\nu}$, which in adapted coordinates reads $n^\mu = -g^{\mu\nu} \delta^0_\nu = -g^{\mu 0}$; for flat space-time the latter would be just δ^μ_0).

An immediate conclusion is that in a vacuum where Ricci vanishes, the above result also vanishes, so the integral (28.44) is independent of how the surface S is chosen. However, the same is true – as ensured by the field equation $\Delta\Phi = 4\pi\rho$ – for the Gaussian integral

$$M = \frac{1}{4\pi} \oint\limits_S \Phi^{,i} r_i \, \mathrm{d}S$$

which defines the total mass enclosed in a surface S in the Newtonian gravity. And, note that $\Phi^{,i}$ is exactly the force that has to be exerted on a unit test mass for the latter to stay at rest, which means that $4\pi M$ stands for the total outward force needed to support the whole shell of mass distributed over S (with unit surface density). Regarding the above correspondence, one is led to define, *in stationary space-times*, the mass by the $1/4\pi$-multiple of (28.44),

$$M = \frac{1}{4\pi} \oint\limits_S \frac{t^\mu{}_{;\nu} t^\nu}{\sqrt{-g_{tt}}} r_\mu \, \mathrm{d}S = -\frac{1}{8\pi} \oint\limits_S t^{\mu;\nu} \, \mathrm{d}S_{\nu\mu} \; . \tag{28.46}$$

Usually the surface S is taken as *asymptotic*, or at least securely surrounding all sources present in the space-time.

Using Einstein equations, it is also possible to rewrite the "volume-form" result (28.45),

$$M = \frac{1}{4\pi} \int\limits_V R^\mu{}_\nu t^\nu \, n_\mu \, \mathrm{d}V = 2 \int\limits_V \left(T^\mu{}_\nu - \frac{1}{2} T \delta^\mu_\nu \right) t^\nu \, n_\mu \, \mathrm{d}V \, ,$$

which is known as the **Tolman formula** for mass. In the special case when the entire space-time is vacuum but contains a black hole, the original surface integral has to be used, with the contributions computed at infinity (or just far away) and over the black-hole horizon.

If the space-time is *axially symmetric* as well, angular momentum can be defined in exactly the same manner as mass, just with t^μ replaced by the axial Killing field ϕ^μ and with the extra factor $-1/2$ in front, that is,

$$J = -\frac{1}{8\pi} \oint\limits_S \frac{\phi^\mu{}_{;\nu} \phi^\nu}{\sqrt{g_{\phi\phi}}} r_\mu \, \mathrm{d}S = \frac{1}{16\pi} \oint\limits_S \phi^{\mu;\nu} \, \mathrm{d}S_{\nu\mu} \; .$$

Let us test the formula on the Kerr metric.[17] The area element is given by "vector product" of the angular vectors $\partial x^\mu / \partial\theta$ and $\partial x^\mu / \partial\phi$ which are tangent to the 2D surfaces $\{t = \mathrm{const}, r = \mathrm{const}\}$ (and "generate" them),

$$\mathrm{d}S_{\mu\nu} = \epsilon_{\mu\nu\kappa\lambda} \frac{\partial x^\kappa}{\partial\theta} \frac{\partial x^\kappa}{\partial\phi} \, \mathrm{d}\theta \, \mathrm{d}\phi = \sqrt{-g} \, [\mu\nu\theta\phi] \, \mathrm{d}\theta \, \mathrm{d}\phi \, ,$$

hence (ϕ^μ is Killing, so $\phi^{r;t} - \phi^{t;r} = -2\phi^{t;r}$)

$$J = \frac{1}{16\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} (\phi^{r;t} - \phi^{t;r}) \sqrt{-g} \, \mathrm{d}\theta \, \mathrm{d}\phi = -\frac{1}{4} \int\limits_0^{\pi} \phi^{t;r} \sqrt{-g} \, \mathrm{d}\theta \, .$$

[17] There is an unproved conjecture that the Komar integrals *necessarily* yield results *with wrong signs*. We will try to give a counter-example.

Recalling the metric from Section 16.2 and taking $\phi^\mu = \delta^\mu_\phi$ (in the BL coordinates), we compute

$$\begin{aligned}
-g &= \left[-g_{tt}g_{\phi\phi} + (g_{t\phi})^2\right] g_{rr}g_{\theta\theta} = g_{\phi\phi}(-g_{tt} - g_{t\phi}\omega)\, g_{rr}g_{\theta\theta} = N^2 g_{\phi\phi}g_{rr}g_{\theta\theta} = \\
&= \frac{\Sigma\Delta}{\mathcal{A}}\,\frac{\mathcal{A}}{\Sigma}\sin^2\theta\,\frac{\Sigma}{\Delta}\,\Sigma = \Sigma^2\sin^2\theta\,, \\
\phi^{\mu;\nu} &= g^{\nu\iota}(\cancel{\phi^\mu_{\ ,\iota}} + \Gamma^\mu{}_{\iota\kappa}\phi^\kappa) = \frac{1}{2}\, g^{\nu\iota}g^{\mu\lambda}(g_{\lambda\iota,\kappa} + g_{\kappa\lambda,\iota} - g_{\iota\kappa,\lambda})\,\delta^\kappa_\phi = \\
&= \frac{1}{2}\, g^{\nu\iota}g^{\mu\lambda}(g_{\phi\lambda,\iota} - g_{\iota\phi,\lambda}) = \frac{1}{2}\, g^{\nu\iota}\left(g^{\mu t}g_{\phi t,\iota} + g^{\mu\phi}g_{\phi\phi,\iota} - g^{\mu r}g_{\iota\phi,r} - g^{\mu\theta}g_{\iota\phi,\theta}\right) = \\
&= \frac{1}{2}\left[g^{\mu t}(g^{\nu r}g_{\phi t,r} + g^{\nu\theta}g_{\phi t,\theta}) + g^{\mu\phi}(g^{\nu r}g_{\phi\phi,r} + g^{\nu\theta}g_{\phi\phi,\theta}) - \right. \\
&\qquad \left. - g^{\mu r}(g^{\nu t}g_{t\phi,r} + g^{\nu\phi}g_{\phi\phi,r}) - g^{\mu\theta}(g^{\nu t}g_{t\phi,\theta} + g^{\nu\phi}g_{\phi\phi,\theta})\right], \\
\phi^{t;r} &= \frac{1}{2}\left(g^{tt}g^{rr}g_{\phi t,r} + g^{t\phi}g^{rr}g_{\phi\phi,r}\right) = -\frac{1}{2\Sigma^2}\left(\mathcal{A}g_{t\phi,r} + 2Mar\,g_{\phi\phi,r}\right) = \\
&= \frac{2Ma\sin^2\theta}{2\Sigma^4}\left[\mathcal{A}(\Sigma - \cancel{2r^2}) + \cancel{2\mathcal{A}r^2} - r\Sigma\mathcal{A}_{,r}\right] = \\
&= -\frac{Ma\sin^2\theta}{\Sigma^3}\left[\Sigma(3r^2 - a^2) + 2r^2a^2\sin^2\theta\right],
\end{aligned}$$

hence

$$J = \frac{Ma}{4}\int_0^\pi \frac{\sin^3\theta}{\Sigma^2}\left[\Sigma(3r^2 - a^2) + 2r^2a^2\sin^2\theta\right] \mathrm{d}\theta\,.$$

If the integration surface is at asymptotic radius, $\Sigma \simeq r^2$ and the integrand reduces to $3\sin^3\theta$, so one indeed obtains

$$J = \frac{3Ma}{4}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = Ma\,.$$

Interestingly, however, the integral in fact provides *the same result for any* r, because

$$\int_0^\pi \frac{\sin^3\theta}{\Sigma^2}\left[\Sigma(3r^2 - a^2) + 2r^2a^2\sin^2\theta\right] \mathrm{d}\theta = 4\,,$$

so it is independent of the choice of the integration "sphere". (This is not a *general* property of the Komar integrals. In particular, in space-times which are not asymptotically flat, the integral has to typically be computed over the horizon.)

CHAPTER 29

Relativistic strings

29.1 Messages from the beyond

The standard model of micro-world is based on the concept of **particles** as fundamental objects. The particles interact via three fundamental interactions, also mediated by particles – the spin-1 quanta of gauge fields subjected to $U(1)$, $SU(2)$ and $SU(3)$ local symmetries (called photons, $W^{\pm}$ and Z bosons, and eight gluons, respectively). The model has been extremely successful, yet still it is clear it is *not* a final, "complete" theory. The first evidence one notices is that particle physicists use, as a settled abbreviation, the **BSM** – *beyond standard model*. "Beyond" are (usually considered) the neutrino oscillations, interpreted as indicating a non-zero rest mass of neutrinos. Beyond is the matter-antimatter asymmetry in the Universe, and – if cosmological models based on GR are plausible – the yet missing explanation of what constitutes the "dark energy" and the "dark matter". Beyond also are certain puzzling numerical relations such as that between the lepton masses.

Yet more interesting from the theoretical side are conceptual, consistency and completeness problems. First, the standard model is not much *elegant*, the less for a relativist, because it contains 19 free parameters which have to be determined by measurements (the masses of 3 leptons, of 6 quarks and of the Higgs boson, plus gauge and Yukawa couplings). Theorists ask, what is the sufficient reason for such particular structure of gauge fields and for the particular values of the parameters? Also, why certain parameters have to be *fine tuned* in a special way in order to lead to the observed reality? The word *unnatural* is being voiced in connection with such *improbable* fine tunings. It concerns, for example, the Higgs field: the standard model predicts huge quantum-fluctuation corrections for its mass, which should *naturally* result in a very small or (rather) a very large mass, perhaps of the order of the Planck mass (unless there is a delicate cancellation between the "bare" value and the quantum-correction contribution). Yet the Higgs' rest energy of 125 GeV is 10^{17} times smaller than E_{Planck}. This discrepancy is important for the ratio between the gravitational constant and the Fermi constant, i.e. for how weak the gravitational interaction is with respect to the weak interaction. The query thus arising is more specifically called **the hierarchy problem**: why certain scales of nature are so vastly different, although it would seem *natural* for them to be close to each other.

Last but mainly: this text is about **gravitation**, the weakest interaction, which is *not* covered by the standard model. When trying to incorporate gravitation in the scheme of the quantum theory, a non-renormalizable field theory is obtained, which standardly is being taken as indication that new physics (BSM, and also beyond GR) appears at very high energy (at very small space-time scales). A simple hint of this feature is being added: the gravitational interaction scales with Newton's gravitational constant G. When computing the graviton-exchange corrections to an amplitude of some process, the ratio of the correction to the original amplitude has to be given by a dimensionless combination of the fundamental constants of the theory, i.e. G, $\hbar$ and c (the cosmological constant is totally irrelevant at micro-scales, and it may not be fundamental), and energy E. The only such dimensionless combination is

$$\frac{GE^2}{\hbar c^5} \equiv \frac{E^2}{(M_{\rm Planck}c^2)^2} \equiv \frac{E^2}{(E_{\rm Planck})^2} \doteq \left(\frac{E}{1.22\cdot 10^{16}\,\mathtt{TeV}}\right)^2 .$$

Indeed, the Planck energy is extremely large, about $2\cdot 10^9$ joules (!). The ratio clearly shows that the gravitational correction is absolutely negligible in usual particle processes (the maximum energy of LHC is of the order of 10 TeV). On the other hand, there is a principal problem at very high energies ($E^2 > E_{\rm Planck}$), because every higher-order correction (involving more and more graviton exchanges) is then $E^2/(E_{\rm Planck})^2$ times *larger*. Practically, this feature is a problem for the theory of very early universe.

Several plausible ways have been suggested how to overcome the weak points of the standard model. Grand-unification theories (unifying the three non-gravitational interactions, mostly employing one single group of gauge symmetry), the idea of extra spatial dimensions (sufficiently curved not to be detectable at moderate energies) and the idea of supersymmetry (which combines the symmetries of fields with different spins into a richer group) are the most important ones. However, none of them has brought a substantially simpler and less arbitrary picture than the standard model.

Two major routes still remain promising: the efforts to "quantize general relativity" (yet probably using different variables, as e.g. in the loop quantum gravity) and thus to infer how to incorporate it in the quantum world of other interactions, and the effort to base the explanation of matter and interactions on strings (and possibly membranes) rather than points as fundamental objects. Actually, the more-dimensional are the objects, the less singular generally are their interactions. On the other hand, complications arising from the growing number of their internal degrees of freedom narrows the practical range down to 1D and 2D objects. J. Polchinski [36] offers a meticulous *sufficient reason* why to study strings: *"Perhaps we merely suffer from a lack of imagination, and there are many other consistent theories of gravity with a short-distance cutoff. However, experience has shown that divergence problems in quantum field theory are not easily resolved, so if we have even one solution we should take it very seriously. Indeed, we are fortunate that consistency turns out to be such a restrictive principle, since the unification of gravity with the other interactions takes place at such high energy, $M_{\rm Planck}$, that experimental tests will be difficult and indirect."*

29.2 Particles and strings in general relativity

We already have some experience that the gravitational field may be a complicated thing. Yet worse are material sources. Point particles either are automatically curvature singularities (if treated *exactly*), or they have to be very lightweight (in order to be adequately described as *test* ones). For extended bodies (stars, clouds, galaxies, ...) , the compass of exact solutions is very restricted, to just extremely symmetric cases. At least a bit realistic, *rotating* star is out of reach. And then, in-between, there are (1D) line sources and (2D) thin shells. As shown notably in [12], the dividing line just goes between them: while for shells of matter it is generally possible to find metrics whose curvature tensors are well defined in terms of distributions, the contrary is true for line sources (and certainly for particles). The shells clearly are important as discs (accretion, planetary, galactic) or spheroidal envelopes left from explosions, while the linear structures might describe thin rings or filaments. And, the field theories offer topological defects – in particular, domain walls and cosmic strings – which arise, between mutually uncorrelated regions of fields, due to symmetry breaking in phase transitions accompanying the cosmological cooling.

However, similarly as particles, the line sources can at least be treated as *test bodies*, in a given space-time background. While particles move along world-lines, line sources ("strings") move along world-sheets. In particular, close strings (loops) have tubular surfaces as their histories, while finite open strings trace out strips in a space-time. In special-relativity course or when asking about motion in the Kerr-Newman space-time, we employed the Lagrangian approach and asked about a stationary value of an action representing the proper time spent along a world-line between given two events. For strings, a natural generalization is to consider an action representing the proper area of a world-sheet. This resembles another variational exercise of that type: the search for minimal surfaces. Actually, that exercise is known to yield (hyper)surfaces with zero mean curvature (trace of the shape operator, see Section 25), of which soap films, or marginally trapped surfaces and horizons, are examples.

29.3 Towards string action: the point-particle case revisited

When studying the motion of a free test particle (of rest mass m), we used two different actions (Sections 17.3.2 and 17.3.3): first,

$$S = -m\int_{\tau_{\rm in}}^{\tau_{\rm fin}} \mathrm{d}\tau = -m\int_{\tau_{\rm in}}^{\tau_{\rm fin}} \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} = -m\int_{\tau_{\rm in}}^{\tau_{\rm fin}} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,\mathrm{d}\tau =$$

$$= -m\int_{p_{\rm in}}^{p_{\rm fin}} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\frac{\mathrm{d}x^\nu}{\mathrm{d}p}}\,\frac{\mathrm{d}p}{\mathrm{d}\tau}\,\mathrm{d}\tau = -m\int_{p_{\rm in}}^{p_{\rm fin}} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\frac{\mathrm{d}x^\nu}{\mathrm{d}p}}\,\mathrm{d}p\,,$$

where also shown is the "automatic" invariance of S with respect to reparameterization $\tau \to p(\tau)$. As it is clear from the comparison of the first and the third expressions, in this formulation one a priori assumes that $-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 1$, i.e. that $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} =: u^\mu$ is normalized to -1, in other words, that τ has *on all the virtual world-lines* the meaning of proper time (these equivalent properties hold "off-shell"). However, this means that τ has to be subjected to variation, because each such world-line has its specific proper time.

The above "length-functional" action cannot be used for massless particles. It is *not* because of the factor m (this can be omitted or – quite naturally – absorbed in the square root by writing $g_{\mu\nu}p^\mu p^\nu$ there inside), but because massless particles move along null world-lines, i.e. those for which both $g_{\mu\nu}p^\mu p^\nu = 0$ and $\mathrm{d}\tau = 0$. Well, even *this* would *not* be a problem, IF the integrand only vanished along the actual world-line. But *in this formulation* it is supposed to vanish on *every* kinematically possible ($\equiv$ "virtual") world-line. Therefore, one would be searching for an extremal path among those which *all* have zero length... And, when making variation of the square root, the latter would get to the denominator...

We know of an alternative: to parameterize *all* the virtual world-lines by proper time of the *actual*, dynamically realized one. We denote it by τ again, but handle it differently, namely we do not vary it (when "looking around" over the neighbouring virtual paths). This implies that the tangent's normalization only holds "on-shell", along the actual world-line, so it cannot be used to simplify the action which now is represented by "energy functional"

$$S = \frac{1}{2}\, m \int_{\tau_{\rm in}}^{\tau_{\rm fin}} g_{\mu\nu} u^\mu u^\nu \, \mathrm{d}\tau = \frac{1}{2} \int_{\lambda_{\rm in}}^{\lambda_{\rm fin}} g_{\mu\nu} p^\mu p^\nu \, \mathrm{d}\lambda = \frac{1}{2} \int_{\lambda_{\rm in}}^{\lambda_{\rm fin}} g_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \, \mathrm{d}\lambda \, ,$$

where λ is a dimensionless parameter normalized so that $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = p^\mu$. The variation proceeds similarly, but there is no problem with the square root (in the denominator). And the geodesic equation of course comes out. Hamiltonian is the same as Lagrangian in this case. This formulation can be applied to time-like as well as null or even space-like world-lines, with the latter options selected by adding, as a constraint, the normalization of the tangent.

29.3.1 The Polyakov-type action

There is yet another possibility, elegant from the Hamiltonian point of view. In it, one upgrades the reparameterization freedom to the level of *independent* "gauge" variable, by considering a certain **internal metric** $\gamma \equiv \gamma_{\lambda\lambda}(\lambda) < 0$[1] which scales the parameterization along the world-lines ($\mathrm{d}\lambda \to \sqrt{-\gamma}\, \mathrm{d}\lambda$),

$$S = \frac{1}{2} \int\limits_{\lambda_{\rm in}}^{\lambda_{\rm fin}} \left(g_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\sqrt{-\gamma}\, \mathrm{d}\lambda} \, \frac{\mathrm{d}x^\nu}{\sqrt{-\gamma}\, \mathrm{d}\lambda} - m^2 \right) \sqrt{-\gamma}\, \mathrm{d}\lambda \;\equiv\; \int\limits_{\lambda_{\rm in}}^{\lambda_{\rm fin}} \mathcal{L}(x^\mu(\lambda), \gamma(\lambda)) \sqrt{-\gamma}\, \mathrm{d}\lambda. \quad (29.1)$$

The internal metric γ is not (necessarily) linked to the space-time metric $g_{\mu\nu}$, in particular, it is *not* necessarily induced on the world-line by $g_{\mu\nu}$. Note that γ indeed only depends directly on λ, not on x^μ.

By variation of $(-\gamma\, \mathcal{L})$ with respect to x^μ (while assuming fixed endpoints and right crossing out the boundary term), one obtains

$$\delta\, (-\gamma\, \mathcal{L}) = \frac{1}{2}\, \delta \left(g_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \right) = \frac{1}{2}\, g_{\mu\nu,\rho} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\, \delta x^\rho + g_{\mu\nu} \, \frac{\mathrm{d}(\delta x^\mu)}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} =$$

[1] The sign may in fact be arbitrary (as it will reveal itself before long), but we anticipate that the *string* world-sheets will be time-like, so we already write it in the same notation, as $\sqrt{-\gamma}$. In passing, this implies that we assume to be scaling $\mathrm{d}\lambda$ by a *positive* function (the square root is always taken as positive).

$$
\begin{aligned}
&= \frac{1}{2}\, g_{\mu\nu,\rho} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\,\delta x^\rho + \cancel{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\delta x^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)\delta x^\mu = \\
&= \frac{1}{2}\, g_{\mu\nu,\rho} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\,\delta x^\rho - g_{\mu\nu,\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,\delta x^\mu - g_{\mu\nu}\,\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2}\,\delta x^\mu = \\
&= \left[-g_{\mu\nu}\,\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} - \left(g_{\mu(\nu,\rho)} - \frac{1}{2}\, g_{\nu\rho,\mu}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\right]\delta x^\mu = \\
&= \left[-g_{\mu\alpha}\,\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} - \Gamma_{\mu\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\right]\delta x^\mu = -g_{\mu\alpha}\left[\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\right]\delta x^\mu,
\end{aligned}
$$

namely the corresponding Euler-Lagrange equations yield the geodesic equation. By variation of $4\mathcal{L}\sqrt{-\gamma} = \frac{2}{\sqrt{-\gamma}}\, g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - 2\sqrt{-\gamma}\, m^2$ with respect to γ, we have

$$
\delta(4\mathcal{L}\sqrt{-\gamma}) = \frac{\delta\gamma}{(-\gamma)^{3/2}}\, g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + \frac{\delta\gamma}{\sqrt{-\gamma}}\, m^2 = \frac{\delta\gamma}{(-\gamma)^{3/2}}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - \gamma m^2\right) ,
$$

so the corresponding Euler-Lagrange equations yield the normalization condition

$$
g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \gamma m^2 .
$$

Let us also add the Hamiltonian view. Defining the momenta conjugated to x^μ and γ,

$$
\Pi_\alpha := \frac{\partial(\mathcal{L}\sqrt{-\gamma})}{\partial\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right)} = \frac{1}{\sqrt{-\gamma}}\, g_{\alpha\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} , \qquad \Pi^{(\gamma)} := \frac{\partial(\mathcal{L}\sqrt{-\gamma})}{\partial\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right)} = 0 \quad \text{(primary constraint)} ,
$$

we get the Hamiltonian

$$
\mathcal{H} = \Pi_\alpha\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} - \mathcal{L}\sqrt{-\gamma} = \frac{1}{2\sqrt{-\gamma}}\, g_{\alpha\nu}\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + \frac{\sqrt{-\gamma}}{2}\, m^2 = \frac{\sqrt{-\gamma}}{2}\left(g^{\alpha\kappa}\Pi_\alpha\Pi_\kappa + m^2\right) .
$$

The canonical equations appear as

$$
\begin{aligned}
\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} &= \frac{\partial\mathcal{H}}{\partial\Pi_\mu} = \sqrt{-\gamma}\,\Pi^\mu , \\
\frac{\mathrm{d}\Pi_\mu}{\mathrm{d}\lambda} &= -\frac{\partial\mathcal{H}}{\partial x^\mu} = -\frac{\sqrt{-\gamma}}{2}\, g^{\alpha\kappa}{}_{,\mu}\Pi_\alpha\Pi_\kappa = -\frac{1}{2\sqrt{-\gamma}}\, g^{\alpha\kappa}{}_{,\mu}\, g_{\alpha\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\, g_{\kappa\sigma}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = \\
&= \frac{1}{2\sqrt{-\gamma}}\, g^{\alpha\kappa} g_{\alpha\rho} g_{\kappa\sigma,\mu}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = \frac{1}{2\sqrt{-\gamma}}\, g_{\rho\sigma,\mu}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} ,
\end{aligned}
$$

the former just repeating the definition of Π^μ and the latter providing the geodesic equation. Actually,

$$
\frac{\mathrm{d}\Pi_\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\sqrt{-\gamma}}\, g_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right) = \frac{\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\, g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}{2(-\gamma)^{3/2}} + \frac{1}{\sqrt{-\gamma}}\, g_{\mu\nu,\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} + \frac{1}{\sqrt{-\gamma}}\, g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} ,
$$

so the equation $\frac{\mathrm{d}\Pi_\mu}{\mathrm{d}\lambda} = -\frac{\partial\mathcal{H}}{\partial x^\mu}$ yields, after multiplication by $\sqrt{-\gamma}\, g^{\iota\mu}$,

$$
\frac{\mathrm{d}^2x^\iota}{\mathrm{d}\lambda^2} + g^{\iota\mu}\left(g_{\mu(\nu,\alpha)} - \frac{1}{2}\, g_{\alpha\nu,\mu}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{1}{2\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\iota}{\mathrm{d}\lambda} ,
$$

which is the geodesic equation, *in a non-affine parameterization* in general. Further, the equation for $\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}$ is of course singular due to the primary constraint $\Pi^{(\gamma)} = 0$ (γ has no *dynamics*, it is a "pure gauge"), while the remaining equation reads

$$\frac{\mathrm{d}\Pi^{(\gamma)}}{\mathrm{d}\lambda} = -\frac{\partial \mathcal{H}}{\partial \gamma} = \frac{\mathcal{H}}{2\gamma} \,.$$

In order that the primary constraint $\Pi^{(\gamma)} = 0$ be propagated, we thus obtain the secondary constraint

$$\mathcal{H} = 0 \,, \qquad \text{i.e.} \quad g^{\alpha\kappa}\Pi_\alpha \Pi_\kappa = -m^2 \quad (\text{“mass shell”}) \,.$$

The Hamiltonian thus itself represents a constraint – "on-shell" it vanishes.

29.4 The Nambu-Goto action

Following standard notation (e.g. Chapter 6 of [54]), let us denote by (τ, σ) two intrinsic coordinates on a **string world-sheet**, and let $X^\mu(\tau, \sigma)$ denote the **string coordinates** in the background space-time.[2] We will understand σ as labelling the position along the string (it parameterizes the space-like direction), while τ as labelling the moments of time (it parameterizes the time-like direction). It is being stressed that one is not supposed to be able to keep track of the individual points constituting the string (the string is structureless), which makes us start from a humble demand: we assume that the causal past $J^-(p)$ of every point p of any "given-time imprint" of a string on an arbitrary time-like congruence has a non-empty intersection with every such preceding imprint.

Denote by $g_{\mu\nu}$ the host space-time metric, by $\frac{\partial X^\mu}{\partial \tau} =: X^\mu_{,\tau}$ and $\frac{\partial X^\mu}{\partial \sigma} =: X^\mu_{,\sigma}$ the "spatial" and "temporal" vectors tangent to the world-sheet, and by

$$h_{\tau\tau} = g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\tau} \,, \quad h_{\tau\sigma} = g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\sigma} \,, \quad h_{\sigma\sigma} = g_{\mu\nu} X^\mu_{,\sigma} X^\nu_{,\sigma}$$

the metric induced by $g_{\mu\nu}$ on the world-sheet. Analogously as we used the proper world-line length as the particle action, we will use the proper world-sheet area as the string action, with element

$$\begin{aligned} \mathrm{d}A &= \sqrt{-h}\,\mathrm{d}\tau\,\mathrm{d}\sigma = \sqrt{(h_{\tau\sigma})^2 - h_{\tau\tau}h_{\sigma\sigma}}\,\mathrm{d}\tau\,\mathrm{d}\sigma = \\ &= \sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}\,\mathrm{d}\tau\,\mathrm{d}\sigma \,. \end{aligned}$$

We know the action can be multiplied by any constant, so let us multiply it by *some* real T_0, finally:

$$\boxed{S = -T_0 \int_{\tau_{\rm in}}^{\tau_{\rm fin}} \int_0^{\Delta\sigma} \sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}\,\mathrm{d}\sigma\,\mathrm{d}\tau} \,. \tag{29.2}$$

It is called the **Nambu-Goto action**. (T_0 will be interpreted as the string tension, see later.)

[2] One thus *must not* denote by τ or σ any (other) indices...

29.5 Momenta and equations of string motion

Variation of the corresponding Lagrangian density ($\mathcal{L} = -T_0$)

$$(\mathcal{L}\sqrt{-h})(X^\mu_{,\tau}, X^\mu_{,\sigma}, g_{\alpha\beta}(X^\mu)) = -T_0\,\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}$$

yields

$$\bar\delta(\mathcal{L}\sqrt{-h}) = \frac{\partial(\mathcal{L}\sqrt{-h})}{\partial X^\alpha_{,\tau}}\,\bar\delta(X^\alpha_{,\tau}) + \frac{\partial(\mathcal{L}\sqrt{-h})}{\partial X^\alpha_{,\sigma}}\,\bar\delta(X^\alpha_{,\sigma}) + \frac{\partial(\mathcal{L}\sqrt{-h})}{\partial g_{\alpha\beta}}\,g_{\alpha\beta,\iota}\bar\delta X^\iota\,.$$

We assume to perform the variation at given τ and σ (so the “barred” one), with only the “field variables” (here meaning the space-time configuration X^μ) changing, so we are allowed to commute

$$\bar\delta(X^\mu_{,\tau}) = (\bar\delta X^\mu)_{,\tau}\ , \quad \bar\delta(X^\mu_{,\sigma}) = (\bar\delta X^\mu)_{,\sigma}\ .$$

Let us introduce the pertinent conjugate momenta,

$$\begin{aligned}
\Pi^{(\tau)}_\alpha := \frac{\partial(\mathcal{L}\sqrt{-h})}{\partial X^\alpha_{,\tau}} &= -T_0\,\frac{\left(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma}\right)g_{\alpha\iota}X^\iota_{,\sigma} - \left(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma}\right)g_{\alpha\kappa}X^\kappa_{,\tau}}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}} = \\
&= -\frac{T_0}{\sqrt{-h}}\,g_{\alpha\iota}\left(h_{\tau\sigma}X^\iota_{,\sigma} - h_{\sigma\sigma}X^\iota_{,\tau}\right), \qquad (29.3)\\
\Pi^{(\sigma)}_\alpha := \frac{\partial(\mathcal{L}\sqrt{-h})}{\partial X^\alpha_{,\sigma}} &= -T_0\,\frac{\left(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma}\right)g_{\iota\alpha}X^\iota_{,\tau} - \left(g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau}\right)g_{\alpha\lambda}X^\lambda_{,\sigma}}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}} = \\
&= -\frac{T_0}{\sqrt{-h}}\,g_{\alpha\iota}\left(h_{\tau\sigma}X^\iota_{,\tau} - h_{\tau\tau}X^\iota_{,\sigma}\right), \qquad (29.4)
\end{aligned}$$

and we also need to calculate the variation with respect to the metric,

$$\begin{aligned}
\frac{\partial(\mathcal{L}\sqrt{-h})}{\partial g_{\alpha\beta}}\,g_{\alpha\beta,\iota} &= -\frac{T_0}{2}\,\frac{2h_{\tau\sigma}X^\alpha_{,\tau}X^\beta_{,\sigma} - X^\alpha_{,\tau}X^\beta_{,\tau}h_{\sigma\sigma} - h_{\tau\tau}X^\alpha_{,\sigma}X^\beta_{,\sigma}}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}}\,g_{\alpha\beta,\iota} = \\
&= -\frac{T_0}{2\sqrt{-h}}\left(h_{\tau\sigma}X^\alpha_{,\tau}X^\beta_{,\sigma} - h_{\sigma\sigma}X^\alpha_{,\tau}X^\beta_{,\tau}\right)g_{\alpha\beta,\iota} - \\
&\quad - \frac{T_0}{2\sqrt{-h}}\left(h_{\tau\sigma}X^\alpha_{,\tau}X^\beta_{,\sigma} - h_{\tau\tau}X^\alpha_{,\sigma}X^\beta_{,\sigma}\right)g_{\alpha\beta,\iota} = \\
&= \frac{1}{2}\,\Pi^{(\tau)\alpha}X^\beta_{,\tau}g_{\alpha\beta,\iota} + \frac{1}{2}\,\Pi^{(\sigma)\alpha}X^\beta_{,\sigma}g_{\alpha\beta,\iota} = \left(\Pi^{(\tau)\alpha}X^\beta_{,\tau} + \Pi^{(\sigma)\alpha}X^\beta_{,\sigma}\right)\Gamma_{\alpha\beta\iota}\ .
\end{aligned}$$

The last form has been obtained from observation that the expressions $\Pi^{(\tau)\alpha}X^\beta_{,\tau}$ and $\Pi^{(\sigma)\alpha}X^\beta_{,\sigma}$ are symmetric in (α,β), so, e.g.,

$$\Gamma^\alpha{}_{\beta\iota}\Pi^{(\tau)}_\alpha X^\beta_{,\tau} = \Gamma_{\alpha\beta\iota}\Pi^{(\tau)\alpha}X^\beta_{,\tau} = \frac{1}{2}\left(g_{\alpha\beta,\iota} + g_{\iota\alpha,\beta} - g_{\beta\iota,\alpha}\right)\Pi^{(\tau)\alpha}X^\beta_{,\tau} = \frac{1}{2}\,g_{\alpha\beta,\iota}\Pi^{(\tau)\alpha}X^\beta_{,\tau}\ .$$

The total variation thus appears as

$$\begin{aligned}\bar{\delta}(\mathcal{L}\sqrt{-h}) &= \Pi^{(\tau)}_{\mu}\,\frac{\partial(\bar{\delta}X^{\mu})}{\partial\tau} + \Pi^{(\sigma)}_{\mu}\,\frac{\partial(\bar{\delta}X^{\mu})}{\partial\sigma} + \Gamma^{\alpha}{}_{\beta\iota}\big(\Pi^{(\tau)}_{\alpha}X^{\beta}_{,\tau} + \Pi^{(\sigma)}_{\alpha}X^{\beta}_{,\sigma}\big)\,\bar{\delta}X^{\iota} = \\ &= \partial_{\tau}\big(\Pi^{(\tau)}_{\mu}\bar{\delta}X^{\mu}\big) + \partial_{\sigma}\big(\Pi^{(\sigma)}_{\mu}\bar{\delta}X^{\mu}\big) - \big[\partial_{\tau}(\Pi^{(\tau)}_{\mu}) + \partial_{\sigma}(\Pi^{(\sigma)}_{\mu})\big]\,\bar{\delta}X^{\mu} + \\ &\quad + \Gamma^{\alpha}{}_{\beta\iota}\big(\Pi^{(\tau)}_{\alpha}X^{\beta}_{,\tau} + \Pi^{(\sigma)}_{\alpha}X^{\beta}_{,\sigma}\big)\,\bar{\delta}X^{\iota} = \\ &= \partial_{\tau}\big(\Pi^{(\tau)}_{\mu}\bar{\delta}X^{\mu}\big) + \partial_{\sigma}\big(\Pi^{(\sigma)}_{\mu}\bar{\delta}X^{\mu}\big) - \big(\overline{\nabla}_{\tau}\Pi^{(\tau)}_{\mu} + \overline{\nabla}_{\sigma}\Pi^{(\sigma)}_{\mu}\big)\,\bar{\delta}X^{\mu}\,,\end{aligned}$$

where we have denoted, for $\{A,B\}=\{\tau,\sigma\}$,

$$\overline{\nabla}_{A}\Pi^{(A)}_{\mu} := \partial_{A}\Pi^{(A)}_{\mu} - \Gamma^{\alpha}{}_{\beta\mu}\Pi^{(A)}_{\alpha}X^{\beta}_{,A}\ .$$

Plugging this to $\bar{\delta}S$ and performing integration over τ in the first term while over σ in the second term, one has

$$\begin{aligned}\bar{\delta}S = -T_0\int_{0}^{\Delta\sigma}\big[\Pi^{(\tau)}_{\mu}\bar{\delta}X^{\mu}\big]^{\tau_{\rm fin}}_{\tau_{\rm in}}\,{\rm d}\sigma - T_0\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\big[\Pi^{(\sigma)}_{\mu}\bar{\delta}X^{\mu}\big]^{\Delta\sigma}_{0}\,{\rm d}\tau\, - \\ + T_0\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\int_{0}^{\Delta\sigma}\big(\overline{\nabla}_{\tau}\Pi^{(\tau)}_{\mu} + \overline{\nabla}_{\sigma}\Pi^{(\sigma)}_{\mu}\big)\,\bar{\delta}X^{\mu}\,{\rm d}\sigma\,{\rm d}\tau\ .\end{aligned}$$

Now, first, we assume $\bar{\delta}X^{\mu}(\tau_{\rm in},\sigma)=0$, $\bar{\delta}X^{\mu}(\tau_{\rm fin},\sigma)=0$, so the first term is out. Second, the last term has to vanish for any $\bar{\delta}X^{\mu}(\tau,\sigma)$, so the main equations of the motion read

$$\boxed{\overline{\nabla}_{\tau}\Pi^{(\tau)}_{\mu} + \overline{\nabla}_{\sigma}\Pi^{(\sigma)}_{\mu} = 0} \qquad \longrightarrow \quad \text{in Minkowski:}\quad \partial_{\tau}\Pi^{(\tau)}_{\mu} + \partial_{\sigma}\Pi^{(\sigma)}_{\mu} = 0\ . \tag{29.5}$$

The remaining term $T_0\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\big[\Pi^{(\sigma)}_{\mu}\bar{\delta}X^{\mu}\big]^{\Delta\sigma}_{0}{\rm d}\tau$ requires to specify boundary conditions at the string ends.

29.5.1 Boundary conditions

In order to evaluate $\big[\Pi^{(\sigma)}_{\mu}\bar{\delta}X^{\mu}\big]^{\Delta\sigma}_{0}$, one needs $2d$ conditions, with d the space-time dimension. For any specific end of the string, one may prescribe either the Dirichlet condition (the end point is fixed to some coordinate value), or the free-endpoint condition (the end point is not constrained). The **Dirichlet condition** *can only be prescribed in spatial directions* (of course), it reads $X^{i}(\tau,\sigma_{\rm end}) = {\rm const}$, i.e. $X^{i}_{,\tau}(\tau,\sigma_{\rm end}) = 0$, and so also $\bar{\delta}X^{i}(\tau,\sigma_{\rm end}) = 0$. Speaking only of *one coordinate of one of the endpoints* yet, such a condition only ensures that the term $\Pi^{(\sigma)}_{i}\bar{\delta}X^{i}$ vanishes for *one particular* value of i. The number of X^{i} which are fixed in the Dirichlet manner is complementary to the dimension of the remaining freedom: if $d-1-p$ of X^{i} are fixed ($0\leqslant p\leqslant d-1$), it means that the given endpoint is tied to a p-dimensional set. For example, in a standard 4D space-time of GR ($d=4$) and with $p=2$, the given endpoint would only be fixed in ($d-1-p\,=\,4-1-2\,=$) one dimension while left free in the remaining two spatial dimensions. This would mean that the endpoint is fixed to a certain $p=2$ dimensional surface. The two remaining, more restrictive possibilities (in the 4D spacetime) would be to fix the endpoint to a certain curve ($p=1$ dimension) or even to a certain spatial point ($p=0$ – as on string instruments). Clearly, the more spatial dimensions, the more options for how to constrain the endpoints.

The subsets to which the string endpoints are fixed in a Dirichlet manner are called the **D-branes**; more specifically, one writes Dp-branes if wanting to specify their dimension (p). The special case of $p=3$ corresponds to the end points being *spatially unconstrained* (in 4D, the D3-brane fills the whole space).

The **free-endpoint condition** leaves the string endpoints free, so the only possibility to make the term $\left[\Pi^{(\sigma)}_\mu \bar{\delta} X^\mu\right]_0^{\Delta\sigma}$ safely vanish is to ensure $\Pi^{(\sigma)}_\mu(\tau,\sigma_{\mathrm{end}})=0$. The free-endpoint condition has to hold *including* the time component ($\mu=0$). Since the momenta $\Pi^{(\sigma)}_\mu$ have something to do with velocities, we expect the free-endpoint conditions to actually be of the **Neumann type**.

29.5.2 Automatic constraints

Similarly as in the case of a relativistic particle, one also obtains, automatically, the primary constraint of momentum normalization. Actually, from definitions (29.3) and (29.4), we have

$$\begin{aligned}
g^{\alpha\beta}\Pi^{(\tau)}_\alpha\Pi^{(\tau)}_\beta &= \frac{T_0^2}{-h}\, g_{\iota\kappa}\left(h_{\tau\sigma}X^\iota_{,\sigma} - h_{\sigma\sigma}X^\iota_{,\tau}\right)\left(h_{\tau\sigma}X^\kappa_{,\sigma} - h_{\sigma\sigma}X^\kappa_{,\tau}\right) = -T_0^2 h_{\sigma\sigma}\,,\\
g^{\alpha\beta}\Pi^{(\sigma)}_\alpha\Pi^{(\sigma)}_\beta &= \frac{T_0^2}{-h}\, g_{\iota\kappa}\left(h_{\tau\sigma}X^\iota_{,\tau} - h_{\tau\tau}X^\iota_{,\sigma}\right)\left(h_{\tau\sigma}X^\kappa_{,\tau} - h_{\tau\tau}X^\kappa_{,\sigma}\right) = -T_0^2 h_{\tau\tau}\,,\\
g^{\alpha\beta}\Pi^{(\tau)}_\alpha\Pi^{(\sigma)}_\beta &= \frac{T_0^2}{-h}\, g_{\iota\kappa}\left(h_{\tau\sigma}X^\iota_{,\sigma} - h_{\sigma\sigma}X^\iota_{,\tau}\right)\left(h_{\tau\sigma}X^\kappa_{,\tau} - h_{\tau\tau}X^\kappa_{,\sigma}\right) = T_0^2 h_{\tau\sigma}\,,
\end{aligned}$$

from where also

$$\frac{1}{2}\,h_{AB}\left(g^{\alpha\beta}\Pi^{(A)}_\alpha\Pi^{(B)}_\beta\right) = T_0^2(-h) \qquad \text{with} \quad \{A,B\}=\{\tau\sigma\}\,.$$

Also worth noticing are the following automatic properties:

$$\begin{aligned}
\Pi^{(\tau)}_\alpha X^\alpha_{,\sigma} &= -\frac{T_0}{\sqrt{-h}}\, g_{\alpha\iota}\left(h_{\tau\sigma}X^\iota_{,\sigma} - h_{\sigma\sigma}X^\iota_{,\tau}\right)X^\alpha_{,\sigma} = -\frac{T_0}{\sqrt{-h}}\left(h_{\tau\sigma}h_{\sigma\sigma} - h_{\sigma\sigma}h_{\sigma\tau}\right) = 0\,,\\
\Pi^{(\sigma)}_\alpha X^\alpha_{,\tau} &= -\frac{T_0}{\sqrt{-h}}\, g_{\iota\alpha}\left(h_{\tau\sigma}X^\iota_{,\tau} - h_{\tau\tau}X^\iota_{,\sigma}\right)X^\alpha_{,\tau} = -\frac{T_0}{\sqrt{-h}}\left(h_{\tau\sigma}h_{\tau\tau} - h_{\tau\tau}h_{\tau\sigma}\right) = 0\,,\\
\Pi^{(\tau)}_\alpha X^\alpha_{,\tau} + \Pi^{(\sigma)}_\alpha X^\alpha_{,\sigma} &= -\frac{T_0}{2\sqrt{-h}}\,2\left[(h_{\tau\sigma})^2 - h_{\tau\tau}h_{\sigma\sigma}\right] = -T_0\sqrt{-h}\,.
\end{aligned}$$

29.5.3 Equations of motion once more

Equations of motion can also be derived in a more elegant way, from the "intrinsic" form of the action

$$S = -T_0\int\limits_{\tau_{\mathrm{in}}}^{\tau_{\mathrm{fin}}}\int\limits_0^{\Delta\sigma}\sqrt{-h}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

Using the knowledge (23.8) from the variational derivation of Einstein equations, we have

$$
\begin{aligned}
&\bar\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\; h^{AB}\bar\delta h_{AB} = \frac{1}{2}\sqrt{-h}\; h^{AB}\,\bar\delta\big(g_{\mu\nu}X^\mu_{,A}X^\nu_{,B}\big) = \\
&= \frac{1}{2}\sqrt{-h}\; h^{AB}\left[g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\bar\delta X^\alpha + 2g_{\mu\nu}(\bar\delta X^\mu)_{,A}X^\nu_{,B}\right] = \\
&= \frac{1}{2}\sqrt{-h}\,h^{AB}g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\bar\delta X^\alpha + \cancel{\big(\sqrt{-h}\,h^{AB}g_{\mu\nu}\bar\delta X^\mu X^\nu_{,B}\big)_{,A}} - \big(\sqrt{-h}\,h^{AB}g_{\mu\nu}X^\nu_{,B}\big)_{,A}\bar\delta X^\mu = \\
&= \frac{1}{2}\sqrt{-h}\,h^{AB}g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\bar\delta X^\alpha - \sqrt{-h}\,h^{AB}g_{\mu\nu,\alpha}X^\alpha_{,A}X^\nu_{,B}\bar\delta X^\mu - g_{\mu\nu}\big(\sqrt{-h}\,h^{AB}X^\nu_{,B}\big)_{,A}\bar\delta X^\mu \\
&= \sqrt{-h}\,h^{AB}\left(\frac{1}{2}\,g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}\right)X^\mu_{,A}X^\nu_{,B}\bar\delta X^\alpha - g_{\mu\nu}\big(\sqrt{-h}\,h^{AB}X^\nu_{,B}\big)_{,A}\bar\delta X^\mu = \\
&= -\sqrt{-h}\,h^{AB}\Gamma_{\alpha\mu\nu}X^\mu_{,A}X^\nu_{,B}\bar\delta X^\alpha - g_{\alpha\nu}\big(\sqrt{-h}\,h^{AB}X^\nu_{,B}\big)_{,A}\bar\delta X^\alpha\,.
\end{aligned}
$$

This vanishes for any $\bar\delta X^\alpha$ if $g_{\alpha\iota}\left[\partial_A\big(\sqrt{-h}\,h^{AB}X^\iota_{,B}\big) + \Gamma^\iota{}_{\mu\nu}\sqrt{-h}\,h^{AB}X^\mu_{,A}X^\nu_{,B}\right] = 0$, i.e.

$$
\boxed{\overline{\nabla}_A\big(\sqrt{-h}\,h^{AB}X^\iota_{,B}\big) := \partial_A\big(\sqrt{-h}\,h^{AB}X^\iota_{,B}\big) + \Gamma^\iota{}_{\mu\nu}\sqrt{-h}\,h^{AB}X^\mu_{,A}X^\nu_{,B} = 0}\,, \tag{29.6}
$$

where we have denoted $\overline{\nabla}_A V^{A\iota} := \partial_A V^{A\iota} + \Gamma^\iota{}_{\mu\nu}V^{A\nu}X^\mu_{,A}$, similarly as in equations (29.5). The boxed equations are equivalent to (29.5).

