# KILLING HORIZONS

*notes for BH-thermodynamics course*[1]

[1] For a rigorous yet well-readable account on basic properties of Killing horizons, see B. Carter, J. Math. Phys. 10 (1969) 70. (It does not involve *"thermodynamics"* yet.)

Some people think that everything important lives on a boundary. In physics, the idea that behaviour of the "bulk" can be described by theories formulated on its boundary is called the **holographic principle**. After a lunch, I tend to doubt about it, yet it might at least work for black holes. Actually, the holographic viewpoint was inspired by the proposal (in 1972) by Jacob Bekenstein to connect the entropy of a black hole with the proper area of the horizon. In terms of such a quantity, he formulated the second law of black-hole thermodynamics. Were Bekenstein right, the usual first-law term $T\mathrm{d}S$ would require that the black hole have **temperature**, proportional to the quantity called **surface gravity**. Bardeen, Carter and Hawking subsequently (1973) wrote the paper "The four laws of black hole mechanics" where they added, for a stationary and axisymmetric black hole, the other three laws.

As it is clear from Carter's thorough lecture on stationary black holes in the proceedings of the famous Les Houches '72 Summer School, the authors of the paper themselves did *not* think their laws were more than an *analogy* of thermodynamics. In particular, Hawking embarked on showing that black holes do *not* emit thermal radiation, in order to demonstrate that the horizon "temperature" does not have its usual sense, and so neither the area *really* represents the black-hole entropy. Employing the quantum-field theory on a (classical) curved background, he managed to show (1974) just the opposite: black holes do emit radiation, in accord with the black-body formula.[1] The road to black-hole thermodynamics was open.

Yet, wait, *thermodynamics of pure geometry?* Nothing *that strange*, actually: according to quantum field theory, an accelerated observer sees thermal bath, even in a flat space-time where an inertial observer sees pure vacuum (so called Unruh effect). And acceleration (inertial field) is equivalent to gravity, isn't it?

Today, the thermodynamics of black holes – of something "built from pure geometry" – is considered a key to deep connections between gravitational, quantum and statistical physics. Since the gravitational degrees of freedom should also contribute to the relevant quantities, a complete picture also requires to quantize these (not only the other fields "living on a background"). The area is thus being studied heavily as one of the promising targets – and tests – of any attempt to quantize gravitation.

Below, as a starting point, we mention several notions and results from the geometry and physics of black-hole horizons. We naturally restrict to non-dynamical, stationary horizons. In the electro-vacuum and asymptotically flat case, the uniqueness theorems say it leads to the Kerr-Newman family of solutions, but we will be slightly more general and will consider stationary and axisymmetric setting (often so-called circular in addition).

Although many definitions of **black-hole horizons** exist, in a generic situation it is not entirely clear what "a black hole" actually means. Anyway, the **laws of black-hole thermodynamics** start from a *stationary and axially symmetric* situation: the zeroth law states that on a stationary horizon the **surface gravity** is everywhere the same, and the first law fixes the "first thermodynamic law" for changes of basic quantities (mass, horizon area, charge and spin) characterizing the transition of the black hole between two close stationary states. The word "axisymmetric" has disappeared somehow, yet it is there automatically: if a stationary black hole is not static, i.e. if it is *rotating*, then it *has to be* axisymmetric, since otherwise – with some bump on it – it would emit gravitational waves, so it could not be stationary.

[1] Black hole is quite an ideal black body, isn't it...