Equation (29.6) shows that the operator $\overline{\nabla}_A$ is kind-of "doubly covariant derivative" (divergence, specifically) – the first term is a covariant divergence determined by the induced metric h^{AB}, while the second term is the contribution from the background-space-time (Levi-Civita) connection. Later such an operator will be presented within a more geometric picture, and we will see it represents a covariant derivative projected to the tangent plane of the world-sheet.

29.5.4 Nambu-Goto Hamiltonian

For the Nambu-Goto Lagrangian, the Legendre transformation with (29.3) and (29.4) inserted yields $\mathcal{H} = \mathcal{L}\sqrt{-h}$,

$$
\begin{aligned}
\mathcal{H} &= \Pi^{(\tau)}_\alpha X^\alpha_{,\tau} + \Pi^{(\sigma)}_\alpha X^\alpha_{,\sigma} - \mathcal{L}\sqrt{-h} = \\
&= -T_0\,\frac{\big(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma}\big)\big(g_{\alpha\iota}X^\iota_{,\sigma}X^\alpha_{,\tau}\big) - \big(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma}\big)\big(g_{\alpha\kappa}X^\kappa_{,\tau}X^\alpha_{,\tau}\big)}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}} - \\
&\quad - T_0\,\frac{\big(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma}\big)\big(g_{\iota\alpha}X^\iota_{,\tau}X^\alpha_{,\sigma}\big) - \big(g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau}\big)\big(g_{\alpha\lambda}X^\lambda_{,\sigma}X^\alpha_{,\sigma}\big)}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}} + \\
&\quad + T_0\,\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})} = \\
&= -T_0\,\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})} = \mathcal{L}\sqrt{-h}\,.
\end{aligned}
$$

This Hamiltonian is sometimes called the *extended Hamiltonian*. Namely, classically one would only take

$$
\mathcal{H} = \Pi^{(\tau)}_\alpha X^\alpha_{,\tau} - \mathcal{L}\sqrt{-h} = 0\,.
$$

Note that this vanishing is *off-shell!*, so, in this picture, the dynamics is fully governed by constraints.

29.5.5 Suitable coordinates

Assuming that the string at any constant τ is a smooth space-like segment, it should be convenient to choose the space-time time coordinate so that within the string it coincides with τ, i.e. $X^0(\tau,\sigma) := \tau$. Such an adaptation may also be understood in an opposite way: having *some* space-time coordinates, the $t = \mathrm{const}$ hypersurface is intersected by the string world-sheet in a certain curve; the latter represents an instantaneous configuration of the string as taken with respect to the coordinate time t. The relation $\tau := X^0(\tau,\sigma) \equiv t(\tau,\sigma)$ then defines a certain intrinsic time parameterizing the string world-sheet. In the described "static gauge", we anyway have

$$X^\mu_{,\tau} = (\tau_{,\tau}, X^i_{,\tau}) = (1, X^i_{,\tau})\,, \quad X^\mu_{,\sigma} = (\tau_{,\sigma}, X^i_{,\sigma}) = (0, X^i_{,\sigma})\,.$$

The other intrinsic coordinate σ need not in general be much specified, the only requirement one definitely has is uniqueness – that the curves $\sigma = \mathrm{const}$ are smooth and do not intersect. An additional requirement might be that they be causal. A certain special attention is necessary if the string is closed, because then the world-sheet has cylindrical topology, so σ has to be cyclic, with the points $X^\mu(\tau,0)$ and $X^\mu(\tau,\Delta\sigma)$ identified.

29.5.6 Tension and energy of the string

Consider an open string to which the coordinates are adapted in such a way that it stretches along one of the spatial coordinates only (X^1, say), with the endpoints fixed to the values $X^1(\tau,0) = 0$ and $X^1(\tau,\Delta\sigma) = a > 0$. Using the time coordinate t adapted to the string "proper time" τ, we thus have

$$X^0(\tau,\sigma) = \tau\,, \quad X^1(\tau,\sigma) = s(\sigma)\,, \quad X^2(\tau,\sigma) = 0\,, \quad X^3(\tau,\sigma) = 0\,, \quad (...)\,,$$

where $s(\sigma)$ is assumed to be continuously increasing from $s(0)=0$ to $s(\Delta\sigma)=a$. Then

$$X^\mu_{,\tau} = (1,0,0,0,...)\,, \quad X^\mu_{,\sigma} = (0, s_{,\sigma}, 0, 0, ...) \quad (s_{,\sigma} > 0)$$
$$\Longrightarrow \quad h_{\tau\tau} \equiv g_{\mu\kappa} X^\mu_{,\tau} X^\kappa_{,\tau} = g_{00}\,, \quad h_{\sigma\sigma} \equiv g_{\nu\lambda} X^\nu_{,\sigma} X^\lambda_{,\sigma} = g_{11}(s_{,\sigma})^2, \quad h_{\tau\sigma} \equiv g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\sigma} = g_{01} s_{,\sigma}\,.$$

Substituting to the action (29.2), one obtains

$$S = -T_0 \int_{\tau_{\rm in}}^{\tau_{\rm fin}} \int_0^{\Delta\sigma} \sqrt{-h}\,\mathrm{d}\sigma\,\mathrm{d}\tau = -T_0 \int_{\tau_{\rm in}}^{\tau_{\rm fin}} \int_0^{\Delta\sigma} \sqrt{(g_{01})^2 - g_{00}g_{11}}\; s_{,\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

In the simple case of Minkowski as the "target" space-time ($g_{\mu\nu} = \eta_{\mu\nu}$), one thus has

$$S = -T_0 \int_{\tau_{\rm in}}^{\tau_{\rm fin}} \int_0^{\Delta\sigma} \frac{\mathrm{d}s}{\mathrm{d}\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau = -T_0 \int_{\tau_{\rm in}}^{\tau_{\rm fin}} [s(\Delta\sigma) - s(0)]\,\mathrm{d}\tau = -T_0 a \int_{\tau_{\rm in}}^{\tau_{\rm fin}} \mathrm{d}\tau = -T_0 a\,(\tau_{\rm fin} - \tau_{\rm in})\,.$$

Comparing this with a general action prescription $S=\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\mathcal{L}\,{\rm d}\tau$, we infer that T_0 represents the tension of the string. Actually, in our setting the string is static, so there would be no kinetic term in the Lagrangian $\mathcal{L}$, just a potential one (which has an opposite sign than the kinetic term) – and that just corresponds to $T_0 a$ standing for potential energy of the string (with T_0 assumed constant, independent of time, position and length of the string). The tension also represents the rest mass of the string per unit proper length (μ_0), according to $m_0c^2=\mu_0 ac^2=T_0 a$ (this means that the string is "massless", its rest mass is fully induced by its tension).

Remark 1: The proper length of such a fixed string in Minkowski would be constant. However, in general this is *not* the case – the proper length may be changing in time.

Remark 2: we should check whether the assumed string configuration can actually be realized in accord with the equations of motion (29.5). The momenta (29.3) and (29.4) reduce, respectively, to

$$\Pi^{(\tau)}_\alpha = -T_0\,\frac{g_{01}g_{\alpha 1}(s_{,\sigma})^2-g_{11}g_{\alpha 0}(s_{,\sigma})^2}{\sqrt{(g_{01}s_{,\sigma})^2-g_{00}g_{11}(s_{,\sigma})^2}} = -T_0\sqrt{(g_{01})^2-g_{00}g_{11}}\;s_{,\sigma}\,\delta^0_\alpha \overset{\rm Mink}{=} -T_0 s_{,\sigma}\delta^0_\alpha\,,$$
$$\Pi^{(\sigma)}_\alpha = -T_0\,\frac{g_{01}g_{0\alpha}s_{,\sigma}-g_{00}g_{\alpha 1}s_{,\sigma}}{\sqrt{(g_{01}s_{,\sigma})^2-g_{00}g_{11}(s_{,\sigma})^2}} = -T_0\sqrt{(g_{01})^2-g_{00}g_{11}}\;\delta^1_\alpha \overset{\rm Mink}{=} -T_0\,\delta^1_\alpha\,.$$

Hence, in Minkowski space-time, both terms of the equations of motion $\partial_\tau\Pi^{(\tau)}_\mu+\partial_\sigma\Pi^{(\sigma)}_\mu=0$ are zero, because both momenta are independent of τ and $\Pi^{(\sigma)}_\alpha$ is also independent of σ.

29.6 String velocity

Imagining the world-sheet as a 2D congruence of world-lines "numbered" by σ and parameterized by τ, one would naturally define a kind-of four-velocity as $U^\mu:=\frac{{\rm d}X^\mu}{{\rm d}\tau}\equiv X^\mu_{,\tau}$. In the "static gauge" where $t\equiv\tau$, it coincides with the coordinate velocity $V^\mu:=\frac{{\rm d}X^\mu}{{\rm d}t}$. However, as stressed already, such a quantity could not be interpreted as the four-velocity of a particular material point of the string, because we do not know how to mark and keep track of any such individual point (except the endpoints), and so we have almost no control over the longitudinal motion *within the string*. Mathematically, this reflects in reparameterization invariance: we can choose different "intrinsic" parameters (τ,σ), which would naturally change the "velocities".

Yet it *is* still possible to ascribe to the string a certain velocity, irrespectively of the above ignorance. At some (arbitrary) moment, choose an arbitrary point p of the string. Imagine to erect a (local) hyperplane orthogonal to the string at that point. After an infinitesimal lapse of τ or t, watch how the intersection point has shifted and thus introduce a velocity. Mathematically, it means to project the above local "four-velocity" to the local hyperplane orthogonal to the string at the given point.

Let us first consider, besides the rather arbitrary parameter σ, the **proper length along the string** (arc length), call it $s(\sigma)$:

$$1=h_{ss}:=g_{\mu\nu}\frac{\partial X^\mu}{\partial s}\frac{\partial X^\nu}{\partial s}=g_{\mu\nu}X^\mu_{,\sigma}X^\nu_{,\sigma}\left(\frac{{\rm d}\sigma}{{\rm d}s}\right)^2.$$

The derivative $X^\mu_{,\sigma}$ is naturally understood to be taken at $\tau = \text{const}$, so $X^\mu_{,s} = X^\mu_{,\sigma}\sigma_{,s}$ is the **unit vector tangent to the string**, let us denote it by $S^\mu := X^\mu_{,s}$. In terms of this vector, one standardly introduces the projector to the local orthogonal hyperplane by

$$h^{\mu\nu} := g^{\mu\nu} - S^\mu S^\nu \ .$$

Using the latter, one defines the **transverse velocity of the string** (at the given point) as

$$U^\mu_\perp := h^\mu_\nu U^\nu \equiv \left(\delta^\mu_\nu - S^\mu g_{\nu\lambda} S^\lambda\right) U^\nu = U^\mu - \left(g_{\nu\lambda} S^\lambda U^\nu\right) S^\mu \equiv U^\mu - U^\mu_{||} \ .$$

Its square amounts to

$$\begin{aligned} g_{\alpha\beta} U^\alpha_\perp U^\beta_\perp &= g_{\alpha\beta} U^\alpha U^\beta - 2\left(g_{\alpha\beta} U^\alpha S^\beta\right)\left(g_{\nu\lambda} S^\lambda U^\nu\right) + \left(g_{\nu\lambda} S^\lambda U^\nu\right)^2 \left(g_{\alpha\beta} S^\alpha S^\beta\right) = \\ &= g_{\alpha\beta} U^\alpha U^\beta - \left(g_{\alpha\beta} U^\alpha S^\beta\right)^2 \ . \end{aligned}$$

(The uncertainty in the identification of the world-lines of individual "string points" makes also the meaning of τ uncertain, so we better do not claim that U^μ is necessarily normalized to -1.)

Let us return to the Nambu-Goto action (29.2) now and rewrite the determinant under the square root,

$$\begin{aligned} &(g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\sigma})^2 - (g_{\mu\kappa} X^\mu_{,\tau} X^\kappa_{,\tau})(g_{\nu\lambda} X^\nu_{,\sigma} X^\lambda_{,\sigma}) = \\ &= \left[(g_{\mu\nu} U^\mu S^\nu)^2 - (g_{\mu\kappa} U^\mu U^\kappa)(g_{\nu\lambda} S^\nu S^\lambda\right] (s_{,\sigma})^2 = (-g_{\alpha\beta} U^\alpha_\perp U^\beta_\perp)(s_{,\sigma})^2 \ . \end{aligned}$$

It is thus clear how to write $\mathcal{L}\sqrt{-h} = \mathcal{H}$ and the action,

$$S = -T_0 \int\limits_{\tau_{\rm in}}^{\tau_{\rm fin}} \int\limits_0^{\Delta\sigma} \sqrt{-g_{\alpha\beta} U^\alpha_\perp U^\beta_\perp}\; s_{,\sigma}\, \mathrm{d}\sigma\, \mathrm{d}\tau \ ,$$

which very much resembles the action for a relativistic particle. (Note that it would seem suitable to write the integration in terms of $\mathrm{d}s = s_{,\sigma}\mathrm{d}\sigma$, but the integration over σ has the advantage that its limits are *fixed*, whereas the upper bound for s – i.e. the proper length of the string – may be changing.)

The transversal four-velocity also enables to rewrite in another form the momenta (29.3) and (29.4) (original forms are repeated for convenience)

$$\begin{aligned} \Pi^{(\tau)}_\alpha &= -T_0\, \frac{\left(g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\sigma}\right) g_{\alpha\iota} X^\iota_{,\sigma} - \left(g_{\nu\lambda} X^\nu_{,\sigma} X^\lambda_{,\sigma}\right) g_{\alpha\kappa} X^\kappa_{,\tau}}{\sqrt{(g_{\mu\nu} X^\mu_{,\tau} X^\nu_{,\sigma})^2 - (g_{\mu\kappa} X^\mu_{,\tau} X^\kappa_{,\tau})(g_{\nu\lambda} X^\nu_{,\sigma} X^\lambda_{,\sigma})}} = \\ &= -T_0\, s_{,\sigma}\, \frac{\left(g_{\mu\nu} U^\mu S^\nu\right) g_{\alpha\iota} S^\iota - g_{\alpha\kappa} U^\kappa}{\sqrt{-g_{\mu\nu} U^\mu_\perp U^\nu_\perp}} = -T_0\, s_{,\sigma}\, \frac{g_{\alpha\iota} U^\iota_{||} - g_{\alpha\kappa} U^\kappa}{\sqrt{-g_{\mu\nu} U^\mu_\perp U^\nu_\perp}} = \\ &= T_0\, s_{,\sigma} \frac{g_{\alpha\kappa} U^\kappa_\perp}{\sqrt{-g_{\mu\nu} U^\mu_\perp U^\nu_\perp}} \ , \end{aligned} \tag{29.7}$$

$$\Pi_\alpha^{(\sigma)} = -T_0\, \frac{\left(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma}\right) g_{\iota\alpha}X^\iota_{,\tau} - \left(g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau}\right) g_{\alpha\lambda}X^\lambda_{,\sigma}}{\sqrt{(g_{\mu\nu}X^\mu_{,\tau}X^\nu_{,\sigma})^2 - (g_{\mu\kappa}X^\mu_{,\tau}X^\kappa_{,\tau})(g_{\nu\lambda}X^\nu_{,\sigma}X^\lambda_{,\sigma})}} = \\ = -T_0\, \frac{(g_{\mu\nu}U^\mu S^\nu)\, g_{\iota\alpha}U^\iota - (g_{\mu\kappa}U^\mu U^\kappa)\, g_{\alpha\lambda}S^\lambda}{\sqrt{-g_{\mu\nu}U^\mu_\perp U^\nu_\perp}}\,. \tag{29.8}$$

The second momentum $\Pi_\alpha^{(\sigma)}$ does not simplify that much, but look at it once more: if factoring out $(-g_{\mu\kappa}U^\mu U^\kappa)$ and thus normalizing U-s in the first term (call the normalized version $\tilde{U}^\mu$), we have

$$\Pi_\alpha^{(\sigma)} = -T_0\,(-g_{\mu\kappa}U^\mu U^\kappa)\, \frac{(g_{\mu\nu}\tilde{U}^\mu S^\nu)\, g_{\iota\alpha}\tilde{U}^\iota + g_{\alpha\nu}S^\nu}{\sqrt{-g_{\mu\nu}U^\mu_\perp U^\nu_\perp}} = \\ = -T_0\, \frac{-g_{\mu\kappa}U^\mu U^\kappa}{\sqrt{-g_{\mu\nu}U^\mu_\perp U^\nu_\perp}}\, (g_{\alpha\nu} + \tilde{U}_\alpha \tilde{U}_\nu)\, S^\nu\,. \tag{29.9}$$

Clearly $(g_{\alpha\nu} + \tilde{U}_\alpha \tilde{U}_\nu)\, S^\nu$ is the projection of S^μ to the hyperplane locally orthogonal to U^μ.

29.6.1 Motion of free endpoints of an open string

Have an open string with free endpoints, so with boundary conditions $\Pi_\mu^{(\sigma)}(\tau, \sigma_{\text{end}}) = 0$. Look at the expression (29.9): the projection $(g_{\alpha\nu} + \tilde{U}_\alpha \tilde{U}_\nu)\, S^\nu$ is certainly non-zero (otherwise S^ν would have to be proportional to U^ν), so the only possibility is that $g_{\mu\kappa}U^\mu U^\kappa = 0$. Hence, the free endpoints move with the speed of light. Wait: we have been admitting that the string points are indistinguishable, so also U^μ is a somewhat vague quantity. However, this does not apply to endpoints – the endpoints do have clear identity, and U^μ is the tangent to their world-lines.

Return now to (29.8) with the above finding, thus with the second term vanishing. In order that $\Pi_\mu^{(\sigma)}(\tau, \sigma_{\text{end}})$ vanish, the first term has to be zero as well, which means that $g_{\mu\nu}U^\mu S^\nu = 0$ – the motion of free endpoints is purely transversal. Such a circumstance in fact requires a bit of back-checking, because $g_{\mu\nu}U^\mu S^\nu = 0$ means (look through the formulas of Section 29.6 once more) that really $U^\mu_{||} \equiv \left(g_{\nu\lambda}S^\lambda U^\nu\right) S^\mu = 0$, so, at the endpoints,

$$-g_{\mu\nu}U^\mu_\perp U^\nu_\perp \to -g_{\mu\nu}U^\mu U^\nu \to 0\,.$$

The query is that the latter appears in denominators at times. However, everything works well: the momentum then becomes null as well, because (29.7) yields

$$g^{\alpha\beta}\Pi_\alpha^{(\tau)}\Pi_\beta^{(\tau)} = (T_0 s_{,\sigma})^2 \sqrt{-g_{\kappa\lambda}U^\kappa_\perp U^\lambda_\perp} \to 0$$

and $\Pi_\mu^{(\sigma)}(\tau, \sigma_{\text{end}}) = 0$ by the boundary constraint. And, from (29.9) it is clear that the constraint is fulfilled at the very endpoints too, $\Pi_\alpha^{(\sigma)} \to -T_0 \sqrt{-g_{\mu\kappa}U^\mu U^\kappa}\; S_\alpha \to 0\,.$

29.7 "Energy-functional" actions and the energy-momentum tensor

The square root in the Nambu-Goto action is somewhat annoying.[3] Fortunately, similar options to those shown in the particle case (Section 29.3) also exist for strings, based on "energy functionals" (those quadratic in velocities). So consider

$$S = \int_{\tau_{\text{in}}}^{\tau_{\text{fin}}} \int_{0}^{\Delta\sigma} \mathfrak{L} \, \mathrm{d}\sigma \, \mathrm{d}\tau = \frac{T_0}{2} \int_{\tau_{\text{in}}}^{\tau_{\text{fin}}} \int_{0}^{\Delta\sigma} \left(g_{\mu\nu} X^{\mu}_{,\tau} X^{\nu}_{,\tau} - g_{\mu\nu} X^{\mu}_{,\sigma} X^{\nu}_{,\sigma} \right) \mathrm{d}\sigma \, \mathrm{d}\tau \, . \tag{29.10}$$

The canonical momenta read

$$\Pi^{(\tau)}_{\mu} := \frac{\partial \mathfrak{L}}{\partial X^{\mu}_{,\tau}} = T_0 \, g_{\mu\nu} X^{\nu}_{,\tau} \, , \qquad \Pi^{(\sigma)}_{\mu} := \frac{\partial \mathfrak{L}}{\partial X^{\mu}_{,\sigma}} = -T_0 \, g_{\mu\nu} X^{\nu}_{,\sigma} \, ,$$

clearly satisfying $\Pi^{(\tau)}_{\mu} X^{\mu}_{,\sigma} + \Pi^{(\sigma)}_{\mu} X^{\mu}_{,\tau} = 0$. The equations of motion follow as

$$\begin{aligned} \bar{\delta}\mathfrak{L} &= \frac{\partial \mathfrak{L}}{\partial X^{\alpha}_{,\tau}} (\bar{\delta} X^{\alpha})_{,\tau} + \frac{\partial \mathfrak{L}}{\partial X^{\alpha}_{,\sigma}} (\bar{\delta} X^{\alpha})_{,\sigma} + \frac{\partial \mathfrak{L}}{\partial g_{\alpha\beta}} g_{\alpha\beta,\iota} \bar{\delta} X^{\iota} = \\ &= \Pi^{(\tau)}_{\alpha} (\bar{\delta} X^{\alpha})_{,\tau} + \Pi^{(\sigma)}_{\alpha} (\bar{\delta} X^{\alpha})_{,\sigma} + \frac{T_0}{2} \left(X^{\alpha}_{,\tau} X^{\beta}_{,\tau} - X^{\alpha}_{,\sigma} X^{\beta}_{,\sigma} \right) g_{\alpha\beta,\iota} \bar{\delta} X^{\iota} = \\ &= \left(\Pi^{(A)}_{\alpha} \bar{\delta} X^{\alpha} \right)_{,A} - \left(\Pi^{(A)}_{\alpha} \right)_{,A} \bar{\delta} X^{\alpha} + \left(\Pi^{(\tau)\alpha} X^{\beta}_{,\tau} + \Pi^{(\sigma)\alpha} X^{\beta}_{,\sigma} \right) \Gamma_{\alpha\beta\iota} \bar{\delta} X^{\iota} \, . \end{aligned}$$

Omitting the first, divergence term, we thus find the same equations as from Nambu-Goto,

$$0 = \partial_A \Pi^{(A)}_{\mu} - \Gamma^{\alpha}{}_{\beta\mu} \Pi^{(A)}_{\alpha} X^{\beta}_{,A} =: \overline{\nabla}_A \Pi^{(A)}_{\mu} \, .$$

The above equations of motion assume the wave-equation form if the momenta are substituted in them. In particular, in Minkowski one obtains

$$\partial_\tau \Pi^{(\tau)}_{\mu} + \partial_\sigma \Pi^{(\sigma)}_{\mu} = 0 \qquad \Longleftrightarrow \qquad \boxed{X^{\mu}_{,\tau\tau} - X^{\mu}_{,\sigma\sigma} = 0} \, . \tag{29.11}$$

For the above Lagrangian, the *extended* Hamiltonian reads

$$\mathcal{H} = \Pi^{(\tau)}_{\mu} X^{\mu}_{,\tau} + \Pi^{(\sigma)}_{\mu} X^{\mu}_{,\sigma} - \mathfrak{L} = T_0 \, g_{\mu\nu} X^{\mu}_{,\tau} X^{\nu}_{,\tau} - T_0 \, g_{\mu\nu} X^{\mu}_{,\sigma} X^{\nu}_{,\sigma} - \mathfrak{L} = \mathfrak{L} \, .$$

A "classical" Hamiltonian yields a different expression, however,

$$\mathcal{H} = \Pi^{(\tau)}_{\mu} X^{\mu}_{,\tau} - \mathfrak{L} = T_0 \, g_{\mu\nu} X^{\mu}_{,\tau} X^{\nu}_{,\tau} - \mathfrak{L} = \frac{T_0}{2} \left(g_{\mu\nu} X^{\mu}_{,\tau} X^{\nu}_{,\tau} + g_{\mu\nu} X^{\mu}_{,\sigma} X^{\nu}_{,\sigma} \right) = \frac{\Pi^{(A)}_{\mu} \Pi^{\mu}_{(A)}}{2T_0} \, .$$

It is easy to check that the canonical equations yield what expected. Yet still the constraints (similar to the normalization of the tangent to the particle world-line) have to be added by hand.

[3] Have seen on the web that MTW is "a somewhat thicker book".

29.7.1 The Polyakov action

The above "energy-type" action is practically the simplest option, similarly as in the case of particles. Yet we showed that *conceptually* the most elegant option was the action (29.1). That introduced, as an independent "gauge" degree of freedom, an internal metric which scaled the parameter along the particle world-line. Let us proceed similarly here, only that the internal metric has now to be 2x2, describing the intrinsic geometry of the world-sheet, we denote it $\gamma_{AB}(\tau,\sigma)$ (again). This actually is a natural way how to generally frame the question "which parameterization of the world-sheet to choose". The action thus is written as

$$S = -\frac{T_0}{2}\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\int_0^{\Delta\sigma} g_{\mu\nu}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma}\;{\rm d}\sigma\,{\rm d}\tau\,, \qquad \{A,B\}=\{\tau,\sigma\}\,. \tag{29.12}$$

It is usually called the **Polyakov action** after A. Polyakov who first employed it for quantization. Note that the mass term is absent – the string is "massless".

Let us vary the 1/2 of the integrand with respect to X^α, while right crossing out the divergence terms. Bearing in mind that γ_{AB} does *not* depend on X^α, we have

$$\begin{aligned}
&\frac{1}{2}\,g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma}\;\bar\delta X^\alpha + g_{\mu\nu}(\bar\delta X^\mu)_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} = \\
&= \frac{1}{2}\,g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\gamma^{AB}\sqrt{-\gamma}\;\bar\delta X^\alpha + \cancel{\left(g_{\mu\nu}\bar\delta X^\mu X^\nu_{,B}\gamma^{AB}\sqrt{-\gamma}\right)_{,A}} - \left(g_{\mu\nu}X^\nu_{,B}\gamma^{AB}\sqrt{-\gamma}\right)_{,A}\bar\delta X^\mu = \\
&= \left[\frac{1}{2}\,g_{\mu\nu,\alpha}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} - g_{\alpha\nu,\mu}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} - g_{\alpha\nu}\partial_A\!\left(X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma}\right)\right]\bar\delta X^\alpha\,,
\end{aligned}$$

so the equations of motion read

$$\begin{aligned}
0 &= g_{\alpha\nu,\mu}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} + g_{\alpha\nu}\partial_A\!\left(X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma}\right) - \frac{1}{2}\,g_{\mu\nu,\alpha}\,X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} = \\
&= g_{\alpha\nu}\partial_A\!\left(X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma}\right) + \Gamma_{\alpha\mu\nu}X^\mu_{,A}X^\nu_{,B}\,\gamma^{AB}\sqrt{-\gamma} = \\
&= g_{\alpha\iota}\left[\partial_A\!\left(\sqrt{-\gamma}\,\gamma^{AB}X^\iota_{,B}\right) + \Gamma^\iota{}_{\mu\nu}\sqrt{-\gamma}\,\gamma^{AB}X^\mu_{,A}X^\nu_{,B}\right].
\end{aligned}$$

Using the same notation $\overline{\nabla}_A V^{A\iota} := \partial_A V^{A\iota} + \Gamma^\iota{}_{\mu\nu}V^{A\nu}X^\mu_{,A}$ as in the field equations (29.5) and (29.6), we could write the result as

$$\overline{\nabla}_A\!\left(\sqrt{-\gamma}\,\gamma^{AB}X^\iota_{,B}\right) = 0\,. \tag{29.13}$$

Good to notice that the first term represents a covariant derivative associated with the internal metric γ^{AB} which one would standardly denote as $\partial_A\!\left(\sqrt{-\gamma}\,\gamma^{AB}X^\iota_{,B}\right) =: \sqrt{-\gamma}\,\nabla_A(\gamma^{AB}X^\iota_{,B})$. The $\sqrt{-\gamma}$ can thus be factored out and the field equations assume the form

$$\boxed{\nabla^B X^\iota_{,B} + \Gamma^\iota{}_{\mu\nu}\gamma^{AB}X^\mu_{,A}X^\nu_{,B} = 0}\,, \tag{29.14}$$

known as the **Virasoro constraints**. One can in fact write them as a wave equation,

$$\Box\, X^\iota + \Gamma^\iota{}_{\mu\nu}\gamma^{AB}X^\mu_{,A}X^\nu_{,B} = 0 \qquad \text{where} \quad \Box\, X^\iota := \nabla_A(\gamma^{AB}X^\iota_{,B})\,.$$

As in the particle case, γ_{AB} naturally does not obey any evolution equation (since it represents a gauge freedom), yet we expect to obtain a certain *constraint* by varying the action with respect to it. Using again the determinant-derivative rule (23.9), we have

$$\bar{\delta}\left(g_{\mu\nu}X^{\mu}_{,A}X^{\nu}_{,B}\,\gamma^{AB}\sqrt{-\gamma}\right) \equiv \bar{\delta}\left(h_{AB}\gamma^{AB}\sqrt{-\gamma}\right) =$$
$$= h_{AB}\sqrt{-\gamma}\,\bar{\delta}\gamma^{AB} - h_{AB}\gamma^{AB}\,\frac{\sqrt{-\gamma}}{2}\,\gamma_{CD}\bar{\delta}\gamma^{CD} = \left(h_{CD} - \frac{1}{2}\,h_{AB}\gamma^{AB}\gamma_{CD}\right)\sqrt{-\gamma}\;\bar{\delta}\gamma^{CD}\,.$$

Should this vanish, it has to hold $h_{CD} = \frac{1}{2}\,h_{AB}\gamma^{AB}\gamma_{CD}$, so the metrics have to be proportional. The determinant of this relation yields $h = \left(\frac{1}{2}\,h_{AB}\gamma^{AB}\right)^2\gamma$, hence $\sqrt{-h} = \frac{1}{2}\,h_{AB}\gamma^{AB}\sqrt{-\gamma}$ (we assume $h_{AB}\gamma^{AB} > 0$). By dividing the matrix proportionality by the last equation, we have

$$\frac{h_{CD}}{\sqrt{-h}} = \frac{\gamma_{CD}}{\sqrt{-\gamma}} \quad\Rightarrow\quad \frac{h_{CD}\gamma^{CD}}{\sqrt{-h}} = \frac{\gamma_{CD}\gamma^{CD}}{\sqrt{-\gamma}} = \frac{2}{\sqrt{-\gamma}} \quad\Rightarrow\quad h_{CD}\gamma^{CD}\sqrt{-\gamma} = 2\sqrt{-h}\,.$$

Therefore, introducing this in the action (29.12) reveals that the latter "on-shell" (with the constraint satisfied) equals the Nambu-Goto action,

$$S = -\frac{T_0}{2}\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\int_0^{\Delta\sigma} h_{AB}\gamma^{AB}\sqrt{-\gamma}\;\mathrm{d}\sigma\,\mathrm{d}\tau = -T_0\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\int_0^{\Delta\sigma}\sqrt{-h}\;\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

29.7.2 Open strings: conditions at the endpoints

In the derivation of the Virasoro equations (29.14), we crossed out $\cancel{\left(g_{\mu\nu}\bar{\delta}X^{\mu}X^{\nu}_{,B}\,\gamma^{AB}\sqrt{-\gamma}\right)_{,A}}$ as a boundary term. Yet it requires more attention. Integrating it partially over the respective variables while assuming that $\delta X^{\mu}(\tau_{\rm in},\sigma) = 0$ and $\delta X^{\mu}(\tau_{\rm fin},\sigma) = 0$, we have (omitting the $-T_0$ factor)

$$\int_{\tau_{\rm in}}^{\tau_{\rm fin}}\int_0^{\Delta\sigma}\left[\left(g_{\mu\nu}\bar{\delta}X^{\mu}X^{\nu}_{,B}\,\gamma^{\tau B}\sqrt{-\gamma}\right)_{,\tau} + \left(g_{\mu\nu}\bar{\delta}X^{\mu}X^{\nu}_{,B}\,\gamma^{\sigma B}\sqrt{-\gamma}\right)_{,\sigma}\right]\mathrm{d}\sigma\,\mathrm{d}\tau =$$
$$= \int_0^{\Delta\sigma}\cancel{\left[g_{\mu\nu}\bar{\delta}X^{\mu}X^{\nu}_{,B}\,\gamma^{\tau B}\sqrt{-\gamma}\right]_{\tau_{\rm in}}^{\tau_{\rm fin}}}\mathrm{d}\sigma + \int_{\tau_{\rm in}}^{\tau_{\rm fin}}\left[g_{\mu\nu}\bar{\delta}X^{\mu}X^{\nu}_{,B}\,\gamma^{\sigma B}\sqrt{-\gamma}\right]_0^{\Delta\sigma}\mathrm{d}\tau\,.$$

For closed strings (loops), $\sigma = 0$ and $\sigma = \Delta\sigma$ represent the same point, so the result of course vanishes, while for open strings the additional condition arises

$$X^{\nu}_{,B}\,\gamma^{\sigma B} = 0 \quad \text{at the endpoints}\ \ (\sigma = 0 \text{ and } \sigma = \Delta\sigma)\,.$$

These are Neumann conditions requiring that the σ component of $X^{\nu}_{,B}$ – i.e. the one tangent to the world-sheet while normal to its endpoint boundary – should vanish.

29.7.3 Energy-momentum tensor

In the variational derivation of Einstein equations we defined the energy-momentum tensor by differentiation of the non-gravitational Lagrangian with respect to the metric. In the case when the Lagrangian did not depend on the metric *derivatives*, we had (in Section 23.4.4)

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\,\frac{\partial(\sqrt{-g}\,\mathcal{L}_{\rm ng})}{\partial g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\,\mathcal{L}_{\rm ng} - 2\,\frac{\partial\mathcal{L}_{\rm ng}}{\partial g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}_{\rm ng} - 2\frac{\partial\mathcal{L}_{\rm ng}}{\partial g^{\mu\nu}}$$

(where $\mathcal{L}_{\rm ng}$ is the *invariant* Lagrangian density). With our invariant world-sheet Lagrangian $\mathcal{L} = -\frac{T_0}{2}\,h_{CD}\gamma^{CD}$, it means

$$\begin{aligned} T_{AB} &:= \gamma_{AB}\mathcal{L} - 2\,\frac{\partial\mathcal{L}}{\partial\gamma^{AB}} = -\frac{T_0}{2}\,h_{CD}\gamma^{CD}\gamma_{AB} + T_0 h_{AB} = \\ &= T_0\left(h_{AB} - \frac{1}{2}\,h_{CD}\gamma^{CD}\,\gamma_{AB}\right) = T_0 h_{AB} + \gamma_{AB}\mathcal{L}\,. \end{aligned}$$

This tensor is traceless, $\gamma^{AB}T_{AB} = 0$, and it satisfies the covariant conservation laws. The conservation follows, in exactly the same manner as in the space-time case treated in Section 23.4.5, from the diffeomorphism invariance of the action – here under the infinitesimal shift of the parameterization (τ,σ). Let us denote, for a while, $w^A := (\tau,\sigma)$ and let us perform the shift $w^A \to w^A + \xi^A$ (with ξ infinitesimal). The internal metric changes in it as usual,

$$\gamma_{AB} \to \gamma_{AB} - \nabla_A\xi_B - \nabla_B\xi_A \;\equiv\; \gamma_{AB} + \bar\delta\gamma_{AB}\,.$$

Under such a special variation, the action is required to stay unchanged,

$$\begin{aligned} 0 = \bar\delta\int\mathcal{L}\,\sqrt{-\gamma}\;{\rm d}\sigma\,{\rm d}\tau &= \int\frac{\partial(\mathcal{L}\sqrt{-\gamma})}{\partial\gamma^{AB}}\,\bar\delta\gamma^{AB}\,{\rm d}\sigma\,{\rm d}\tau = -\frac{1}{2}\int T_{AB}\bar\delta\gamma^{AB}\sqrt{-\gamma}\;{\rm d}\sigma\,{\rm d}\tau = \\ &= \int T^{AB}\nabla_A\xi_B\,\sqrt{-\gamma}\;{\rm d}\sigma\,{\rm d}\tau = \int\Big[\cancel{\nabla_A(T^{AB}\xi_B)} - (\nabla_A T^{AB})\,\xi_B\Big]\sqrt{-\gamma}\;{\rm d}\sigma\,{\rm d}\tau\,. \end{aligned}$$

The first, divergence term vanishes because ξ_B is supposed to vanish on the boundaries, and the second has to vanish for any ξ_B satisfying such boundary behaviour (but non-zero over the world-sheet), so, necessarily, $\nabla_A T^{AB} = 0$.

The energy-momentum tensor thus conserves *off-shell*, namely the conservation follows just from the diffeomorphism invariance of the action. We stress this, because *on-shell* it is rather trivial: from the variation of S with respect to γ^{AB}, we even showed that $T^{AB} = 0$.

There is also a "partially on-shell" view. The T_{AB} has been introduced solely from the (off-shell) behaviour of $\mathcal{L}$ with respect to γ_{AB}, without any information from its behaviour with respect to $X^\mu_{\ ,A}$. One may thus speculate that the conservation might follow if this missing information is added. Actually, take the definition of T_{AB} and the Virasoro constraints (29.14) which, exactly, were obtained from the variation of S with respect to $X^\mu_{\ ,A}$. Omitting the factor T_0 in T_{AB} and realizing that ∇^A is the connection of γ_{AB}, we have

$$\nabla^A T_{AB} = \nabla^A h_{AB} - \frac{1}{2}\,\nabla^A(h_{CD}\gamma^{CD}\gamma_{AB}) = \gamma^{CD}\nabla_D h_{CB} - \frac{1}{2}\,\gamma^{CD}\nabla_B h_{CD} =$$

$$
\begin{aligned}
&= \gamma^{CD}\nabla_D\left(g_{\mu\kappa}X^{\mu}_{,C}X^{\kappa}_{,B}\right) - \frac{1}{2}\gamma^{CD}\nabla_B\left(g_{\mu\nu}X^{\mu}_{,C}X^{\nu}_{,D}\right) = \\
&= \gamma^{CD}g_{\mu\kappa,\nu}X^{\mu}_{,C}X^{\kappa}_{,B}X^{\nu}_{,D} + \gamma^{CD}g_{\mu\kappa}(\nabla_D X^{\mu}_{,C})X^{\kappa}_{,B} + \gamma^{CD}g_{\mu\kappa}X^{\mu}_{,C}\nabla_D X^{\kappa}_{,B} - \\
&\quad -\frac{1}{2}\gamma^{CD}g_{\mu\nu,\kappa}X^{\mu}_{,C}X^{\nu}_{,D}X^{\kappa}_{,B} - \gamma^{CD}g_{\mu\nu}X^{\mu}_{,C}\nabla_B X^{\nu}_{,D} = \\
&= \cancel{\Gamma_{\kappa\mu\nu}\gamma^{CD}X^{\kappa}_{,B}X^{\mu}_{,C}X^{\nu}_{,D}} \cancel{+ g_{\mu\kappa}(\nabla^{C}X^{\mu}_{,C})X^{\kappa}_{,B}} + \\
&\quad + \gamma^{CD}g_{\mu\kappa}X^{\mu}_{,C}\nabla_D X^{\kappa}_{,B} - \gamma^{CD}g_{\mu\nu}X^{\mu}_{,C}\nabla_B X^{\nu}_{,D}\,,
\end{aligned}
$$

where the cancellation follows from the Virasoro constraint $\nabla^{C}X^{\mu}_{,C} = -\Gamma^{\mu}{}_{\lambda\nu}X^{\lambda}_{,C}X^{\nu}_{,D}$. The remaining two terms mutually cancel as well, due to the Levi-Civita nature of ∇,

$$
\gamma^{CD}g_{\mu\nu}X^{\mu}_{,C}\left(\nabla_D X^{\nu}_{,B} - \nabla_B X^{\nu}_{,D}\right) = \gamma^{CD}g_{\mu\nu}X^{\mu}_{,C}\left[X^{\nu}_{,BD} - X^{\nu}_{,DB} - (\Gamma^{A}{}_{DB} - \Gamma^{A}{}_{BD})X^{\nu}_{,A}\right] = 0\,.
$$

29.7.4 Scale invariance

Besides the symmetries of the Nambu-Goto action (with respect to the reparameterization of the world-sheet and with respect to the transformation of space-time coordinates), the Polyakov action (29.12) has one more symmetry which only holds in 2D – the Weyl symmetry, i.e. the one with respect to the conformal rescaling $\gamma_{AB} \to \Omega^2\gamma_{AB}$. Indeed, the $\gamma^{AB}\sqrt{-\gamma}$ term of the action remains invariant, because $\sqrt{-\gamma}$ scales as Ω^2 and γ^{AB} scales as γ_{AB}/γ, that is, as Ω^2/Ω^4, so altogether the Ω factors cancel out.

The scale invariance (often called the **Weyl symmetry**) has turned out to be crucial in the quantization of the action (serving as the path integral). Three items are at place in connection with the scale invariance:

- Scale invariance (with respect to a constant rescaling), or, more generally, conformal invariance (with respect to rescaling by a *function of position*) is the crucial feature of **conformal field theories**, used in various branches of physics, most notably in statistical physics and solid-state physics. Let us show that it is in fact *the* deep reason why the energy-momentum tensor is traceless. Consider the conformal transformation $\gamma_{AB} \to \Omega^2(\tau,\sigma)\gamma_{AB}$, so $\bar{\delta}\gamma_{AB} = (\Omega^2 - 1)\gamma_{AB}$. The thus induced variation of the Lagrangian reads

$$
\bar{\delta}\left(\mathcal{L}\sqrt{-\gamma}\right) = \frac{\partial(\mathcal{L}\sqrt{-\gamma})}{\partial\gamma_{AB}}\,\bar{\delta}\gamma_{AB} = \frac{\sqrt{-\gamma}}{2}\,T^{AB}\bar{\delta}\gamma_{AB} = \frac{\sqrt{-\gamma}}{2}\,T^{AB}(\Omega^2 - 1)\gamma_{AB}\,.
$$

  Should this vanish, for a general Ω^2, it requires $T^{AB}\gamma_{AB} = 0$ which is the traceless property.

- String theory is *not* fully scale invariant, because its actions contain T_0 which is connected to the string mass-energy density, so it does bring a certain scale. This is well understandable actually: a strict scale invariance forbids the occurrence of any length, time or mass parameters in the theory, hence, in particular, the theory can only involve massless excitations of the pertinent field. This would of course be too restrictive for a string theory.

- Even if the classical field theory is scale-invariant, its quantum version may violate this. Actually, in renormalization procedures standardly employed to treat divergences in the perturbation theory, the counter-terms necessary to regularize the quantum corrections often introduce a certain length scale. Generally, such a loss of symmetry is being called an **anomaly** (it even occasionally occurs within classical physics itself). In string theory, this type of anomaly is called the **Weyl anomaly**. It manifests itself on the energy-momentum tensor: classically, it is traceless, but quantum fluctuations (the Casimir effect) generate a certain vacuum mean value of the trace, specifically, $\langle T_A^A \rangle = -\frac{C}{12} R$, where $R = R(\gamma_{AB})$ is the Ricci scalar of the world-sheet and C is a constant.[4]

29.8 World-sheet curvature in action?

Experience from the Hilbert action of GR suggests the question: what about curvature of the world-sheet, i.e. the one associated with the metric γ_{AB}? Why have not we considered the Hilbert-type term

$$\int R(\gamma_{AB}) \sqrt{-\gamma}\, \mathrm{d}\sigma\, \mathrm{d}\tau$$

in designing the string action? Although we have devoted a separate section to this important question, the answer is pretty short. Remember from Section 23.4.3 that the variation of the Hilbert term with respect to the associated metric lead to the Einstein tensor,

$$\bar{\delta} \int R \sqrt{-g}\, \mathrm{d}^4 x = \int \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \bar{\delta} g^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x\,,$$

at least provided that the corresponding Christoffel-symbol variations vanished on the boundary (together with the metric variations themselves). However, due to the Bianchi identities, in 2D the Riemann tensor only has *one* independent component, satisfying

$$R_{ABCD} = \frac{R}{2} \left(\gamma_{AC}\gamma_{BD} - \gamma_{AD}\gamma_{BC} \right) \qquad \Longrightarrow \qquad R_{BD} \equiv \gamma^{AC} R_{ABCD} = \frac{R}{2} \gamma_{BD}\,.$$

Hence, the Einstein tensor vanishes. The Hilbert-type term thus is irrelevant for variations of the action.

However, although it does not contribute to the dynamics, the Hilbert term does contribute to the action itself, thus affecting global quantities computed from it. And, we also saw in the Lagrangian formulation of GR that if the variation of the Christoffel symbols does *not* vanish on the boundary, the Hilbert action in addition brings a non-trivial boundary term (Section 23.4.6). Specifically, it requires that the term

$$2\epsilon \oint_{\partial\Omega} K \sqrt{h}\, \mathrm{d}^3 y\,, \qquad \text{where} \quad K = h^{\alpha\beta} n_{\alpha;\beta}\,, \quad h^{\alpha\beta} = g^{\alpha\beta} - \epsilon\, n^\alpha n^\beta\,, \quad \epsilon := g_{\mu\nu} n^\mu n^\nu\,,$$

[4] This constant is called the central charge of the theory and it kind-of measures the number of degrees of freedom in the theory. It is the Noether charge induced by symmetry with respect to a certain “central” subgroup of the whole symmetry group which commutes with all the rest of the group. In the conformal field theories, it is represented by an operator which commutes with all the other operators, thus necessarily being a “c-number”. The constant C then is referred to as the central charge of the Virasoro algebra spanned by generators of the conformal transformations of the world-sheet.

be included in the action (with $\partial\Omega$ the boundary, $h^{\alpha\beta}$ its metric, n^α its outward unit normal and K the trace of its extrinsic curvature). Analogously, in the case of an open string, where the world-sheet (W) *has* boundaries represented by the string-endpoint histories, the whole curvature contribution to the action thus reads

$$\int_W R(\gamma_{AB})\sqrt{-\gamma}\,\mathrm{d}\sigma\,\mathrm{d}\tau + 2\epsilon\int_{\partial W} k\,\mathrm{d}s\,, \tag{29.15}$$

where $k := t^A t^B \nabla_A n_B$, with t^A the tangent to the boundary, n^A the outward unit normal orthogonal to t^A, and $\epsilon := \gamma_{AB} t^A t^B$. If the string endpoints are free, they move along null geodesics; along such parts of the boundary the second integral apparently has no contribution (sure, $\mathrm{d}s = 0$ then, but also $k = t^A t^B \nabla_A n_B = -t^B \nabla_B t^A n_B = 0$, because t^A is geodesic).

The expression (29.15) exactly appears in the **Gauss-Bonnet theorem**: for a compact 2D (pseudo-)Riemannian manifold with a boundary, the above expression yields the number $4\pi\chi$, where $\chi = 2 - 2h - b - c$ (with h the number of handles and b the number of boundaries, i.e. of "legs", and c the number of cross-caps) is the Euler characteristic of the surface. (If $b = 0$, the second integral is absent.) The Euler characteristic is a topological invariant, so it does not change under smooth deformations of the surface (the world-sheet in our case).

The last point is very important for the string perturbation theory, since the string processes which appear in the "Feynman diagrams" geometrically correspond to the various changes of the above characteristics, so the pertinent world-sheets can be classified according to their values. For example, one open string follows, within any finite time interval, a world-sheet with no handles, one boundary and no cross-caps. If it emits and reabsorbs another open string, the resulting world-sheet has one boundary more (there appears a hole), so χ changes by $\Delta\chi = -1$. If a *closed* string is emitted and reabsorbed, χ changes by $\Delta\chi = -2$, because there appears a handle on the world-sheet. In the quantum string theory, the Hilbert-type, "topological" part of the action stands multiplied by a factor Φ which physically represents the vacuum expectation value of one of the massless excitations of the string (the so called dilaton field). And the number e^Φ stands for the string coupling constant: the amplitude of every next-order correction is weighted by $e^{-\Phi\Delta\chi}$, i.e. by e^Φ for open-string loops ($\Delta\chi = -1$) while by $e^{2\Phi}$ for closed-string loops ($\Delta\chi = -2$).

- We have been speaking of the conformal invariance as the central property of the theory. The Hilbert action term does not seem to have this property! Actually, the Ricci scalar is known to transform, in a general space-time of dimension d, as

$$R' = \frac{R}{\Omega^2} - 2(d-1)\,\frac{g^{ab}\Omega_{;ab}}{\Omega^3} - (d-1)(d-4)\,\frac{g^{ab}\Omega_{,a}\Omega_{,b}}{\Omega^4}$$

(the derivation of this result is one of the most "intriguing" procedures in GR, so let us leave it to the reader). It is convenient – the more in 2D – to parameterize the transformation as $\Omega^2 = e^{2\omega}$ which leads to

$$R' = e^{-2\omega}\left[R - 2(d-1)\,g^{ab}\omega_{;ab} - (d-1)(d-2)\,g^{ab}\omega_{,a}\omega_{,b}\right].$$

In 2D, the formula simplifies to just $R' = e^{-2\omega}(R - 2\nabla^2\omega)$. Hence, since $\sqrt{-\gamma}$ in 2D conformally transforms as $\sqrt{-\gamma'} = \sqrt{-\gamma}\,e^{2\omega}$, the whole Lagrangian term $R\sqrt{-\gamma}$ transforms to $R'\sqrt{-\gamma'} = (R - 2\nabla^2\omega)\sqrt{-\gamma}$. There *is* the extra term $2\sqrt{-\gamma}\,\nabla^2\omega$, but that term can be written as a divergence,

$$\sqrt{-\gamma}\,\nabla^2\omega \equiv \sqrt{-\gamma}\,\nabla_A(\gamma^{AB}\nabla_B\omega) = \partial_A\left(\gamma^{AB}\omega_{,B}\right).$$

The extra term thus does not contribute to the dynamics and, for closed-string world-sheets (without boundary), it leaves the action integral conformally invariant. For open-string world-sheets, the above boundary term has to be included.

29.8.1 And cosmological term?

A natural additional footnote is to query the possibility of an analogue of a cosmological term in the action, $\int \lambda\,\sqrt{-\gamma}\,\mathrm{d}\sigma\,\mathrm{d}\tau$. However, we immediately refuse it, because – if λ is a constant (not subject to any transformation) – the term is not conformally invariant, transforming as $\sqrt{-\gamma'} = \sqrt{-\gamma}\,e^{2\omega}$.

29.9 The world-sheet 2+2 splitting

Finally, good to read what B. Carter thinks about what we have tried to outline above:
"... Even in the most recent literature there are still (under Eisenhart's uninspiring influence) many examples of insufficient effort to sort out the messy clutter of indices of different kinds (Greek or Latin, early or late,[5] small or capital) that arise in this way by grouping the various contributions into simple tensorially covariant combinations. Another inconvenient feature of many publications is that results have been left in a form that depends on some particular gauge choice (such as the conformal gauge for internal string coordinates) which obscures the relationship with other results concerning the same system but in a different gauge. The strategy adopted here aims at minimising such problems (they can never be entirely eliminated) by working as far as possible with a single kind of tensor index, which must of course be the one that is most fundamental, namely that of the background coordinates, x^μ."

The quotation is from one of the versions of Carter's arXiv notes on Classical dynamics of strings and branes. One more reflection from there: Of the cases of particles, strings and thin layers, only the string case is non-trivial (in a 4D spacetime) in the strong sense of having a world-sheet with both dimension and codimension greater than one. The following section largely follows Carter's notes.

29.9.1 The first and the second fundamental tensors

The Σ_t hypersurfaces of the 3+1 decomposition were 3D and space-like, whereas the world-sheet is 2D and time-like. If the world-sheet is smooth (if its induced metric is differentiable), it is possible, at its any particular point, to consider an orthonormal tetrad consisting of two tangent and two normal vectors. The background picture is to have *some* global time coordinate t, fix it to some constant and thus obtain the string as the $\{t = \mathrm{const}\}$ section of the

[5] Comment by OS: "early" and "late" does not indicate a historical period (when the given font had been used), but whether the given letter is close to the beginning or to the end of the alphabet.

world-sheet. Parameterize the world-sheet by $\tau := t$ and by σ going along the $\{\tau = \mathrm{const}\}$ string. Let s^μ denote the unit tangent vector to the string (one has two options how to orient it, of course) – in the previous notation, it means the normalized version of $X^\mu_{,\sigma}$. Second, take, at the same point, the unit world-sheet tangent which is orthogonal to s^μ and future-oriented, call it (perhaps too suggestively) u^μ. (It *need not* be proportional to $X^\mu_{,\tau}$! Actually, $X^\mu_{,\tau}$ may *not* be orthogonal to $X^\mu_{,\sigma}$, namely $g_{\mu\nu}X^\mu_{,\tau}X^\mu_{,\sigma} = h_{\tau\sigma}$ may not be zero.) The vector s^μ is supposed to be space-like while the u^μ time-like, so, denoting the whole pair as w^μ_A, with $w^\mu_0 := u^\mu$ and $w^\mu_1 := s^\mu$, we have

$$g_{\mu\nu}w^\mu_A w^\nu_B = \eta_{AB} \equiv \mathrm{diag}(-1,1)\,, \qquad \eta^{AB}w^\mu_A w^\nu_B \equiv -u^\mu u^\nu + s^\mu s^\nu = h^{\mu\nu}\,.$$

Let us denote the two mutually orthogonal unit normals to the world-sheet by n^μ_2 and n^μ_3, together n^μ_M $(M = 2,3)$. They are given, up to a rotation within their plane, by

$$g_{\mu\nu}n^\mu_M n^\nu_N = \delta_{MN} \equiv \mathrm{diag}(1,1)\,, \qquad \delta^{MN}n^\mu_M n^\nu_N = g^{\mu\nu} - h^{\mu\nu}\,, \qquad g_{\mu\nu}w^\mu_A n^\nu_N = 0\,.$$

The induced metric $h_{AB} = g_{\mu\nu}X^\mu_{,A}X^\nu_{,B} = h_{\mu\nu}X^\mu_{,A}X^\nu_{,B}$ (these are same, because $g_{\mu\nu}-h_{\mu\nu}$ is normal whereas $X^\mu_{,A}$ tangent to the world-sheet) used in preceding sections is related to the present picture by $h^{AB}X^\mu_{,A}X^\nu_{,B} = h^{\mu\nu}$ (only that in the case when we use an *orthonormal* tangent dyad w^μ_A, the role of h_{AB} is played by Minkowski, $h_{AB} = \eta_{AB}$). Geometrically, this represents a pullback of the inverse of h_{AB} “back” to space-time. One may check that the combined relation yields what expected,

$$h^{\mu\nu} = h^{AB}X^\mu_{,A}X^\nu_{,B} = h^{AC}h^{BD}h_{CD}X^\mu_{,A}X^\nu_{,B} = h^{AC}h^{BD}g_{\kappa\lambda}X^\kappa_{,C}X^\lambda_{,D}X^\mu_{,A}X^\nu_{,B} = h^{\mu\kappa}h^{\nu\lambda}g_{\kappa\lambda}\,.$$

The induced metric $h^{\mu\nu}$ is called the **first fundamental tensor** of the world-sheet. Together with the projector h^μ_ν (satisfying $h^\mu_\iota h^\iota_\nu = h^\mu_\nu$),[6] one automatically has the complementary, orthogonal projector $\delta^\mu_\nu - h^\mu_\nu$ which obviously satisfies $h^\mu_\iota(\delta^\iota_\nu - h^\iota_\nu) = 0$.

The tensor fields only living on the world-sheet one cannot differentiate in *arbitrary* space-time direction; only meaningful is the tangential derivative

$$\overline{\nabla}^\mu := h^{\mu\nu}\nabla_\nu = h^{AB}X^\mu_{,A}X^\nu_{,B}\nabla_\nu = h^{AB}X^\mu_{,A}\nabla_B\,,$$

where ∇_ν is the standard space-time LC connection and ∇_B is its world-sheet counterpart. For example, for a scalar field φ only defined on the world-sheet, one computes the tangential gradient as $\overline{\nabla}^\mu\varphi = h^{AB}X^\mu_{,A}\varphi_{,B}$.

In 3+1, one defined the second fundamental form (extrinsic curvature) of the hypersurface (with metric $h^{\alpha\beta}$) by tangential gradient of its unit normal n^μ, that is, $K_{\mu\nu} = h^\alpha_\mu h^\beta_\nu n_{\alpha;\beta} = h^\beta_\nu n_{\mu;\beta}$. For a 2D surface, it suggests to define a corresponding tensor by tangential gradient of its orthogonal-complement metric $\delta^\mu_\nu - h^\mu_\nu$. As the world-sheet is time-like, let us do it with a minus sign,

$$K_{\mu\nu}{}^\alpha := h^\iota_\mu h^\kappa_\nu \nabla_\kappa(h^\alpha_\iota - \delta^\alpha_\iota) = h^\iota_\mu h^\kappa_\nu \nabla_\kappa h^\alpha_\iota \equiv h^\iota_\mu \overline{\nabla}_\nu h^\alpha_\iota\,. \tag{29.16}$$

[6] When playing with the metrics with indices here and there, it is good to remember that in the mixed version (as a projector), the d-dimensional metric is only the $d\times d$ unit matrix *in the adapted, “intrinsic”, “d-dimensional” coordinates*. For example, the 2D metric h satisfies $h^A_B = \delta^A_B$, but the latter does *not* hold in the “extrinsic”, 4D “bulk” coordinates, $h^\mu_\alpha \neq \delta^\mu_\alpha$.

This is the **second fundamental tensor** of the world-sheet. It shares, with the 3+1 extrinsic curvature, the Weingarten-Frobenius property of being symmetric. Indeed, one expresses

$$\begin{aligned}K_{\mu\nu}{}^{\alpha} &= h_{\mu\iota}h_{\nu\kappa}h^{\kappa\lambda}\nabla_\lambda h^{\iota\alpha} = h_{\mu\iota}h_{\nu\kappa}\eta^{AB}w^\kappa_A w^\lambda_B\nabla_\lambda h^{\iota\alpha} = h_{\mu\iota}h_{\nu\kappa}\eta^{AB}w^\kappa_A\nabla_B(\eta^{CD}w^\iota_C w^\alpha_D) = \\ &= 2h_{\mu\iota}h_{\nu\kappa}\eta^{AB}\eta^{CD}w^\kappa_A w^\iota_C\nabla_B w^\alpha_D = 2w^B_\nu w^D_\mu\nabla_B w^\alpha_D\,,\end{aligned}$$

so the behaviour in the indices $[\mu,\nu]$ translates through $[\iota,\kappa]$ and $[A,C]$ to the behaviour in $[B,D]$. Anti-symmetrization in the latter yields

$$w^B_\nu w^D_\mu(\nabla_B w^\alpha_D - \nabla_D w^\alpha_B) = w^B_\nu w^D_\mu(w^\alpha_{D,B} - w^\alpha_{B,D}) = (w^0_\mu w^1_\nu - w^1_\mu w^0_\nu)(w^\alpha_{0,1} - w^\alpha_{1,0})\,.$$

However, the vectors w^α_0 and w^α_1 have to be surface-forming, because they are the generators of the world-sheet. In our parameterization they even commute, because $w^\alpha_A \sim X^\alpha_{,A}$ and

$$\left(X^\alpha_{,0}\right)_{,\beta}X^\beta_{,1} - \left(X^\alpha_{,1}\right)_{,\beta}X^\beta_{,0} = X^\alpha_{,01} - X^\alpha_{,10} = 0\,.$$

Below we mention some other properties of $K_{\mu\nu}{}^{\alpha}$:

- The second fundamental tensor is "mixed" in the sense that it is tangential in the first two indices while orthogonal in the last one,

$$\begin{aligned}&(\delta^\mu_\lambda - h^\mu_\lambda)K_{\mu\nu}{}^{\alpha} = (\delta^\mu_\lambda - h^\mu_\lambda)h^\iota_\mu h^\kappa_\nu\nabla_\kappa h^\alpha_\iota = 0\,,\\ &K_{\mu\nu}{}^{\alpha}h^\beta_\alpha = h^\iota_\mu h^\kappa_\nu h^\beta_\alpha\nabla_\kappa h^\alpha_\iota = h^\iota_\mu h^\kappa_\nu\left[\nabla_\kappa(h^\beta_\alpha h^\alpha_\iota) - h^\alpha_\iota\nabla_\kappa h^\beta_\alpha\right] = K_{\mu\nu}{}^{\beta} - K_{\mu\nu}{}^{\beta} = 0\,.\end{aligned}$$

- $K_{\mu\nu}{}^{\alpha}$ fully determines the tangential derivative of $h_{\mu\nu}$. Indeed, the definition (29.16) says

$$K_{\mu\nu}{}^{\alpha} \equiv h^\iota_\mu\overline{\nabla}_\nu h^\alpha_\iota = \overline{\nabla}_\nu(h^\iota_\mu h^\alpha_\iota) - h^\alpha_\iota\overline{\nabla}_\nu h^\iota_\mu = \overline{\nabla}_\nu h^\alpha_\mu - h^{\alpha\iota}\overline{\nabla}_\nu h_{\mu\iota} = \overline{\nabla}_\nu h^\alpha_\mu - K^\alpha{}_{\nu\mu}\,,$$

  which yields, if multiplied by $g_{\rho\alpha}$,

$$\overline{\nabla}_\nu h_{\mu\rho} = K_{\mu\nu\rho} + K_{\rho\nu\mu} = K_{\nu\mu\rho} + K_{\nu\rho\mu} \equiv 2K_{\nu(\mu\rho)}\,.$$

- Have some vector tangent to the world-sheet, $v^\mu = h^\mu_\nu v^\nu$. The normal component of its "acceleration" $\dot{v}^\mu := v^\nu\nabla_\nu v^\mu$ satisfies

$$\begin{aligned}(\delta^\rho_\mu - h^\rho_\mu)\,\dot{v}^\mu &= (\delta^\rho_\mu - h^\rho_\mu)\,v^\nu\nabla_\nu v^\mu = v^\nu\nabla_\nu v^\rho - v^\nu h^\rho_\mu\nabla_\nu v^\mu = \\ &= \cancel{\dot{v}^\rho} - \cancel{v^\nu\nabla_\nu(h^\rho_\mu v^\mu)} + v^\nu v^\mu\nabla_\nu h^\rho_\mu = h^\nu_\beta v^\beta h^\mu_\alpha v^\alpha\nabla_\nu h^\rho_\mu = v^\alpha v^\beta K_{\alpha\beta}{}^{\rho}\,.\end{aligned}$$

- Let us try to express $K_{\mu\nu}{}^{\alpha}$ totally explicitely. From its first definition (29.16), we have

$$\begin{aligned}K_{\mu\nu}{}^{\alpha} &\equiv h^\iota_\mu h^\kappa_\nu\nabla_\kappa(h^\alpha_\iota - \delta^\alpha_\iota) = (\delta^\alpha_\iota - h^\alpha_\iota)h^\kappa_\nu\nabla_\kappa h^\iota_\mu = \\ &= (\delta^\alpha_\iota - h^\alpha_\iota)h^\kappa_\nu\left(h^\iota_{\mu,\kappa} + \Gamma^\iota{}_{\kappa\lambda}h^\lambda_\mu - \Gamma^\lambda{}_{\kappa\mu}h^\iota_\lambda\right) = (\delta^\alpha_\iota - h^\alpha_\iota)h^\kappa_\nu\left(h^\iota_{\mu,\kappa} + \Gamma^\iota{}_{\kappa\lambda}h^\lambda_\mu\right).\end{aligned}$$

  In the first part, one writes

$$h^\kappa_\nu h^\iota_{\mu,\kappa} = h_{\nu\sigma}h^{\sigma\kappa}(h_{\mu\rho}h^{\rho\iota})_{,\kappa} = h_{\nu\sigma}h^{\sigma\kappa}\left(h_{\mu\rho,\kappa}h^{\rho\iota} + h_{\mu\rho}h^{\rho\iota}{}_{,\kappa}\right),$$

of which the 1st term does not contribute since $(\delta^\alpha_\iota - h^\alpha_\iota)h^{\rho\iota} = 0$, and the 2nd term yields

$$\begin{aligned} & h_{\mu\rho} h_{\nu\sigma} h^{\sigma\kappa} h^{\rho\iota}{}_{,\kappa} = h_{\mu\rho} h_{\nu\sigma} h^{BD} X^\sigma_{,B} X^\kappa_{,D} \left(h^{AC} X^\rho_{,A} X^\iota_{,C} \right)_{,\kappa} = \\ & = h_{\mu\rho} h_{\nu\sigma} h^{BD} X^\sigma_{,B} \left(h^{AC} X^\rho_{,A} X^\iota_{,C} \right)_{,D} = \\ & = h_{\mu\rho} h_{\nu\sigma} h^{BD} X^\sigma_{,B} \left(h^{AC} X^\rho_{,A} \right)_{,D} X^\iota_{,C} + h_{\mu\rho} h_{\nu\sigma} h^{BD} X^\sigma_{,B} h^{AC} X^\rho_{,A} X^\iota_{,CD} \, . \end{aligned}$$

After multiplication by the orthogonal projector $(\delta^\alpha_\iota - h^\alpha_\iota)$, the first term does not contribute, because $X^\iota_{,C}$ is *tangent* to the world-sheet, i.e. $(\delta^\alpha_\iota - h^\alpha_\iota) X^\iota_{,C} = 0$. Hence, we arrive at

$$K_{\mu\nu}{}^\alpha = (\delta^\alpha_\iota - h^\alpha_\iota) \left(h_{\mu\rho} h_{\nu\sigma} h^{AC} h^{BD} X^\rho_{,A} X^\sigma_{,B} X^\iota_{,CD} + h^\kappa_\nu h^\lambda_\mu \Gamma^\iota{}_{\kappa\lambda} \right) . \tag{29.17}$$

Since $(\delta^\alpha_\iota - h^\alpha_\iota) X^\iota_{,C} = 0$, the derivative by "$D$" can alternatively be extended to the whole first term $h_{\mu\rho} h_{\nu\sigma} h^{AC} h^{BD} X^\rho_{,A} X^\sigma_{,B} X^\iota_{,C} = h_{\mu\rho} h_{\nu\sigma} h^{\rho\iota} h^{BD} X^\sigma_{,B} = h^\iota_\mu h_{\nu\sigma} h^{BD} X^\sigma_{,B}$,

$$K_{\mu\nu}{}^\alpha = (\delta^\alpha_\iota - h^\alpha_\iota) \left[\left(h^\iota_\mu h_{\nu\sigma} h^{BD} X^\sigma_{,B} \right)_{,D} + h^\kappa_\nu h^\lambda_\mu \Gamma^\iota{}_{\kappa\lambda} \right] . \tag{29.18}$$

The result is manifestly orthogonal to the world-sheet in the upper index while tangent in the bottom ones, and its "Weingarten property" is also clear: in the first term of (29.17), note (from the end) the symmetry in (CD), implying the symmetry in (AB), which in turn implies symmetry in $(\rho\sigma)$ and thus in $(\mu\nu)$; symmetry of the second term is even more obvious.

29.9.2 The mean-curvature vector

The tangential covariant derivative which appears in the definition of $K_{\mu\nu}{}^\alpha$ was actually employed in (29.5), (29.6) and (29.13) already. Here we show more how things go together.

Since $K_{\mu\nu}{}^\alpha$ is tangential in the first two indices and transversal in the last one, its only non-trivial contraction is that over the first two indices,

$$K^\alpha := g^{\mu\nu} K_{\mu\nu}{}^\alpha = h^{\mu\nu} K_{\mu\nu}{}^\alpha \equiv h^{\mu\nu} h^\iota_\mu \overline{\nabla}_\nu h^\alpha_\iota \equiv h^{\mu\nu} h^\iota_\mu h^\kappa_\nu \nabla_\kappa h^\alpha_\iota \equiv h^{\iota\kappa} \nabla_\kappa h^\alpha_\iota \equiv \overline{\nabla}_\iota h^{\iota\alpha} \, .$$

This vector, clearly orthogonal to the world-sheet, is called the world-sheet **mean-curvature vector** (or **extrinsic-curvature vector**). For its explicit form, one may just contract (29.18): since $(\delta^\alpha_\iota - h^\alpha_\iota) h^\iota_\mu = 0$, one may enter the parenthesis with $g^{\mu\nu}$ and directly perform the contraction there, $g^{\mu\nu} h^\iota_\mu h_{\nu\sigma} h^{BD} X^\sigma_{,B} = h^\iota_\sigma h^{BD} X^\sigma_{,B} = h^{BD} X^\iota_{,B}$, hence

$$K^\alpha = (\delta^\alpha_\iota - h^\alpha_\iota) \left[\left(h^{BD} X^\iota_{,B} \right)_{,D} + h^{\kappa\lambda} \Gamma^\iota{}_{\kappa\lambda} \right] . \tag{29.19}$$

$\boxed{\text{Lemma:}}$ The mean-curvature vector can also be expressed as

$$K^\alpha = \frac{1}{\sqrt{h}} \left(\sqrt{h}\, h^{BD} X^\alpha_{,B} \right)_{,D} + h^{\kappa\lambda} \Gamma^\alpha{}_{\kappa\lambda} \equiv \Box X^\alpha + h^{\kappa\lambda} \Gamma^\alpha{}_{\kappa\lambda} \, . \tag{29.20}$$

<u>Proof</u>: Were the above expression normal to the world-sheet (as it should be), it would make no difference to multiply it by the orthogonal projector $(\delta^\alpha_\iota - h^\alpha_\iota)$. This would yield

$$(\delta^\alpha_\iota - h^\alpha_\iota)\, K^\iota = \frac{(\sqrt{h})_{,D}}{\sqrt{h}}\, h^{BD} \cancel{X^\iota_{,B} (\delta^\alpha_\iota - h^\alpha_\iota)} + (\delta^\alpha_\iota - h^\alpha_\iota) \left[\left(h^{BD} X^\iota_{,B} \right)_{,D} + h^{\kappa\lambda} \Gamma^\iota{}_{\kappa\lambda} \right] ,$$

which is exactly the desired result. Therefore, it is sufficient to show that (29.20) *is* really normal to any world-sheet tangent $X^{\gamma}_{,C}$.

Let us first compute $\frac{(\sqrt{h})_{,D}}{\sqrt{h}}\, h^{BD} X^{\alpha}_{,B}$:

$$(\sqrt{h})_{,D} = \frac{\partial\sqrt{h}}{\partial h_{AC}}\, h_{AC,D} = \frac{1}{2}\sqrt{h}\, h^{AC} h_{AC,D}$$

with

$$h_{AC,D} = \left(g_{\kappa\lambda} X^{\kappa}_{,A} X^{\lambda}_{,C}\right)_{,D} = g_{\kappa\lambda,\delta} X^{\kappa}_{,A} X^{\lambda}_{,C} X^{\delta}_{,D} + 2 g_{\kappa\lambda} X^{\kappa}_{,A} X^{\lambda}_{,CD}\ ,$$

hence

$$\begin{aligned}\frac{(\sqrt{h})_{,D}}{\sqrt{h}}\, h^{BD} X^{\alpha}_{,B} &= \frac{1}{2}\, h^{AC} h^{BD} X^{\alpha}_{,B}\left(g_{\kappa\lambda,\delta} X^{\kappa}_{,A} X^{\lambda}_{,C} X^{\delta}_{,D} + 2 g_{\kappa\lambda} X^{\kappa}_{,A} X^{\lambda}_{,CD}\right) = \\ &= \frac{1}{2}\, g_{\kappa\lambda,\delta} h^{\kappa\lambda} h^{\alpha\delta} + g_{\kappa\lambda} h^{AC} h^{BD} X^{\kappa}_{,A} X^{\alpha}_{,B} X^{\lambda}_{,CD}\ . \qquad (29.21)\end{aligned}$$

Using the latter, let us multiply (29.20) by $g_{\alpha\gamma} X^{\gamma}_{,C}$:

$$\begin{aligned}&\frac{1}{\sqrt{h}}\left(\sqrt{h}\, h^{BD} X^{\alpha}_{,B}\right)_{,D} g_{\alpha\gamma} X^{\gamma}_{,C} + h^{\kappa\lambda}\Gamma^{\alpha}{}_{\kappa\lambda} g_{\alpha\gamma} X^{\gamma}_{,C} = \\ &= \frac{1}{2}\, g_{\kappa\lambda,\delta} h^{\kappa\lambda} h^{\alpha\delta} g_{\alpha\gamma} X^{\gamma}_{,C} + g_{\kappa\lambda} h^{AC} h^{BD} X^{\kappa}_{,A} X^{\alpha}_{,B} X^{\lambda}_{,CD} g_{\alpha\gamma} X^{\gamma}_{,C} + \\ &\quad + \left(h^{BD} X^{\alpha}_{,B}\right)_{,D} g_{\alpha\gamma} X^{\gamma}_{,C} + h^{\kappa\lambda}\Gamma_{\gamma\kappa\lambda} X^{\gamma}_{,C} = \\ &= \frac{1}{2}\, g_{\kappa\lambda,\delta} h^{\kappa\lambda} X^{\delta}_{,C} + h^{\kappa\lambda}\Gamma_{\gamma\kappa\lambda} X^{\gamma}_{,C} + g_{\kappa\lambda} h^{AC} h^{BD} h_{BC} X^{\kappa}_{,A} X^{\lambda}_{,CD} + \\ &\quad + \left(h^{BD} X^{\alpha}_{,B} g_{\alpha\gamma} X^{\gamma}_{,C}\right)_{,D} - h^{BD} X^{\alpha}_{,B}\left(g_{\alpha\gamma} X^{\gamma}_{,C}\right)_{,D} = \\ &= \frac{1}{2}\left(\cancel{g_{\kappa\lambda,\gamma}} + g_{\gamma\kappa,\lambda} + g_{\lambda\gamma,\kappa} - \cancel{g_{\kappa\lambda,\gamma}}\right) h^{\kappa\lambda} X^{\gamma}_{,C} + g_{\kappa\lambda} h^{AD} X^{\kappa}_{,A} X^{\lambda}_{,CD} + \\ &\quad + (h^{BD} h_{BC})_{,D} - h^{BD} X^{\alpha}_{,B}\left(g_{\alpha\gamma,\delta} X^{\delta}_{,D} X^{\gamma}_{,C} + g_{\alpha\gamma} X^{\gamma}_{,CD}\right) = \\ &= \cancel{g_{\gamma\kappa,\lambda} h^{\kappa\lambda} X^{\gamma}_{,C}} + \cancel{g_{\kappa\lambda} h^{AD} X^{\kappa}_{,A} X^{\lambda}_{,CD}} + \cancel{(h^{D}_{C} = \delta^{D}_{C})_{,D}} - \cancel{h^{\alpha\delta} g_{\alpha\gamma,\delta} X^{\gamma}_{,C}} - \cancel{h^{BD} X^{\alpha}_{,B} g_{\alpha\gamma} X^{\gamma}_{,CD}}\ .\end{aligned}$$

That's it.

$\boxed{\text{Corollary:}}$ Comparing (29.20) with (29.6) or (29.13), we see that the string equations of motion can be expressed in an elegant geometrical way [sic!]: as the vanishing of the mean-curvature vector, $\boxed{K^{\alpha} = 0}$.

29.9.3 Gauss-Codazzi equations

Having described the world-sheet extrinsic curvature, one asks whether Gauss-Codazzi–type equations exist, similarly as in the case of the 3+1 space-time splitting. Let us check it.

Exactly as in Section 25.4, we compute the second tangent covariant derivative of an arbitrary (co-)tangent of the world-sheet (V_μ),

$$\left(V_{\mu;\gamma}h^\mu_\alpha h^\gamma_\rho\right)_{;\sigma} h^\alpha_\nu h^\rho_\kappa h^\sigma_\lambda = V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_{\alpha;\sigma}h^\gamma_\kappa h^\alpha_\nu h^\sigma_\lambda + V_{\mu;\gamma}h^\mu_\nu h^\gamma_{\rho;\sigma}h^\rho_\kappa h^\sigma_\lambda =$$
$$= V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_{\mu;\gamma}K_{\nu\lambda}{}^\mu h^\gamma_\kappa + V_{\mu;\gamma}h^\mu_\nu K_{\kappa\lambda}{}^\gamma\,,$$

where now, naturally, h^μ_ν stands for the 2D world-sheet projector. The aim is to obtain the world-sheet Riemann tensor by commutator of the above expression in $[\kappa\lambda]$. The last term is *symmetric* in $(\kappa\lambda)$, so it will not contribute. In the middle term, we may use the trivial fact $V_\mu = h^\alpha_\mu V_\alpha$ and write

$$V_{\mu;\gamma}K_{\nu\lambda}{}^\mu = (h^\alpha_\mu V_\alpha)_{;\gamma}K_{\nu\lambda}{}^\mu = h^\alpha_{\mu;\gamma}V_\alpha K_{\nu\lambda}{}^\mu + h^\alpha_\mu V_{\alpha;\gamma}K_{\nu\lambda}{}^\mu\,,$$

where however the second term vanishes because $h^\alpha_\mu K_{\nu\lambda}{}^\mu = 0$. So we have

$$V_{\mu;\gamma}K_{\nu\lambda}{}^\mu h^\gamma_\kappa = h^\alpha_{\mu;\gamma}V_\alpha K_{\nu\lambda}{}^\mu h^\gamma_\kappa = h^\alpha_{\mu;\gamma}h^\iota_\alpha V_\iota K_{\nu\lambda}{}^\mu h^\gamma_\kappa = V_\iota K^\iota{}_{\kappa\mu}K_{\nu\lambda}{}^\mu\,.$$

Finally,

$$2V_{\mu;\gamma\sigma}h^\mu_\nu h^\gamma_{[\kappa}h^\sigma_{\lambda]} = 2V_{\mu;[\gamma\sigma]}h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda = R^\beta{}_{\mu\gamma\sigma}V_\beta h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda = R^\beta{}_{\mu\gamma\sigma}h^\iota_\beta V_\iota h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda\,.$$

The commutator in $[\kappa\lambda]$ thus reads

$$2\left(V_{\mu;\gamma}h^\mu_\alpha h^\gamma_\rho\right)_{;\sigma}h^\alpha_\nu h^\rho_{[\kappa}h^\sigma_{\lambda]} \equiv {}^{(2)}R^\iota{}_{\nu\kappa\lambda}V_\iota = R^\beta{}_{\mu\gamma\sigma}h^\iota_\beta V_\iota h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda + V_\iota K^\iota{}_{\kappa\mu}K_{\nu\lambda}{}^\mu - V_\iota K^\iota{}_{\lambda\mu}K_{\nu\kappa}{}^\mu.$$

Since V_ι has been arbitrary, we arrive at the **Gauss equation**

$$\boxed{R^\beta{}_{\mu\gamma\sigma}h^\iota_\beta h^\mu_\nu h^\gamma_\kappa h^\sigma_\lambda = {}^{(2)}R^\iota{}_{\nu\kappa\lambda} + K^\iota{}_{\kappa\mu}K_{\nu\lambda}{}^\mu - K^\iota{}_{\lambda\mu}K_{\nu\kappa}{}^\mu} \qquad (29.22)$$

(remember in passing that the 2D Riemann has but one independent component).

The Codazzi equation we derived, in the 3+1 decomposition, from the Ricci identity for the normal n^μ of the 3D hypersurface, $R^\alpha{}_{\beta\gamma\delta}n_\alpha = n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma}$. Here we have not introduced the world-sheet normal, rather working in terms of the orthogonal projector $(\delta^\alpha_\beta - h^\alpha_\beta)$, so let us start from the Ricci identity for the latter,

$$(\delta^\alpha_\beta - h^\alpha_\beta)_{;\gamma\delta} - (\delta^\alpha_\beta - h^\alpha_\beta)_{;\delta\gamma} = R_\iota{}^\alpha{}_{\gamma\delta}(\delta^\iota_\beta - h^\iota_\beta) + R^\iota{}_{\beta\gamma\delta}(\delta^\alpha_\iota - h^\alpha_\iota)\,.$$

On the l.h. side, the covariant derivative of the metric (δ^α_β) is zero, and on the r.h. side,

$$R^\iota{}_{\beta\gamma\delta}\delta^\alpha_\iota + R_\iota{}^\alpha{}_{\gamma\delta}\delta^\iota_\beta = R^\alpha{}_{\beta\gamma\delta} + R_\beta{}^\alpha{}_{\gamma\delta} = 0\,,$$

so the identity – expectably – applies to the projector h^α_β itself,

$$h^\alpha_{\beta;\gamma\delta} - h^\alpha_{\beta;\delta\gamma} = R^\iota{}_{\beta\gamma\delta}h^\alpha_\iota - R^\alpha{}_{\iota\gamma\delta}h^\iota_\beta\,.$$

The left-hand side likely can be related to a suitable subtraction of derivatives of the extrinsic curvature. To this aim, consider a tensor called the **third fundamental tensor**[7]

$$O_{\mu\nu\iota}{}^\alpha := h^\rho_\mu h^\sigma_\nu(\delta^\alpha_\beta - h^\alpha_\beta)\overline{\nabla}_\iota K_{\rho\sigma}{}^\beta \equiv h^\rho_\mu h^\sigma_\nu(\delta^\alpha_\beta - h^\alpha_\beta)h^\kappa_\iota\nabla_\kappa K_{\rho\sigma}{}^\beta\,. \qquad (29.23)$$

[7] Carter uses the notation $\Xi_{\mu\nu\iota}{}^\alpha$, which however seems uncomfortable. Sometimes this tensor is being denoted by C, but we use C for the Weyl tensor.

This tensor inherits the symmetry in (μ,ν) from $K_{\mu\nu}{}^{\alpha}$. Substituting $K_{\rho\sigma}{}^{\beta}=h^{\gamma}_{\rho}h^{\delta}_{\sigma}h^{\beta}_{\gamma;\delta}$, one has

$$\begin{aligned}O_{\mu\nu\iota}{}^{\alpha} &= h^{\rho}_{\mu}h^{\sigma}_{\nu}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\kappa}_{\iota}\big(h^{\gamma}_{\rho}h^{\delta}_{\sigma}h^{\beta}_{\gamma;\delta}\big)_{;\kappa} = \\ &= h^{\rho}_{\mu}h^{\sigma}_{\nu}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\kappa}_{\iota}\big(h^{\gamma}_{\rho;\kappa}h^{\delta}_{\sigma}h^{\beta}_{\gamma;\delta}+h^{\gamma}_{\rho}h^{\delta}_{\sigma;\kappa}h^{\beta}_{\gamma;\delta}+h^{\gamma}_{\rho}h^{\delta}_{\sigma}h^{\beta}_{\gamma;\delta\kappa}\big) = \\ &= \big(K_{\mu\iota}{}^{\gamma}h^{\delta}_{\nu}+K_{\nu\iota}{}^{\delta}h^{\gamma}_{\mu}\big)(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma;\delta}+h^{\gamma}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma;\delta\kappa}\,.\end{aligned}$$

Let us evaluate the commutator $O_{\mu\nu\iota}{}^{\alpha}-O_{\mu\iota\nu}{}^{\alpha}$. The last term contributes by

$$\begin{aligned}2h^{\gamma}_{\mu}h^{\delta}_{[\nu}h^{\kappa}_{\iota]}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma;\delta\kappa} &= h^{\gamma}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})\big(h^{\beta}_{\gamma;\delta\kappa}-h^{\beta}_{\gamma;\kappa\delta}\big) = \\ &= h^{\gamma}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})\big(R_{\lambda}{}^{\beta}{}_{\delta\kappa}h^{\lambda}_{\gamma}+R^{\lambda}{}_{\gamma\delta\kappa}h^{\beta}_{\lambda}\big) = \\ &= h^{\gamma}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})R_{\lambda}{}^{\beta}{}_{\delta\kappa}h^{\lambda}_{\gamma} = -h^{\gamma}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})R^{\beta}{}_{\lambda\delta\kappa}h^{\lambda}_{\gamma} = \\ &= -h^{\lambda}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})R^{\beta}{}_{\lambda\delta\kappa}\,.\end{aligned}$$

The term $K_{\nu\iota}{}^{\delta}h^{\gamma}_{\mu}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma;\delta}$ does not contribute at all since it is *symmetric* in (ν,ι). And in the first term, since $(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma}=0$, we can swap the derivative from the latter to the former (with minus),

$$K_{\mu\iota}{}^{\gamma}h^{\delta}_{\nu}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})h^{\beta}_{\gamma;\delta} = -K_{\mu\iota}{}^{\gamma}h^{\delta}_{\nu}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})_{;\delta}h^{\beta}_{\gamma} = 0$$

(because $K_{\mu\iota}{}^{\gamma}h^{\beta}_{\gamma}=0$). Hence, we arrive at the **Codazzi equation**

$$\boxed{O_{\mu\nu\iota}{}^{\alpha}-O_{\mu\iota\nu}{}^{\alpha} = -h^{\lambda}_{\mu}h^{\delta}_{\nu}h^{\kappa}_{\iota}(\delta^{\alpha}_{\beta}-h^{\alpha}_{\beta})R^{\beta}{}_{\lambda\delta\kappa}}\,. \tag{29.24}$$

—

The string problem definitely suggests itself for further research. One might compare the 2+2 approach with treating the world-sheet as a 2D time-like congruence. Or one might study the string configurations "at given time", as a sequence of space-like curves, using (e.g.) the Frenet-Serret formalism, if not to even consider the internal physics of the string (elasticity), making perturbations and studying transversal and longitudinal vibrations... These directions however are out of the scope of the present course.