## Notation

$V^\mu$ ... a generic vector field
$\xi^\mu$ ... a (generic) Killing vector field
$t^\mu$ ... time-symetry Killing vector field
$\phi^\mu$ ... axial-symmetry Killing vector field
$t$, $\phi$ ... Killing coordinates (chosen as parameters along the respective symmetries),
$t^\mu =: \frac{\partial x^\mu}{\partial t}$, $\phi^\mu =: \frac{\partial x^\mu}{\partial \phi}$
$g_{tt} \equiv g_{\mu\nu}t^\mu t^\nu$, $g_{t\phi} \equiv g_{\mu\nu}t^\mu \phi^\nu$, $g_{\phi\phi} \equiv g_{\mu\nu}\phi^\mu\phi^\nu$ ... Killing part of the metric
$\omega := -\frac{g_{t\phi}}{g_{\phi\phi}}$ ... dragging angular velocity; $\omega_{\rm H}$ ... its horizon value
$N$ ... lapse function ($N^2 = -g_{tt} - g_{t\phi}\omega$)
$h^\mu := t^\mu + \omega_{\rm H}\phi^\mu$ ... *the* Killing vector field which generates the horizon
$\mathcal{n}^\mu := Nn^\mu := t^\mu + \omega\phi^\mu$ ... vector field orthogonal to hypersurfaces $t = \text{const}$
(on a Killing horizon, $\mathcal{n}^\mu$ and $h^\mu$ coincide, including their gradient)
$n^\mu := \frac{1}{N}\mathcal{n}^\mu$ ... unit version of $\mathcal{n}^\nu$ (four-velocity of zero-angular-momentum congruence)
$\omega^\mu[V]$ (sometimes without $[V]$) ... vorticity vector of a (co)vector field $V^\mu$
$\Theta_{\mu\nu}$, $\sigma_{\mu\nu}$ ... expansion and shear tensors (of a given vector field)
$\kappa$, $\kappa_{\rm H}$ ... acceleration scalar and its horizon value $\equiv$ surface gravity

## 1.1 Selected properties of Killing vector fields

The Killing vector fields $\xi^\mu$ fix directions along which the space-time metric does not change. Such a property is naturally expressed in terms of the Lie derivative. The Lie derivative is a very "low level" operation, it needs neither the connection and nor the metric, yet if we speak of the Lie derivative *of the metric*, the latter has to be there, right? Let us also assume the space-time is equipped with the Levi-Civita connection as it is standard in GR. Then the (Killing) equation for the Killing fields writes

$$0 = \pounds_\xi g_{\mu\nu} = g_{\mu\nu,\iota}\xi^\iota + \xi^\iota{}_{,\mu}g_{\iota\nu} + \xi^\iota{}_{,\nu}g_{\mu\iota} = \cancel{g_{\mu\nu;\iota}}\xi^\iota + \xi^\iota{}_{;\mu}g_{\iota\nu} + \xi^\iota{}_{;\nu}g_{\mu\iota} = \xi_{\nu;\mu} + \xi_{\mu;\nu} =: 2\xi_{(\mu;\nu)}\,.$$

The Killing vector fields have many simple properties, of which we only mention the clearly vanishing expansion scalar, i.e. $\xi^\mu{}_{;\mu} = 0$. Next, let us mention an important relation between Killing vector fields and curvature. In the Ricci identity for $\xi_\nu$,

$$\xi_{\nu;\kappa\lambda} - \xi_{\nu;\lambda\kappa} = R^\sigma{}_{\nu\kappa\lambda}\xi_\sigma,$$

we anti-commute, by the Killing equation, $\nu$ and $\lambda$ in the second term, and then we write the relation together with its cyclic permutations:

$$\xi_{\nu;\kappa\lambda} + \xi_{\lambda;\nu\kappa} = R^\sigma{}_{\nu\kappa\lambda}\xi_\sigma\,,$$
$$\xi_{\lambda;\nu\kappa} + \xi_{\kappa;\lambda\nu} = R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma\,,$$
$$\xi_{\kappa;\lambda\nu} + \xi_{\nu;\kappa\lambda} = R^\sigma{}_{\kappa\lambda\nu}\xi_\sigma\,.$$

Now add the first and the last equation, while subtracting the middle one (for example),

$$2\xi_{\nu;\kappa\lambda} = (R^\sigma{}_{\nu\kappa\lambda} + R^\sigma{}_{\kappa\lambda\nu} - R^\sigma{}_{\lambda\nu\kappa})\xi_\sigma = \left(\cancel{R^\sigma{}_{\{\nu\kappa\lambda\}}} - 2R^\sigma{}_{\lambda\nu\kappa}\right)\xi_\sigma = -2R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma$$
$$\Longrightarrow \quad \xi_{\nu;\kappa\lambda} = -R^\sigma{}_{\lambda\nu\kappa}\xi_\sigma = R_{\nu\kappa\lambda\sigma}\xi^\sigma\,. \tag{1.1}$$