CHAPTER 30

Algebraic classification of tensor fields

If acting, by a tensor of rank r, on $r-1$ vectors and/or covectors, it yields a vector or a covector. Hence, besides mapping from tangent and cotangent spaces to numbers, tensors can also be understood as mappings "back" to either tangent or cotangent spaces. In such an action, a tensor may "favour" certain (co)vectors in the sense that it does not rotate them – it only (at most) changes their length. Such (co)vectors are called eigen-(co)vectors of a given tensor. Therefore, besides various coordinate-independent properties we already know (e.g. symmetries or congruence properties), a vector field may also be privileged in that it is everywhere an eigen-vector of some important tensor field. In the case of gravitation, mainly important is the Riemann (or Weyl) tensor. We will analyse the gravitational eigen-problem after "warming up" on the EM-field tensor. First, however, we shall recall several generic properties of second-rank tensors.

30.1 Eigen-problem for second-rank tensors

Every second-rank tensor can be decomposed into the antisymmetric and symmetric parts, and the symmetric part can further be decomposed in a trace-free part and a pure-trace part proportional to the metric tensor,

$$T_{\mu\nu} = T_{[\mu\nu]} + T_{(\mu\nu)} = T_{[\mu\nu]} + \left\{ T_{(\mu\nu)} - \frac{1}{4}\,T g_{\mu\nu} \right\} + \frac{1}{4}\,T g_{\mu\nu}\,; \qquad T := T^{\iota}{}_{\iota}\,. \tag{30.1}$$

We will thus specifically focus on algebraic properties of tensors of the above types.

The **eigen-value equation**

$$T^{\mu}{}_{\nu} V^{\nu} = \lambda V^{\mu} \qquad \Longleftrightarrow \qquad \left(T^{\mu}{}_{\nu} - \lambda \delta^{\mu}_{\nu}\right) V^{\nu} = 0 \tag{30.2}$$

is, for both symmetric and skew-symmetric tensors, independent of which index pair is being summed over, modulo sign change in the skew-symmetric case.[1] It is a system of four

[1] Multiplication of the equation by V^{α} yields a symmetric tensor on the right-hand side, so the equation can also be written as $V^{[\alpha}T^{\mu]}{}_{\nu}V^{\nu} = 0$.

linear algebraic equations for V^ν. The system has a solution iff the **characteristic (secular) equation** is satisfied

$$\det(T^\mu{}_\nu - \lambda\delta^\mu_\nu) = 0 \qquad \Longleftrightarrow \qquad \lambda^4 + I_{(3)}\lambda^3 + I_{(2)}\lambda^2 + I_{(1)}\lambda + I_{(0)} = 0\,, \tag{30.3}$$

the coefficients of which $I_{(\iota)}$ are determined by algebraic operations from the components of $T^\mu{}_\nu$ (and of the metric tensor). Equation (30.2) being a tensor (here vector) equation, the eigen-values λ have to be invariants. Should they, at the same time, come out as the roots of the characteristic equation, the latter's coefficients $I_{(\iota)}$ have to be invariant as well. Indeed, submitting the expressions

$$\begin{aligned}
M \ &= T^\mu{}_\nu - \lambda\delta^\mu_\nu\,,\\
M^2 &= (T^\mu{}_\iota - \lambda\delta^\mu_\iota)(T^\iota{}_\nu - \lambda\delta^\iota_\nu) = T^\mu{}_\iota T^\iota{}_\nu - 2\lambda T^\mu{}_\nu + \lambda^2\delta^\mu_\nu\,,\\
M^3 &= \cdots = T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\nu - 3\lambda T^\mu{}_\iota T^\iota{}_\nu + 3\lambda^2 T^\mu{}_\nu - \lambda^3\delta^\mu_\nu\,,\\
M^4 &= \cdots = T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\lambda T^\lambda{}_\nu - 4\lambda T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\nu + 6\lambda^2 T^\mu{}_\iota T^\iota{}_\nu - 4\lambda^3 T^\mu{}_\nu + \lambda^4\delta^\mu_\nu\,,
\end{aligned}$$

to the formula (A.10) for calculation of the determinant from traces of the matrix products, therefore

$$\begin{aligned}
\mathrm{Tr}\,M \ &= T^\mu{}_\mu - 4\lambda \ \ \equiv \ T - 4\lambda\,,\\
\mathrm{Tr}\,M^2 &= T^\mu{}_\iota T^\iota{}_\mu - 2\lambda T + 4\lambda^2,\\
\mathrm{Tr}\,M^3 &= T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\mu - 3\lambda T^\mu{}_\iota T^\iota{}_\mu + 3\lambda^2 T - 4\lambda^3,\\
\mathrm{Tr}\,M^4 &= T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\lambda T^\lambda{}_\mu - 4\lambda T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\mu + 6\lambda^2 T^\mu{}_\iota T^\iota{}_\mu - 4\lambda^3 T + 4\lambda^4,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\det(T^\mu{}_\nu - \lambda\delta^\mu_\nu) =\\
&\lambda^4 - T\lambda^3 + \frac{1}{2}(T^2 - T^\mu{}_\iota T^\iota{}_\mu)\lambda^2 - \frac{1}{6}(T^3 - 3T\,T^\mu{}_\iota T^\iota{}_\mu + 2T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\mu)\lambda + \det(T^\mu{}_\nu)\,.
\end{aligned} \tag{30.4}$$

This so-called **characteristic polynomial** always has 4 roots – more generally, a $d\times d$ matrix always has d eigen-values, of which some (or even all possibly) may be multiple, "degenerate". The number of eigen-vectors may range from 0 to d. The values of λ, their multiplicity and the number of independent eigen-vectors V^ν provide the starting point for identifying several different algebraic types of tensors.

Let us recall main knowledge on eigen-problem for a square matrix M:

- $\sum\limits_i \lambda_i = \mathrm{Tr}(M)\ldots$ sum of the eigen-values equals trace of the matrix
- $\prod\limits_i \lambda_i = \det(M)\ldots$ product of the eigen-values equals determinant of the matrix
- eigen-vectors corresponding to different eigen-values are linearly independent
- the dimension of the space of eigen-vectors corresponding to a one particular eigen-value (i.e. the *geometrical multiplicity* of the eigen-value) is less or equal to the multiplicity of that eigen-value as the root of the characteristic equation (i.e. its *algebraic multiplicity*)

- eigen-values and eigen-vectors of a real *symmetric* $d\times d$ matrix are real, there exactly exist d of them, and the eigen-vectors corresponding to different eigen-values are orthogonal; in a Euclidean (positive semi-definite) metric, every such matrix can be diagonalized thanks to that, because $M = V\cdot\Lambda\cdot V^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_i)$ and V is the matrix whose columns are independent eigen-vectors of M; unfortunately, the same is not true in a Lorentzian (indefinite) metric
- eigen-values and eigen-vectors of a real *antisymmetric* $d\times d$ matrix may be both real and imaginary, with eigen-values grouped in pairs having the same magnitude but opposite sign; there may exist from 0 to d independent eigen-vectors
- a real *symmetric* matrix may have any rank (from 0 to d), whereas a real *anti-symmetric* matrix can only have even rank (Jacobi's theorem), because, for M anti-symmetric $d\times d$,

  $$\det(M) = \det(M^{\mathrm{T}}) = \det(-M) = \det(-I\cdot M) = \det(-I)\,\det(M) = (-1)^d\det(M)\,,$$

  so for odd d the determinant has to vanish, i.e. the matrix' rank has to be lower; in the $d=4$ case it means that non-trivial bivectors can only have rank 4 or 2.

30.1.1 Metric tensor

Trivial example first: metric tensor has *every* vector as its eigen-vector – the equation

$$g_{\mu\nu}V^\nu = \lambda V_\mu$$

is always, "trivially" satisfied for $\lambda = 1$. Hence, the metric tensor does not determine any privileged direction. This is why tensors proportional to the metric are called *isotropic*. For example, isotropic is the cosmological term $\Lambda g_{\mu\nu}$, and thus the Ricci tensor of the vacuum yet $\Lambda\neq 0$ space-times, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. The pressure term in the ideal-fluid energy-momentum tensor, $Ph_{\mu\nu}$, is isotropic in the 3D space locally orthogonal to the fluid's flow (described by the metric $h_{\mu\nu}$).

30.1.2 Antisymmetric tensor (bivector)

An antisymmetric tensor (bivector) we denote suggestively by $F_{\mu\nu}$ and its Hodge dual by

$${}^*F^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

as usual. The properties of the duality mapping are given by the relations (A.9) for the products of Levi-Civita pseudo-tensors, specifically by the relations

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\lambda}\epsilon_{\mu\nu\kappa\lambda} &= -\delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\kappa - \delta^\gamma_\mu\delta^\alpha_\nu\delta^\beta_\kappa - \delta^\beta_\mu\delta^\gamma_\nu\delta^\alpha_\kappa + \delta^\alpha_\mu\delta^\gamma_\nu\delta^\beta_\kappa + \delta^\beta_\mu\delta^\alpha_\nu\delta^\gamma_\kappa + \delta^\gamma_\mu\delta^\beta_\nu\delta^\alpha_\kappa\,,\\ \epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda} &= -2\delta^\alpha_\mu\delta^\beta_\nu + 2\delta^\beta_\mu\delta^\alpha_\nu\,.\end{aligned}$$

The general formula for dual of a dual, (A.24), implies that in the case of a bivector the dual of a dual equals the original tensor with minus,

$${}^{**}F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}{}^*F^{\kappa\lambda} = \frac{1}{4}\epsilon_{\mu\nu\kappa\lambda}\epsilon^{\kappa\lambda\alpha\beta}F_{\alpha\beta} = \frac{1}{2}(-F_{\mu\nu} + F_{\nu\mu}) = -F_{\mu\nu}\,, \tag{30.5}$$

where the second of the above $\epsilon\epsilon$ products has been employed. The first of the products provides the identity

$$ {}^*F_{\mu\lambda}{}^*F^{\alpha\lambda} - F_{\mu\lambda}F^{\alpha\lambda} = -\frac{1}{2}\delta^\alpha_\mu F_{\iota\lambda}F^{\iota\lambda}\,, \tag{30.6} $$

which in turn yields, by tracing,

$$ {}^*F_{\mu\lambda}{}^*F^{\mu\lambda} = -F_{\mu\lambda}F^{\mu\lambda}\,, \tag{30.7} $$

and it also provides a similar identity for the mixed product,

$$ {}^*F_{\mu\lambda}F^{\alpha\lambda} = F_{\mu\lambda}{}^*F^{\alpha\lambda} = \frac{1}{4}\delta^\alpha_\mu F_{\iota\lambda}{}^*F^{\iota\lambda}\,. \tag{30.8} $$

Invariant content of a bivector is encoded in two independent scalars, $F_{\mu\nu}F^{\mu\nu}$ and $(F_{\mu\nu}{}^*F^{\mu\nu})^2$ (without the square, the second one is a pseudo-scalar). It might seem that another invariants follow by higher products $F_\mu{}^\nu F_\nu{}^\iota F_\iota{}^\kappa \dots F_\lambda{}^\tau F_\tau{}^\mu$, but it is not so: *matrix* products involving even number of Fs – i.e. those in which the "boundary" indices are not summed over – are symmetric in these free indices, so all products involving odd number of Fs vanish; and even-number products can be expressed in terms of the above two invariants – for example, by multiplication of equation (30.6) by $F_{\alpha\sigma}F^{\mu\sigma}$, we have

$$ {}^*F_{\mu\lambda}{}^*F^{\alpha\lambda}F_{\alpha\sigma}F^{\mu\sigma} - F_{\mu\lambda}F^{\alpha\lambda}F_{\alpha\sigma}F^{\mu\sigma} = -\frac{1}{2}(F_{\iota\lambda}F^{\iota\lambda})^2\,, $$

which yields, by substitution to the first term from (30.8),

$$ F_{\mu\lambda}F^{\alpha\lambda}F_{\alpha\sigma}F^{\mu\sigma} = \frac{1}{2}(F_{\iota\lambda}F^{\iota\lambda})^2 + {}^*F_{\mu\lambda}{}^*F^{\alpha\lambda}F_{\alpha\sigma}F^{\mu\sigma} = \frac{1}{2}(F_{\iota\lambda}F^{\iota\lambda})^2 + \frac{1}{4}(F_{\iota\lambda}{}^*F^{\iota\lambda})^2 \tag{30.9} $$

(the value and the whole relation remain exactly the same if $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ are swapped).

Invariant as well is the determinant of the mixed second-rank tensor, yet it neither brings anything new: from the general relation (A.10) we obtain, in the antisymmetric case, only

$$ \det(F_\mu{}^\nu) = \frac{1}{4!}\left[3\,(\mathrm{Tr}\,F^2)^2 - 6\,\mathrm{Tr}\,F^4\right] = -\left(\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}\right)^2 \quad \left[\,=\det({}^*F_\mu{}^\nu)\right], \tag{30.10} $$

because $\mathrm{Tr}\,F = F_\mu{}^\mu = 0$ and in the bracket we substitute from (30.9), i.e.

$$ (\mathrm{Tr}\,F^2)^2 = 2\,\mathrm{Tr}\,F^4 - \frac{1}{2}\left[\mathrm{Tr}(F{}^*F)\right]^2 \quad \Rightarrow \quad \frac{3}{4!}\left[(\mathrm{Tr}\,F^2)^2 - 2\,\mathrm{Tr}\,F^4\right] = -\frac{1}{4^2}\left[\mathrm{Tr}(F{}^*F)\right]^2. $$

Now we turn to the **eigen-value equation** $F_{\mu\nu}V^\nu = \lambda V_\mu$. Multiplication by V^μ leads to

$$ 0 = F_{\mu\nu}V^\nu V^\mu = \lambda V_\mu V^\mu\,, \tag{30.11} $$

so either the eigen-value is zero, or the eigen-vector is light-like, or both. Further, thanks to the skew symmetry of $F_{\mu\nu}$, to the even space-time dimension and to the fact that a determinant does not change under swapping matrix rows for columns, it holds

$$ \det(F_{\mu\nu}-\lambda g_{\mu\nu}) = \det(-F_{\nu\mu}-\lambda g_{\nu\mu}) = \det\left[-I\cdot(F_{\nu\mu}+\lambda g_{\nu\mu})\right] = \det(-I)\,\det(F_{\nu\mu}+\lambda g_{\nu\mu}) $$

$$= (-1)^4 \det(F_{\nu\mu} + \lambda g_{\nu\mu}) = \det(F_{\nu\mu} + \lambda g_{\nu\mu}) = \det(F_{\mu\nu} + \lambda g_{\mu\nu}) \,,$$

that is, the characteristic equation is independent of the sign of λ. It thus cannot contain odd powers of λ. Actually, for a bivector, the determinant (A.10) is simplified due to the vanishing trace, and the "triple" term in the coefficient at λ vanishes as well, so the characteristic equation (30.4) assumes the form

$$\det(F_\mu{}^\nu + \lambda\delta^\nu_\mu) = \lambda^4 + \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)\lambda^2 - \left(\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}\right)^2 = 0 \,. \tag{30.12}$$

It has two real, mutually opposite solutions ($\lambda_- = -\lambda_+$) given by the plus-root

$$\lambda^2 = (\lambda_\pm)^2 = \frac{1}{4}\left[-F_{\mu\nu}F^{\mu\nu} + \sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}{}^*F^{\mu\nu})^2}\right] \tag{30.13}$$

(the root with minus in front of the square root is clearly negative, so the other two solutions are imaginary).

For the dual bivector, the same exercise leads to the characteristic equation

$$\det({}^*F_\mu{}^\nu + {}^*\lambda\delta^\nu_\mu) = {}^*\lambda^4 + \left(\frac{1}{2}{}^*F_{\mu\nu}{}^*F^{\mu\nu}\right){}^*\lambda^2 - \left(\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}\right)^2 = 0 \,, \tag{30.14}$$

differing in sign of the middle term (because ${}^*F_{\mu\nu}{}^*F^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$). Again two real, mutually opposite solutions exist (${}^*\lambda_- = -{}^*\lambda_+$), given by the plus-root

$${}^*\lambda^2 = ({}^*\lambda_\pm)^2 = \frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} + \sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}{}^*F^{\mu\nu})^2}\right] \tag{30.15}$$

(the other two solutions are imaginary again).

The eigen-values $\lambda_\pm$ and ${}^*\lambda_\pm$ are related by the simple rules

$$(\lambda_\pm)^2 - ({}^*\lambda_\pm)^2 = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} \,, \qquad (\lambda_\pm)^2 + ({}^*\lambda_\pm)^2 = \frac{1}{2}\sqrt{(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}{}^*F^{\mu\nu})^2} \,,$$
$$(\lambda_\pm)^2({}^*\lambda_\pm)^2 = \left(\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}\right)^2 .$$

30.2 Algebraic classification of bivectors

Three distinct **algebraic types** of bivectors exist, as distinguished by several equivalent (or overlapping) criteria:

- according to whether the invariants $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}{}^*F^{\mu\nu}$ vanish or not
- according to eigen-values and the number of respective eigen-vectors, in particular, according to whether $F_{\mu\nu}$ (or ${}^*F_{\mu\nu}$) has an eigen-vector corresponding to a zero eigen-value, $F_{\mu\nu}V^\nu = 0$

- according to the algebraic structure of $F_{\mu\nu}$ (or $^*F_{\mu\nu}$), namely according to whether it can be decomposed as an anti-symmetrized (co)vector dyad
- according to the rank of the bivector

Below we list the basic properties of the three algebraic types, proceeding from the generic one to the most special.

30.2.1 Algebraic type I, generic case

Let a bivector be represented by a matrix of maximal rank, i.e. 4. It means that neither rows nor columns of the matrix are bound by any constraint of the $F_{\mu\nu}V^\nu = 0$ type, i.e., $F_{\mu\nu}$ does not have any eigen-vector corresponding to a zero eigen-value. In such a case, the determinant of $F_{\mu\nu}$ – and thus the invariant $F_{\mu\nu}{}^*F^{\mu\nu}$ – is non-zero. Hence, both $F_{\mu\nu}$ and $^*F_{\mu\nu}$ have two distinct, non-zero eigen-values, $\lambda_\pm$ and $^*\lambda_\pm$. Then, however, the corresponding eigen-vectors have to be null. More specifically, it turns out that they are two and common for $F_{\mu\nu}$ and $^*F_{\mu\nu}$, one (k^μ, say) corresponding to λ_+ and $^*\lambda_-$ (i.e., $F^\mu{}_\nu k^\nu = \lambda_+ k^\mu$ and $^*F^\mu{}_\nu k^\nu = {}^*\lambda_- k^\mu$), while the other ($l^\mu$, say) corresponding to λ_- and $^*\lambda_+$ (i.e., $F^\mu{}_\nu l^\nu = \lambda_- l^\mu$ and $^*F^\mu{}_\nu l^\nu = {}^*\lambda_+ l^\mu$). Indeed, assume that $F^\mu{}_\nu k^\nu = \lambda k^\mu$. Multiplying this by $^*F^\alpha{}_\mu$ and using equation (30.8), we have

$$\text{l.h.side} = {}^*F^\alpha{}_\mu F^\mu{}_\nu k^\nu = -\frac{1}{4}\delta^\alpha_\nu F_{\iota\lambda}{}^*F^{\iota\lambda}\, k^\nu = -\frac{1}{4}F_{\iota\lambda}{}^*F^{\iota\lambda}\, k^\alpha \,, \qquad \text{r.h.side} = \lambda{}^*F^\alpha{}_\mu k^\mu \,,$$

so it must hold

$$^*F^\alpha{}_\mu k^\mu = -\frac{1}{4\lambda}F_{\iota\lambda}{}^*F^{\iota\lambda}\, k^\alpha$$

– every eigen-vector k^μ of $F^\mu{}_\nu$ (associated with the eigen-value λ) is simultaneously the eigen-vector of $^*F^\mu{}_\nu$, associated with the eigen-value $^*\lambda = -\frac{1}{4\lambda}F_{\iota\lambda}{}^*F^{\iota\lambda}$. (And obviously it also holds conversely.)

Note that the case $F_{\mu\nu}F^{\mu\nu} = 0$ belongs to the generic ones (if $F_{\mu\nu}{}^*F^{\mu\nu} \neq 0$). The eigen-values (30.13,30.15) are equal in pairs in that case, $(\lambda_\pm)^2 = ({}^*\lambda_\pm)^2 = \frac{1}{4}\,|F_{\mu\nu}{}^*F^{\mu\nu}|\ (\neq 0)$.

30.2.2 Algebraic type I, simple case

The opposite case, $F_{\mu\nu}{}^*F^{\mu\nu} = 0$ and $F_{\mu\nu}F^{\mu\nu} \neq 0$, is somewhat more special. First, the eigen-values are degenerate here: if $F_{\mu\nu}F^{\mu\nu} > 0$, they read $(\lambda_\pm)^2 = 0$, $({}^*\lambda_\pm)^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, whereas if $F_{\mu\nu}F^{\mu\nu} < 0$, they read $(\lambda_\pm)^2 = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, $({}^*\lambda_\pm)^2 = 0$. Hence, the eigen-value equation reduces to $F_{\mu\nu}V^\nu = 0$ in the first case, while to $^*F_{\mu\nu}{}^*V^\nu = 0$ in the second case.

Second, $F_{\mu\nu}{}^*F^{\mu\nu} = 0$ is clearly special in that $F_{\mu\nu}$ has zero determinant, so it cannot have rank 4. If the rank is 2, the bivector is called **simple** (also **decomposable** or **degenerate**). In such a case, the columns (or rows) of $F_{\mu\nu}$ are dependent – they have to be bound by two independent constraints $F_{\mu\nu}V^\nu = 0$. These exactly represent equations for eigen-vectors associated with zero eigen-value. From the other side, if there exists such a vector for which $F_{\mu\nu}V^\nu = 0$, the bivector must have rank 2, so yet another independent vector (W^μ) must exist for which $F_{\mu\nu}W^\nu = 0$ holds as well. Therefore, a bivector is simple iff it has two independent eigen-vectors associated with zero eigen-value. Any linear combination of the eigen-vectors also satisfies the eigen-value equation, so the zero eigen-value is in fact associated with the whole **eigen-plane**. This plane may be time-like, space-like or null; the two independent eigen-directions which span it can be chosen as orthogonal to each other.

Equation $F_{\mu\nu}V^\nu = 0$ may also be interpreted in a different way, namely that the vectors represented by the rows of $F_{\mu\nu}$ are orthogonal to the eigen-plane. This implies that the rows have to be a combination of vectors from the plane orthogonal to the eigen-plane, p_ν and q_ν say: $F_{\mu\nu} = a_\mu p_\nu + b_\mu q_\nu$. Owing to the antisymmetry, the same has to also hold for columns, that is, the "coefficients" a_μ and b_μ must lie within the orthogonal complement of the eigen-plane as well: $a_\mu = \alpha p_\mu + \beta q_\mu$, $b_\mu = \gamma p_\mu + \delta q_\mu$. By submitting this into $F_{\mu\nu} + F_{\nu\mu} = 0$, we find that necessarily $\alpha = 0 = \delta$, $\beta + \gamma = 0$. Absorbing the only remaining constant $\beta = -\gamma$ in p_ν or q_μ, we reach the expression

$$F_{\mu\nu} = q_\mu p_\nu - p_\mu q_\nu \,. \tag{30.16}$$

From here, one further gets

$$^*F_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}(q^\rho p^\sigma - p^\rho q^\sigma) = \epsilon_{\mu\nu\rho\sigma} q^\rho p^\sigma \,, \tag{30.17}$$

and consequently

$$^*F_{\mu\nu} q^\nu = {}^*F_{\mu\nu} p^\nu = 0 \; : \tag{30.18}$$

the independent vectors q^μ, p^μ which span the plane complementary to the eigen-plane of $F_{\mu\nu}$ are eigen-vectors of its dual. This means that the dual is also simple and can be decomposed similarly,

$$^*F_{\mu\nu} = {}^*q_\mu {}^*p_\nu - {}^*p_\mu {}^*q_\nu \,, \tag{30.19}$$

where $^*q_\mu$, $^*p_\mu$ are (some) two independent vectors orthogonal to its eigen-plane (being eigen-vectors of $F_{\mu\nu}$ at the same time). To summarize, $F_{\mu\nu}$ is simple iff $^*F_{\mu\nu}$ is simple, with their eigen-planes being complementary to each other.

Now we show the equivalence with the other properties. From (30.16) and (30.17) it follows immediately that

$$^*F_{\mu\lambda} F^{\alpha\lambda} = 0 \qquad \Longrightarrow \qquad {}^*F_{\iota\lambda} F^{\iota\lambda} = 0 \,; \tag{30.20}$$

thanks to the relation (30.8), these two equations are even equivalent. Also straightforwardly – by (30.16) and (30.19), respectively – one obtains

$$F_{\mu\{\nu} F_{\rho\sigma\}} = 0 \,, \qquad {}^*F_{\mu\{\nu} {}^*F_{\rho\sigma\}} = 0 \,. \tag{30.21}$$

Conversely, if a non-trivial $F_{\mu\nu}$ satisfies the latter property, it is simple to show that it has to be simple. First, there certainly exist some vectors Q^μ, P^μ for which $F_{\rho\sigma}Q^\rho P^\sigma \neq 0$. Multiplying the equation (30.21) by their dyad $Q^\rho P^\sigma$, we get

$$F_{\mu\nu} = \frac{F_{\mu\rho}Q^\rho F_{\nu\sigma}P^\sigma - F_{\mu\sigma}P^\sigma F_{\nu\rho}Q^\rho}{F_{\rho\sigma}Q^\rho P^\sigma} \,,$$

which however is exactly the decomposition (30.16), if identifying $\frac{F_{\mu\rho}Q^\rho F_{\nu\sigma}P^\sigma}{F_{\rho\sigma}Q^\rho P^\sigma} =: q^\mu p^\nu$.

Similarly, multiplying equation (30.20) by some (co)vector of this kind (e.g. Q_α), we obtain ${}^*F_{\mu\lambda}F^{\alpha\lambda}Q_\alpha = 0$, which means that ${}^*F_{\mu\lambda}$ is simple as well.

Summary and terminology: For a simple bivector, $F_{\mu\nu} = q_\mu p_\nu - p_\mu q_\nu$, the surface spanned by the vectors q^μ and p^μ is called the **blade** of the bivector. The blade can be space-like, light-like or time-like, and the bivector is called respectively. The blade is orthogonal to the eigen-plane of the bivector. The eigen-plane of $F_{\mu\nu}$ coincides with the blade of ${}^*F_{\mu\nu}$, and vice versa. Hence, $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ have complementary space-time characters. If they are both null, the blades intersect in their common null eigen-vector – see the following.

30.2.3 Algebraic type N ("null" case)

Let both the invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}{}^*F^{\mu\nu}$ vanish, thus also the determinants of $F_\mu{}^\nu$ as well as of its dual ${}^*F_\mu{}^\nu$. Should it still represent a non-trivial case, the rank of the matrices $F_\mu{}^\nu$, ${}^*F_\mu{}^\nu$ has to be 2 – the tensors must be simple. According to (30.13,30.15), the eigen-values of both matrices are zero. This means that eigen-vectors need not be light-like. Let us learn how many they are.

Let us demand that a vector exist (denote it k^μ) within the eigen-plane of $F_{\mu\nu}$ which simultaneously is an eigen-vector of ${}^*F_{\mu\nu}$, i.e. for which it holds

$$0 = {}^*F_{\mu\nu}k^\nu = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}(q^\kappa p^\lambda - p^\kappa q^\lambda)k^\nu = \epsilon_{\mu\nu\kappa\lambda}q^\kappa p^\lambda k^\nu \quad \Rightarrow \quad q^{[\kappa}p^\lambda k^{\nu]} = 0\,. \tag{30.22}$$

The vectors p^μ, q^μ and k^μ thus have to be linearly dependent, which implies that $F_{\mu\nu}$ can be expressed, besides in terms of p_μ and q_μ, equally well as $F_{\mu\nu} = k_\mu p_\nu - p_\mu k_\nu$. Since k^μ belongs to its eigen-plane while p^μ belongs to the complementary plane, it necessarily holds $p_\nu k^\nu = 0$. Then, however, plugging k^μ to the eigen-value equation leads to the conclusion that it has to be light-like:

$$0 = F_{\mu\nu}k^\nu = (k_\mu p_\nu - p_\mu k_\nu)k^\nu = -p_\mu\, k_\nu k^\nu \qquad \Rightarrow \quad k_\nu k^\nu = 0\,. \tag{30.23}$$

So, in the case of the null field, both $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ have the whole planes of eigen-vectors. These two planes (thus their blades at the same time) intersect in a null direction k^μ which is their only common eigen-direction. Both $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ are simple – they may be written as commutators (and thus have the meaning of surface elements spanned by the commuted vectors)

$$F_{\mu\nu} = k_\mu p_\nu - p_\mu k_\nu\,, \quad {}^*F_{\mu\nu} = k_\mu {}^*p_\nu - {}^*p_\mu k_\nu\,, \quad \text{with} \quad k_\nu k^\nu = p_\nu k^\nu = {}^*p_\nu k^\nu = 0\,. \tag{30.24}$$

Introducing this expression into (30.8) – in which the r.h. side is zero in the null case –, we obtain one more condition,

$$0 = {}^*F_{\mu\lambda}F^{\alpha\lambda} = (k_\mu {}^*p_\lambda - {}^*p_\mu k_\lambda)(k^\alpha p^\lambda - p^\alpha k^\lambda) = k_\mu k^\alpha\, {}^*p_\lambda p^\lambda \quad \Longrightarrow \quad {}^*p_\lambda p^\lambda = 0\,.$$

Since both p^μ and ${}^*p^\mu$ are space-like (to a light-like vector, namely k^μ, only space-like vectors can be normal – besides k^μ itself, of course), we see that all the remaining eigen-vectors –

those associated with only one of the bivectors $F_{\mu\nu}$, $^*F_{\mu\nu}$ (denoted V^μ and $^*V^\mu$, respectively) – are space-like: they have to satisfy $(k_\mu p_\nu - p_\mu k_\nu)V^\nu = 0$, where however $p_\nu V^\nu = 0$ (p_ν is from the eigen-plane complement), so necessarily $k_\nu V^\nu = 0$; and similarly for $^*V^\nu$.

Definition Light-like directions determined by the common null eigen-vectors k^μ of tensors $F_{\mu\nu}$ and $^*F_{\mu\nu}$ are called their **principal null directions** (PND); they form **principal null congruences** (PNC).

30.2.4 Algebraic typ O (trivial case: no field)

$F_{\mu\nu}$ only has one, zero eigen-value, and any vector is its eigen-vector. This may only happen if $F_{\mu\nu} = 0$.

30.3 Algebraic classification of symmetric tensors

For symmetric tensors the classification is more complicated, because in the Lorentzian (indefinite) metric it is *not* so that every real symmetric matrix was diagonalizable. Let us at least give some partial results:

- At every point, there exist two mutually orthogonal invariant planes of a symmetric tensor, namely such mutually orthogonal 2D subspaces of the tangent space whose every vector V^μ satisfies that $T^\mu{}_\nu V^\nu$ again belongs to that subspace.

- There always exist 4 eigen-vectors. If an invariant plane is not light-like, it may contain 2 distinct real eigen-vectors, one double real eigen-vector, or no real eigen-vector. If at least one of the invariant planes is light-like, it always contains a triple null eigen-vector; the fourth eigen-vector is either space-like or null as well.

Algebraically special are again called the situations when the characteristic equation has multiple elementary factors $(\lambda - \lambda_k)$, and thus multiple ("degenerate") eigen-values – such that are associated with more eigen-vectors. In the opposite case, the situation is **algebraically generic**.

30.4 Algebraic classification of electromagnetic fields

It suffices to take over the results derived above for a generic bivector. In the case of an EM field $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$ in Minkowski space-time (or in LIFE),

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{30.25}$$

and the Hodge dual

$$^*F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}, \quad ^*F^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}, \tag{30.26}$$

the two independent invariants are

$$F_{\mu\nu}F^{\mu\nu} = 2B^2 - 2E^2, \qquad F_{\mu\nu}{}^*F^{\mu\nu} = 4\vec{E}\cdot\vec{B} \ \ (\geqslant 0)\,. \tag{30.27}$$

The pertinent characteristic equations thus have two pairs of real solutions,

$$\begin{aligned} (\lambda_\pm)^2 &= \frac{1}{2}\left[E^2 - B^2 + \sqrt{(E^2-B^2)^2 + 4(\vec{E}\cdot\vec{B})^2}\right], \\ ({}^*\lambda_\pm)^2 &= \frac{1}{2}\left[-E^2 + B^2 + \sqrt{(E^2-B^2)^2 + 4(\vec{E}\cdot\vec{B})^2}\right], \end{aligned} \tag{30.28}$$

related by

$$(\lambda_\pm)^2({}^*\lambda_\pm)^2 = (\vec{E}\cdot\vec{B})^2\,. \tag{30.29}$$

According to their values and according to the number of the associated eigen-vectors, one recognizes three possible algebraic types of EM fields – I, N and O.

30.4.1 Some properties of null EM fields

The EM-field energy-momentum tensor can be written in two forms,

$$T^{\mu\nu} = \frac{1}{8\pi}\left(F^{\mu\iota}F^{\nu}{}_{\iota} + {}^*F^{\mu\iota}{}^*F^{\nu}{}_{\iota}\right) = \frac{1}{4\pi}\left(F^{\mu\iota}F^{\nu}{}_{\iota} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{30.30}$$

linked through the relation (30.6). In the case of the null (N) field, we know that $F_{\rho\sigma}F^{\rho\sigma} = 0$ and that the $F_{\mu\nu}$, ${}^*F^{\mu\iota}$ tensors can be written as (30.24), i.e.

$$F_{\mu\nu} = k_\mu p_\nu - p_\mu k_\nu\,, \quad {}^*F_{\mu\nu} = k_\mu{}^*p_\nu - {}^*p_\mu k_\nu\,, \quad \text{with} \ \ k_\nu k^\nu = p_\nu k^\nu = {}^*p_\nu k^\nu = {}^*p_\nu p^\nu = 0\,,$$

so the corresponding two forms of $T_{\mu\nu}$ read

$$\begin{aligned} 8\pi T^{\mu\nu} &= F^{\mu\iota}F^{\nu}{}_{\iota} + {}^*F^{\mu\iota}{}^*F^{\nu}{}_{\iota} = \\ &= (k^\mu p^\iota - p^\mu k^\iota)(k^\nu p_\iota - p^\nu k_\iota) + (k^\mu{}^*p^\iota - {}^*p^\mu k^\iota)(k^\nu{}^*p_\iota - {}^*p^\nu k_\iota) = \\ &= k^\mu k^\nu (p^\iota p_\iota + {}^*p^\iota{}^*p_\iota)\,, \\ 8\pi T^{\mu\nu} &= 2F^{\mu\iota}F^{\nu}{}_{\iota} = 2(k^\mu p^\iota - p^\mu k^\iota)(k^\nu p_\iota - p^\nu k_\iota) = 2k^\mu k^\nu p^\iota p_\iota\,. \end{aligned}$$

This immediately implies

$$p^\iota p_\iota = {}^*p^\iota{}^*p_\iota\,.$$

In Minkowski, or in LIFE in a general space-time, the energy density $w = T^{00}$ and the Poynting vector $S^i = T^{0i}$ are related by

$$\frac{S_i S^i}{w^2} = \frac{(p^\iota p_\iota)^2 (k^0)^2 k_i k^i}{(p^\iota p_\iota)^2 (k^0)^4} = \frac{k^2}{(k^0)^2} = 1 \quad (= c^2 \text{ in physical units})\,. \tag{30.31}$$

The energy-flux density $\vec{S}$ thus corresponds to a propagation of all the energy density w with the speed of light in the direction $\vec{k}$: the null case represents a "**pure radiative field**", with $\vec{k}$ playing the role of the wave vector.

The simplest example of the null field is the plane harmonic wave, described, in the Minkowski space-time, by

$$A_\mu = \hat{A}_\mu \cos(k_\iota x^\iota) \implies F_{\mu\nu} = (\hat{A}_\mu k_\nu - k_\mu \hat{A}_\nu)\sin(k_\iota x^\iota)\,; \;\; k_\iota k^\iota = 0\,, \;\; \hat{A}_\iota k^\iota = 0\,. \tag{30.32}$$

The light-like eigen-vector k^μ has several special properties:

$\boxed{\text{Mariot-Robinson theorem}}$ (1954/1961):
Principal null vector of a null EM field is, in a charge-free region, geodesic and shear-free.

Proof: Make a covariant divergence of the "degenerate" eigen-equation $F^{\mu\nu}k_\mu = 0$. Employing the source-less first pair of Maxwell equations $F^{\mu\nu}{}_{;\nu} = 0$, the algebraically special form of the EM bivector $F^{\mu\nu} = k^\mu p^\nu - p^\mu k^\nu$, and the null character of k^μ (which implies $k^\mu k_{\mu;\nu} = 0$), we obtain

$$0 = (F^{\mu\nu}k_\mu)_{;\nu} = \cancel{F^{\mu\nu}{}_{;\nu}k_\mu} + F^{\mu\nu}k_{\mu;\nu} = (k^\mu p^\nu - p^\mu k^\nu)k_{\mu;\nu} = -p^\mu k^\nu k_{\mu;\nu}\,. \tag{30.33}$$

Similarly for ${}^*F^{\mu\nu}$ we get $0 = -{}^*p^\mu k^\nu k_{\mu;\nu}$ as well. Besides that, $k^\mu k^\nu k_{\mu;\nu} = 0$ due to the null character of k^μ. Therefore, the (co)vector $k^\nu k_{\mu;\nu}$ is orthogonal to the three independent vectors p^μ, ${}^*p^\mu$ and k^μ. Since p^μ and ${}^*p^\mu$ are space-like, $k^\nu k_{\mu;\nu}$ cannot be space-like, because the direction orthogonal to three mutually orthogonal space-like vectors would have to be time-like (it could not be k^μ). Neither can the vector $k^\nu k_{\mu;\nu}$ be time-like, because in such a case it could not be orthogonal to k^μ. Hence, it has to be null. But the only null vector orthogonal to k^μ is proportional to k^μ itself.[2] So, $k^\nu k_{\mu;\nu}$ has to be proportional to k_μ, i.e. the

[2] Orthogonality is a local property, and one can at every point work in a locally Minkowskian frame where $g_{\mu\nu} = \eta_{\mu\nu}$. So, have a non-trivial null vector k^μ and some other non-trivial vector V^μ orthogonal to k^μ:

$$0 = \eta_{\mu\nu}k^\mu k^\nu = -(k^0)^2 + k^2 \quad \Rightarrow \quad (k^0)^2 = k^2\,, \qquad 0 = \eta_{\mu\nu}k^\mu V^\nu = -k^0 V^0 + \vec{k}\cdot\vec{V} \quad \Rightarrow \quad V^0 = \frac{\vec{k}\cdot\vec{V}}{k^0}\,,$$

where $k^2 := \vec{k}\cdot\vec{k} \equiv \eta_{ij}k^i k^j = \delta_{ij}k^i k^j$ and likewise for V^2. Then

$$\eta_{\mu\nu}V^\mu V^\nu = -(V^0)^2 + V^2 = -\frac{(\vec{k}\cdot\vec{V})^2}{(k^0)^2} + V^2 = -\frac{(\vec{k}\cdot\vec{V})^2}{k^2} + V^2 = V^2 - \frac{(kV\cos\alpha)^2}{k^2} = V^2\sin^2\alpha\,,$$

with α the angle between $\vec{k}$ and $\vec{V}$. Therefore, V^μ is space-like in general ($\sin\alpha \neq 0$), with the special exception of $\sin\alpha = 0$ when it is null. In the latter case, moreover, $\vec{V} = \lambda\vec{k}$ (with λ some constant), which enforces

$$V^0 \equiv \frac{\vec{k}\cdot\vec{V}}{k^0} = \frac{\lambda k^2}{k^0} = \frac{\lambda(k^0)^2}{k^0} = \lambda k^0 \quad \implies \quad V^\mu = \lambda k^\mu\,.$$

vector field determined at each and every point by the principal null vector k_μ is *geodesic*.

The shear-free property also follows from the Maxwell equations. Write out the first and the second one, $F^{\mu\nu}{}_{;\nu}=0$ and $F^{\{\mu\nu;\kappa\}}=0$, for our bivector $F^{\mu\nu}=k^\mu p^\nu-p^\mu k^\nu$:

$$k^\mu{}_{;\nu}p^\nu+k^\mu p^\nu{}_{;\nu}-p^\mu{}_{;\nu}k^\nu-p^\mu k^\nu{}_{;\nu}=0\,, \tag{30.34}$$
$$k^{[\mu;\kappa]}p^\nu+k^{[\kappa;\nu]}p^\mu+k^{[\nu;\mu]}p^\kappa+p^{[\nu;\kappa]}k^\mu+p^{[\mu;\nu]}k^\kappa+p^{[\kappa;\mu]}k^\nu=0\,. \tag{30.35}$$

Multiply (30.35) by $p_\mu k_\nu$, while regarding $k_\nu p^\nu=0$, $k_\nu k^{\nu;\mu}=0$, $k_\nu k^\nu=0$, and assuming affine parametrization of the geodesic k^μ (i.e. $k_\nu k^{\mu;\nu}=0$): only the 5th term survives,

$$(p^{\mu;\nu}-p^{\nu;\mu})k^\kappa p_\mu k_\nu=0 \qquad\Longrightarrow\qquad p^{\mu;\nu}p_\mu k_\nu=p^{\nu;\mu}p_\mu k_\nu=-p^\nu p_\mu k_\nu{}^{;\mu}\,.$$

Multiplication of (30.34) by p_μ yields

$$p_\mu k^\mu{}_{;\nu}p^\nu+\cancel{p_\mu k^\mu p^\nu{}_{;\nu}}-p_\mu p^\mu{}_{;\nu}k^\nu-p_\mu p^\mu k^\nu{}_{;\nu}=0 \quad\Rightarrow\quad p_\mu p^\mu k^\nu{}_{;\nu}=p_\mu k^\mu{}_{;\nu}p^\nu-p_\mu p^\mu{}_{;\nu}k^\nu,$$

hence, by substituting for the last term from the preceding implication, we have

$$p_\mu p^\mu\,k^\nu{}_{;\nu}=2k_{\mu;\nu}p^\mu p^\nu=2k_{(\mu;\nu)}p^\mu p^\nu\equiv(k_{\mu;\nu}+k_{\nu;\mu})p^\mu p^\nu\,.$$

Exactly the same result also follows for the ${}^*p^\mu{}^*p^\nu$ projection (by multiplying, respectively by ${}^*p_\mu$ and by ${}^*p_\mu k_\nu$, the Maxwell equations written using ${}^*F^{\mu\nu}$),

$${}^*p_\mu{}^*p^\mu\,k^\nu{}_{;\nu}=(k_{\mu;\nu}+k_{\nu;\mu}){}^*p^\mu{}^*p^\nu\,.$$

Finally, combining (30.34) multiplied by ${}^*p_\mu$ and (30.35) multiplied by ${}^*p_\mu k_\nu$, i.e.

$${}^*p_\mu k^\mu{}_{;\nu}p^\nu={}^*p_\mu p^\mu{}_{;\nu}k^\nu\,, \qquad\text{and}\qquad p^{\mu;\nu}{}^*p_\mu k_\nu=p^{\nu;\mu}{}^*p_\mu k_\nu=-p^\nu{}^*p_\mu k_\nu{}^{;\mu}\,,$$

one finds

$$(k_{\mu;\nu}+k_{\nu;\mu})\,{}^*p^\mu p^\nu=0\,.$$

Recall now, from equation (24.31), the shear scalar of a light-like field k^μ,

$$8\sigma^2=(k_{\mu;\nu}+k_{\nu;\mu})(k^{\mu;\nu}+k^{\nu;\mu})-2(k^\iota{}_{;\iota})^2\,.$$

In our case of the null-EM field, take the three mutually orthogonal vectors k^μ, p^μ and ${}^*p^\mu$, and add to them some light-like vector l^μ also orthogonal to p^μ and ${}^*p^\mu$ while independent of k^μ; such a vector can always be normalized so that $k_\mu l^\mu=-1$. In terms of such four vectors, the metric can at any point be decomposed as

$$g^{\mu\nu}=-k^\mu l^\nu-l^\mu k^\nu+\frac{p^\mu p^\nu}{p_\kappa p^\kappa}+\frac{{}^*p^\mu{}^*p^\nu}{{}^*p_\kappa{}^*p^\kappa}\,,$$

so the first term in the expression for σ^2 can be written as

$$(k_{\mu;\nu}+k_{\nu;\mu})(k^{\mu;\nu}+k^{\nu;\mu})=g^{\mu\alpha}g^{\nu\beta}(k_{\mu;\nu}+k_{\nu;\mu})(k_{\alpha;\beta}+k_{\beta;\alpha})=$$

$$= \left(-k^\mu l^\alpha - l^\mu k^\alpha + \frac{p^\mu p^\alpha}{p_\kappa p^\kappa} + \frac{{}^*\!p^\mu\, {}^*\!p^\alpha}{{}^*\!p_\kappa\, {}^*\!p^\kappa}\right)\left(-k^\nu l^\beta - l^\nu k^\beta + \frac{p^\nu p^\beta}{p_\kappa p^\kappa} + \frac{{}^*\!p^\nu\, {}^*\!p^\beta}{{}^*\!p_\kappa\, {}^*\!p^\kappa}\right)(k_{\mu;\nu} + k_{\nu;\mu})(\ldots) =$$
$$= \frac{1}{(p_\kappa p^\kappa)^2}\left(p^\mu p^\alpha + {}^*\!p^\mu\, {}^*\!p^\alpha\right)\left(p^\nu p^\beta + {}^*\!p^\nu\, {}^*\!p^\beta\right)(k_{\mu;\nu} + k_{\nu;\mu})(k_{\alpha;\beta} + k_{\beta;\alpha}) =$$
$$= \frac{1}{(p_\kappa p^\kappa)^2}\left[(p_\mu p^\mu k^\nu{}_{;\nu})^2 + ({}^*\!p_\mu\, {}^*\!p^\mu k^\nu{}_{;\nu})^2\right] = 2(k^\nu{}_{;\nu})^2\,,$$

where we have first used that $k^\mu k_{\mu;\nu} = 0 = k^\nu k_{\mu;\nu}$ and $p_\kappa p^\kappa = {}^*\!p_\kappa\, {}^*\!p^\kappa$, and then what we found above from the Maxwell equations, namely

$$(k_{\mu;\nu} + k_{\nu;\mu})p^\mu p^\nu = p_\mu p^\mu\, k^\nu{}_{;\nu}\,, \quad (k_{\mu;\nu} + k_{\nu;\mu}){}^*\!p^\mu\, {}^*\!p^\nu = {}^*\!p_\mu\, {}^*\!p^\mu\, k^\nu{}_{;\nu}\,, \quad (k_{\mu;\nu} + k_{\nu;\mu})\, {}^*\!p^\mu p^\nu = 0\,.$$

The shear scalar σ^2 thus vanishes for a principal null vector field. □

Remark: The theorem in fact also adds that the host space-time (where the given EM field lives) itself has to be algebraically special, having the vector k^μ as the principal null direction of the Weyl tensor. (See below for classification of gravitational fields.)

30.5 Algebraic (Petrov) classification of the Weyl tensor

Now we turn to the gravitational problem. The algebraic classification proceeds in a similar way as above, but it concerns the Riemann (actually Weyl) tensor which is more complicated than the EM-field tensor. We thus only outline three main methods of tensor-type classification. (The discussion is shorter – though only applicable in the standard 4D case – in terms of *spinors*; an interested reader is referred to geometrical-method courses.)

30.5.1 Weyl tensor and its eigen-problem

In formula (8.5), we decomposed the Riemann tensor in three parts, calling the one independent of sources (of the Ricci tensor) the Weyl tensor, $C^\mu{}_{\nu\kappa\lambda}$. This tensor has the same symmetries as Riemann, but is traceless in addition, thus having just 10 independent components. In analogy with the electric and magnetic components of $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ with respect to some observer, we defined the "electric" and "magnetic" parts of the Weyl tensor in Section 8.3.1.

Consider now an eigen-problem for the Weyl tensor, more specifically, the task to find eigen-bivectors $V^{\kappa\lambda}$ of the Weyl tensor and the respective eigen-values λ, as solutions of the equation

$$\frac{1}{2}C_{\mu\nu\kappa\lambda}V^{\kappa\lambda} = \lambda V_{\mu\nu}\,. \tag{30.36}$$

Since the left and the right duals of Weyl equal each other, one has

$$C_{\mu\nu\kappa\lambda}{}^*V^{\kappa\lambda} = \frac{1}{2}C_{\mu\nu\kappa\lambda}\epsilon^{\kappa\lambda}{}_{\alpha\beta}V^{\alpha\beta} = {}^*C_{\mu\nu\alpha\beta}V^{\alpha\beta}\,,$$

so the multiplication of equation (30.36) by $\frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}$ yields

$$\frac{1}{4}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\kappa\lambda}V^{\kappa\lambda} = \frac{1}{2}{}^{*}C_{\alpha\beta\kappa\lambda}V^{\kappa\lambda} = \frac{1}{2}C_{\alpha\beta\kappa\lambda}{}^{*}V^{\kappa\lambda} \;\; ... \; = \lambda\,\frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}V_{\mu\nu} = \lambda{}^{*}V_{\alpha\beta}\,, \tag{30.37}$$

and further multiplication of that by $\frac{1}{2}\epsilon_{\rho\sigma}{}^{\alpha\beta}$ leads to

$$\frac{1}{2}{}^{*}C_{\rho\sigma\kappa\lambda}{}^{*}V^{\kappa\lambda}\left(= -\frac{1}{2}C_{\rho\sigma\kappa\lambda}V^{\kappa\lambda}\right) = -\lambda V_{\rho\sigma}\,. \tag{30.38}$$

Consequently, the eigen-problem is equivalently represented by any of the equations

$$\frac{1}{4}C_{\rho\sigma\kappa\lambda}V^{\kappa\lambda} - \frac{1}{4}{}^{*}C_{\rho\sigma\kappa\lambda}{}^{*}V^{\kappa\lambda} = \lambda V_{\rho\sigma}, \qquad \frac{1}{4}C_{\alpha\beta\kappa\lambda}{}^{*}V^{\kappa\lambda} + \frac{1}{4}{}^{*}C_{\alpha\beta\kappa\lambda}V^{\kappa\lambda} = \lambda{}^{*}V_{\alpha\beta}\,. \tag{30.39}$$

30.5.2 Eigen-problem for the "electric" and "magnetic" parts of Weyl

The Weyl-tensor bivector eigen-problem can also be formulated in terms of the Weyl-tensor electric and magnetic parts. Recall first how the EM-field tensor can be "reconstructed" from its electric and magnetic parts $\hat{E}_\mu = F_{\mu\nu}\hat{u}^\nu$ and $\hat{B}_\mu = -{}^{*}F_{\mu\nu}\hat{u}^\nu$ – see equation (7.12):

$$F_{\mu\nu} = \hat{u}_\mu\hat{E}_\nu - \hat{E}_\mu\hat{u}_\nu + \epsilon_{\mu\nu\rho\sigma}\hat{u}^\rho\hat{B}^\sigma\,.$$

For any bivector ($C_{\mu\nu}$), it holds, analogously,

$$C_{\mu\nu} = \hat{u}_\mu C_{\nu\iota}\hat{u}^\iota - \hat{u}_\nu C_{\mu\iota}\hat{u}^\iota - \epsilon_{\mu\nu\rho\sigma}\hat{u}^\rho{}^{*}C^{\sigma\iota}\hat{u}_\iota\,. \tag{30.40}$$

The Weyl tensor can be decomposed accordingly, thanks to that it has the structure $C_{\mu\nu}C_{\kappa\lambda}$:

$$\begin{aligned} C_{\mu\nu}C_{\kappa\lambda} &= (\hat{u}_\mu C_{\nu\iota}\hat{u}^\iota - \hat{u}_\nu C_{\mu\iota}\hat{u}^\iota - \epsilon_{\mu\nu\rho\sigma}\hat{u}^\rho{}^{*}C^{\sigma\iota}\hat{u}_\iota)(\hat{u}_\kappa C_{\lambda\gamma}\hat{u}^\gamma - \hat{u}_\lambda C_{\kappa\gamma}\hat{u}^\gamma - \epsilon_{\kappa\lambda\alpha\beta}\hat{u}^{\alpha}{}^{*}C^{\beta\gamma}\hat{u}_\gamma) = \\ &= \hat{u}_\mu\hat{u}_\kappa\hat{E}_{\nu\lambda} - \hat{u}_\mu\hat{u}_\lambda\hat{E}_{\nu\kappa} + \hat{u}_\nu\hat{u}_\lambda\hat{E}_{\mu\kappa} - \hat{u}_\nu\hat{u}_\kappa\hat{E}_{\mu\lambda} - \epsilon_{\mu\nu\rho\sigma}\epsilon_{\kappa\lambda\alpha\beta}\hat{u}^\rho\hat{u}^\alpha\hat{E}^{\sigma\beta} - \\ &\quad - \hat{u}_\mu\hat{u}^\alpha\epsilon_{\kappa\lambda\alpha\beta}\hat{B}^\beta_\nu + \hat{u}_\nu\hat{u}^\alpha\epsilon_{\kappa\lambda\alpha\beta}\hat{B}^\beta_\mu - \hat{u}_\kappa\hat{u}^\rho\epsilon_{\mu\nu\rho\sigma}\hat{B}^\sigma_\lambda + \hat{u}_\lambda\hat{u}^\rho\epsilon_{\mu\nu\rho\sigma}\hat{B}^\sigma_\kappa\,; \end{aligned} \tag{30.41}$$

we have employed

$$C_{\nu\iota}{}^{*}C^{\beta\gamma} = {}^{*}C_{\nu\iota}C^{\beta\gamma} = {}^{*}(C_{\nu\iota}C^{\beta\gamma})\,, \quad {}^{*}C^{\sigma\iota}{}^{*}C^{\beta\gamma} = {}^{**}(C^{\sigma\iota}C^{\beta\gamma}) = -C^{\sigma\iota}C^{\beta\gamma}\,,$$

with notation $C_{\mu\nu}C_{\kappa\lambda}\hat{u}^\nu\hat{u}^\lambda =: \hat{E}_{\mu\kappa}$, ${}^{*}(C_{\mu\nu}C_{\kappa\lambda})\hat{u}^\nu\hat{u}^\lambda =: \hat{B}_{\mu\kappa}$. Enjoy now the use of (A.4), i.e.,

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma}\epsilon_{\kappa\lambda\alpha\beta} = &- g_{\mu\kappa}g_{\nu\lambda}g_{\rho\alpha}g_{\sigma\beta} - g_{\mu\kappa}g_{\nu\beta}g_{\rho\lambda}g_{\sigma\alpha} - g_{\mu\kappa}g_{\nu\alpha}g_{\rho\beta}g_{\sigma\lambda} \\ &+ g_{\mu\kappa}g_{\nu\alpha}g_{\rho\lambda}g_{\sigma\beta} + g_{\mu\kappa}g_{\nu\beta}g_{\rho\alpha}g_{\sigma\lambda} + g_{\mu\kappa}g_{\nu\lambda}g_{\rho\beta}g_{\sigma\alpha} \\ &+ g_{\mu\lambda}g_{\nu\kappa}g_{\rho\alpha}g_{\sigma\beta} + g_{\mu\lambda}g_{\nu\beta}g_{\rho\kappa}g_{\sigma\alpha} + g_{\mu\lambda}g_{\nu\alpha}g_{\rho\beta}g_{\sigma\kappa} \\ &- g_{\mu\lambda}g_{\nu\alpha}g_{\rho\kappa}g_{\sigma\beta} - g_{\mu\lambda}g_{\nu\beta}g_{\rho\alpha}g_{\sigma\kappa} - g_{\mu\lambda}g_{\nu\kappa}g_{\rho\beta}g_{\sigma\alpha} \\ &- g_{\mu\alpha}g_{\nu\kappa}g_{\rho\lambda}g_{\sigma\beta} - g_{\mu\alpha}g_{\nu\beta}g_{\rho\kappa}g_{\sigma\lambda} - g_{\mu\alpha}g_{\nu\lambda}g_{\rho\beta}g_{\sigma\kappa} \\ &+ g_{\mu\alpha}g_{\nu\lambda}g_{\rho\kappa}g_{\sigma\beta} + g_{\mu\alpha}g_{\nu\beta}g_{\rho\lambda}g_{\sigma\kappa} + g_{\mu\alpha}g_{\nu\kappa}g_{\rho\beta}g_{\sigma\lambda} \end{aligned}$$

$$+ g_{\mu\beta}g_{\nu\kappa}g_{\rho\lambda}g_{\sigma\alpha} + g_{\mu\beta}g_{\nu\alpha}g_{\rho\kappa}g_{\sigma\lambda} + g_{\mu\beta}g_{\nu\lambda}g_{\rho\alpha}g_{\sigma\kappa}$$
$$- g_{\mu\beta}g_{\nu\lambda}g_{\rho\kappa}g_{\sigma\alpha} - g_{\mu\beta}g_{\nu\alpha}g_{\rho\lambda}g_{\sigma\kappa} - g_{\mu\beta}g_{\nu\kappa}g_{\rho\alpha}g_{\sigma\lambda}\,,$$

and thus compute – while regarding that $\hat{E}^{\sigma\beta}$ is symmetric, traceless and orthogonal to $\hat{u}^\mu$ (hence omitting all terms containing $g_{\sigma\beta}$, $g_{\sigma\alpha}$ and $g_{\rho\beta}$) –

$$\begin{aligned}-\epsilon_{\mu\nu\rho\sigma}\epsilon_{\kappa\lambda\alpha\beta}\hat{u}^\rho\hat{u}^\alpha\hat{E}^{\sigma\beta} = {} & \hat{u}_\mu\hat{u}_\kappa\hat{E}_{\nu\lambda} - \hat{u}_\mu\hat{u}_\lambda\hat{E}_{\nu\kappa} + \hat{u}_\nu\hat{u}_\lambda\hat{E}_{\mu\kappa} - \hat{u}_\nu\hat{u}_\kappa\hat{E}_{\mu\lambda} \\ & + g_{\mu\kappa}\hat{E}_{\nu\lambda} - g_{\mu\lambda}\hat{E}_{\nu\kappa} + g_{\nu\lambda}\hat{E}_{\mu\kappa} - g_{\nu\kappa}\hat{E}_{\mu\lambda}\,.\end{aligned}$$

Using this in (30.41), we finally have

$$\frac{1}{2}C_{\mu\nu\kappa\lambda} = 4\hat{u}_{[\mu}\hat{E}_{\nu][\lambda}\hat{u}_{\kappa]} + g_{\mu[\kappa}\hat{E}_{\lambda]\nu} - g_{\nu[\kappa}\hat{E}_{\lambda]\mu} + \hat{u}^\alpha\epsilon_{\mu\nu\alpha\beta}\hat{B}^\beta{}_{[\kappa}\hat{u}_{\lambda]} + \hat{u}^\alpha\epsilon_{\kappa\lambda\alpha\beta}\hat{B}^\beta{}_{[\mu}\hat{u}_{\nu]}\,. \quad (30.42)$$

Take the above decomposition and introduce it in the eigen-equation $\frac{1}{2}C_{\mu\nu\kappa\lambda}V^{\kappa\lambda} = \lambda V_{\mu\nu}$. Multiplication by the reference-observer four-velocity $\hat{u}^\nu$ yields

$$\frac{1}{2}C_{\mu\nu\kappa\lambda}V^{\kappa\lambda}\hat{u}^\nu = \left(\hat{u}_\kappa\hat{E}_{\mu\lambda} - \frac{1}{2}\hat{u}^\alpha\epsilon_{\kappa\lambda\alpha\beta}\hat{B}^\beta{}_\mu\right)V^{\kappa\lambda} = \lambda V_{\mu\nu}\hat{u}^\nu\,,$$

which means

$$\left(\hat{E}_{\mu\lambda}V^{\kappa\lambda} - \hat{B}_{\mu\lambda}{}^*V^{\kappa\lambda}\right)\hat{u}_\kappa = \lambda V_{\mu\nu}\hat{u}^\nu\,. \quad (30.43)$$

If proceeding, instead, via the equivalent eigen-equation $\frac{1}{2}C_{\mu\nu\kappa\lambda}{}^*V^{\kappa\lambda} = \lambda{}^*V_{\mu\nu}$, we would have obtained, similarly,

$$\left(\hat{E}_{\mu\lambda}{}^*V^{\kappa\lambda} + \hat{B}_{\mu\lambda}V^{\kappa\lambda}\right)\hat{u}_\kappa = \lambda{}^*V_{\mu\nu}\hat{u}^\nu\,. \quad (30.44)$$

The Weyl-tensor eigen-problem has thus been translated into an equivalent problem for its electric and magnetic parts.

Let us remark that in the literature the problem is mostly being formulated in a complex language – one introduces complex matrices

$$-Q_{\mu\nu} := \hat{E}_{\mu\nu} + \mathrm{i}\hat{B}_{\mu\nu}\,, \qquad V_\mu := (V_{\mu\nu} + \mathrm{i}{}^*V_{\mu\nu})\hat{u}^\nu \quad (30.45)$$

and writes the eigen-equation as

$$Q_{\mu\nu}V^\nu = \lambda V_\mu \quad (30.46)$$

whose real and imaginary parts are (30.43) and (30.44), respectively.

The $Q^\mu{}_\nu$ matrix satisfies $Q^\mu{}_\nu\hat{u}^\nu = 0$ – it possesses an "automatic" eigen-vector $\hat{u}^\nu$ tied to the null eigen-value, so the characteristic equation for its eigen-values is of the 3rd order only. Indeed, $Q^\mu{}_\nu\hat{u}^\nu = 0$ means that the rows/columns of $Q^\mu{}_\nu$ are dependent $\Rightarrow \det(Q^\mu{}_\nu) = 0$, so the corresponding characteristic equation lacks the last term of (30.4) and it can be divided by λ (whose zero value has already been accounted for). Regarding also that $Q^\mu{}_\nu$ is traceless, the equation for the remaining eigen-values reduces to

$$\lambda^3 - \frac{1}{2}Q^\mu{}_\iota Q^\iota{}_\mu \lambda - \frac{1}{3}Q^\mu{}_\iota Q^\iota{}_\kappa Q^\kappa{}_\mu = 0\,. \tag{30.47}$$

The coefficients of this equation

$$\mathcal{I} := Q^\mu{}_\iota Q^\iota{}_\mu\,, \qquad \mathcal{J} := Q^\mu{}_\iota Q^\iota{}_\kappa Q^\kappa{}_\mu \tag{30.48}$$

are the only invariants which can be composed out of $Q^\mu{}_\nu$. By comparing the characteristic equation with the factorization of its left-hand side,

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3\,,$$

simpler relations are obtained for the eigen-values than if one tried to solve the equation directly:

$$\lambda_1+\lambda_2+\lambda_3 = 0\,, \quad (\lambda_1)^2+(\lambda_2)^2+(\lambda_3)^2 = \mathcal{I}\,, \quad (\lambda_1)^3+(\lambda_2)^3+(\lambda_3)^3 = \mathcal{J}\ (= 3\lambda_1\lambda_2\lambda_3)\,. \tag{30.49}$$

The first of them follows immediately by the above comparison, while the third one does so by summing the characteristic equations (30.47) for $\lambda=\lambda_1$, $\lambda=\lambda_2$ and $\lambda=\lambda_3$ (and employing the first relation);[3] finally, the first relation implies

$$0 = (\lambda_1+\lambda_2+\lambda_3)^2 = (\lambda_1)^2+(\lambda_2)^2+(\lambda_3)^2 + 2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\,,$$

where, however, the last term has to be $=-\mathcal{I}$ by comparison, and hence the second relation.

Relations (30.49) tell which basic situations may occur:

- Gravitational field of type I (algebraically general):
  All eigen-values λ and all eigen-vectors V^μ are different.

- Gravitational fields of type II and of type D:
  Two of the eigen-values are equal – say, $\lambda_3 \neq \lambda_1 = \lambda_2 \neq 0$. From (30.49) we then get

$$\begin{aligned} &\mathcal{I} = 6(\lambda_1)^2 = 6(\lambda_2)^2 = \frac{3}{2}(\lambda_3)^2, \qquad \mathcal{J} = -6(\lambda_1)^3 = -6(\lambda_2)^3 = \frac{3}{4}(\lambda_3)^3 \\ &\Longrightarrow \quad \mathcal{I}^3 = 6\mathcal{J}^2 \neq 0\,. \end{aligned} \tag{30.50}$$

  Eigen-vectors associated with $\lambda_1=\lambda_2$ are different for type II whereas same for type D.

- Gravitational fields of types III, N and O:
  All eigen-values are equal, thus being zero. Consequently, both the invariants $\mathcal{I}$ and $\mathcal{J}$ have to be zero as well. The respective three eigen-vectors are different in type III while two of them coincide in type N; the type O is "trivial", $Q^\mu{}_\nu = 0$.

Note that in the original formulation of the problem, $\frac{1}{2}C_{\mu\nu\kappa\lambda}V^{\kappa\lambda} = \lambda V_{\mu\nu}$, there exist three different eigen-bivectors in the I, D and O cases, while only two exist in the II and N cases, and just a single one in the III case.

[3] The final expression added in parenthesis again stems from direct comparison.

30.5.3 Newman-Penrose tetrads, Debever-Penrose equation, and principal null directions of the Weyl tensor

In analytical work in GR, the **Newman-Penrose (NP) formalism** has proved very useful. It is a particular type of tetrad formalism, in which one uses tetrad projections of quantities rather than their coordinate components (the approach thus depends on how the tetrad field is selected, but it is independent of coordinates). Tetrad projections (of the quantities as well as of differential operators) are many, so many new symbols have to be introduced, connected by many equations... But there is a benefit: all the differential equations relevant in the formulation of GR (Einstein equations, Bianchi identities) are of the first order only.

The NP formalism is tied to the **NP tetrads**, namely tetrads composed of two real and two complex, mutually complex-conjugate null vectors, $\{k^\mu,l^\mu,m^\mu,\bar{m}^\mu\}$, determined by the following relations:

$$\begin{aligned}
& k_\sigma l^\sigma = -1\,, \quad m_\sigma \bar{m}^\sigma = 1\,, \qquad \text{all other products are zero} \\
& \iff \quad g_{\mu\nu} = -k_\mu l_\nu - k_\nu l_\mu + m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu\,.
\end{aligned} \tag{30.51}$$

Weyl tensor is represented then by five complex projections

$$\begin{aligned}
& \Psi_0 := C_{\mu\nu\kappa\lambda} k^\mu m^\nu k^\kappa m^\lambda\,, \qquad \Psi_1 := C_{\mu\nu\kappa\lambda} k^\mu l^\nu k^\kappa m^\lambda\,, \\
& \Psi_2 := C_{\mu\nu\kappa\lambda} k^\mu m^\nu \bar{m}^\kappa l^\lambda = \frac{1}{2} C_{\mu\nu\kappa\lambda} k^\mu l^\nu (k^\kappa l^\lambda - m^\kappa \bar{m}^\lambda)\,, \\
& \Psi_3 := C_{\mu\nu\kappa\lambda} k^\mu l^\nu \bar{m}^\kappa l^\lambda\,, \qquad \Psi_4 := C_{\mu\nu\kappa\lambda} \bar{m}^\mu l^\nu \bar{m}^\kappa l^\lambda\,.
\end{aligned} \tag{30.52}$$

The NP tetrad is not unique – at any point, one can perform an arbitrary Lorentz transformation. Any such transformation can be decomposed in three simple operations:

- Rotation about k^μ, described by
  $k'^\mu = k^\mu, \quad l'^\mu = l^\mu + \kappa\bar{\kappa}\, k^\mu + \bar{\kappa} m^\mu + \kappa \bar{m}^\mu, \quad m'^\mu = m^\mu + \kappa k^\mu, \quad \bar{m}'^\mu = \bar{m}^\mu + \bar{\kappa} k^\mu,$
  where κ is a complex parameter. Under such a transformation, the Weyl scalars behave as

$$\begin{aligned}
& \Psi'_0 = \Psi_0\,, \quad \Psi'_1 = \Psi_1 + \bar{\kappa}\Psi_0\,, \quad \Psi'_2 = \Psi_2 + 2\bar{\kappa}\Psi_1 + \bar{\kappa}^2\Psi_0\,, \\
& \Psi'_3 = \Psi_3 + 3\bar{\kappa}\Psi_2 + 3\bar{\kappa}^2\Psi_1 + \bar{\kappa}^3\Psi_0\,, \quad \Psi'_4 = \Psi_4 + 4\bar{\kappa}\Psi_3 + 6\bar{\kappa}^2\Psi_2 + 4\bar{\kappa}^3\Psi_1 + \bar{\kappa}^4\Psi_0\,.
\end{aligned}$$

- Rotation about l^μ, described by
  $l'^\mu = l^\mu, \quad k'^\mu = k^\mu + \lambda\bar{\lambda}\, l^\mu + \bar{\lambda} m^\mu + \lambda \bar{m}^\mu, \quad m'^\mu = m^\mu + \lambda l^\mu, \quad \bar{m}'^\mu = \bar{m}^\mu + \bar{\lambda} l^\mu,$
  where λ is a complex parameter. Under such a transformation, the Weyl scalars behave as

$$\begin{aligned}
& \Psi'_0 = \Psi_0 + 4\lambda\Psi_1 + 6\lambda^2\Psi_2 + 4\lambda^3\Psi_3 + \lambda^4\Psi_4\,, \quad \Psi'_1 = \Psi_1 + 3\lambda\Psi_2 + 3\lambda^2\Psi_3 + \lambda^3\Psi_4\,, \\
& \Psi'_2 = \Psi_2 + 2\lambda\Psi_3 + \lambda^2\Psi_4\,, \quad \Psi'_3 = \Psi_3 + \lambda\Psi_4\,, \quad \Psi'_4 = \Psi_4\,.
\end{aligned} \tag{30.53}$$

- Boost in the (k^μ,l^μ) plane + spatial rotation in the $(m^\mu,\bar{m}^\mu)$ plane, described by
  $k'^\mu = bk^\mu, \quad l'^\mu = b^{-1} l^\mu, \quad m'^\mu = e^{\mathrm{i}\varphi} m^\mu, \quad \bar{m}'^\mu = e^{-\mathrm{i}\varphi} \bar{m}^\mu,$
  with b and φ real parameters. Under such a transformation, the Weyl scalars behave as

$$\Psi'_0 = \frac{e^{2\mathrm{i}\varphi}}{b^2}\,\Psi_0\,, \quad \Psi'_1 = \frac{e^{\mathrm{i}\varphi}}{b}\,\Psi_1\,, \quad \Psi'_2 = \Psi_2\,, \quad \Psi'_3 = \frac{b}{e^{\mathrm{i}\varphi}}\,\Psi_3\,, \quad \Psi'_4 = \frac{b^2}{e^{2\mathrm{i}\varphi}}\,\Psi_4\,.$$

It turns out that the behaviour of the scalars under the above transformations provide further – and *equivalent* – option for the algebraic classification of curvature. Let us focus on the rotation about l^μ. First, at least one of the Ψ scalars can always be made vanish by it. If any further scalar can be nullified as well, the associated vector k^μ defines the **principal null direction** of the Weyl tensor. Such directions may have various properties (multiplicities) related to how many of the Ψ scalars can be made vanish.

Let us proceed from Ψ_0. This can apparently be nullified by rotation about l^μ with λ satisfying the quartic equation

$$\Psi_0 + 4\lambda\Psi_1 + 6\lambda^2\Psi_2 + 4\lambda^3\Psi_3 + \lambda^4\Psi_4 = 0\,. \tag{30.54}$$

Five cases may occur according to the number and multiplicity of the roots. Consider first a *generic case* – let the equation have four different non-zero solutions. The corresponding eigen-vector equation is the requirement $\Psi_0 \equiv C_{\mu\nu\kappa\lambda}k^\mu m^\nu k^\kappa m^\lambda = 0$ itself. It can be understood as a requirement on the decomposition of the tensor $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa$ in the basis of dyadic products made of the NP-tetrad vectors. As the tensor $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa$ is real, symmetric (thanks to the symmetry of Weyl in the two *pairs* of its indices), traceless and satisfying $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa k^\lambda = 0$, the decomposition can only involve the dyadic terms

$$k_\nu k_\lambda\,, \qquad k_{(\nu}\mathrm{Re}(m_{\lambda)}) = \frac{1}{2}k_{(\nu}(m_{\lambda)} + \bar{m}_{\lambda)}) = k_{(\nu}\mathrm{Re}(\bar{m}_{\lambda)})\,.$$

Indeed, l_λ cannot be involved at all, because $l_\lambda k^\lambda = -1$, so $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa k^\lambda$ would not vanish, and $\mathrm{Re}(m_\nu m_\lambda) = \frac{1}{2}(m_\nu\bar{m}_\lambda + \bar{m}_\nu m_\lambda) = \mathrm{Re}(\bar{m}_\nu\bar{m}_\lambda)$ are not traceless. Hence the decomposition

$$C_{\mu\nu\kappa\lambda}k^\mu k^\kappa = \alpha\, k_\nu k_\lambda + \beta\left[k_\nu(m_\lambda + \bar{m}_\lambda) + k_\lambda(m_\nu + \bar{m}_\nu)\right], \tag{30.55}$$

where multiplications by $l^\nu l^\lambda$ and by $l^\nu m^\lambda$ reveal, respectively, that

$$\alpha = C_{\mu\nu\kappa\lambda}k^\mu l^\nu k^\kappa l^\lambda\,, \qquad \beta = -C_{\mu\nu\kappa\lambda}k^\mu l^\nu k^\kappa m^\lambda \equiv -\Psi_1\,.$$

This tells that "at least in one of the indices" the tensor is spanned by the k-vector, which can alternatively be expressed in the form (it's suitable to swap the indices $\mu \leftrightarrow \nu$ at $C_{\mu\nu\kappa\lambda}$)

$$\boxed{k_{[\alpha}C_{\nu]\mu\kappa[\lambda}k_{\beta]}k^\mu k^\kappa = 0} \qquad \text{... called the } \textbf{Debever-Penrose equation}\,. \tag{30.56}$$

The Weyl tensor is classified according to whether and how many of the four eigen-vector solutions of this equation coincide. Below we proceed towards more and more special cases.

- Suppose we have already rotated the NP tetrad so that $\Psi_0 = 0$. Compute the remaining Ψ scalars in that tetrad, and write down the equation (30.54) once more. It now has the simple root $\lambda = 0$ and three further solutions given by equation

$$4\Psi_1 + 6\lambda\Psi_2 + 4\lambda^2\Psi_3 + \lambda^3\Psi_4 = 0\,. \tag{30.57}$$

If these three are non-zero and different, it is still a generic case (just with the NP tetrad already rotated in a special way).

- If some of the three roots is zero, Ψ_1 has to be zero as well. Vice versa, if it is possible to reach, besides $\Psi_0=0$, also $\Psi_1=0$, i.e. to solve, *simultaneously*, the equations

$$\Psi_0 + 4\lambda\Psi_1 + 6\lambda^2\Psi_2 + 4\lambda^3\Psi_3 + \lambda^4\Psi_4 = 0\,, \qquad \Psi_1 + 3\lambda\Psi_2 + 3\lambda^2\Psi_3 + \lambda^3\Psi_4 = 0\,,$$

then, after rotating the tetrad accordingly, $\lambda = 0$ will be a double root of the former and a simple root of the latter. Consider now what the conditions $\Psi_0 = 0$, $\Psi_1 = 0$ imply for the associated double-degenerate eigen-vector k^μ. Should the tensor (30.55) satisfy $\Psi_1 \equiv C_{\mu\nu\kappa\lambda}k^\mu k^\kappa l^\nu m^\lambda = 0$, one has to set in it $\beta=0$, being thus left with $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa = \alpha\, k_\nu k_\lambda$. This can equivalently be written as

$$C_{\nu\mu\kappa[\lambda}k_{\beta]}k^\mu k^\kappa = 0\,. \tag{30.58}$$

- We can proceed further in a similar manner. A simultaneous nullifying of Ψ_0, Ψ_1 and Ψ_2 is possible if and only if $\lambda=0$ is a triple root of equation (30.54). Equation for the corresponding triple-degenerate eigen-vector can be obtained by realizing which triple products of the NP-tetrad vectors may occur in the tensor $C_{\mu\nu\kappa\lambda}k^\mu$, in order that it satisfy the conditions

$$C_{\mu\nu\kappa\lambda}k^\mu k^\kappa = \alpha\, k_\nu k_\lambda\,, \qquad \Psi_2 \equiv C_{\mu\nu\kappa\lambda}k^\mu m^\nu \bar{m}^\kappa l^\lambda = 0\,,$$

while being real, skew-symmetric in $[\kappa,\lambda]$, having vanishing contraction in ${}^{\nu}_{\lambda}$ and vanishing cyclic permutation $C_{\mu\{\nu\kappa\lambda\}}k^\mu$. One concludes that it has to be proportional to $(\ldots)_{\nu\kappa}k_\lambda$, thus satisfying the condition

$$C_{\nu\mu\kappa[\lambda}k_{\beta]}k^\mu = 0\,. \tag{30.59}$$

- Simultaneous nullifying of Ψ_0, Ψ_1, Ψ_2 and Ψ_3 is possible if and only if $\lambda=0$ is a quadruple root of equation (30.54). In order to satisfy, simultaneously, the conditions

$$C_{\mu\nu\kappa\lambda}k^\mu k^\kappa = \alpha\, k_\nu k_\lambda\,, \quad \Psi_2 \equiv C_{\mu\nu\kappa\lambda}k^\mu m^\nu \bar{m}^\kappa l^\lambda = 0\,, \quad \Psi_3 \equiv C_{\mu\nu\kappa\lambda}k^\mu l^\nu \bar{m}^\kappa l^\lambda = 0\,,$$

the corresponding quadruple-degenerate eigen-vector has to satisfy the equation

$$C_{\nu\mu\kappa\lambda}k^\mu = 0\,. \tag{30.60}$$

- Finally, the last possible case, $\Psi_0=\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$, is equivalent to $C_{\nu\mu\kappa\lambda}=0$, i.e. the space-time has to be conformally flat.

The above possibilities correspond, respectively, to the algebraic types I, II or D, III, N, and O. In the general, I case, the equation (30.54) has 4 different simple roots, associated with 4 different principal null directions k^μ. In the II case, equation (30.54) has 1 double root equal to zero, and 2 further different simple roots; two of the four PNDs coincide. In the D case, equation (30.54) has 1 double root equal to zero, and 1 further double root; the four PNDs coincide in pairs. In the III case, equation (30.54) has 1 triple root equal to zero, and 1 another simple root; three of the four PNDs coincide. In the N case, equation (30.54) has just 1 quadruple root equal to zero; all the four PNDs coincide. In the O case, the Weyl tensor vanishes, which means that the space-time is conformally flat (metric is flat modulo a multiplicative scalar factor).

In some of the cases, it is in addition possible to make such a "null rotation" about k^μ (the other of the two NP-tetrad real vectors) that some further Ψ scalars vanish (besides those listed above). Just as a footnote, let us specify that it is possible to reach $\Psi_2 = 0$ for type I (besides $\Psi_0 = 0$), $\Psi_3 = 0$ for type II (besides $\Psi_0 = 0$ and $\Psi_1 = 0$), $\Psi_3 = 0 = \Psi_4$ for type D (besides $\Psi_0 = 0$ and $\Psi_1 = 0$), and $\Psi_4 = 0$ for type III (besides $\Psi_0 = 0$, $\Psi_1 = 0$ and $\Psi_2 = 0$). In an asymptotically flat case, it is natural that the far field is "radial", so the two radial directions (outgoing and ingoing) are also "eigen" to the Weyl tensor. Within the NP tetrad $\{k^\mu, l^\mu, m^\mu, \bar{m}^\mu\}$, they are represented by the k^μ and l^μ vectors.

Let us add that the invariants made of the matrix $Q^\mu{}_\nu$ in the preceding section can be expressed in terms of the Weyl scalars as

$$\mathcal{I} = 2\left[\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3(\Psi_2)^2\right], \qquad \mathcal{J} = 6\,\det\begin{pmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{pmatrix}. \tag{30.61}$$

It is easy to check that the invariants are

... non-zero and independent in the I case
... $\mathcal{I} = 6(\Psi_2)^2, \ \mathcal{J} = -6(\Psi_2)^3 \quad \Longrightarrow \quad \mathcal{I}^3 = 6\mathcal{J}^2 \neq 0$ in the II or D cases
... zero in all the other cases.

Extra relations between the Weyl-tensor projections

Looking at the Ψ scalars, one may ask whether and how the other conceivable projections of Weyl onto the NP tetrad relate to them. Some follow immediately from the Riemann-type (anti)symmetries, plus there is the extra Weyl's property of vanishing trace which is worth exploiting. Without loss of generality, let us consider the trace over the 2nd and the 4th indices,

$$0 = C_{\mu\nu\kappa\lambda}g^{\nu\lambda} = C_{\mu\nu\kappa\lambda}(-k^\nu l^\lambda - l^\nu k^\lambda + m^\nu\bar{m}^\lambda + \bar{m}^\nu m^\lambda) = C_{\kappa\nu\mu\lambda}g^{\nu\lambda},$$

for the projections onto $k^\mu k^\kappa$, $k^\mu l^\kappa$, $k^\mu m^\kappa$, $k^\mu\bar{m}^\kappa$, $l^\mu l^\kappa$, $l^\mu m^\kappa$, $l^\mu\bar{m}^\kappa$, $m^\mu m^\kappa$, $m^\mu\bar{m}^\kappa$ and $\bar{m}^\mu\bar{m}^\kappa$. In obvious notation (e.g., $C_{\mu\nu\kappa\lambda}l^\mu k^\nu\bar{m}^\kappa l^\lambda =: C_{lk\bar{m}l}$) and right omitting the terms which vanish "trivially", we obtain, respectively, zero for the following combinations:

$$\begin{aligned}
&C_{kmk\bar{m}} + C_{k\bar{m}km}\,, \quad -C_{kllk} + C_{kml\bar{m}} + C_{k\bar{m}lm}\,, \quad -C_{klmk} + C_{kmm\bar{m}}\,, \quad -C_{kl\bar{m}k} + C_{k\bar{m}\bar{m}m}\,, \\
&C_{lml\bar{m}} + C_{l\bar{m}lm}\,, \qquad -C_{lkml} + C_{lmm\bar{m}}\,, \qquad -C_{lk\bar{m}l} + C_{l\bar{m}\bar{m}m}\,, \\
&-C_{mkml} - C_{mlmk}\,, \qquad -C_{mk\bar{m}l} - C_{ml\bar{m}k} + C_{m\bar{m}\bar{m}m}\,, \qquad -C_{\bar{m}k\bar{m}l} - C_{\bar{m}l\bar{m}k}\,,
\end{aligned}$$

that is, when identifying the canonical scalars according to (30.52),

$$\begin{aligned}
&C_{kmk\bar{m}} + C_{k\bar{m}km}\,, \quad -C_{kllk} - \Psi_2 - \bar{\Psi}_2\,, \quad \Psi_1 + C_{kmm\bar{m}}\,, \quad \bar{\Psi}_1 + C_{k\bar{m}\bar{m}m}\,, \\
&C_{lml\bar{m}} + C_{l\bar{m}lm}\,, \qquad \bar{\Psi}_3 + C_{lmm\bar{m}}\,, \qquad \Psi_3 + C_{l\bar{m}\bar{m}m}\,,
\end{aligned}$$

$$-C_{mkml}-C_{mlmk}\,, \qquad \Psi_2+\bar{\Psi}_2+C_{m\bar{m}\bar{m}m}\,, \qquad -C_{\bar{m}k\bar{m}l}-C_{\bar{m}l\bar{m}k}\,.$$

Using elementary symmetries, we thus find

$$0=C_{kmk\bar{m}}=C_{lml\bar{m}}=C_{mkml}=C_{\bar{m}k\bar{m}l}\,,$$
$$C_{klkl}=C_{m\bar{m}m\bar{m}}=\Psi_2+\bar{\Psi}_2\,, \qquad C_{km\bar{m}m}=\Psi_1\,, \qquad C_{l\bar{m}m\bar{m}}=\Psi_3\,.$$

Plus it is worth to add that from the first Bianchi identity it follows

$$C_{klm\bar{m}}=-C_{km\bar{m}l}-C_{k\bar{m}lm}=-\Psi_2+\bar{\Psi}_2\,,$$

and that the function $\alpha\equiv C_{\mu\nu\kappa\lambda}k^\mu l^\nu k^\kappa l^\lambda$ from the algebraically special property $C_{\mu\nu\kappa\lambda}k^\mu k^\kappa=\alpha\,k_\nu k_\lambda$ we now see to equal $\alpha=\Psi_2+\bar{\Psi}_2$.