- Corollary: by contraction of this equation, one has

$$\boxed{\xi_{\nu;}{}^{\kappa}{}_{\kappa} \equiv \Box\xi_\nu = -R^\sigma{}_\nu \xi_\sigma} \,. \tag{1.2}$$

If the Ricci tensor vanishes, this corresponds to the (de Rham) wave equation for the EM four-potential (otherwise the signs at the curvature terms are opposite). Regarding also that the Killing fields automatically satisfy the "Lorenz condition" $\xi^\mu{}_{;\mu}=0$, one infers the following: in space-times with $R^{\mu\nu}=0$, the knowledge of a Killing vector implies the knowledge of a possible EM four-potential. (Note that the corresponding EM field must be a *test* field, because otherwise the Ricci tensor would be $R^{\mu\nu}=8\pi T^{\mu\nu}_{\rm EM}$ rather than zero.)

- Another corollary: projecting (1.1) twice on a tangent vector $u^\mu=\frac{{\rm d}x^\mu}{{\rm d}\tau}$ of any geodesic ($\frac{{\rm D}u^\mu}{{\rm d}\tau}=0$), we find

$$\xi_{\nu;\kappa\lambda}u^\kappa u^\lambda = \frac{{\rm D}\xi_{\nu;\kappa}}{{\rm d}\tau}\,u^\kappa = \frac{{\rm D}(\xi_{\nu;\kappa}u^\kappa)}{{\rm d}\tau} = \frac{{\rm D}^2\xi_\nu}{{\rm d}\tau^2} = R_{\nu\kappa\lambda\sigma}u^\kappa u^\lambda \xi^\sigma \,. \tag{1.3}$$

That means, the Killing vectors satisfy the geodesic-deviation equation.

## 1.2 A useful formula

Have a vector field $V^\mu$ (imagine it is time-like, though it is not necessary). Denote

$$g_{\mu\nu}V^\mu V^\nu =: -N^2 \quad\Rightarrow\quad (-N^2)_{;\alpha} = 2V_{\mu;\alpha}V^\mu \,, \quad \text{and} \;\; \omega^\mu[V] := \frac{1}{2}\,\epsilon^{\mu\nu\kappa\lambda}V_{\nu;\kappa}V_\lambda \,.$$

Using the famous relation

$$\epsilon_{\mu\nu\kappa\lambda}\epsilon^{\mu\alpha\beta\gamma} = -3!\,\delta^{[\alpha}_\nu \delta^\beta_\kappa \delta^{\gamma]}_\lambda \,, \tag{1.4}$$

we obtain, for the square of the vorticity (twist) $\omega^\mu[V]$,

$$\begin{aligned}
4\omega_\mu\omega^\mu &= \epsilon_{\mu\nu\kappa\lambda}V^{\nu;\kappa}V^\lambda\,\epsilon^{\mu\alpha\beta\gamma}V_{\alpha;\beta}V_\gamma = \\
&= -\left(\delta^\alpha_\nu\delta^\beta_\kappa\delta^\gamma_\lambda + \delta^\gamma_\nu\delta^\alpha_\kappa\delta^\beta_\lambda + \delta^\beta_\nu\delta^\gamma_\kappa\delta^\alpha_\lambda - \delta^\beta_\nu\delta^\alpha_\kappa\delta^\gamma_\lambda - \delta^\gamma_\nu\delta^\beta_\kappa\delta^\alpha_\lambda - \delta^\alpha_\nu\delta^\gamma_\kappa\delta^\beta_\lambda\right) V^{\nu;\kappa}V^\lambda V_{\alpha;\beta}V_\gamma = \\
&= V^{\alpha;\beta}V_{\alpha;\beta}N^2 - V^{\gamma;\alpha}V^\beta V_{\alpha;\beta}V_\gamma - V^{\beta;\gamma}V^\alpha V_{\alpha;\beta}V_\gamma \\
&\quad - V^{\beta;\alpha}V_{\alpha;\beta}N^2 + V^{\gamma;\beta}V^\alpha V_{\alpha;\beta}V_\gamma + V^{\alpha;\gamma}V^\beta V_{\alpha;\beta}V_\gamma = \\
&= 2N^2V^{[\alpha;\beta]}V_{\alpha;\beta} - 2V^{[\gamma;\alpha]}V_\gamma V_{\alpha;\beta}V^\beta + 2V^{[\gamma;\beta]}V_\gamma V_{\alpha;\beta}V^\alpha \,.
\end{aligned}$$

Relabelling $\alpha \leftrightarrow \beta$ in the last term, we reach a very useful formula

$$4\omega_\mu\omega^\mu = 2N^2V^{[\alpha;\beta]}V_{\alpha;\beta} + 4V^{[\gamma;\alpha]}V_\gamma V_{[\beta;\alpha]}V^\beta \,. \tag{1.5}$$

Corollary for Killing fields: If $V^\mu$ is Killing, $V^\mu \to \xi^\mu$, i.e. if $\xi_{[\alpha;\beta]}=\xi_{\alpha;\beta}$ , it becomes

$$\boxed{4\omega_\mu\omega^\mu = 2N^2\xi^{\alpha;\beta}\xi_{\alpha;\beta} + (N^2)^{;\alpha}(N^2)_{;\alpha}} \,. \tag{1.6}$$

Also, writing $(N^2)^{;\alpha}(N^2)_{;\alpha} = 4N^2 N^{;\alpha}N_{;\alpha}$ and

$$2\xi^{\alpha;\beta}\xi_{\alpha;\beta} = 2(\xi^{\alpha;\beta}\xi_\alpha)_{;\beta} - 2\xi^{\alpha;\beta}{}_{\beta}\xi_\alpha = (-N^2)^{;\beta}{}_{\beta} + 2R_{\alpha\beta}\xi^\alpha\xi^\beta = \\ = -2N_{;\beta}N^{;\beta} - 2N\,\Box N + 2R_{\alpha\beta}\xi^\alpha\xi^\beta\,, \tag{1.7}$$

one brings curvature into the formula through equation (1.2),

$$2\omega_\mu\omega^\mu = N^2\left(N^{;\alpha}N_{;\alpha} - N\,\Box N + R_{\alpha\beta}\xi^\alpha\xi^\beta\right)\,. \tag{1.8}$$

## 1.3 Killing horizon: Vishveshwara, otherwise vacuum

We will now focus on such Killing fields which *somewhere* become light-like (while *not* being light-like around). More specifically, we will be interested in the situation when they become light-like on a connected null hypersurface (which will be called a Killing horizon then). The Killing vector field which thus "generates" the horizon will be denoted by $h^\mu$.

Let us remark there exist different *horizons*, typically being null hypersurfaces, sometimes even having Killing generators. Very loosely speaking, they usually represent causal boundaries of certain regions which are "dragged with respect to the rest of space-time in a superluminal speed", either by strong gravity (black-hole horizons), by "dark energy" (cosmological horizons) or simply by boost/acceleration of a family of observers (acceleration horizons). In order to represent a *black-hole* event horizon, the Killing horizon should be non-singular (geodesically complete) and closed in the sense that its topology is $S^2\times R$.

$\boxed{\text{Definition}}$ The set $\{g_{\mu\nu}h^\mu h^\nu \equiv -N^2 = 0,\ \text{with}\, h^\mu\ \text{Killing and}\ h^\mu \neq 0\}$, is called a **Killing horizon**, if it is a connected null hypersurface (or a union of such).

$\boxed{\text{Proposition}}$ Two orthogonal null vectors are necessarily proportional to each other.

Proof: Orthogonality is a local property, and one can at every point work in a locally Minkowskian frame where $g_{\mu\nu}=\eta_{\mu\nu}$. So, have a non-trivial null vector $k^\mu$ and some other non-trivial vector $V^\mu$ orthogonal to $k^\mu$:

$$0 = \eta_{\mu\nu}k^\mu k^\nu = -(k^0)^2 + k^2 \quad\Rightarrow\quad (k^0)^2 = k^2\,,$$

$$0 = \eta_{\mu\nu}k^\mu V^\nu = -k^0V^0 + \vec{k}\cdot\vec{V} \quad\Rightarrow\quad V^0 = \frac{\vec{k}\cdot\vec{V}}{k^0} \quad\Rightarrow\quad (V^0)^2 = \frac{(\vec{k}\cdot\vec{V})^2}{k^2}\,,$$

where $k^2 := \vec{k}\cdot\vec{k} \equiv \eta_{ij}k^ik^j = \delta_{ij}k^ik^j$ and likewise for $V^2$. Then