To conclude this section, note the following important link between the algebraic character of the Weyl tensor and kinematic properties of its principal null directions (counter-part of the Mariot-Robinson theorem from electromagnetism):

30.5.4 The Goldberg-Sachs theorem

A non-flat vacuum ($R_{\mu\nu}=\Lambda g_{\mu\nu}$) space-time is algebraically special (*not* of type I) if and only if it admits a shear-free geodesic null congruence. (The latter is integral to the repeated Weyl-tensor principal null direction.)

Proof: projections of the vacuum Bianchi identities

The proof follows from the vacuum Bianchi identities, $C_{\mu\nu\kappa\lambda}{}^{;\lambda}=0$, by projections onto the NP tetrad. Since the statement concerns Ψ_0 and Ψ_1, one tries to select such projections which do contain Ψ_0 and Ψ_1, and – if possible – do not contain many other Ψs. The right choices turn out to be $C_{\mu\nu\kappa\lambda}{}^{;\lambda}k^\mu m^\nu k^\kappa$ and $C_{\mu\nu\kappa\lambda}{}^{;\lambda}k^\mu m^\nu m^\kappa$. The first of them:

$$C_{\mu\nu\kappa\lambda}{}^{;\lambda}k^\mu m^\nu k^\kappa=\left[C_{\mu\nu\kappa\delta}k^\mu m^\nu k^\kappa\delta^\delta_\lambda\right]^{;\lambda}-C_{\mu\nu\kappa\lambda}(k^\mu m^\nu k^\kappa)^{;\lambda}=$$
$$=\Big[C_{\mu\nu\kappa\delta}k^\mu m^\nu k^\kappa(-\cancel{k^\delta l_\lambda}-l^\delta k_\lambda+m^\delta\bar{m}_\lambda+\bar{m}^\delta m_\lambda)\Big]^{;\lambda}-C_{\mu\nu\kappa\lambda}(k^\mu m^\nu k^\kappa)^{;\lambda}=$$
$$=[-C_{kmkl}k_\lambda+C_{kmkm}\bar{m}_\lambda+C_{kmk\bar{m}}m_\lambda]^{;\lambda}-C_{\mu\nu\kappa\lambda}(k^\mu m^\nu k^\kappa)^{;\lambda}\,,$$

where $C_{kmkl}=\Psi_1$, $C_{kmkm}=\Psi_0$ and $C_{kmk\bar{m}}=0$, so we have

$$(0=)\;C_{\mu\nu\kappa\lambda}{}^{;\lambda}k^\mu m^\nu k^\kappa=-(\Psi_1 k_\lambda)^{;\lambda}+(\Psi_0\bar{m}_\lambda)^{;\lambda}-C_{\mu\nu\kappa\lambda}(k^\mu m^\nu k^\kappa)^{;\lambda}\,. \tag{30.62}$$

The last term one writes out by Leibniz, and works patiently,

$$-C_{\mu\nu\kappa\lambda}(k^\mu m^\nu k^\kappa)^{;\lambda}=-C_{\mu\nu\kappa\lambda}k^{\mu;\lambda}m^\nu k^\kappa-C_{\mu\nu\kappa\lambda}k^\mu m^{\nu;\lambda}k^\kappa-C_{\mu\nu\kappa\lambda}k^\mu m^\nu k^{\kappa;\lambda}=$$
$$=-C_{\alpha\nu\kappa\delta}m^\nu k^\kappa(-k^\alpha l_\mu-\cancel{l^\alpha k_\mu}+\cancel{m^\alpha\bar{m}_\mu}+\bar{m}^\alpha m_\mu)(-\cancel{k^\delta l_\lambda}-l^\delta k_\lambda+m^\delta\bar{m}_\lambda+\bar{m}^\delta m_\lambda)\,k^{\mu;\lambda}-$$
$$-\,C_{\mu\beta\kappa\delta}k^\mu k^\kappa(-\cancel{k^\beta l_\nu}-l^\beta k_\nu+m^\beta\bar{m}_\nu+\cancel{\bar{m}^\beta m_\nu})(-\cancel{k^\delta l_\lambda}-l^\delta k_\lambda+m^\delta\bar{m}_\lambda+\bar{m}^\delta m_\lambda)\,m^{\nu;\lambda}-$$

$$
\begin{aligned}
&- C_{\mu\nu\gamma\delta}k^\mu m^\nu(-k^\gamma l_\kappa - \cancel{l^\gamma k_\kappa} + m^\gamma \bar m_\kappa + \bar m^\gamma m_\kappa)(-k^\delta l_\lambda - l^\delta k_\lambda + m^\delta \bar m_\lambda + \bar m^\delta m_\lambda)\, k^{\kappa;\lambda} = \\
&= -\left[C_{kmkl} l_\mu k_\lambda - C_{kmkm} l_\mu \bar m_\lambda - \cancel{C_{kmk\bar m}} l_\mu m_\lambda - C_{\bar m mkl} m_\mu k_\lambda + C_{\bar m mkm} m_\mu \bar m_\lambda + \right.\\
&\quad \left. + C_{\bar m mk\bar m} m_\mu m_\lambda\right] k^{\mu;\lambda} - \\
&\quad - \left[C_{klkl} k_\nu k_\lambda - C_{klkm} k_\nu \bar m_\lambda - C_{klk\bar m} k_\nu m_\lambda - C_{kmkl} \bar m_\nu k_\lambda + C_{kmkm} \bar m_\nu \bar m_\lambda + \right.\\
&\quad \left. + \cancel{C_{kmk\bar m}} \bar m_\nu m_\lambda\right] m^{\nu;\lambda} - \\
&\quad - \left[C_{kmkl} l_\kappa k_\lambda - C_{kmkm} l_\kappa \bar m_\lambda - \cancel{C_{kmk\bar m}} l_\kappa m_\lambda - C_{kmmk} \bar m_\kappa l_\lambda - \cancel{C_{kmml}} \bar m_\kappa k_\lambda + \right.\\
&\quad \left. + C_{kmm\bar m} \bar m_\kappa m_\lambda - \cancel{C_{km\bar m k}} m_\kappa l_\lambda - C_{km\bar m l} m_\kappa k_\lambda + C_{km\bar m m} m_\kappa \bar m_\lambda\right] k^{\kappa;\lambda} = \\
&= -\left[\Psi_1 l_\mu k_\lambda - \Psi_0 l_\mu \bar m_\lambda - (\Psi_2 - \bar\Psi_2) m_\mu k_\lambda + \Psi_1 m_\mu \bar m_\lambda - \bar\Psi_1 m_\mu m_\lambda\right] k^{\mu;\lambda} - \\
&\quad - \left[(\Psi_2 + \bar\Psi_2) k_\nu k_\lambda - \Psi_1 k_\nu \bar m_\lambda - \bar\Psi_1 k_\nu m_\lambda - \Psi_1 \bar m_\nu k_\lambda + \Psi_0 \bar m_\nu \bar m_\lambda\right] m^{\nu;\lambda} - \\
&\quad - \left[\Psi_1 l_\kappa k_\lambda - \Psi_0 l_\kappa \bar m_\lambda + \Psi_0 \bar m_\kappa l_\lambda - \Psi_1 \bar m_\kappa m_\lambda - \Psi_2 m_\kappa k_\lambda + \Psi_1 m_\kappa \bar m_\lambda\right] k^{\kappa;\lambda} = \\
&= \Psi_1(-\cancel{k_\mu l_\lambda} - l_\mu k_\lambda + m_\mu \bar m_\lambda + \bar m_\mu m_\lambda)\, k^{\mu;\lambda} - \Psi_1(l_\mu k_\lambda + 4 m_\mu \bar m_\lambda)\, k^{\mu;\lambda} + \Psi_1 \bar m_\nu k_\lambda m^{\nu;\lambda} \\
&\quad - \Psi_0(-k_\mu l_\lambda - l_\mu k_\lambda + m_\mu \bar m_\lambda + \cancel{\bar m_\mu m_\lambda})\, \bar m^{\mu;\lambda} - \Psi_0 l_\mu k_\lambda \bar m^{\mu;\lambda} - 2\Psi_0 \bar m_\mu \bar m_\lambda m^{\mu;\lambda} + \\
&\quad + 2\Psi_0 l_\mu \bar m_\lambda k^{\mu;\lambda} + 3\Psi_2 m_\mu k_\lambda k^{\mu;\lambda} = \\
&= \Psi_1 k_\lambda{}^{;\lambda} - \Psi_0 \bar m_\lambda{}^{;\lambda} - \Psi_1(l_\mu k_\lambda + 4 m_\mu \bar m_\lambda)\, k^{\mu;\lambda} + \Psi_1 \bar m_\nu k_\lambda m^{\nu;\lambda} + \\
&\quad + \Psi_0(\bar m_\mu k_\lambda - 2 k_\mu \bar m_\lambda)\, l^{\mu;\lambda} - 2\Psi_0 \bar m_\mu \bar m_\lambda m^{\mu;\lambda} + 3\Psi_2 m_\mu k_\lambda k^{\mu;\lambda}\,.
\end{aligned}
$$

Substituting this to (30.62), we find that the (kmk) projection of the Bianchi identities yields

$$
\begin{aligned}
\Psi_1{}^{;\lambda} k_\lambda - \Psi_0{}^{;\lambda} \bar m_\lambda &= \Psi_0(\bar m_\mu k_\lambda - 2 k_\mu \bar m_\lambda)\, l^{\mu;\lambda} - 2\Psi_0 \bar m_\mu \bar m_\lambda m^{\mu;\lambda} + \Psi_1 \bar m_\nu k_\lambda m^{\nu;\lambda} - \\
&\quad - \Psi_1(l_\mu k_\lambda + 4 m_\mu \bar m_\lambda)\, k^{\mu;\lambda} + 3\Psi_2 m_\mu k_\lambda k^{\mu;\lambda}\,.
\end{aligned}
\tag{30.63}
$$

Let's proceed to the second useful Bianchi-identities projection, $C_{\mu\nu\kappa\lambda}{}^{;\lambda} k^\mu m^\nu m^\kappa$:

$$
\begin{aligned}
C_{\mu\nu\kappa\lambda}{}^{;\lambda} k^\mu m^\nu m^\kappa &= \left[C_{\mu\nu\kappa\delta} k^\mu m^\nu m^\kappa \delta^\delta_\lambda\right]^{;\lambda} - C_{\mu\nu\kappa\lambda}(k^\mu m^\nu m^\kappa)^{;\lambda} = \\
&= \left[C_{\mu\nu\kappa\delta} k^\mu m^\nu m^\kappa(-k^\delta l_\lambda - l^\delta k_\lambda + \cancel{m^\delta \bar m_\lambda} + \bar m^\delta m_\lambda)\right]^{;\lambda} - C_{\mu\nu\kappa\lambda}(k^\mu m^\nu m^\kappa)^{;\lambda} = \\
&= \left[-C_{kmmk} l_\lambda - C_{kmml} k_\lambda + C_{kmm\bar m} m_\lambda\right]^{;\lambda} - C_{\mu\nu\kappa\lambda}(k^\mu m^\nu m^\kappa)^{;\lambda}\,,
\end{aligned}
$$

where $C_{kmmk} = -\Psi_0$, $C_{kmml} = 0$ and $C_{kmm\bar m} = -\Psi_1$, so we have

$$
(0 =)\; C_{\mu\nu\kappa\lambda}{}^{;\lambda} k^\mu m^\nu m^\kappa = (\Psi_0 l_\lambda)^{;\lambda} - (\Psi_1 m_\lambda)^{;\lambda} - C_{\mu\nu\kappa\lambda}(k^\mu m^\nu m^\kappa)^{;\lambda}\,. \tag{30.64}
$$

Similarly as above, we tackle the last term,

$$
\begin{aligned}
&- C_{\mu\nu\kappa\lambda}(k^\mu m^\nu m^\kappa)^{;\lambda} = -C_{\mu\nu\kappa\lambda} k^{\mu;\lambda} m^\nu m^\kappa - C_{\mu\nu\kappa\lambda} k^\mu m^{\nu;\lambda} m^\kappa - C_{\mu\nu\kappa\lambda} k^\mu m^\nu m^{\kappa;\lambda} = \\
&= -C_{\alpha\nu\kappa\delta} m^\nu m^\kappa(-k^\alpha l_\mu - \cancel{l^\alpha k_\mu} + \cancel{m^\alpha \bar m_\mu} + \bar m^\alpha m_\mu)(-k^\delta l_\lambda - l^\delta k_\lambda + \cancel{m^\delta \bar m_\lambda} + \bar m^\delta m_\lambda)\, k^{\mu;\lambda} - \\
&\quad - C_{\mu\beta\kappa\delta} k^\mu m^\kappa(-\cancel{k^\beta l_\nu} - l^\beta k_\nu + m^\beta \bar m_\nu + \cancel{\bar m^\beta m_\nu})(-k^\delta l_\lambda - l^\delta k_\lambda + \cancel{m^\delta \bar m_\lambda} + \bar m^\delta m_\lambda)\, m^{\nu;\lambda} - \\
&\quad - C_{\mu\nu\gamma\delta} k^\mu m^\nu(-k^\gamma l_\kappa - l^\gamma k_\kappa + m^\gamma \bar m_\kappa + \cancel{\bar m^\gamma m_\kappa})(-k^\delta l_\lambda - l^\delta k_\lambda + m^\delta \bar m_\lambda + \bar m^\delta m_\lambda)\, m^{\kappa;\lambda} = \\
&= -\left[C_{kmmk} l_\mu l_\lambda + \cancel{C_{kmml}} l_\mu k_\lambda - C_{kmm\bar m} l_\mu m_\lambda - C_{\bar m mmk} m_\mu l_\lambda - C_{\bar m mml} m_\mu k_\lambda + \right.\\
&\quad \left. + C_{\bar m mm\bar m} m_\mu m_\lambda\right] k^{\mu;\lambda} -
\end{aligned}
$$

$$
\begin{aligned}
&- \left[C_{klmk}k_\nu l_\lambda + C_{klml}k_\nu k_\lambda - C_{klm\bar m}k_\nu m_\lambda - C_{kmmk}\bar m_\nu l_\lambda - \cancel{C_{kmml}}\bar m_\nu k_\lambda + \right.\\
&\quad \left. + C_{kmm\bar m}\bar m_\nu m_\lambda\right] m^{\nu;\lambda} - \\
&- \left[C_{kmkl}l_\kappa k_\lambda - C_{kmkm}l_\kappa \bar m_\lambda - \cancel{C_{kmk\bar m}}l_\kappa m_\lambda + C_{kmlk}k_\kappa l_\lambda - \cancel{C_{kmlm}}k_\kappa \bar m_\lambda - \right.\\
&\quad \left. - C_{kml\bar m}k_\kappa m_\lambda - C_{kmmk}\bar m_\kappa l_\lambda - \cancel{C_{kmml}}\bar m_\kappa k_\lambda + C_{kmm\bar m}\bar m_\kappa m_\lambda\right] m^{\kappa;\lambda} = \\
= &- \left[-\Psi_0 l_\mu l_\lambda + \Psi_1 l_\mu m_\lambda + \Psi_1 m_\mu l_\lambda + \bar\Psi_3 m_\mu k_\lambda - (\Psi_2 + \bar\Psi_2) m_\mu m_\lambda\right] k^{\mu;\lambda} - \\
&- \left[-\Psi_1 k_\nu l_\lambda + \bar\Psi_3 k_\nu k_\lambda + (\Psi_2 - \bar\Psi_2) k_\nu m_\lambda + \Psi_0 \bar m_\nu l_\lambda - \Psi_1 \bar m_\nu m_\lambda\right] m^{\nu;\lambda} - \\
&- \left[\Psi_1 l_\kappa k_\lambda - \Psi_0 l_\kappa \bar m_\lambda - \Psi_1 k_\kappa l_\lambda + \Psi_2 k_\kappa m_\lambda + \Psi_0 \bar m_\kappa l_\lambda - \Psi_1 \bar m_\kappa m_\lambda\right] m^{\kappa;\lambda} = \\
= &-\Psi_0(-k_\mu l_\lambda - \cancel{l_\mu k_\lambda} + m_\mu \bar m_\lambda + \bar m_\mu m_\lambda)\, l^{\mu;\lambda} - \Psi_0 (l_\mu m_\lambda - 2 m_\mu l_\lambda)\, \bar m^{\mu;\lambda} - 2\Psi_0 k_\mu l_\lambda l^{\mu;\lambda} \\
&+ \Psi_1(-k_\mu l_\lambda - l_\mu k_\lambda + \cancel{m_\mu \bar m_\lambda} + \bar m_\mu m_\lambda)\, m^{\mu;\lambda} - \Psi_1 (l_\mu m_\lambda + 4 m_\mu l_\lambda)\, k^{\mu;\lambda} + \\
&+ \Psi_1 \bar m_\mu m_\lambda m^{\mu;\lambda} + 3\Psi_2 m_\mu m_\lambda k^{\mu;\lambda} = \\
= &-\Psi_0 {l_\lambda}^{;\lambda} + \Psi_1 {m_\lambda}^{;\lambda} - \Psi_0 (l_\mu m_\lambda - 2 m_\mu l_\lambda)\, \bar m^{\mu;\lambda} - 2\Psi_0 k_\mu l_\lambda l^{\mu;\lambda} - \\
&- \Psi_1 (l_\mu m_\lambda + 4 m_\mu l_\lambda)\, k^{\mu;\lambda} + \Psi_1 \bar m_\mu m_\lambda m^{\mu;\lambda} + 3\Psi_2 m_\mu m_\lambda k^{\mu;\lambda} \,.
\end{aligned}
$$

Substituting this to (30.64), we find that the (kmm) projection of the Bianchi identities yields

$$
\begin{aligned}
{\Psi_0}^{;\lambda} l_\lambda - \Psi_1^{;\lambda} m_\lambda = {}& \Psi_0 (l_\mu m_\lambda - 2 m_\mu l_\lambda)\, \bar m^{\mu;\lambda} - 2\Psi_0 l_\mu l_\lambda k^{\mu;\lambda} + \Psi_1 m_\nu m_\lambda \bar m^{\nu;\lambda} + \\
&+ \Psi_1 (l_\mu m_\lambda + 4 m_\mu l_\lambda)\, k^{\mu;\lambda} - 3\Psi_2 m_\mu m_\lambda k^{\mu;\lambda} \,.
\end{aligned}
\tag{30.65}
$$

Proof $\Rightarrow$

If $\Psi_0 = 0$ and $\Psi_1 = 0$, equations (30.63) and (30.65) reduce, respectively, to

$$
0 = \Psi_2 m_\mu k_\lambda k^{\mu;\lambda} \,, \qquad 0 = \Psi_2 m_\mu m_\lambda k^{\mu;\lambda} \,.
$$

Should Ψ_2 be non-zero (which it should in general), the expressions behind have to vanish.

- First, $m_\mu k_\lambda k^{\mu;\lambda} = 0$ tells that $k_\lambda k^{\mu;\lambda}$ has to lie within the (k, l) plane. However, if it was (at least partially) proportional to l^μ, it would have non-zero scalar product with k_μ – which is not the case, because $k_\mu k^{\mu;\lambda} = 0$. Hence, $k_\lambda k^{\mu;\lambda}$ has to be solely proportional to k^μ (or zero), which means that k^μ is geodesic.
- In order to infer what $m_\mu m_\lambda k^{\mu;\lambda} = 0$ means, let us first multiply it by its complex conjugate, $\bar m_\mu \bar m_\kappa k^{\mu;\kappa} = 0$, to arrive at $(m_\nu m_\lambda k^{\nu;\lambda})(\bar m_\mu \bar m_\kappa k^{\mu;\kappa}) = 0$.

  The next step is hard to call otherwise than Cimrman's step-aside.[4] Consider three possible projections of $k^{\mu;\kappa} k^{\nu;\lambda}$:

$$
\begin{aligned}
&g_{\mu\nu} g_{\kappa\lambda} k^{\mu;\kappa} k^{\nu;\lambda} = \\
&= (-k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar m_\nu + \bar m_\mu m_\nu)(-k_\kappa l_\lambda - l_\kappa k_\lambda + m_\kappa \bar m_\lambda + \bar m_\kappa m_\lambda)\, k^{\mu;\kappa} k^{\nu;\lambda} = \\
&= 2 m_\mu \bar m_\nu m_\kappa \bar m_\lambda k^{\mu;\kappa} k^{\nu;\lambda} + 2 m_\mu \bar m_\nu \bar m_\kappa m_\lambda k^{\mu;\kappa} k^{\nu;\lambda} \,, \\
&g_{\mu\lambda} g_{\nu\kappa} k^{\mu;\kappa} k^{\nu;\lambda} =
\end{aligned}
$$

[4] See Wikipedia: Jára (da) Cimrman and his philosophy of externism.

$$
\begin{aligned}
&= (-k_\mu l_\lambda - l_\mu k_\lambda + m_\mu \bar{m}_\lambda + \bar{m}_\mu m_\lambda)(-k_\nu l_\kappa - l_\nu k_\kappa + m_\nu \bar{m}_\kappa + \bar{m}_\nu m_\kappa)\, k^{\mu;\kappa} k^{\nu;\lambda} = \\
&= (m_\mu \bar{m}_\lambda m_\nu \bar{m}_\kappa + \bar{m}_\mu m_\lambda \bar{m}_\nu m_\kappa)\, k^{\mu;\kappa} k^{\nu;\lambda} + 2 m_\mu \bar{m}_\lambda \bar{m}_\nu m_\kappa k^{\mu;\kappa} k^{\nu;\lambda}\ , \\
&g_{\mu\kappa} g_{\nu\lambda} k^{\mu;\kappa} k^{\nu;\lambda} = \\
&= (-k_\mu l_\kappa - l_\mu k_\kappa + m_\mu \bar{m}_\kappa + \bar{m}_\mu m_\kappa)(-k_\nu l_\lambda - l_\nu k_\lambda + m_\nu \bar{m}_\lambda + \bar{m}_\nu m_\lambda)\, k^{\mu;\kappa} k^{\nu;\lambda} = \\
&= (m_\mu \bar{m}_\kappa m_\nu \bar{m}_\lambda + \bar{m}_\mu m_\kappa \bar{m}_\nu m_\lambda)\, k^{\mu;\kappa} k^{\nu;\lambda} + 2 m_\mu \bar{m}_\kappa \bar{m}_\nu m_\lambda k^{\mu;\kappa} k^{\nu;\lambda}\ ,
\end{aligned}
$$

where the terms containing k_μ do not contribute due to $k_\mu k^{\mu;\kappa} = 0$ and $k_\nu k^{\nu;\lambda} = 0$. Labelling the above six terms by A to F, it is easily seen that A=D, B=F and C=E, which leads to the simple result A+B+C+D-E-F = A+D = 2A,

$$
(g_{\mu\nu} g_{\kappa\lambda} + g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda})\, k^{\mu;\kappa} k^{\nu;\lambda} = 4 m_\mu m_\kappa \bar{m}_\nu \bar{m}_\lambda k^{\mu;\kappa} k^{\nu;\lambda}\ .
$$

This is exactly what we previously found to be zero. Hence, we obtain

$$
0 = m_\mu m_\kappa \bar{m}_\nu \bar{m}_\lambda k^{\mu;\kappa} k^{\nu;\lambda} = \frac{1}{4}\left(k_{\nu;\lambda} k^{\nu;\lambda} + k_{\lambda;\nu} k^{\nu;\lambda} - k_\kappa{}^{;\kappa} k_\lambda{}^{;\lambda}\right) = \frac{1}{2}\, k_{(\nu;\lambda)} k^{\nu;\lambda} - \frac{1}{4}\,(k^\kappa{}_{;\kappa})^2.
$$

Checking equation (24.31), one sees this vanishing expression is exactly the shear of k^μ.[5]

Proof $\Leftarrow$

Have a vacuum space-time and let a geodesic shear-free congruence k^μ exist in it. From equation (24.45), we see that in suitable coordinates (where only physical degrees of freedom in shear and vorticity are manifested) the vanishing of shear implies $h^\rho_\mu h^\sigma_\nu C_{\rho\iota\sigma\lambda} k^\iota k^\lambda = 0$. Writing out

$$
\begin{aligned}
h^\rho_\mu h^\sigma_\nu C_{\rho\iota\sigma\lambda} k^\iota k^\lambda &= (\delta^\rho_\mu + k^\rho l_\mu + l^\rho k_\mu)(\delta^\sigma_\nu + k^\sigma l_\nu + l^\sigma k_\nu)\, C_{\rho\iota\sigma\lambda} k^\iota k^\lambda = \\
&= (m^\rho \bar{m}_\mu + \bar{m}^\rho m_\mu)(m^\sigma \bar{m}_\nu + \bar{m}^\sigma m_\nu)\, C_{\rho\iota\sigma\lambda} k^\iota k^\lambda = \\
&= C_{mkmk} \bar{m}_\mu \bar{m}_\nu + \cancel{C_{mk\bar{m}k}} \bar{m}_\mu m_\nu + \cancel{C_{\bar{m}kmk}} m_\mu \bar{m}_\nu + C_{\bar{m}k\bar{m}k} m_\mu m_\nu = \\
&= \Psi_0 \bar{m}_\mu \bar{m}_\nu + \bar{\Psi}_0 m_\mu m_\nu\ ,
\end{aligned}
$$

we see Ψ_0 has to vanish. The non-trivial part remains, however: to show that Ψ_1 vanishes as well, i.e. that k^μ is a *repeated* eigen-vector of Weyl.

We again rely on the same projections of the Bianchi identities as above, (30.63) and (30.65). With $\Psi = 0$, with $k_\lambda k^{\mu;\lambda} = 0$ (without loss of generality, k^μ can be affinely parameterized) and with vanishing shear, $m_\mu m_\lambda k^{\mu;\lambda} = 0$, the two equations appear as

$$
\Psi_1{}^{;\lambda} k_\lambda = \Psi_1 \bar{m}_\nu k_\lambda m^{\nu;\lambda} - 4\Psi_1 m_\mu \bar{m}_\lambda k^{\mu;\lambda}\ , \tag{30.66}
$$

$$
\Psi_1^{;\lambda} m_\lambda = -\Psi_1 m_\nu m_\lambda \bar{m}^{\nu;\lambda} - \Psi_1 (l_\mu m_\lambda + 4 m_\mu l_\lambda)\, k^{\mu;\lambda}\ . \tag{30.67}
$$

The aim is to show that under the given assumptions these equations *necessarily* lead to the solution $\Psi_1 = 0$.

---

[5] If a magnitude of a complex number is zero, $0 = z\bar{z} = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$, it means that both real and imaginary parts of z have to vanish, i.e. that z itself has to vanish. Therefore, zero shear means $m_\mu m_\kappa k^{\mu;\kappa} = 0$.

Time to consider the freedom in orientation/rotation of the NP basis. We are concerned with the properties of k^μ, so *that* has to be left untouched, yet transformations can be made, without loss of generality, which do have such a property. Two terms of the above equations can thus be eliminated:

- The first term of (30.66), $\bar{m}_\nu k_\lambda m^{\nu;\lambda}$, can be nullified by the rotation $m'^\nu = e^{\mathrm{i}\varphi} m^\nu$:

  $$m'^{\nu;\lambda}\bar{m}'_\nu k_\lambda = (e^{\mathrm{i}\varphi}m^\nu)^{;\lambda}e^{-\mathrm{i}\varphi}\bar{m}_\nu k_\lambda = \varphi^{;\lambda}k_\lambda + m^{\nu;\lambda}\bar{m}_\nu k_\lambda\,.$$

  Indeed, such φ can always be found (possibly accompanied by a linear rescaling of k^μ) which makes this expression vanish.

- The second term which can be "gauged out" is $m_\mu l_\lambda k^{\mu;\lambda}$ in equation (30.67). Actually, make the "null rotation" of the NP tetrad about k^μ, i.e. the transformation described by

  $$k'^\mu = k^\mu, \quad l'^\mu = l^\mu + \kappa\bar{\kappa}\,k^\mu + \bar{\kappa}m^\mu + \kappa\bar{m}^\mu, \quad m'^\mu = m^\mu + \kappa k^\mu, \quad \bar{m}'^\mu = \bar{m}^\mu + \bar{\kappa}k^\mu\,,$$

  and require that property: one obtains easily

  $$0 = m'_\mu l'_\lambda k'^{\mu;\lambda} = (m_\mu + \cancel{\kappa k_\mu})(l_\lambda + \cancel{\kappa\bar{\kappa}\,k_\lambda} + \bar{\kappa}m_\lambda + \kappa\bar{m}_\lambda)\,k^{\mu;\lambda} = m_\mu l_\lambda k^{\mu;\lambda} + \kappa m_\mu \bar{m}_\lambda k^{\mu;\lambda}\,,$$

  because $k_\mu k^{\mu;\lambda} = 0$ (k^μ is null), $k_\lambda k^{\mu;\lambda} = 0$ (k^μ is geodesic, with affine parameterization assumed) and $m_\mu m_\lambda k^{\mu;\lambda} = 0$ (k^μ is shear free). Hence, if $m_\mu l_\lambda k^{\mu;\lambda}$ is non-zero, its elimination is achieved by choosing

  $$\kappa = -\frac{m_\mu l_\lambda k^{\mu;\lambda}}{m_\alpha \bar{m}_\delta k^{\alpha;\delta}}\,.$$

  The only case when this could *not* be done is $m_\alpha \bar{m}_\delta k^{\alpha;\delta} = 0$. However, if that happened, equation (30.66) would read $\Psi_1{}^{;\lambda}k_\lambda = 0$, so Ψ_1 would vanish and the proof would be finished.

- The rotation about k^μ does not harm the other assumptions and results used: $k_\mu k^{\mu;\lambda} = 0$ and $k_\lambda k^{\mu;\lambda} = 0$ *of course* do not change, but neither

  $$m'^{\nu;\lambda}\bar{m}'_\nu k'_\lambda = (m^{\nu;\lambda} + \cancel{\kappa^{;\lambda}k^\nu} + \cancel{\kappa k^{\nu;\lambda}})(\bar{m}_\nu + \bar{\kappa}k_\nu)\,k_\lambda = m^{\nu;\lambda}\bar{m}_\nu k_\lambda - \bar{\kappa}m_\nu \cancel{k^{\nu;\lambda}k_\lambda}\;(=0)\,,$$
  $$m'_\mu m'_\lambda k'^{\mu;\lambda} = (m_\mu + \cancel{\kappa k_\mu})(m_\lambda + \cancel{\kappa k_\lambda})\,k^{\mu;\lambda} = m_\mu m_\lambda k^{\mu;\lambda}\;(=0)\,.$$

  Also, the relevant two Weyl scalars change according to $\Psi'_0 = \Psi_0$, $\Psi'_1 = \Psi_1 + \bar{\kappa}\Psi_0$, which means that $(\Psi_0 = 0, \Psi_1 = 0) \Leftrightarrow (\Psi'_0 = 0, \Psi'_1 = 0)$.

- Note also that the *same* rotation about k^μ cannot at the same time eliminate the other terms of (30.66) and (30.67). Indeed,

  $$m'_\mu \bar{m}'_\lambda k'^{\mu;\lambda} = (m_\mu + \cancel{\kappa k_\mu})(\bar{m}_\lambda + \cancel{\bar{\kappa}k_\lambda})\,k^{\mu;\lambda} = m_\mu \bar{m}_\lambda k^{\mu;\lambda}\,,$$
  $$m'_\mu m'_\lambda \bar{m}'^{\mu;\lambda} + l'_\mu m'_\lambda k'^{\mu;\lambda} =$$
  $$= (m_\mu + \kappa k_\mu)(m_\lambda + \cancel{\kappa k_\lambda})(\bar{m}^{\mu;\lambda} + \cancel{\bar{\kappa}^{;\lambda}k^\mu} + \cancel{\bar{\kappa}k^{\mu;\lambda}}) +$$

$$+ \left(l_\mu + \cancel{\kappa\bar{\kappa}k_\mu} + \bar{\kappa}m_\mu + \kappa\bar{m}_\mu\right)\left(m_\lambda + \cancel{\kappa k_\lambda}\right) k^{\mu;\lambda} =$$
$$= m_\mu m_\lambda \bar{m}^{\mu;\lambda} + \cancel{\kappa k_\mu m_\lambda \bar{m}^{\mu;\lambda}} + l_\mu m_\lambda k^{\mu;\lambda} + \cancel{\kappa \bar{m}_\mu m_\lambda k^{\mu;\lambda}} = m_\mu m_\lambda \bar{m}^{\mu;\lambda} + l_\mu m_\lambda k^{\mu;\lambda}\,.$$

(Warning: the ~~cancellations~~ again have not been meant so that the respective terms were zero, but that they vanished after the pertinent scalar products.)

To summarize, we simplified equations (30.66) and (30.67) to

$$\Psi_1{}^{;\lambda}k_\lambda = -4\Psi_1 m_\mu \bar{m}_\lambda k^{\mu;\lambda}\,, \qquad -\Psi_1^{;\lambda}m_\lambda = \Psi_1 m_\nu m_\lambda \bar{m}^{\nu;\lambda} + \Psi_1 l_\mu m_\lambda k^{\mu;\lambda}\,,$$

which can also be written as

$$(\ln\Psi_1)^{;\lambda}k_\lambda = -4m_\mu\bar{m}_\lambda k^{\mu;\lambda}\,, \qquad (\ln\Psi_1)^{;\lambda}m_\lambda = -m_\nu m_\lambda \bar{m}^{\nu;\lambda} - l_\mu m_\lambda k^{\mu;\lambda}\,. \qquad (30.68)$$

The ultimate point is to consider "integrability condition" for this pair. Let us compute the commutator of the two derivatives in two ways (and compare the results):

- First start from computing it for a generic scalar (S, say),

$$(S^{;\alpha}k_\alpha)^{;\beta}m_\beta - (S^{;\beta}m_\beta)^{;\alpha}k_\alpha = \cancel{\left(S^{;\alpha\beta} - S^{;\beta\alpha}\right)} k_\alpha m_\beta + S^{;\beta}(k_\beta{}^{;\alpha}m_\alpha - m_\beta{}^{;\alpha}k_\alpha) =$$
$$= S^{;\iota}(-k_\iota l^\beta - \cancel{l_\iota k^\beta} + m_\iota\bar{m}^\beta + \bar{m}_\iota m^\beta)(k_\beta{}^{;\alpha}m_\alpha - m_\beta{}^{;\alpha}k_\alpha) =$$
$$= S^{;\iota}k_\iota(-l_\beta m_\alpha k^{\beta;\alpha} + l_\beta k_\alpha m^{\beta;\alpha}) + S^{;\iota}m_\iota(\bar{m}_\beta m_\alpha k^{\beta;\alpha} - \bar{m}_\beta k_\alpha m^{\beta;\alpha}) + S^{;\iota}\bar{m}_\iota \cancel{m_\beta m_\alpha k^{\beta;\alpha}}$$

(in the last term appears the shear, so we already omit it). Cancelling, as above, the term $\bar{m}_\beta k_\alpha m^{\beta;\alpha}$, we thus have

$$(S^{;\alpha}k_\alpha)^{;\beta}m_\beta - (S^{;\beta}m_\beta)^{;\alpha}k_\alpha = -(S^{;\iota}k_\iota)(l_\beta m_\alpha k^{\beta;\alpha} + m_\beta k_\alpha l^{\beta;\alpha}) + (S^{;\iota}m_\iota)\,\bar{m}_\beta m_\alpha k^{\beta;\alpha}\,.$$

Using this for $S = \ln\Psi_1$ and substituting from equations (30.68), we arrive at

$$[(\ln\Psi_1)^{;\alpha}k_\alpha]^{;\beta}m_\beta - [(\ln\Psi_1)^{;\beta}m_\beta]^{;\alpha}k_\alpha =$$
$$= -(\ln\Psi_1)^{;\iota}k_\iota\,(l_\beta m_\alpha k^{\beta;\alpha} + m_\beta k_\alpha l^{\beta;\alpha}) + (\ln\Psi_1)^{;\iota}m_\iota\,\bar{m}_\beta m_\alpha k^{\beta;\alpha} =$$
$$= 4(m_\mu\bar{m}_\lambda k^{\mu;\lambda})(l_\beta m_\alpha k^{\beta;\alpha} + m_\beta k_\alpha l^{\beta;\alpha}) - (m_\nu m_\lambda\bar{m}^{\nu;\lambda} + l_\mu m_\lambda k^{\mu;\lambda})(\bar{m}_\beta m_\alpha k^{\beta;\alpha})\,.$$

This can further be simplified:

$$(m_\mu\bar{m}_\lambda k^{\mu;\lambda})(l_\beta m_\alpha k^{\beta;\alpha}) = (-\cancel{k_\alpha l_\lambda} - \cancel{l_\alpha k_\lambda} + m_\alpha\bar{m}_\lambda + \cancel{\bar{m}_\alpha m_\lambda})\,m_\mu k^{\mu;\lambda}l_\beta k^{\beta;\alpha} =$$
$$= m_\mu k^{\mu;\lambda}l^\beta k_{\beta;\lambda}\,,$$
$$(m_\mu\bar{m}_\lambda k^{\mu;\lambda})(m_\beta k_\alpha l^{\beta;\alpha}) = (-\cancel{k_\beta l_\lambda} - \cancel{l_\beta k_\lambda} + m_\beta\bar{m}_\lambda + \cancel{\bar{m}_\beta m_\lambda})\,m_\mu k^{\mu;\lambda}k_\alpha l^{\beta;\alpha} =$$
$$= m_\mu k^{\mu;\lambda}k^\alpha l_{\lambda;\alpha}\,,$$
$$(m_\nu m_\lambda\bar{m}^{\nu;\lambda})(\bar{m}_\beta m_\alpha k^{\beta;\alpha}) = (-\cancel{k_\lambda l_\beta} - \cancel{l_\lambda k_\beta} + m_\lambda\bar{m}_\beta + \cancel{\bar{m}_\lambda m_\beta})\,m_\nu\bar{m}^{\nu;\lambda}m_\alpha k^{\beta;\alpha} =$$
$$= m_\nu\bar{m}^{\nu;\lambda}m^\alpha k_{\lambda;\alpha}\,,$$
$$(l_\mu m_\lambda k^{\mu;\lambda})(\bar{m}_\beta m_\alpha k^{\beta;\alpha}) = (-\cancel{k_\lambda l_\beta} - \cancel{l_\lambda k_\beta} + m_\lambda\bar{m}_\beta + \cancel{\bar{m}_\lambda m_\beta})\,l_\mu k^{\mu;\lambda}m_\alpha k^{\beta;\alpha} =$$
$$= l_\mu k^{\mu;\lambda}m^\alpha k_{\lambda;\alpha}\,,$$

where the last cancellations ($\cancel{\bar{m}m}$) are thanks to $m_\lambda m_\mu k^{\mu;\lambda} = 0$ (zero shear). Hence,

$$[(\ln\Psi_1)^{;\alpha}k_\alpha]^{;\beta}m_\beta - [(\ln\Psi_1)^{;\beta}m_\beta]^{;\alpha}k_\alpha =$$
$$= 4m_\mu k^{\mu;\lambda}l^\beta k_{\beta;\lambda} + 4m_\mu k^{\mu;\lambda}k^\alpha l_{\lambda;\alpha} - m_\nu\bar{m}^{\nu;\lambda}m^\alpha k_{\lambda;\alpha} - l_\mu k^{\mu;\lambda}m^\alpha k_{\lambda;\alpha}\,. \qquad (30.69)$$

- On the other hand, computing the same commutator directly from (30.68) yields

$$
\begin{aligned}
&[(\ln\Psi_1)^{;\alpha}k_\alpha]^{;\beta}m_\beta - [(\ln\Psi_1)^{;\beta}m_\beta]^{;\alpha}k_\alpha = \\
&= -4(m_\mu\bar{m}_\lambda k^{\mu;\lambda})^{;\beta}m_\beta + (m_\nu m_\lambda \bar{m}^{\nu;\lambda} + l_\mu m_\lambda k^{\mu;\lambda})^{;\alpha}k_\alpha = \\
&= \left[-4m_{\mu;\beta}m^\beta\bar{m}_\lambda k^{\mu;\lambda} - 4m_\mu\bar{m}_{\lambda;\beta}m^\beta k^{\mu;\lambda}\right] - 4m_\mu\bar{m}_\lambda k^{\mu;\lambda\beta}m_\beta + \\
&\quad + \cancel{m_{\nu;\alpha}k^\alpha m_\lambda\bar{m}^{\nu;\lambda}} + \cancel{m_\nu m_{\lambda;\alpha}k^\alpha\bar{m}^{\nu;\lambda}} + m_\nu m_\lambda\bar{m}^{\nu;\lambda\alpha}k_\alpha + \\
&\quad + \cancel{l_{\mu;\alpha}k^\alpha m_\lambda k^{\mu;\lambda}} + \cancel{l_\mu m_{\lambda;\alpha}k^\alpha k^{\mu;\lambda}} + l_\mu m_\lambda k^{\mu;\lambda\alpha}k_\alpha\,.
\end{aligned}
\tag{30.70}
$$

Cancellations:

 – Since $k^{\alpha;\lambda}k_\alpha = 0$, $\bar{m}_\lambda k^{\alpha;\lambda}\bar{m}_\alpha = 0$ (zero shear), $k^{\mu;\delta}k_\delta = 0$ (geodesicity), $m_\mu k^{\mu;\delta}l_\delta = 0$ (rotation of the tetrad) and $m_\mu k^{\mu;\delta}m_\delta = 0$ (zero shear), one decomposes

$$
\begin{aligned}
\bar{m}_\lambda k^{\mu;\lambda} &= \bar{m}_\lambda k^{\alpha;\lambda}(-k_\alpha l^\mu - l_\alpha k^\mu + m_\alpha\bar{m}^\mu + \bar{m}_\alpha m^\mu) = \\
&= -\bar{m}_\lambda k^{\alpha;\lambda}l_\alpha k^\mu + \bar{m}_\lambda k^{\alpha;\lambda}m_\alpha\bar{m}^\mu\,, \\
m_\mu k^{\mu;\lambda} &= m_\mu k^{\mu;\delta}(-k_\delta l^\lambda - l_\delta k^\lambda + m_\delta\bar{m}^\lambda + \bar{m}_\delta m^\lambda) = \\
&= m_\mu k^{\mu;\delta}\bar{m}_\delta m^\lambda\,,
\end{aligned}
$$

 which implies that the first two terms (they are in the bracket) yield zero,

$$
\begin{aligned}
&-4m_{\mu;\beta}m^\beta\bar{m}_\lambda k^{\mu;\lambda} - 4m_\mu\bar{m}_{\lambda;\beta}m^\beta k^{\mu;\lambda} = \\
&= -4m_{\mu;\beta}m^\beta(-\bar{m}_\lambda k^{\alpha;\lambda}l_\alpha k^\mu + \bar{m}_\lambda k^{\alpha;\lambda}m_\alpha\bar{m}^\mu) - 4\bar{m}_{\lambda;\beta}m^\beta m_\mu k^{\mu;\delta}\bar{m}_\delta m^\lambda = \\
&= -4m_{\mu;\beta}m^\beta\bar{m}_\lambda k^{\alpha;\lambda}m_\alpha\bar{m}^\mu + 4\bar{m}^\lambda m^\beta m_\mu k^{\mu;\delta}\bar{m}_\delta m_{\lambda;\beta} = 0\,.
\end{aligned}
$$

 – The term $\cancel{m_\nu m_{\lambda;\alpha}k^\alpha\bar{m}^{\nu;\lambda}}$: decomposing in it, similarly as above,

$$
k^\alpha m_{\lambda;\alpha} = k^\alpha m_{\delta;\alpha}(-k^\delta l_\lambda - l^\delta k_\lambda + m^\delta\bar{m}_\lambda + \bar{m}^\delta m_\lambda) = -k^\alpha m_{\delta;\alpha}l^\delta k_\lambda\,,
$$

 it reads $-m_\nu\bar{m}^{\nu;\lambda}k^\alpha m_{\delta;\alpha}l^\delta k_\lambda$, which contains $m_\nu\bar{m}^{\nu;\lambda}k_\lambda = 0$ (by tetrad rotation).