$$\eta_{\mu\nu}V^\mu V^\nu = -(V^0)^2 + V^2 = -\frac{(\vec{k}\cdot\vec{V})^2}{k^2} + V^2 = -\frac{(kV\cos\alpha)^2}{k^2} + V^2 = V^2\sin^2\alpha\,,$$

with $\alpha$ the angle between $\vec{k}$ and $\vec{V}$. Therefore, $V^\mu$ is space-like in general ($\sin\alpha \neq 0$), with the special exception of $\sin\alpha = 0$ when it is null. However, the latter case means $\vec{V} = \lambda\vec{k}$ (with $\lambda$ some constant), which enforces

$$V^0 \equiv \frac{\vec{k}\cdot\vec{V}}{k^0} = \frac{\lambda k^2}{k^0} = \frac{\lambda(k^0)^2}{k^0} = \lambda k^0 \quad\Longrightarrow\quad V^\mu = \lambda k^\mu\,.$$

$\boxed{\text{Lemma}}$ On its horizon, the Killing field is proportional to the normal of that hypersurface.

Proof: On the Killing horizon, by definition, $h^\mu$ is light-like and the normal $(N^2)^{;\mu}$ as well, while both are non-zero. And they are *orthogonal* to each other, $(-N^2)_{;\alpha}h^\alpha = 2h_{\mu;\alpha}h^\mu h^\alpha = 2h_{(\mu;\alpha)}h^\mu h^\alpha = 0$. Hence, they must be proportional. (That means, $h^\mu$ is a **null generator** of the Killing horizon, being both tangent and orthogonal to it.)

Corollary:

Since, on the Killing horizon, the vectors $h^\mu$ and $(N^2)^{;\mu}$ are proportional to each other, $\frac{1}{2}(N^2)^{;\mu} = -h^{\alpha;\mu}h_\alpha = h^{\mu;\alpha}h_\alpha = \kappa_{\rm H} h^\mu$, it means $h^\mu$ is *geodesic* there; and we have introduced an invariant $\kappa_{\rm H}$ quantifying how much non-affine its parameterization is. It is called the **surface gravity** and will be very important later.

$\boxed{\text{Theorem [Vishveshwara (1968)]}}$

The set $\{g_{\mu\nu}h^\mu h^\nu \equiv -N^2 = 0, \text{ with } h^\mu \text{ Killing and } h^\mu \neq 0\}$ is a null hypersurface (thus a Killing horizon) if and only if $\omega_\mu[h] = 0$ there, with $(N^2)_{;\alpha} \neq 0$.

Proof $\Rightarrow$: First, the above Lemma says that on the horizon $h^\mu$ and $(N^2)^{;\mu}$ are proportional. Hence, if $h^\mu$ is non-trivial, $(N^2)^{;\mu}$ must be as well, and since $(N^2)^{;\mu}$ is hypersurface-orthogonal by construction, $h^\mu$ has to be as well, thus having zero vorticity there. (Note that from the formula (1.6) it only follows that $\omega^\mu$ is null there, $\omega_\mu\omega^\mu = 0$.)

Proof $\Leftarrow$: If $\omega = 0$ on the set $\{N^2 = 0\}$, formula (1.6) says that this set has to be null. Besides that, Frobenius says it is an integral hypersurface. And, if $(N^2)^{;\mu} \neq 0$ there, it implies that $h^\mu$ is non-trivial there as well, because – again – these two vectors have to be proportional there.

$\boxed{\text{Lemma}}$ On a Killing horizon, $R_{\alpha\beta}h^\alpha h^\beta$ (thus $T_{\alpha\beta}h^\alpha h^\beta$) has to vanish.

Proof follows from the Killing property, from the Raychaudhuri equation and from the theorem by Vishveshwara. Consider first that for a geodesic null field (a *generic* one, not necessarily Killing yet) which is *not* affinely parameterized, $k_{\alpha;\beta}k^\beta = \kappa k_\alpha$, the Raychaudhuri equation reads

$$\Theta_{;\lambda}k^\lambda = (k^\alpha{}_{;\beta}k^\beta)_{;\alpha} - \frac{1}{2}\Theta^2 + 2\omega^2 - 2\sigma^2 - R_{\alpha\beta}k^\alpha k^\beta\,.$$

Actually, recall that in a time-like case, when acceleration $a^\alpha := u^\alpha{}_{;\beta}u^\beta$ is taken into account, the Raychaudhuri equation involves the extra term $a^\alpha{}_{;\alpha}$. Here the role of this term is played by $(k^\alpha{}_{;\beta}k^\beta)_{;\alpha} = (\kappa k^\alpha)_{;\alpha}$. Now consider the case of a *Killing* vector field (which somewhere has its horizon), $k^\mu \to h^\mu$. For such, $h_{\mu;\nu} = h_{[\mu;\nu]}$, so the expansion scalar $\Theta \equiv h^\mu{}_{;\mu} = 0$ and (thus) the shear scalar reads

$$\sigma^2 = \frac{1}{2}\,\cancel{h_{(\mu;\nu)}}h^{\mu;\nu} + \frac{1}{4}\,\kappa^2 h_\mu h^\mu - \frac{\cancel{\Theta}^2}{4}\,.$$

On its horizon, $h^\mu$ is null, so the shear scalar vanishes completely, and $h^\mu$ has zero vorticity (Vishveshwara theorem), hence the Raychaudhuri equation reduces to $0 = (h^\alpha{}_{;\beta}h^\beta)_{;\alpha} - R_{\alpha\beta}h^\alpha h^\beta$. Finally, the red term reads

$$(h^\alpha{}_{;\beta}h^\beta)_{;\alpha} = (\kappa h^\alpha)_{;\alpha} = \kappa_{,\alpha}h^\alpha + \kappa_{,\alpha}\cancel{h^\alpha{}_{;\alpha}}\,,$$

of which the second part vanishes (*everywhere*) due to the zero expansion, and the first part has to vanish on the horizon, because the horizon value $\kappa_{\rm H}$ has to be constant along $h^\alpha$ (since the latter is a symmetry generator). Well, let us show the latter formally: application of the covariant derivative along $h^\alpha$ to the equation $\kappa_{\rm H}^2 = -\frac{1}{2}\, h^{\mu;\nu} h_{\mu;\nu}$ yields

$$h^\alpha (\kappa_{\rm H}^2)_{;\alpha} = -h^\alpha h^{\mu;\nu} h_{\mu;\nu\alpha} = -h^{\mu;\nu} R_{\mu\nu\alpha\sigma} h^\alpha h^\sigma = 0\,.$$

Hence, $R_{\alpha\beta} h^\alpha h^\beta = 0$ on the Killing horizon. Using the Einstein equations (in principle even with the cosmological term), this means that $T_{\alpha\beta} h^\alpha h^\beta = 0$ on a Killing horizon. Actually, on the horizon where $h^\mu$ is null, one has

$$8\pi\, T_{\alpha\beta} h^\alpha h^\beta = \left( R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} \right) h^\alpha h^\beta = R_{\alpha\beta} h^\alpha h^\beta\,.$$

Corollary: The condition $T_{\alpha\beta} h^\alpha h^\beta = 0$ can only be satisfied by a vacuum or by a special EM field aligned with $h^\mu$, as it is clear from the EM energy-momentum tensor:

$$4\pi T_{\alpha\beta} h^\alpha h^\beta = (F_{\alpha\iota} F_\beta{}^\iota + {}^*F_{\alpha\iota} {}^*F_\beta{}^\iota)\, h^\alpha h^\beta = F_{\alpha\iota} h^\alpha F_\beta{}^\iota h^\beta + {}^*F_{\alpha\iota} h^{\alpha *} F_\beta{}^\iota h^\beta = E_\iota E^\iota + B_\iota B^\iota\,,$$

where $E_\iota := -F_{\alpha\iota} h^\alpha$, $B_\iota := {}^*F_{\alpha\iota} h^\alpha$. Hence, both the electric-field and magnetic-field vectors have to be light-like, which means they have to be proportional to $h^\mu$ on the horizon; in other words, $h^\mu$ has to be a common principal null vector of both $F_{\alpha\iota}$ and ${}^*F_{\alpha\iota}$.