 – In order to show that the terms $\cancel{m_{\nu;\alpha}k^\alpha m_\lambda\bar{m}^{\nu;\lambda}} + \cancel{l_{\mu;\alpha}k^\alpha m_\lambda k^{\mu;\lambda}}$ cancel against each other, one again writes

$$
m_{\nu;\alpha}k^\alpha = -k^\alpha m_{\beta;\alpha}l^\beta k_\nu\,, \quad m_\lambda k^{\mu;\lambda} = -m_\lambda k^{\beta;\lambda}l_\beta k^\mu + m_\lambda k^{\beta;\lambda}\bar{m}_\beta m^\mu
$$

 and substitutes it into them,

$$
\begin{aligned}
&m_{\nu;\alpha}k^\alpha m_\lambda\bar{m}^{\nu;\lambda} + l_{\mu;\alpha}k^\alpha m_\lambda k^{\mu;\lambda} = \\
&= -k^\alpha m_{\beta;\alpha}l^\beta k_\nu m_\lambda\bar{m}^{\nu;\lambda} + l_{\mu;\alpha}k^\alpha(-\cancel{m_\lambda k^{\beta;\lambda}l_\beta k^\mu} + m_\lambda k^{\beta;\lambda}\bar{m}_\beta m^\mu) = \\
&= -k^\alpha m^\beta l_{\beta;\alpha}k^{\nu;\lambda}m_\lambda\bar{m}_\nu + l_{\mu;\alpha}k^\alpha m_\lambda k^{\beta;\lambda}\bar{m}_\beta m^\mu = 0\,.
\end{aligned}
$$

 – Similarly the term $\cancel{l_\mu m_{\lambda;\alpha}k^\alpha k^{\mu;\lambda}}$: substituting $m_{\lambda;\alpha}k^\alpha = -k^\alpha m_{\delta;\alpha}l^\delta k_\lambda$ from above, it really vanishes (due to $k_\lambda k^{\mu;\lambda} = 0$), $-l_\mu k^\alpha m_{\delta;\alpha}l^\delta k_\lambda k^{\mu;\lambda} = 0$.

Thus only the terms with the second derivatives have remained in (30.70). All these second derivatives can be eliminated.

- First, since the shear $m_\mu m_\beta k^{\mu;\beta}$ is assumed to be zero,

$$\begin{aligned} m_\mu \bar{m}_\lambda k^{\mu;\lambda\beta} m_\beta &= m_\mu \bar{m}_\lambda m_\beta (k^{\mu;\beta\lambda} + C^{\iota\mu\lambda\beta} k_\iota) = \\ &= \cancel{(m_\mu m_\beta k^{\mu;\beta})}^{;\lambda} \bar{m}_\lambda - (m_\mu m_\beta)^{;\lambda} \bar{m}_\lambda k^{\mu;\beta} + m_\mu \bar{m}_\lambda m_\beta C^{\iota\mu\lambda\beta} k_\iota = \\ &= -m_{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\mu;\beta} - \cancel{m_\mu m_{\beta;\lambda} \bar{m}^\lambda k^{\mu;\beta}} + \Psi_1 \,. \end{aligned}$$

  The ~~cancelled~~ term: one again substitutes into it $m_\mu k^{\mu;\beta} = m_\mu k^{\mu;\nu} \bar{m}_\nu m^\beta$, to obtain $m_\mu k^{\mu;\nu} \bar{m}_\nu m^\beta m_{\beta;\lambda} \bar{m}^\lambda = 0$ (because $m^\beta m_{\beta;\lambda} = 0$).

- Second, since one can arrange for $m_\nu k_\alpha \bar{m}^{\nu;\alpha} = 0$ by rotation of the tetrad, one has

$$\begin{aligned} m_\nu m_\lambda \bar{m}^{\nu;\lambda\alpha} k_\alpha &= m_\nu m_\lambda k_\alpha (\bar{m}^{\nu;\alpha\lambda} + C^{\iota\nu\lambda\alpha} \bar{m}_\iota) = \\ &= \cancel{(m_\nu k_\alpha \bar{m}^{\nu;\alpha})}^{;\lambda} m_\lambda - (m_\nu k_\alpha)^{;\lambda} m_\lambda \bar{m}^{\nu;\alpha} + m_\nu m_\lambda k_\alpha C^{\iota\nu\lambda\alpha} \bar{m}_\iota = \\ &= -\cancel{m_{\nu;\lambda} m^\lambda k_\alpha \bar{m}^{\nu;\alpha}} - m_\nu k_{\alpha;\lambda} m^\lambda \bar{m}^{\nu;\alpha} - \Psi_1 \,. \end{aligned}$$

  The ~~cancelled~~ term: substituting in it $k_\alpha \bar{m}^{\nu;\alpha} = -k_\alpha \bar{m}^{\beta;\alpha} l_\beta k^\nu$, one finds it vanishes, $-m_{\nu;\lambda} m^\lambda k_\alpha \bar{m}^{\beta;\alpha} l_\beta k^\nu = 0$ (because of zero shear, $-m_{\nu;\lambda} m^\lambda k^\nu = m_\nu m_\lambda k^{\nu;\lambda} = 0$).

- Third, since $k_\alpha k^{\mu;\alpha} = 0$ (k^μ is geodesic and affine-parameterized),

$$\begin{aligned} l_\mu m_\lambda k^{\mu;\lambda\alpha} k_\alpha &= l_\mu m_\lambda k_\alpha (k^{\mu;\alpha\lambda} + C^{\iota\mu\lambda\alpha} k_\iota) = \\ &= \cancel{(l_\mu k_\alpha k^{\mu;\alpha})}^{;\lambda} m_\lambda - (l_\mu k_\alpha)^{;\lambda} m_\lambda k^{\mu;\alpha} + l_\mu m_\lambda k_\alpha C^{\iota\mu\lambda\alpha} k_\iota = \\ &= -l_{\mu;\lambda} m^\lambda \cancel{k_\alpha k^{\mu;\alpha}} - l_\mu k_{\alpha;\lambda} m^\lambda k^{\mu;\alpha} - \Psi_1 \,. \end{aligned}$$

Substituting all the three results in (30.70) leads to

$$\begin{aligned} &[(\ln \Psi_1)^{;\alpha} k_\alpha]^{;\beta} m_\beta - [(\ln \Psi_1)^{;\beta} m_\beta]^{;\alpha} k_\alpha = \\ &= 4 m_{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\mu;\beta} - m_\nu k_{\alpha;\lambda} m^\lambda \bar{m}^{\nu;\alpha} - l_\mu k_{\alpha;\lambda} m^\lambda k^{\mu;\alpha} - 6\Psi_1 \,. \end{aligned} \tag{30.71}$$

- Big finale: compare the commutator computed by two different routes, (30.69) and (30.71). Equal thus should be the expressions

$$\begin{aligned} &4 m_\mu k^{\mu;\lambda} l^\beta k_{\beta;\lambda} + 4 m_\mu k^{\mu;\lambda} k^\alpha l_{\lambda;\alpha} - \cancel{m_\nu \bar{m}^{\nu;\lambda} m^\alpha k_{\lambda;\alpha}} - \cancel{l_\mu k^{\mu;\lambda} m^\alpha k_{\lambda;\alpha}} = \\ &= 4 m_{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\mu;\beta} - \cancel{m_\nu k_{\alpha;\lambda} m^\lambda \bar{m}^{\nu;\alpha}} - \cancel{l_\mu k_{\alpha;\lambda} m^\lambda k^{\mu;\alpha}} - 6\Psi_1 \,. \end{aligned}$$

Substituting the decomposition $m_\beta k^{\mu;\beta} = -m_\beta k^{\alpha;\beta} l_\alpha k^\mu + m_\beta k^{\alpha;\beta} \bar{m}_\alpha m^\mu$ into the first term on the right, we have

$$-4 m_{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\alpha;\beta} l_\alpha k^\mu + \cancel{4 m_{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\alpha;\beta}} \bar{m}_\alpha m^\mu \,,$$

of which the second term vanishes due to $m_{\mu;\lambda} m^\mu = 0$, while the first term can be rewritten as $4 m_\mu k^{\mu;\lambda} \bar{m}^\lambda m_\beta k^{\alpha;\beta} l_\alpha$ and thus it clearly cancels out with the first term on the left of the above commutator-comparison equation. Hence, the latter reduces to

$$6\Psi_1 = -4 m_\mu k^{\mu;\lambda} k^\alpha l_{\lambda;\alpha} \,. \tag{30.72}$$

- Yet bigger finale:[6] the tetrad has been rotated so that $k_{\mu;\nu}m^\mu l^\nu = 0$, so one obtains from its k-derivative

$$0 = (k_{\mu;\nu}m^\mu l^\nu)_{;\lambda}k^\lambda = k_{\mu;\nu\lambda}m^\mu l^\nu k^\lambda + \cancel{k_{\mu;\nu}m^{\mu;\lambda}l^\nu k_\lambda} + k_{\mu;\nu}m^\mu l^{\nu;\lambda}k_\lambda =$$
$$= (k_{\mu;\lambda\nu} + C^\iota{}_{\mu\nu\lambda}k_\iota)\, m^\mu l^\nu k^\lambda + k_{\mu;\nu}m^\mu l^{\nu;\lambda}k_\lambda = k_{\mu;\lambda\nu}m^\mu l^\nu k^\lambda + C_{kmlk} + k_{\mu;\nu}m^\mu l^{\nu;\lambda}k_\lambda\,,$$

where we have expressed $m^{\mu;\lambda}k_\lambda = -m^{\alpha;\lambda}k_\lambda l_\alpha k^\mu$ and, consequently, cancelled the term $k_{\mu;\nu}m^{\mu;\lambda}l^\nu k_\lambda = -k_{\mu;\nu}l^\nu m^{\alpha;\lambda}k_\lambda l_\alpha k^\mu$ (due to $k_{\mu;\nu}k^\mu = 0$). The second derivative can in fact be omitted as well,

$$k_{\mu;\lambda\nu}m^\mu l^\nu k^\lambda = (\cancel{k_{\mu;\lambda}k^\lambda})_{;\nu}m^\mu l^\nu - \cancel{k_{\mu;\lambda}k^{\lambda;\nu}m^\mu l_\nu}\,,$$

where the second term vanishes because after substituting $k_{\mu;\lambda}m^\mu = k_{\mu;\delta}m^\mu \bar{m}^\delta m_\lambda$ it reads $k_{\mu;\delta}m^\mu \bar{m}^\delta m_\lambda k^{\lambda;\nu}l_\nu$, thus containing $m_\lambda k^{\lambda;\nu}l_\nu = 0$. Therefore, as $C_{kmlk} = -\Psi_1$, we find

$$\Psi_1 = k_{\mu;\nu}m^\mu l^{\nu;\lambda}k_\lambda\,. \tag{30.73}$$

The expression in (30.72) is exactly the same as in (30.73), just with a different numerical factor, so it has to be zero – i.e., $\boxed{\Psi_1 = 0}$.

Concluding remark: By writing $k_{\mu;\nu}m^\mu = k_{\mu;\beta}m^\mu \bar{m}^\beta m_\nu$ again, the vanishing expression reads $(k_{\mu;\beta}m^\mu \bar{m}^\beta)(m_\nu l^{\nu;\lambda}k_\lambda)$. Recall now that in rotating the tetrad in order to ensure $m_\mu l_\lambda k^{\mu;\lambda} = 0$, we needed $k_{\mu;\beta}m^\mu \bar{m}^\beta \neq 0$ (otherwise the whole proof would have been established immediately), which means that the result of the proof actually reads $\boxed{m_\nu l^{\nu;\lambda}k_\lambda = 0}$. Worth to make oneself sure that this is *not* trivial: what can *always* be made vanish by tetrad rotation has been $k_{\mu;\nu}m^\mu l^\nu = -m_{\mu;\nu}k^\mu l^\nu$, whereas here we have found vanishing of $m_\mu l^{\mu;\nu}k_\nu = -m_{\mu;\nu}l^\mu k^\nu$.

One more remark: the proof of the Goldberg-Sachs theorem is usually not being examined in full detail.

30.6 Classification of Ricci and of energy-momentum tensor

Since $R_{\mu\nu}$ and $T_{\mu\nu}$ are bound by Einstein's equations, i.e. they merely differ by terms proportional to $g_{\mu\nu}$, their algebraic classification is equivalent. Actually, if adding a term proportional to $g_{\mu\nu}$ to some second-rank tensor, the eigen-vectors remain the same, while the eigen-values shift accordingly:

$$(T^\mu{}_\nu - f\delta^\mu_\nu)V^\nu = \lambda V^\mu \quad\Longrightarrow\quad T^\mu{}_\nu V^\nu = (\lambda + f)V^\mu\,. \tag{30.74}$$

Various circumstances may occur in the classification of symmetric tensors. Anyway, we will only touch on two most important cases of the energy-momentum tensor.

[6] Another Cimrman's step-aside.

30.6.1 Energy-momentum tensor of EM field

As the EM-field $T_{\mu\nu}$ (30.30) is traceless, its determinant reads, by prescription (A.10),

$$\det(T^{\mu}{}_{\nu}) = \frac{1}{8}\left[(T^{\mu}{}_{\iota}T^{\iota}{}_{\mu})^2 - 2\,T^{\mu}{}_{\iota}T^{\iota}{}_{\kappa}T^{\kappa}{}_{\lambda}T^{\lambda}{}_{\mu}\right]. \tag{30.75}$$

In order to evaluate both terms, it is useful to compute

$$\begin{aligned}(8\pi)^2 T^{\mu}{}_{\iota}T^{\iota}{}_{\nu} &= (F^{\mu\rho}F_{\iota\rho} + {}^*F^{\mu\rho}{}^*F_{\iota\rho})\,(F^{\iota\sigma}F_{\nu\sigma} + {}^*F^{\iota\sigma}{}^*F_{\nu\sigma}) = \\ &= (F^{\mu\rho}F_{\iota\rho} - {}^*F^{\mu\rho}{}^*F_{\iota\rho})\,(F^{\iota\sigma}F_{\nu\sigma} - {}^*F^{\iota\sigma}{}^*F_{\nu\sigma}) + 4F^{\mu\rho}F_{\iota\rho}{}^*F^{\iota\sigma}{}^*F_{\nu\sigma} = \\ &= \frac{1}{4}\delta^{\mu}_{\nu}(F_{\iota\lambda}F^{\iota\lambda})^2 + \frac{1}{4}\delta^{\mu}_{\nu}(F_{\iota\lambda}{}^*F^{\iota\lambda})^2,\end{aligned} \tag{30.76}$$

where we have regarded that in the product of the $F_{\iota\rho}{}^*F^{\iota\sigma}$ type it does not matter where asterisk is placed – see relation (30.8) –, and then equation (30.6) has been used in the parentheses, and equation (30.8) in the last term. Now easily already,

$$(8\pi)^4\left[(T^{\mu}{}_{\iota}T^{\iota}{}_{\mu})^2 - 2\,T^{\mu}{}_{\iota}T^{\iota}{}_{\kappa}T^{\kappa}{}_{\lambda}T^{\lambda}{}_{\mu}\right] = \left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right]^2 - \frac{1}{2}\,[\text{the same}]^2\,,$$

hence

$$\det(T^{\mu}{}_{\nu}) = \frac{1}{16(8\pi)^4}\left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right]^2\,. \tag{30.77}$$

From (30.76) and from tracelessness of $T^{\mu}{}_{\mu}$ it also follows that the "triple" term $T^{\mu}{}_{\iota}T^{\iota}{}_{\nu}T^{\nu}{}_{\mu}$ vanishes as well, so the characteristic equation (30.4)$=0$ reduces to

$$\begin{aligned}&\lambda^4 - \frac{1}{2}T^{\mu}{}_{\iota}T^{\iota}{}_{\mu}\lambda^2 + \det(T^{\mu}{}_{\nu}) = \\ &= \lambda^4 - \frac{1}{2(8\pi)^2}\left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right]\lambda^2 + \frac{1}{16(8\pi)^4}\left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right]^2 = \\ &= \left\{\lambda^2 - \frac{1}{4(8\pi)^2}\left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right]\right\}^2 = 0\,.\end{aligned} \tag{30.78}$$

The eigen-values are thus given by

$$(\lambda^{\mathrm{T}}_{\pm})^2 = \frac{1}{(16\pi)^2}\left[(F_{\iota\lambda}F^{\iota\lambda})^2 + (F_{\iota\lambda}{}^*F^{\iota\lambda})^2\right], \tag{30.79}$$

both of them being double degenerate.

Comparison of this result with the eigen-values (30.13) and (30.15) of $F^{\mu}{}_{\nu}$ and ${}^*F^{\mu}{}_{\nu}$ reveals the simple relation

$$\pm\lambda^{\mathrm{T}}_{\pm} = \frac{1}{8\pi}\left[(\lambda_{\pm})^2 + ({}^*\lambda_{\pm})^2\right] \tag{30.80}$$

which perfectly mirrors the structure of $T_{\mu\nu}$. Sure, one can verify this straightforwardly: being k^{ν} the common eigen-vector of $F^{\mu}{}_{\nu}$ and ${}^*F^{\mu}{}_{\nu}$, one has

$$8\pi T^{\mu}{}_{\nu}k^{\nu} = (F^{\mu\iota}F_{\nu\iota} + {}^*F^{\mu\iota}{}^*F_{\nu\iota})\,k^{\nu} = -F^{\mu\iota}\lambda_{\pm}k_{\iota} - {}^*F^{\mu\iota}{}^*\lambda_{\mp}k_{\iota} =$$

$$= -\left[(\lambda_\pm)^2 + ({}^*\lambda_\mp)^2\right] k^\mu \equiv \mp\lambda^{\mathrm{T}}_\pm\, k^\mu \,. \tag{30.81}$$

In summary, the tensors $F^\mu{}_\nu$, ${}^*F^\mu{}_\nu$ and $T^\mu{}_\nu$ share two null eigen-vectors which are different in general whereas they coincide for a null (N) field (in that case, all the eigen-values are zero).

30.6.2 Energy-momentum tensor of ideal fluid

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu_\nu = \rho u^\mu u_\nu + P h^\mu_\nu \tag{30.82}$$

has

$$\begin{aligned} &T \equiv T^\mu{}_\mu = -\rho + 3P\,, \qquad T^\mu{}_\iota T^\iota{}_\mu = \rho^2 + 3P^2\,,\\ &T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\mu = -\rho^3 + 3P^3\,, \qquad T^\mu{}_\iota T^\iota{}_\kappa T^\kappa{}_\lambda T^\lambda{}_\mu = \rho^4 + 3P^4\,, \end{aligned} \tag{30.83}$$

so (A.10) yields the determinant

$$\det(T^\mu{}_\nu) = -\rho P^3 \tag{30.84}$$

and from (30.4) we get the characteristic equation

$$\lambda^4 + (\rho - 3P)\lambda^3 - 3P(\rho - P)\lambda^2 + P^2(3\rho - P)\lambda - \rho P^3 = (\lambda + \rho)(\lambda - P)^3 = 0\,. \tag{30.85}$$

Therefore, the eigen-value $\lambda = P$ is triple while the eigen-value $\lambda = -\rho$ is simple.

The eigen-vectors also follow very easily: from the equation $(\rho + P)u^\mu u_\nu V^\nu + PV^\mu = -\rho V^\mu$ we have $(\rho + P)h^\mu_\nu V^\nu = 0$, so the eigen-direction associated with $\lambda = -\rho$ is the fluid's four-velocity u^μ itself; on the other hand, the equation $(\rho + P)u^\mu u_\nu V^\nu + PV^\mu = PV^\mu$ implies $(\rho + P)u^\mu u_\nu V^\nu = 0$, so the eigen-value $\lambda = P$ is accorded with the whole eigen-hyperplane orthogonal to u^μ (three independent eigen-vectors can be identified within it at any point).

APPENDIX A

Tensor densities, duals and volumes

In GR, one is celebrating tensors due to their invariant character, but not all quantities can of course be represented by tensors (like not all scalars are invariants – see mass-energy, for example). Neither the principle of general covariance demands that it be so – it only demands that fundamental physical equations *as a whole* be covariant. Below, we note one particular class of non-tensorial (though "almost tensorial") quantities, the so-called **tensor densities**.

Motivation for the study of densities stems from volume integrations. One certainly likes to integrate *tensors* over *proper* volumes, areas, etc., yet some mathematical rules – such as the Gauss theorem – apply to integration over *coordinate* domains. Below, we will see the proper volumes differ from the coordinate ones by a factor given by the metric determinant, which is why the transformation property of the latter will be our starting point:

A.1 Metric determinant and tensor densities

Since (determinant of a product) = (product of determinants), the determinant of the covariant metric tensor $g := |g_{\mu\nu}|$ transforms as

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma} \quad \Longrightarrow \quad g' = \left|\frac{\partial x}{\partial x'}\right|^2 g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g\,,$$

where the second arrangement follows from

$$\frac{\partial x'^\alpha}{\partial x^\iota}\frac{\partial x^\iota}{\partial x'^\beta} = \delta^\alpha_\beta \quad \Longrightarrow \quad \left|\frac{\partial x'}{\partial x}\right|\left|\frac{\partial x}{\partial x'}\right| = 1\,.$$

We say that g is a **scalar density of weight** $w=2$ – it transforms like an invariant, except that the w-th power of the inverse-transform Jacobian appears there.[1] The nomenclature naturally generalizes to multi-component quantities:

[1] Some authors use the opposite sign, $w=-2$, referring to the power of the *direct*-transform Jacobian which appears in the transformation formula. We adhere to the choice which implies that the quantities containing $(\sqrt{-g})^w$ are of weight w. Actually, the latter is the most important case, specifically with $w=1$, because it occurs in the study of integrals of tensors over proper volumes – see below.

Definition We call **tensor density of weight** w (w integer) the quantity[2] $\mathfrak{T}^{\cdots}_{\cdots}$ which transforms as a tensor (of the respective type), except that the transformation also involves Jacobi determinant of the inverse coordinate change in the w-th power, i.e.

$$\mathfrak{T}'^{\cdots}_{\cdots}(x') = \left|\frac{\partial x}{\partial x'}\right|^{w} \frac{\partial x'^{\cdot}}{\partial x^{\cdot}} \cdots \frac{\partial x^{\cdot}}{\partial x'^{\cdot}} \cdots \; \mathfrak{T}^{\cdots}_{\cdots}(x) = \left|\frac{\partial x'}{\partial x}\right|^{-w} \frac{\partial x'^{\cdot}}{\partial x^{\cdot}} \cdots \frac{\partial x^{\cdot}}{\partial x'^{\cdot}} \cdots \; \mathfrak{T}^{\cdots}_{\cdots}(x)\,. \tag{A.1}$$

Just by looking at the above definition and the transformation of g, we infer the following

Lemma From an arbitrary tensor density, one can make a tensor (of the respective type) by multiplying it by $g^{-w/2}$.
Proof: One simply multiplies the transformation (A.1) by $(g')^{-w/2} = \left|\frac{\partial x}{\partial x'}\right|^{-w} g^{-w/2}$.
Note: in the case of Lorentzian (indefinite) metric, one has to take $(-g)^{-w/2}$ actually (at least for odd w); one often writes $|g|$ for generality.

A.1.1 Levi-Civita pseudo-tensor and determinants

Be $[\alpha\beta\gamma\delta]$ the permutation symbol, i.e. the object which is anti-symmetric in all its indices and whose all components are, in any coordinate system, only 0, $+1$ and -1; and let the sign be fixed by $[0123]=+1$. Recalling the definition of a Jacobian,

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{\partial x'^0}{\partial x^\mu}\frac{\partial x'^1}{\partial x^\nu}\frac{\partial x'^2}{\partial x^\kappa}\frac{\partial x'^3}{\partial x^\lambda}\,[\mu\nu\kappa\lambda]\,,$$

and using the fact that a determinant is "anti-symmetric" with respect to exchange of any two columns or rows, we may also write

$$\left|\frac{\partial x'}{\partial x}\right|[\alpha\beta\gamma\delta] = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x'^\gamma}{\partial x^\kappa}\frac{\partial x'^\delta}{\partial x^\lambda}\,[\mu\nu\kappa\lambda]\,.$$

Therefore, $[\alpha\beta\gamma\delta]$ transforms as a (4,0)-type tensor density of weight $w=1$. Make a tensor out of it according to the above Lemma:

$$\epsilon^{\alpha\beta\gamma\delta} \equiv -\frac{1}{\sqrt{-g}}\,[\alpha\beta\gamma\delta]\,. \tag{A.2}$$

This is the covariant Levi-Civita tensor. Minus in front is just a convention; this our one fixes $\epsilon^{0123}=-\epsilon^{123}\equiv -1$ (thus $\epsilon_{0123}=\epsilon_{123}\equiv +1$) for the Minkowski metric.

In the same way as for the Jacobian, we obtain, from the formula for the determinant of the covariant metric, $g = g_{0\mu}g_{1\nu}g_{2\kappa}g_{3\lambda}[\mu\nu\kappa\lambda]$, the relation

$$g\,[\mu\nu\kappa\lambda] = g_{\mu\alpha}g_{\nu\beta}g_{\kappa\gamma}g_{\lambda\delta}[\alpha\beta\gamma\delta]\,.$$

Owing to that, we get for the covariant Levi-Civita tensor, by just lowering indices of $\epsilon^{\alpha\beta\gamma\delta}$,

$$\epsilon_{\mu\nu\kappa\lambda} \equiv g_{\mu\alpha}g_{\nu\beta}g_{\kappa\gamma}g_{\lambda\delta}\,\epsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}\,g_{\mu\alpha}g_{\nu\beta}g_{\kappa\gamma}g_{\lambda\delta}[\alpha\beta\gamma\delta] = \sqrt{-g}\,[\mu\nu\kappa\lambda]\,. \tag{A.3}$$

[2] It is a habit in the literature to write the tensor densities in gothic fonts, so we also adhere to it.

Often useful (and always welcome :-)) are inner products of two epsilons over various possible number of indices,

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\kappa\lambda} = -4!\,\delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\kappa}\delta^{\delta]}_{\lambda}\,, \tag{A.4}$$
$$\epsilon^{\alpha\beta\gamma\lambda}\epsilon_{\mu\nu\kappa\lambda} = -3!\,\delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma]}_{\kappa}\,, \tag{A.5}$$
$$\epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda} = -(2!)^2\,\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\,, \tag{A.6}$$
$$\epsilon^{\alpha\nu\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda} = -3!\,\delta^{\alpha}_{\mu}\,, \tag{A.7}$$
$$\epsilon^{\mu\nu\kappa\lambda}\epsilon_{\mu\nu\kappa\lambda} = -4!\,. \tag{A.8}$$

In general, they can be summarized as

$$\epsilon^{\alpha_1\dots\alpha_k\lambda_1\dots\lambda_{4-k}}\,\epsilon_{\mu_1\dots\mu_k\lambda_1\dots\lambda_{4-k}} = -k!\,(4-k)!\,\delta^{[\alpha_1}_{\mu_1}\dots\delta^{\alpha_{4-k}]}_{\mu_{4-k}} \qquad (k \leqslant 4)\,. \tag{A.9}$$

For antisymmetrized products of Kronecker deltas, often introduced is the *generalized Kronecker symbol*, $\delta^{\alpha_1\dots\alpha_n}_{\mu_1\dots\mu_n} \equiv n!\,\delta^{[\alpha_1}_{\mu_1}\delta^{\cdot}_{\cdot}\delta^{\cdot}_{\cdot}\delta^{\cdot}_{\cdot}\delta^{\alpha_n]}_{\mu_n}$. Using this symbol, one can easily write the determinant of a generic matrix (4x4); by evaluating the individual terms (they are 24), one may also express such a determinant in terms of traces of M and of its powers:

$$\begin{aligned}
&\det(M\,\mathrm{mixed}) = \\
&= \frac{1}{4!}\,M_\mu{}^\alpha M_\nu{}^\beta M_\kappa{}^\gamma M_\lambda{}^\delta\;\delta^{\mu\nu\kappa\lambda}_{\alpha\beta\gamma\delta} = \\
&= M_\mu{}^\alpha M_\nu{}^\beta M_\kappa{}^\gamma M_\lambda{}^\delta\;\delta^{[\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\kappa}_{\gamma}\delta^{\lambda]}_{\delta} = M_{[\mu}{}^\alpha M_\nu{}^\beta M_\kappa{}^\gamma M_{\lambda]}{}^\delta\;\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\kappa}_{\gamma}\delta^{\lambda}_{\delta} = \\
&= \frac{1}{4!}\left[(\mathrm{Tr}\,M)^4 - 6\,(\mathrm{Tr}\,M)^2\,\mathrm{Tr}\,M^2 + 8\,\mathrm{Tr}\,M\;\mathrm{Tr}\,M^3 - 6\,\mathrm{Tr}\,M^4 + 3\,(\mathrm{Tr}\,M^2)^2\right].
\end{aligned} \tag{A.10}$$

Since, again, determinant of a matrix product = product of matrix determinants, the determinant of a (corresponding) fully covariant or fully contravariant matrix is obtained (respectively) by multiplying or dividing the above result by g. Note that if the mixed matrix represents a mixed tensor, the traces of all its powers are invariants, so the determinant is invariant as well. Consequently, the determinant of a fully covariant, resp. fully contravariant second-rank tensor is a scalar density of weight $+2$, resp. -2.

Why "*pseudo*-tensor"?

The Levi-Civita tensor, strictly speaking, is a *pseudo*-tensor – a quantity which behaves as a tensor, just that in its transformation appears the sign of the transformation Jacobian. This is because the term $\sqrt{-g}$ in the definition: the latter transforms like $g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$, hence

$$\frac{1}{\sqrt{-g'}} = \mathrm{abs}\left|\frac{\partial x'}{\partial x}\right|\frac{1}{\sqrt{-g}}\,,$$

and so, after division by the Jacobian which appears in the transformation of the permutation symbol itself, the sign is really left in the overall transformation,

$$\frac{\mathrm{abs}\left|\frac{\partial x'}{\partial x}\right|}{\left|\frac{\partial x'}{\partial x}\right|} = \mathrm{sign}\left|\frac{\partial x'}{\partial x}\right|\,.$$

This implies that pseudo-tensors are also all other tensor-like quantities introduced using $\sqrt{-g}$, in particular those containing $\epsilon^{\alpha\beta\gamma\delta}$ (linearly), as e.g. the duals to antisymmetric tensors.

A.1.2 Differentiation of tensor densities

Let $\mathfrak{T}^{\cdots}_{\cdots}$ be a tensor density of some tensor type and weight w, and let $T^{\cdots}_{\cdots}$ be the corresponding tensor; we know their relation is $T = \mathfrak{T}\,|g|^{-w/2}$. Since derivatives should satisfy the Leibniz rule, and since $|g|_{;\alpha} = 0$ while $|g|_{,\alpha} = 2|g|\Gamma^{\mu}{}_{\mu\alpha}$ by equation (5.15), we have (indices are suppressed)

$$
\begin{aligned}
T_{;\alpha} &= \mathfrak{T}_{;\alpha}\,|g|^{-w/2} - \mathfrak{T}\,\frac{w}{2|g|}\,|g|^{-w/2}\cancel{|g|_{;\alpha}} = \mathfrak{T}_{;\alpha}\,|g|^{-w/2}, \\
T_{,\alpha} &= \mathfrak{T}_{,\alpha}\,|g|^{-w/2} - \mathfrak{T}\,\frac{w}{2|g|}\,|g|^{-w/2}|g|_{,\alpha} = \mathfrak{T}_{,\alpha}\,|g|^{-w/2} - \mathfrak{T}w\,|g|^{-w/2}\,\Gamma^{\mu}{}_{\mu\alpha} \\
&\Longrightarrow\;\; \mathfrak{T}_{,\alpha} = T_{,\alpha}\,|g|^{w/2} + \mathfrak{T}w\,\Gamma^{\mu}{}_{\mu\alpha}\,. \qquad \text{(A.11)}
\end{aligned}
$$

The Lie-derivative action on tensor densities can be inferred from its action on the metric determinant, with the latter in turn derived from the formula (23.8) for a generic variation of $\sqrt{-g}$, i.e. $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\;g^{\mu\nu}\delta g_{\mu\nu}$. Actually, different derivatives of quantities are determined by the latter's variations under different types of transport. Similarly as the formula yields $(\sqrt{-g})_{,\rho} = \frac{1}{2}\sqrt{-g}\;g^{\mu\nu}g_{\mu\nu,\rho} = \sqrt{-g}\,\Gamma^{\mu}{}_{\mu\iota}$ for the partial derivative of $\sqrt{-g}$ (and zero for its Levi-Civita covariant derivative), for the Lie derivative one has

$$
\pounds_{\xi}\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\;g^{\mu\nu}\pounds_{\xi}g_{\mu\nu} = \frac{1}{2}\sqrt{|g|}\;g^{\mu\nu}(\xi_{\mu;\nu}+\xi_{\nu;\mu}) = \sqrt{|g|}\,\xi^{\mu}{}_{;\mu} = \left(\sqrt{|g|}\,\xi^{\mu}\right)_{,\mu}. \quad \text{(A.12)}
$$

The knowledge of $\pounds_{\xi}\sqrt{|g|}$ allows to fix the Lie derivative of any tensor density: using again the relation $\mathfrak{T} = T\,|g|^{w/2}$, with $\mathfrak{T}$ the density and T the tensor (indices omitted), one has

$$
\begin{aligned}
\pounds_{\xi}\mathfrak{T} &= |g|^{w/2}\pounds_{\xi}T + T\,\frac{w}{2|g|}\,|g|^{w/2}\pounds_{\xi}|g| = |g|^{w/2}\pounds_{\xi}T + T\,\frac{w}{|g|}\,|g|^{w/2}\sqrt{|g|}\,\pounds_{\xi}\sqrt{|g|} = \\
&= |g|^{w/2}\pounds_{\xi}T + Tw\,|g|^{w/2}\xi^{\iota}{}_{;\iota} = |g|^{w/2}\pounds_{\xi}T + \mathfrak{T}w\,\xi^{\iota}{}_{;\iota}\,. \qquad \text{(A.13)}
\end{aligned}
$$

Similarly as for tensors, this result assumes the same form whether written in terms of covariant or partial derivatives. Actually, expressing above $\pounds_{\xi}T$ in terms of covariant derivatives,

$$
\pounds_{\xi}T^{\mu\ldots}_{\alpha\ldots} = T^{\mu\ldots}_{\alpha\ldots;\iota}\xi^{\iota} - \sum\xi^{\mu}{}_{;\iota}T^{\iota\ldots}_{\alpha\ldots} + \sum\xi^{\iota}{}_{;\alpha}T^{\mu\ldots}_{\iota\ldots}\,,
$$

one has

$$
\begin{aligned}
\pounds_{\xi}\mathfrak{T}^{\mu\ldots}_{\alpha\ldots} &= \mathfrak{T}^{\mu\ldots}_{\alpha\ldots;\iota}\xi^{\iota} - \sum\xi^{\mu}{}_{;\iota}\mathfrak{T}^{\iota\ldots}_{\alpha\ldots} + \sum\xi^{\iota}{}_{;\alpha}\mathfrak{T}^{\mu\ldots}_{\iota\ldots} + \mathfrak{T}^{\mu\ldots}_{\alpha\ldots}w\,\xi^{\iota}{}_{;\iota} = \\
&= [\text{same covariant-derivative formula as for tensor}] + \mathfrak{T}^{\mu\cdots}_{\alpha\ldots}w\,\xi^{\iota}{}_{;\iota}\,, \qquad \text{(A.14)}
\end{aligned}
$$

while expressing $\pounds_{\xi}T$ in terms of partial derivatives,

$$
\pounds_{\xi}T^{\mu\ldots}_{\alpha\ldots} = T^{\mu\ldots}_{\alpha\ldots,\iota}\xi^{\iota} - \sum\xi^{\mu}{}_{,\iota}T^{\iota\ldots}_{\alpha\ldots} + \sum\xi^{\iota}{}_{,\alpha}T^{\mu\ldots}_{\iota\ldots}\,,
$$

leads to

$$\begin{aligned}\pounds_\xi \mathfrak{T}^{\mu\ldots}_{\alpha\ldots} &= \mathfrak{T}^{\mu\ldots}_{\alpha\ldots,\iota}\xi^\iota - \mathfrak{T}^{\mu\ldots}_{\alpha\ldots} w\,\Gamma^\kappa{}_{\kappa\iota}\xi^\iota - \sum \xi^\mu{}_{,\iota}\mathfrak{T}^{\iota\ldots}_{\alpha\ldots} + \sum \xi^\iota{}_{,\alpha}\mathfrak{T}^{\mu\ldots}_{\iota\ldots} + \mathfrak{T}^{\mu\ldots}_{\alpha\ldots} w\,\xi^\iota{}_{;\iota} = \\ &= [\text{same partial-derivative formula as for tensor}] + \mathfrak{T}^{\mu\ldots}_{\alpha\ldots} w\,\xi^\iota{}_{,\iota}\,, \qquad \text{(A.15)}\end{aligned}$$

where the relation (A.11) has been employed in the first term, i.e.

$$T^{\mu\ldots}_{\alpha\ldots,\iota}|g|^{w/2} = \mathfrak{T}^{\mu\ldots}_{\alpha\ldots,\iota} - \mathfrak{T}^{\mu\ldots}_{\alpha\ldots} w\,\Gamma^\kappa{}_{\kappa\iota}\,.$$

Specifically for the $w=1$ densities, the first and the last terms can be joined, obtaining

$$\begin{aligned}\pounds_\xi \mathfrak{T} &= (\mathfrak{T}\,\xi^\iota)_{;\iota} + [\text{products of } \mathfrak{T} \text{ with covariant gradients of } \xi^\mu] \qquad \text{(A.16)}\\ &= (\mathfrak{T}\,\xi^\iota)_{,\iota} + [\text{products of } \mathfrak{T} \text{ with partial gradients of } \xi^\mu]\,, \qquad \text{(A.17)}\end{aligned}$$

and, still more specifically, for the $w=1$ *scalar* densities this implies the very useful result

$$\pounds_\xi \mathfrak{T} = (\mathfrak{T}\,\xi^\iota)_{;\iota} = (\mathfrak{T}\,\xi^\iota)_{,\iota}\,. \qquad \text{(A.18)}$$

Clearly $\mathfrak{T}\,\xi^\iota$ is a *vector* density, so the last sub-result can also be voiced such that for vector densities of weight $w=1$ the covariant and partial divergences are equal.

Last general observation: it is obvious from the formulas that – similarly as with tensors – *the Lie differentiation does not change the type of tensor densities* (it leaves both their tensor type and weight unchanged).

A.1.3 Differentiation of the Levi-Civita tensor and of metric determinant by a generic connection

The Levi-Civita tensor is solely given by the metric and by pure numerical factors $[\mu\nu\kappa\lambda]$, so its covariant derivative has to vanish. Let us remind that this is however a special property of the Levi-Civita connection (for which $g_{\mu\nu;\rho}=0$). How would the differentiation work for a *generic* affine connection (not related to the metric)? According to the covariant differentiation prescription,

$$\begin{aligned}\epsilon_{\mu\nu\kappa\lambda;\rho} &\equiv \epsilon_{\mu\nu\kappa\lambda,\rho} - \Gamma^\iota{}_{\rho\mu}\epsilon_{\iota\nu\kappa\lambda} - \Gamma^\iota{}_{\rho\nu}\epsilon_{\mu\iota\kappa\lambda} - \Gamma^\iota{}_{\rho\kappa}\epsilon_{\mu\nu\iota\lambda} - \Gamma^\iota{}_{\rho\lambda}\epsilon_{\mu\nu\kappa\iota} = \\ &= (\sqrt{-g})_{,\rho}[\mu\nu\kappa\lambda] - \sqrt{-g}\,\left(\Gamma^\iota{}_{\rho\mu}[\iota\nu\kappa\lambda] + \Gamma^\iota{}_{\rho\nu}[\mu\iota\kappa\lambda] + \Gamma^\iota{}_{\rho\kappa}[\mu\nu\iota\lambda] + \Gamma^\iota{}_{\rho\lambda}[\mu\nu\kappa\iota]\right).\end{aligned}$$

For the relation to be non-trivial, (μ,ν,κ,λ) must assume different values. Consequently, in all the Γ terms, ι has to assume the same value as the index it has substituted in the permutation symbol (i.e., respectively, μ, ν, κ and λ). And, in each of the four terms, ι substituted a *different* index, so it must just run through all the four possible different values in total. The relation can thus be written as

$$\epsilon_{\mu\nu\kappa\lambda;\rho} = \left[(\sqrt{-g})_{,\rho} - \sqrt{-g}\,\Gamma^\iota{}_{\rho\iota}\right][\mu\nu\kappa\lambda]\,.$$

Should the Leibniz rule be satisfied,

$$\epsilon_{\mu\nu\kappa\lambda;\rho} = (\sqrt{-g})_{;\rho}[\mu\nu\kappa\lambda] + \sqrt{-g}\,\cancel{[\mu\nu\kappa\lambda]_{;\rho}}\,,$$

the metric determinant has to be differentiated according to

$$(\sqrt{-g})_{;\rho} = (\sqrt{-g})_{,\rho} - \sqrt{-g}\,\Gamma^{\iota}{}_{\rho\iota}\,. \tag{A.19}$$

We will show now that such a rule can also be obtained directly from the generic formula for the differentiation of a determinant (of a square matrix M),

$$\frac{(\det M)_{,\rho}}{\det M} = \mathrm{Tr}\left(M^{-1}\cdot M_{,\rho}\right)\,,$$

if applied for the *covariant* derivative rather than the partial one. For the metric determinant, one thus has

$$(-g)_{;\rho} = (-g)g^{\alpha\beta}g_{\alpha\beta;\rho} \qquad \Longrightarrow \qquad (\sqrt{-g})_{;\rho} = \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta;\rho}\,. \tag{A.20}$$

Actually, by directly writing out $g_{\alpha\beta;\rho}$, one finds

$$\begin{aligned}(\sqrt{-g})_{;\rho} &= \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\left(g_{\alpha\beta,\rho} - \Gamma^{\iota}{}_{\rho\alpha}g_{\iota\beta} - \Gamma^{\iota}{}_{\rho\beta}g_{\alpha\iota}\right) = \\ &= \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta,\rho} - \frac{1}{2}\sqrt{-g}\left(\delta^{\alpha}_{\iota}\Gamma^{\iota}{}_{\rho\alpha} + \delta^{\beta}_{\iota}\Gamma^{\iota}{}_{\rho\beta}\right) = (\sqrt{-g})_{,\rho} - \sqrt{-g}\,\Gamma^{\iota}{}_{\rho\iota}\,,\end{aligned}$$

where we used the similar partial-derivative relation $(\sqrt{-g})_{,\rho} = \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta,\rho}$ obtained in Section 23.4.1, equation (23.8). Hence the generic correspondence

$$(\sqrt{-g})_{,\rho} = \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta,\rho} \qquad \longleftrightarrow \qquad (\sqrt{-g})_{;\rho} = \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}g_{\alpha\beta;\rho}\,. \tag{A.21}$$

The above finding implies that for a torsion-free, symmetric connection, one has

$$\begin{aligned}(V^{\rho}\sqrt{-g})_{;\rho} &= V^{\rho}{}_{;\rho}\sqrt{-g} + V^{\rho}(\sqrt{-g})_{;\rho} = \\ &= (V^{\rho}{}_{,\rho} + \Gamma^{\rho}{}_{\rho\iota}V^{\iota})\sqrt{-g} + V^{\rho}\left[(\sqrt{-g})_{,\rho} - \sqrt{-g}\,\Gamma^{\iota}{}_{\rho\iota}\right] = \\ &= V^{\rho}{}_{,\rho}\sqrt{-g} + V^{\rho}(\sqrt{-g})_{,\rho} + \sqrt{-g}\,V^{\iota}\cancel{(\Gamma^{\rho}{}_{\rho\iota} - \Gamma^{\rho}{}_{\iota\rho})} = (V^{\rho}\sqrt{-g})_{,\rho}\,.\end{aligned} \tag{A.22}$$

This result means that the Gauss theorem is *not* only restricted to the divergence performed by the Levi-Civita connection. For the Levi-Civita connection, one just specifically obtains the usual GR relation $\sqrt{-g}\,V^{\lambda}{}_{;\lambda} = (\sqrt{-g}\,V^{\lambda})_{,\lambda}$, because anything solely made of metric is constant with respect to it.

A.2 Duals to antisymmetric tensors

$\boxed{\text{Definition}}$ Have a totally antisymmetric tensor $T_{\mu_1\dots\mu_k}$ $(2\leqslant k\leqslant 4)$. The (pseudo-)tensor

$${}^*T^{\alpha_1\dots\alpha_{4-k}} := \frac{1}{k!}\,\epsilon^{\alpha_1\dots\alpha_{4-k}\mu_1\dots\mu_k}T_{\mu_1\dots\mu_k} \tag{A.23}$$

we call its (**Hodge**) **dual**. The dual is of the $(4–k)$-th rank and is as well totally antisymmetric.

- Apparently, we might consider **dual of a dual** – but without reaching anything new, because, according to the above definition,

$$
\begin{aligned}
{}^{**}T_{\nu_1\ldots\nu_k} &\equiv \frac{1}{(4-k)!}\,\epsilon_{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{4-k}}\,{}^*T^{\alpha_1\ldots\alpha_{4-k}} = \\
&= \frac{1}{(4-k)!\,k!}\,\epsilon_{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{4-k}}\epsilon^{\alpha_1\ldots\alpha_{4-k}\mu_1\ldots\mu_k}T_{\mu_1\ldots\mu_k} = \\
&= -(-1)^{k(4-k)}T_{\nu_1\ldots\nu_k} = (-1)^{k-1}T_{\nu_1\ldots\nu_k}\ ,
\end{aligned}
\tag{A.24}
$$

  where we have used (A.9) in evaluation of the product of Levi-Civita tensors.

- Particular cases:

$$
k=2: \quad {}^*T^{\alpha\beta} \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\mu\nu}T_{\mu\nu}\ , \qquad {}^{**}T_{\rho\sigma} \equiv \frac{1}{2}\,\epsilon_{\rho\sigma\alpha\beta}\,{}^*T^{\alpha\beta} = -T_{\rho\sigma} \tag{A.25}
$$

$$
k=3: \quad {}^*T^{\alpha} \equiv \frac{1}{6}\,\epsilon^{\alpha\mu\nu\kappa}T_{\mu\nu\kappa}\ , \qquad {}^{**}T_{\rho\sigma\tau} \equiv \epsilon_{\rho\sigma\tau\alpha}\,{}^*T^{\alpha} = T_{\rho\sigma\tau} \tag{A.26}
$$

$$
k=4: \quad {}^*T \equiv \frac{1}{24}\,\epsilon^{\mu\nu\kappa\lambda}T_{\mu\nu\kappa\lambda}\ , \qquad {}^{**}T_{\rho\sigma\tau\omega} \equiv \epsilon_{\rho\sigma\tau\omega}\,{}^*T = -T_{\rho\sigma\tau\omega}\ . \tag{A.27}
$$

  The $k=2$ case is e.g. related to the EM-field tensor. The $k=4$ case is "trivial" actually, since the fourth-rank totally antisymmetric tensor must simply be proportional to the Levi-Civita tensor.

- However, most important for GR is the dual of the Riemann tensor. Since the latter is antisymmetric in two pairs of indices, it is actually possible to define *two* duals (different in general), the **left dual** and the ... **right dual**, right,

$$
{}^*R^{\alpha\beta}{}_{\kappa\lambda} \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}R_{\rho\sigma\kappa\lambda}\ , \qquad R^*{}_{\alpha\beta}{}^{\kappa\lambda} \equiv \frac{1}{2}\,R_{\alpha\beta\rho\sigma}\epsilon^{\rho\sigma\kappa\lambda}\ . \tag{A.28}
$$

- Well, yes, one may thus consider a **double dual of Riemann** as well. Enjoy the application of relation (A.4):

$$
\begin{aligned}
&{}^*R^{*\alpha\beta}{}_{\mu\nu} := \frac{1}{4}\,\epsilon^{\alpha\beta\rho\sigma}R_{\rho\sigma}{}^{\kappa\lambda}\epsilon_{\kappa\lambda\mu\nu} = -\frac{1}{4}\,4!\ \delta^{[\alpha}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\rho}_{\mu}\delta^{\sigma]}_{\nu}\,R_{\rho\sigma}{}^{\kappa\lambda} = \\
&= -\frac{1}{4}\,R_{\rho\sigma}{}^{\kappa\lambda}\ \times \\
&\quad \times\big(\delta^{\alpha}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\alpha}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\beta}_{\mu}\delta^{\rho}_{\nu} + \delta^{\alpha}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\beta}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\rho}_{\mu}\delta^{\alpha}_{\nu} + \delta^{\beta}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\rho}_{\nu} + \\
&\qquad + \delta^{\rho}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\rho}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\rho}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\alpha}_{\nu} + \delta^{\sigma}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\rho}_{\mu}\delta^{\beta}_{\nu} + \delta^{\sigma}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\rho}_{\nu} + \delta^{\sigma}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - \\
&\qquad - \delta^{\alpha}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\rho}_{\nu} - \delta^{\alpha}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\beta}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\alpha}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\rho}_{\mu}\delta^{\beta}_{\nu} - \delta^{\beta}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\alpha}_{\nu} - \delta^{\beta}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\beta}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\rho}_{\nu} - \\
&\qquad - \delta^{\rho}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\sigma}_{\mu}\delta^{\beta}_{\nu} - \delta^{\rho}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\rho}_{\kappa}\delta^{\sigma}_{\lambda}\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - \delta^{\sigma}_{\kappa}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\delta^{\rho}_{\nu} - \delta^{\sigma}_{\kappa}\delta^{\rho}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\sigma}_{\kappa}\delta^{\beta}_{\lambda}\delta^{\rho}_{\mu}\delta^{\alpha}_{\nu}\big) = \\
&= -R^{\alpha\beta}{}_{\mu\nu} + 2R^{[\alpha}_{\mu}\delta^{\beta]}_{\nu} + 2\delta^{[\alpha}_{\mu}R^{\beta]}_{\nu} - R\,\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}\ .
\end{aligned}
\tag{A.29}
$$

  Introducing the traceless part of Ricci $S^{\alpha}_{\mu} := R^{\alpha}_{\mu} - \frac{1}{4}R\delta^{\alpha}_{\mu}$, one can also rewrite the above as

$$
{}^*R^{*\alpha\beta}{}_{\mu\nu} + R^{\alpha\beta}{}_{\mu\nu} = 2S^{[\alpha}_{\mu}\delta^{\beta]}_{\nu} + 2\delta^{[\alpha}_{\mu}S^{\beta]}_{\nu} = 2E^{\alpha\beta}{}_{\mu\nu}\ , \tag{A.30}
$$

where $E^{\alpha\beta}{}_{\mu\nu}$ we know from the Riemann-tensor decomposition (8.5). This relation is called the Ruse-Lanczos identity. It specifically reduces to ${}^{*}R^{*\alpha\beta}{}_{\mu\nu} + R^{\alpha\beta}{}_{\mu\nu} = 0$ if $T_{\mu\nu} = 0$ (whence $S_{\mu\nu} = 0$). The double dual has the same symmetries as Riemann itself. Let us show another two properties:

- By contraction over $\binom{\alpha}{\mu}$ one finds easily that

$$ {}^{*}R^{*\mu\beta}{}_{\mu\nu} = R^{\beta}_{\nu} - \frac{1}{2}R\delta^{\beta}_{\nu} \ \left(\equiv G^{\beta}_{\nu}\right) \qquad \text{... the } \mathbf{Einstein\ tensor}\,. \tag{A.31} $$

- Divergence of the double dual is zero. Actually, by substituting to

$$ {}^{*}R^{*}{}_{\alpha\beta\mu\nu}{}^{;\nu} = -R_{\alpha\beta\mu\nu}{}^{;\nu} + R_{\mu\alpha;\beta} - R_{\mu\beta;\alpha} + R_{\nu\beta}{}^{;\nu}g_{\mu\alpha} - R_{\nu\alpha}{}^{;\nu}g_{\mu\beta} - \frac{1}{2}R_{;\beta}g_{\mu\alpha} + \frac{1}{2}R_{;\alpha}g_{\mu\beta} $$

  for the first term from (6.33), i.e. $-R_{\alpha\beta\mu\nu}{}^{;\nu} = R_{\mu\beta;\alpha} - R_{\mu\alpha;\beta}$, this term exactly cancels out with the second and the third terms, so one is left with

$$ {}^{*}R^{*}{}_{\alpha\beta\mu\nu}{}^{;\nu} = \left(R^{\nu}{}_{\beta;\nu} - \frac{1}{2}R_{;\beta}\right)g_{\mu\alpha} - \left(R^{\nu}{}_{\alpha;\nu} - \frac{1}{2}R_{;\alpha}\right)g_{\mu\beta}\,. $$

  We know, however, from the section about Einstein equations, that double contraction of the Bianchi identities $R^{\mu\nu}{}_{[\alpha\beta;\rho]} = 0$ in indices $\binom{\mu}{\alpha}$ and $\binom{\nu}{\rho}$ implies vanishing of the parentheses. Hence,

$$ {}^{*}R^{*}{}_{\alpha\beta\mu\nu}{}^{;\nu} = 0\,. \tag{A.32} $$

  And, by contraction, one confirms $G_{\beta\nu}{}^{;\nu} = 0$ as well.

- **Duals of the Weyl tensor** and of the other parts of Riemann's decomposition (8.5) now follow quite easily. First, the above Ruse-Lanczos identity (A.29) has been obtained solely on the basis of the Riemann-tensor symmetries, so it actually holds for any tensor which shares them (with the pertinent counterpart of Ricci tensor appearing on the right-hand side, of course). In particular, it holds for the tensors which stand in its decomposition (8.5). Regarding their traces (8.6), i.e.

$$ E^{\kappa}{}_{\nu\kappa\lambda} = S_{\nu\lambda}, \quad G^{\kappa}{}_{\nu\kappa\lambda} = \frac{1}{4}Rg_{\nu\lambda} \qquad \Longrightarrow \qquad C^{\kappa}{}_{\nu\kappa\lambda} = 0\,, $$

  one has (let us write it in lower indices)

$$ {}^{*}C^{*}{}_{\alpha\beta\mu\nu} + C_{\alpha\beta\mu\nu} = 0\,, \tag{A.33} $$
$$ {}^{*}E^{*}{}_{\alpha\beta\mu\nu} - E_{\alpha\beta\mu\nu} = 0\,, \tag{A.34} $$
$$ {}^{*}G^{*}{}_{\alpha\beta\mu\nu} + G_{\alpha\beta\mu\nu} = 0\,. \tag{A.35} $$

  Regarding that dual of a dual in 2 indices is minus the original tensor (A.25), another left dualization of these relations yields

$$ {}^{**}C^{*}{}_{\alpha\beta\mu\nu} + {}^{*}C_{\alpha\beta\mu\nu} = -C^{*}{}_{\alpha\beta\mu\nu} + {}^{*}C_{\alpha\beta\mu\nu} = 0\,, \tag{A.36} $$

$$^{**}E^{*}{}_{\alpha\beta\mu\nu} - {}^{*}E_{\alpha\beta\mu\nu} = -E^{*}{}_{\alpha\beta\mu\nu} - {}^{*}E_{\alpha\beta\mu\nu} = 0\,, \tag{A.37}$$
$$^{**}G^{*}{}_{\alpha\beta\mu\nu} + {}^{*}G_{\alpha\beta\mu\nu} = -G^{*}{}_{\alpha\beta\mu\nu} + {}^{*}G_{\alpha\beta\mu\nu} = 0\,. \tag{A.38}$$

Finally, left-dualizing the Ruse-Lanczos formula for Riemann (A.30), we have

$$^{**}R^{*}{}_{\alpha\beta\mu\nu} + {}^{*}R_{\alpha\beta\mu\nu} = -R^{*}{}_{\alpha\beta\mu\nu} + {}^{*}R_{\alpha\beta\mu\nu} = 2\,{}^{*}E_{\alpha\beta\mu\nu}\,. \tag{A.39}$$

- $\boxed{\text{Lemma}}$

  $$^{*}T^{\dots\nu\alpha}{}_{\nu\dots} = 0 \quad\Longleftrightarrow\quad T_{\dots[\nu\rho\sigma]\dots} = 0\;,$$

  and similarly with T and $^{*}T$ switched. The asterisk means here the dual of T "in the indices $[\rho,\sigma]$".
  Proof: By definition,

  $$^{*}T^{\dots\nu\alpha\beta\dots} \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}T_{\dots}{}^{\nu}{}_{\rho\sigma\dots}\;,$$

  which yields, when multiplied by $g_{\beta\nu}$,

  $$^{*}T^{\dots\nu\alpha}{}_{\nu\dots} = \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}T_{\dots\beta\rho\sigma\dots} = \frac{1}{2}\,\epsilon^{\alpha\nu\rho\sigma}T_{\dots[\nu\rho\sigma]\dots}\;,$$

  and thus the required statement.

- The lemma specifically applies to the derivative of the EM-field tensor:

  $$^{*}F^{\alpha\beta} \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma} \quad\Longrightarrow\quad {}^{*}F^{\alpha\beta}{}_{;\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma;\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}F_{[\rho\sigma;\beta]}\;.$$

  The "duality" of source-free Maxwell equations thus shows itself: vanishing of the divergence of one of the EM tensors is equivalent to vanishing of the cycle-permuted gradient of the other.

- Let us check how the Lemma works for the Riemann tensor, too. For its right and left dual, we have, respectively,

  $$R^{*}{}_{\alpha\beta}{}^{\kappa\lambda} \equiv \frac{1}{2}\,R_{\alpha\beta\rho\sigma}\epsilon^{\rho\sigma\kappa\lambda} \quad\Longrightarrow\quad R^{*}{}_{\iota\beta}{}^{\iota\lambda} = \frac{1}{2}\,R_{\iota\beta\rho\sigma}\epsilon^{\rho\sigma\iota\lambda} = -\frac{1}{2}\,R_{\beta[\iota\rho\sigma]}\epsilon^{\rho\sigma\iota\lambda} = 0\;, \tag{A.40}$$
  $$^{*}R^{\alpha\beta}{}_{\kappa\lambda} \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\rho\sigma}R_{\rho\sigma\kappa\lambda} \quad\Longrightarrow\quad {}^{*}R^{\iota\beta}{}_{\iota\lambda} = \frac{1}{2}\,\epsilon^{\iota\beta\rho\sigma}R_{\rho\sigma\iota\lambda} = -\frac{1}{2}\,\epsilon^{\iota\beta\rho\sigma}R_{\lambda[\iota\rho\sigma]} = 0\;. \tag{A.41}$$

  Therefore, vanishing of $R_{\mu[\iota\rho\sigma]}$ is equivalent to the vanishing of the contraction of dual tensors. In the opposite direction it holds analogously,

  $$\left(R^{**}{}_{\alpha\beta\rho\sigma} \equiv\right)\;\frac{1}{2}\,R^{*}{}_{\alpha\beta}{}^{\kappa\lambda}\epsilon_{\kappa\lambda\rho\sigma} = -R_{\alpha\beta\rho\sigma} \quad\Longrightarrow\quad -R_{\alpha\iota\rho}{}^{\iota} = \frac{1}{2}\,R^{*}{}_{\alpha}{}^{[\iota\kappa\lambda]}\epsilon_{\kappa\lambda\rho\iota}\;, \tag{A.42}$$
  $$\left({}^{**}R_{\rho\sigma\kappa\lambda} \equiv\right)\;\frac{1}{2}\,\epsilon_{\rho\sigma\alpha\beta}{}^{*}R^{\alpha\beta}{}_{\kappa\lambda} = -R_{\rho\sigma\kappa\lambda} \quad\Longrightarrow\quad -R_{\rho\iota\kappa}{}^{\iota} = -\frac{1}{2}\,\epsilon_{\rho\iota\alpha\beta}{}^{*}R^{[\alpha\beta\iota]}{}_{\kappa}\;. \tag{A.43}$$

  Hence, the Ricci tensor is zero if and only if the antisymmetrization of the dual tensors in three indices vanishes (i.e., when they satisfy an analogue of the "cyclic indentity" known from Maxwell equations).

A.3 Integration and general covariance

Physical equations often contain integrals and/or derivatives. Should the equations be covariant, the operations of integration and differentiation should be such as well. Introduction of covariant derivative is a standard part of GR basics, whereas integration is not so often discussed, although it is equally important in geometry (it is most notably being treated in differential forms).

A.3.1 Invariant volume element

The simplest query is how to integrate over volume – in an invariant way (so that by integration of invariant one again gets invariant). Since the coordinate element (in a d-dimensional space)

$$\mathrm{d}^d x \equiv \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^d$$

transforms via Jacobian of the transformation,

$$\mathrm{d}^d x' = \left|\frac{\partial x'}{\partial x}\right| \mathrm{d}^d x\,,$$

it is a scalar density of weight $w = -1$. According to the lemma from the beginning of this Appendix, we make it invariant by multiplying it by $g^{-w/2} \equiv g^{1/2}$, so

$$\sqrt{g'}\;\mathrm{d}^d x' = \sqrt{g}\;\mathrm{d}^d x \tag{A.44}$$

is the invariant volume element. In a Lorentzian space(-time), one has to take *minus* g under the square root, so in GR the element reads

$$\sqrt{-g}\;\mathrm{d}^4 x = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3\,.$$

$\boxed{\text{Example}}$ In $\mathbb{E}^3$, the question reads (for example)

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = \boxed{???}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

Since the corresponding metrics of $\mathbb{E}^3$ read

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

the metric determinant amounts to $g = g_{rr}g_{\theta\theta}g_{\phi\phi} = r^4\sin^2\theta$ (while it is $g_{xx}g_{yy}g_{zz} = 1$ in Cartesian coordinates), so one concludes $\mathrm{d}V = r^2\sin\theta\;\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$.

A.3.2 Volume, surface and line integration of scalars. Stokes theorem

If integrating over some d-dimensional region, its boundaries need not necessarily correspond to constant value of some of the coordinates, so it may be suitable to introduce the pertinent volume/surface/linear element more generally that in the preceding paragraph. In the region in question, one can certainly choose a certain infinitesimal basis out of d independent infinitesimal four-vectors $\left\{\mathrm{d}_{(A)}x^\mu\right\}_{A=1}^{d}$. The volume of an elementary d-dimensional parallelepiped with edges given by shifts $\mathrm{d}_{(A)}x^\mu$ is the pseudo-scalar

$$\epsilon_{\lambda_1\dots\lambda_d}\mathrm{d}_{(1)}x^{\lambda_1}\dots\mathrm{d}_{(d)}x^{\lambda_d}, \tag{A.45}$$

where $\epsilon_{\lambda_1\dots\lambda_d}$ is the d-dimensional Levi-Civita tensor. Integration can now be written covariantly as $\int\tau$, where

$$\tau \equiv T_{\kappa_1\dots\kappa_d}\delta^{\kappa_1\dots\kappa_d}_{\lambda_1\dots\lambda_d}\,\mathrm{d}_{(1)}x^{\lambda_1}\dots\mathrm{d}_{(d)}x^{\lambda_d} = d!\,T_{\kappa_1\dots\kappa_d}\,\mathrm{d}_{(1)}x^{[\lambda_1}\dots\mathrm{d}_{(d)}x^{\lambda_d]} \tag{A.46}$$

is a differential form associated with a chosen tensor $T_{\kappa_1\dots\kappa_d}$. The "oriented" integration element is given by generalized Kronecker symbol, and the antisymmetrization in the indices of all its linear elements can be written as the determinant

$$\mathrm{d}_{(1)}x^{[\lambda_1}\dots\mathrm{d}_{(d)}x^{\lambda_d]} = \frac{1}{d!}\begin{vmatrix} \mathrm{d}_{(1)}x^{\lambda_1} & \dots & \mathrm{d}_{(1)}x^{\lambda_d} \\ \dots & \dots & \dots \\ \mathrm{d}_{(d)}x^{\lambda_1} & \dots & \mathrm{d}_{(d)}x^{\lambda_d} \end{vmatrix}. \tag{A.47}$$

As from the tensor $T_{\kappa_1\dots\kappa_d}$ clearly applies its antisymmetric part only, one may right away assume that it *is* antisymmetric, but then it has to be proportional to the Levi-Civita tensor of the pertinent rank.

- In the $\underline{d=4}$ case appears the symbol $\delta^{\alpha\beta\gamma\delta}_{\mu\nu\kappa\lambda} = -\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\kappa\lambda}$, so if we introduce a pseudo-scalar T by the relation

  $$T_{\alpha\beta\gamma\delta} = T\epsilon_{\alpha\beta\gamma\delta}\,,$$

  the integration assumes the form

  $$\int\limits_\Omega T_{\alpha\beta\gamma\delta}\delta^{\alpha\beta\gamma\delta}_{\mu\nu\kappa\lambda}\,\mathrm{d}_{(1)}x^\mu\mathrm{d}_{(2)}x^\nu\mathrm{d}_{(3)}x^\kappa\mathrm{d}_{(4)}x^\lambda = 4!\int\limits_\Omega T\mathrm{d}\Omega\,, \tag{A.48}$$

  where the 4D element reads

  $$\mathrm{d}\Omega = \epsilon_{\mu\nu\kappa\lambda}\,\mathrm{d}_{(1)}x^\mu\mathrm{d}_{(2)}x^\nu\mathrm{d}_{(3)}x^\kappa\mathrm{d}_{(4)}x^\lambda\,.$$

  In particular, if the infinitesimal elements point in the direction of the coordinate axes, i.e. each and every vector $\mathrm{d}_{(A)}x^\mu$ solely has the A-th component and that is given by $\mathrm{d}x^A$ ($A = 1,\dots,d$), the prescription reduces to $\int\limits_\Omega T\sqrt{-g}\,\mathrm{d}^4x$ from the preceding subsection.

- In the $\underline{d=3}$ case one has $\delta^{\alpha\beta\gamma}_{\mu\nu\kappa} = -\epsilon^{\alpha\beta\gamma\lambda}\epsilon_{\mu\nu\kappa\lambda}$, and the tensor $T_{\alpha\beta\gamma}$ can be associated with a pseudo-vector T^σ according to $T_{\alpha\beta\gamma} = T^\sigma\epsilon_{\alpha\beta\gamma\sigma}$. The integral then becomes

  $$3!\int\limits_V T^\sigma\mathrm{d}V_\sigma\,, \tag{A.49}$$

  where the oriented volume element reads

  $$\mathrm{d}V_\sigma = \epsilon_{\mu\nu\kappa\sigma}\,\mathrm{d}_{(1)}x^\mu\mathrm{d}_{(2)}x^\nu\mathrm{d}_{(3)}x^\kappa\,.$$

- In the $\underline{d=2}$ case we substitute $\delta^{\alpha\beta}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}$, thus arriving at the integral

$$\int_S T_{\alpha\beta}\mathrm{d}S^{\alpha\beta} = \int_S (T_{\alpha\beta} - T_{\beta\alpha})\,\mathrm{d}_{(1)}x^{\alpha}\mathrm{d}_{(2)}x^{\beta} \tag{A.50}$$

over the surface element

$$\mathrm{d}S^{\alpha\beta} = \mathrm{d}_{(1)}x^{\alpha}\mathrm{d}_{(2)}x^{\beta} - \mathrm{d}_{(1)}x^{\beta}\mathrm{d}_{(2)}x^{\alpha}\,.$$

- Finally, the $\underline{d=1}$ case corresponds to the length element $\mathrm{d}l^{\alpha} = \mathrm{d}x^{\alpha}$ and to the line integration

$$\int_{\gamma} T_{\alpha}\mathrm{d}x^{\alpha}\,. \tag{A.51}$$

Stokes theorem

permits to translate the integration (of some tensor) over a boundary of some region into the integration (of the derivative of that tensor) over the enclosed region, and vice versa:

$$\oint_{\partial(\text{region})} \tau = \int_{\text{region}} \mathrm{d}\tau\,. \tag{A.52}$$

Here, τ is the differential form associated with the tensor $T_{\kappa_1\ldots\kappa_{d-1}}$ –

$$\begin{aligned}\tau \equiv T_{\kappa_1\ldots\kappa_{d-1}}\,\delta^{\kappa_1\ldots\kappa_{d-1}}_{\lambda_1\ldots\lambda_{d-1}}\,\mathrm{d}_{(1)}x^{\lambda_1}\ldots\mathrm{d}_{(d-1)}x^{\lambda_{d-1}} =\\ = (d-1)!\,T_{\kappa_1\ldots\kappa_{d-1}}\,\mathrm{d}_{(1)}x^{[\lambda_1}\ldots\mathrm{d}_{(d-1)}x^{\lambda_{d-1}]}\,,\end{aligned}$$

and $\mathrm{d}\tau$ is its exterior differential,

$$\begin{aligned}\mathrm{d}\tau = T_{\kappa_1\ldots\kappa_{d-1},\kappa_d}\,\delta^{\kappa_1\ldots\kappa_{d-1}\kappa_d}_{\lambda_1\ldots\lambda_{d-1}\lambda_d}\,\mathrm{d}_{(1)}x^{\lambda_1}\ldots\mathrm{d}_{(d-1)}x^{\lambda_{d-1}}\mathrm{d}_{(d)}x^{\lambda_d} =\\ = d!\,T_{\kappa_1\ldots\kappa_{d-1},\kappa_d}\,\mathrm{d}_{(1)}x^{[\lambda_1}\ldots\mathrm{d}_{(d-1)}x^{\lambda_{d-1}}\mathrm{d}_{(d)}x^{\lambda_d]}\,.\end{aligned}$$

APPENDIX B

Killing and Killing-Yano tensors

Killing vector fields can be viewed as a subclass of more general objects connected with more "hidden" symmetries of space-time – the so-called Killing tensors and Killing-Yano tensors.

Killing tensors ($\xi_{\mu\ldots\nu}$) are totally symmetric tensors which satisfy the generalized Killing equation

$$\xi_{(\mu\ldots\nu;\alpha)} = 0\,. \tag{B.1}$$

Killing-Yano tensors ($Y_{\mu\ldots\nu}$) are totally antisymmetric tensors which satisfy a different possible generalization of the Killing equation,

$$Y_{\mu\ldots(\nu;\alpha)} = 0\,. \tag{B.2}$$

Several simple properties of these tensors:

- Similarly as for a Killing vector, the existence of the Killing tensor implies conservation of the quantity $\xi_{\mu\ldots\nu}u^\mu\ldots u^\nu$ along geodesics (with tangent vector u^μ):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\xi_{\mu\ldots\nu}u^\mu\ldots u^\nu) = \frac{\mathrm{D}}{\mathrm{d}\tau}(\xi_{\mu\ldots\nu}u^\mu\ldots u^\nu) = \xi_{\mu\ldots\nu;\alpha}u^\mu\ldots u^\nu u^\alpha = \xi_{(\mu\ldots\nu;\alpha)}u^\mu\ldots u^\nu u^\alpha = 0\,.$$

- Obviously, the metric tensor itself is "trivially" the Killing tensor – the corresponding constant of geodesic motion is $g_{\mu\nu}u^\mu u^\nu = -1$. Similarly, trivial Killing tensors are also symmetrized products of Killing vectors, $\xi_{\mu\ldots\nu} = \xi_{(\mu}\cdots\eta_{\nu)}$:

$$\xi_{(\mu\ldots\nu;\alpha)} = \xi_{(\mu;\alpha}\cdots\eta_{\nu)} + \ldots + \xi_{(\mu}\cdots\eta_{\nu;\alpha)} = 0\,,$$

  because gradients of the Killing vectors $\xi_{\mu;\alpha}$, ... , $\eta_{\nu;\alpha}$ are antisymmetric.

- Important relation: the existence of the second-rank Killing-Yano tensor implies the existence of the second-rank Killing tensor – it is given by "square" of the Killing-Yano tensor, $\xi_{\mu\nu} = Y_{\mu\iota}Y_\nu{}^\iota$. Actually, if $Y_{\mu(\nu;\alpha)} = 0$, then

$$3\,\xi_{(\mu\nu;\kappa)} = (Y_{\mu\alpha}Y_\nu{}^\alpha)_{;\kappa} + (Y_{\kappa\alpha}Y_\mu{}^\alpha)_{;\nu} + (Y_{\nu\alpha}Y_\kappa{}^\alpha)_{;\mu} =$$

$$= -2Y_{\alpha(\mu;\kappa)}Y_\nu{}^\alpha - 2Y_{\alpha(\kappa;\nu)}Y_\mu{}^\alpha - 2Y_{\alpha(\nu;\mu)}Y_\kappa{}^\alpha = 0\,.$$

The opposite implication does not hold in general.

- In 4D, the Killing-Yano tensor can maximally be of the 4th rank. However, the 1st-rank and the 4th-rank cases are trivial – Y_μ is just the Killing vector and $Y_{\mu\nu\kappa\lambda}$ is proportional to the Levi-Civita tensor. The corresponding Killing tensors are $Y_\mu Y_\nu$ and $\frac{1}{6}\,\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma}{}_\nu = g_{\mu\nu}$, respectively. The Killing-Yano tensor of the 3rd rank has, due to its total antisymmetry, only 4 independent components, so it can be represented by the vector dual

$$h_\mu := \frac{1}{3!}\,\epsilon_\mu{}^{\alpha\beta\gamma}Y_{\alpha\beta\gamma}\,.$$

Using the equations $Y_{\alpha\beta(\gamma;\nu)} = 0$ and inversion of the definition $Y_{\alpha\beta\nu} = \epsilon_{\alpha\beta\nu\sigma}h^\sigma$, one obtains the equation for h_μ:

$$\begin{aligned}
6h_{\mu;\nu} &= \epsilon_\mu{}^{\alpha\beta\gamma}Y_{\alpha\beta\gamma;\nu} = -\epsilon_\mu{}^{\alpha\beta\gamma}Y_{\alpha\beta\nu;\gamma} = -\epsilon_\mu{}^{\alpha\beta\gamma}\epsilon_{\alpha\beta\nu\sigma}h^\sigma{}_{;\gamma} = 2(g_{\mu\nu}\delta^\gamma_\sigma - g_{\mu\sigma}\delta^\gamma_\nu)h^\sigma{}_{;\gamma} = \\
&= 2g_{\mu\nu}h^\sigma{}_{;\sigma} - 2h_{\mu;\nu} \\
\Longrightarrow \quad & h_{\mu;\nu} = \frac{1}{4}g_{\mu\nu}h^\sigma{}_{;\sigma}\,.
\end{aligned}$$

Duals of the Killing-Yano tensors are called **closed conformal Killing-Yano tensors**. They are of the rank = dimension minus rank of the KY tensor, and the general relation between their properties is indicated by the above example: gradient of the KY tensors has only antisymmetric part, while gradient of their duals has only trace part.

- If it is possible to express the Killing tensor in terms of the Killing-Yano tensor, then the constants of geodesic motion $\xi_{\mu\nu}u^\mu u^\nu$ can also be expressed in terms of the KY-tensor invariants. Namely, a tensor $Y_{\mu\nu}$ is the KY tensor if and only if the vector $Y_{\mu\nu}u^\nu$ parallel transports along arbitrary geodesic (of which u^ν is the tangent vector),

$$\frac{\mathrm{D}}{\mathrm{d}\tau}(Y_{\mu\nu}u^\nu) = (Y_{\mu\nu}u^\nu)_{;\alpha}u^\alpha = Y_{\mu\nu;\alpha}u^\nu u^\alpha = Y_{\mu(\nu;\alpha)}u^\nu u^\alpha = 0 \quad \Longleftrightarrow \quad Y_{\mu(\nu;\alpha)} = 0\,.$$

From here immediately follows conservation of the scalar product $g_{\mu\nu}Y^\mu{}_\alpha u^\alpha Y^\nu{}_\beta u^\beta = \xi_{\alpha\beta}u^\alpha u^\beta$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}Y^\mu{}_\alpha u^\alpha Y^\nu{}_\beta u^\beta\right) = \frac{\mathrm{D}}{\mathrm{d}\tau}\left(g_{\mu\nu}Y^\mu{}_\alpha u^\alpha Y^\nu{}_\beta u^\beta\right) = 0\,.$$

In the case of the 3rd-rank KY tensor, the same holds for the tensor $(h_\mu u_\nu - h_\nu u_\mu)$ composed of its dual h_μ,

$$(h_\mu u_\nu - h_\nu u_\mu)_{;\alpha}u^\alpha = (h_{\mu;\alpha}u_\nu - h_{\nu;\alpha}u_\mu)u^\alpha = \frac{1}{4}\,h^\sigma{}_{;\sigma}(g_{\mu\alpha}u_\nu - g_{\nu\alpha}u_\mu)u^\alpha = 0\,.$$

These properties indicate that the existence of the Killing and Killing-Yano tensors is closely related to the separability of the geodesic equation (to the existence of its separated first integrals), but it turned out that it is also interconnected with separability of other differential equations. In other words, the Killing(-Yano) tensors do not describe direct symmetries of space-time (of metric), but rather symmetries in the dynamics of test particles and fields (thus concerning the *symplectic* structure). However, their existence meets at numerous points with the properties of Killing vectors and also with the properties of space-time curvature.

- Maximal number of independent Killing tensors which can exist in a 4D space-time is 50, while for Killing-Yano tensors it is 10. These maximal numbers exist in "maximally symmetric space-times" of constant curvature.

- Observation: A bivector $Y_{\mu\nu}$ is the KY tensor if and only if ${}^*Y_{\rho\sigma;\kappa} = 2\xi_{[\rho}g_{\sigma]\kappa}$, where ${}^*Y_{\rho\sigma} := \frac{1}{2}\epsilon_{\rho\sigma}{}^{\mu\nu}Y_{\mu\nu}$ is the dual of the tensor $Y_{\mu\nu}$ and $\xi^\beta := \frac{1}{6}\epsilon^{\beta\mu\nu\kappa}Y_{\mu\nu;\kappa}$ is the dual of its gradient $Y_{\mu\nu;\kappa}$.

  Proof: Let a bivector $Y_{\mu\nu}$ be the KY tensor, so let $Y_{\mu(\nu;\kappa)} = 0$. This means that the tensor $Y_{\mu\nu;\kappa}$ is antisymmetric in all the three indices, so it can be expressed in terms of its dual ξ^α as $Y_{\mu\nu;\kappa} = \epsilon_{\mu\nu\kappa\alpha}\xi^\alpha$. Then, for a dual ${}^*Y_{\rho\sigma}$ of the tensor $Y_{\mu\nu}$ we have

  $$\frac{1}{2}\left(\epsilon_{\rho\sigma}{}^{\mu\nu}Y_{\mu\nu}\right)_{;\kappa} = \frac{1}{2}\epsilon_{\rho\sigma}{}^{\mu\nu}Y_{\mu\nu;\kappa} = \frac{1}{2}\epsilon_{\rho\sigma}{}^{\mu\nu}\epsilon_{\mu\nu\kappa\alpha}\xi^\alpha = (g_{\rho\alpha}g_{\sigma\kappa} - g_{\rho\kappa}g_{\sigma\alpha})\xi^\alpha = 2\xi_{[\rho}g_{\sigma]\kappa}\,,$$

  where we used the product formula (A.6).
  Opposite implication: if some bivector $Y_{\mu\nu}$ and some covector ξ_ρ satisfy the relation

  $$\frac{1}{2}\epsilon_{\rho\sigma}{}^{\mu\nu}Y_{\mu\nu;\kappa} = 2\xi_{[\rho}g_{\sigma]\kappa}\,,$$

  then by multiplying the latter by $\frac{1}{2}\epsilon_{\alpha\beta}{}^{\rho\sigma}$ and by employing the "$\epsilon\cdot\cdot\epsilon$" formula again, it follows

  $$-Y_{\alpha\beta;\kappa} = -\epsilon_{\alpha\beta\kappa\sigma}\xi^\sigma\,,$$

  from where clearly $Y_{\alpha(\beta;\kappa)} = 0$. □

- Another observation: In Chapter11, we derived the formula (11.17) for the Lie derivative of affine connection,

  $$\xi^\mu{}_{;\kappa\lambda} = \xi^\mu{}_{;\kappa\lambda} + R^\mu{}_{\kappa\delta\lambda}\xi^\delta\,.$$

  Consider the case when this expression vanishes, so $\xi_{\iota;\kappa\lambda} = R_{\iota\kappa\lambda\delta}\xi^\delta$. Then $\xi_{(\iota;\kappa)}$ is a Killing tensor, because $\xi_{(\iota;\kappa\lambda)} = R_{(\iota\kappa\lambda)\delta}\xi^\delta = 0$.

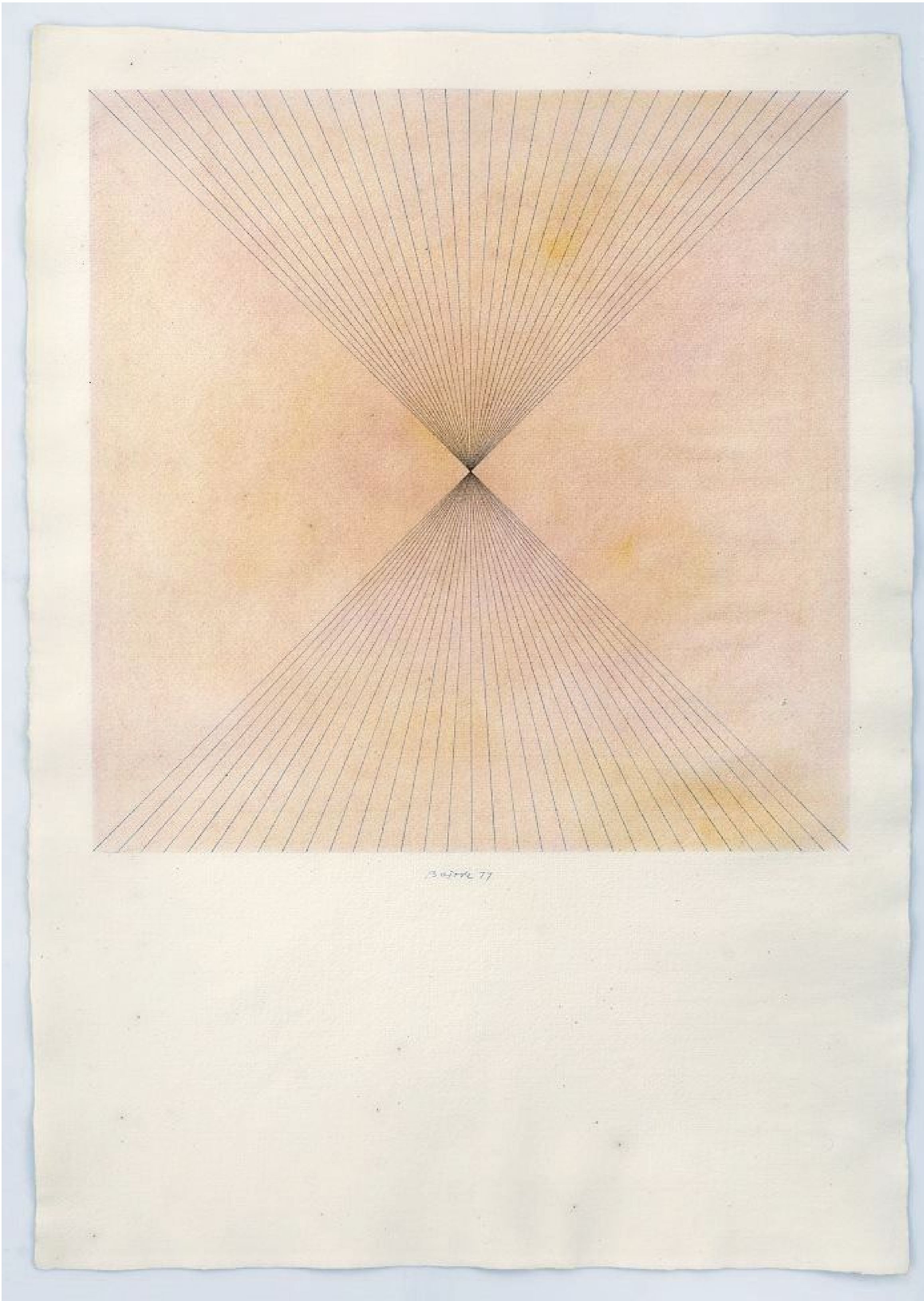
B.1 Other types of symmetries

There exists a number of other, less straightforward and intuitively less accessible symmetries. Let us just touch on several of them. The mapping $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, where Ω is a non-zero scalar function, is called the **conformal isometry**. Its generator ξ^μ, satisfying

$$(\pounds_\xi g_{\mu\nu}) = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\phi g_{\mu\nu} \tag{B.3}$$

(the scalar function ϕ is connected to Ω by a line integral), is called the **conformal Killing vector** (or also the **conformal motion**). If ϕ is constant, the vector ξ^μ is called the **homothetic vector** (the **homothetic motion**); especially if $\phi = 0$, then ξ^μ is the ordinary Killing vector or motion (the mapping is an isometry) from Section 11.4.

In GR, main attention is being devoted to continuous symmetries of space-time, i.e. to the symmetries of the Einstein equations, but privileges of the vector fields may of course follow from symmetries of other equations or quantities of the theory. For example, the mappings which transforms between geodesics are called projective collineations, with those conserving affine parameterization being called affine collineations. Ricci collineations are analogous to isometries, but for the Ricci tensor instead for the metric, and curvature collineations work similarly for the Riemann tensor. Between these symmetries and other space-time features (algebraic type, but also the existence of the integrals of geodesic motion) exist many relations which, however, cannot be summarized in a few brief statements; we refer to the monography [44].



Václav Boštík: bez názvu (without name)

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