Another corollary: The condition $T_{\alpha\beta} h^\alpha h^\beta = 0$ means that $T_{\alpha\beta} h^\beta$ is orthogonal to $h^\alpha$ which however is null on the horizon, so $T_{\alpha\beta} h^\beta$ has to be either space-like or null. But the dominant energy condition requires that $T_{\alpha\beta} h^\beta$ be causal (time-like or null), so the only (limit) option is left that $T_{\alpha\beta} h^\beta$ is light-like. Two orthogonal light-like vectors necessarily being proportional, this means that $T_{\alpha\beta} h^\beta \sim h_\alpha$ on a Killing horizon.

Another expressions of $\kappa_{\rm H}$: By the Vishveshwara theorem, a Killing field has zero vorticity on its horizon, hence it is hypersurface-orthogonal there.[2] This is equivalent to the property (see GR course, Frobenius theorem)

$$0 = h^{\{\alpha;\beta} h^{\lambda\}} \equiv h^{\alpha;\beta} h^\lambda + h^{\lambda;\alpha} h^\beta + h^{\beta;\lambda} h^\alpha\,.$$

Multiplying this equation by $h_{\alpha;\beta}$, one obtains

$$\begin{aligned} 0 &= h^{\alpha;\beta} h_{\alpha;\beta} h^\lambda + h^{\lambda;\alpha} h_{\alpha;\beta} h^\beta + h^{\beta;\lambda} h_{\alpha;\beta} h^\alpha = h^{\alpha;\beta} h_{\alpha;\beta} h^\lambda + h^{\lambda;\alpha} \kappa_{\rm H} h_\alpha - h^{\beta;\lambda} \kappa_{\rm H} h_\beta = \\ &= h^{\alpha;\beta} h_{\alpha;\beta} h^\lambda + \kappa_{\rm H}^2 h^\lambda + \kappa_{\rm H}^2 h^\lambda = (h^{\alpha;\beta} h_{\alpha;\beta} + 2\kappa_{\rm H}^2)\, h^\lambda \\ &\Longrightarrow \quad \kappa_{\rm H}^2 = -\frac{1}{2}\, h^{\alpha;\beta} h_{\alpha;\beta}\,. \end{aligned}$$

Yet another expression of $\kappa_{\rm H}$: On the horizon (where $R_{\alpha\beta} \xi^\alpha \xi^\beta = 0$), equation (1.7) implies (for $\xi^\mu \to h^\mu$)

$$\kappa_{\rm H}^2 \equiv -\frac{1}{2}\, h^{\alpha;\beta} h_{\alpha;\beta} = -\frac{1}{4} \Box (-N^2)\,.$$

[2] Actually, we have just shown that, on the horizon, $\frac{1}{2}(N^2)^{;\mu} = \kappa_{\rm H} h^\mu$, which means that $h^\mu$ is hypersurface-orthogonal there since $h_\alpha = \frac{1}{2\kappa_{\rm H}}(N^2)_{,\alpha}$.

## 1.4 Stationary (and axisymmetric) black holes

Most notably it was Roger Penrose who, at the end of 1960s, was arguing that total gravitational collapse is a *generic* tendency of large masses, rather than anything just tied to very high symmetries or otherwise special initial conditions. He outlined the picture of such a collapse, accompanied by radiating away most of the features of the collapsing object, and leaving a relatively simple black hole which in a correspondingly short dynamical time (scaled by $GM/c^3$) settles to a quasi-stationary state (if not strongly perturbed further, of course).

Intuitively, a stationary yet non-static, i.e. *rotating* body should be axisymmetric, since otherwise it would generate gravitational waves (and possibly EM waves as well), so it *could not stay stationary*. This was really subsequently proved by Stephen Hawking within his **strong rigidity theorem** [1972]. He showed that the stationary horizon has to be either static or axially symmetric, and that the boundaries of stationary black holes represent Killing horizons of the asymptotically time-like Killing vector fields. Indeed, however generic our previous section may have sound, it factually concerned the **stationary and axisymmetric space-times**. Leaving this fundamental part outside the scope of these notes, let us at least sketch the notions and results which are necessary.

Definition Complement of the causal past of future null infinity is called a **black hole**. By this definition, black hole is the region from where there is no escape to future null infinity. The boundary of a black hole is called the **(future) event horizon**.

Remark: The space-time in which these notions are defined is supposed to be **strongly asymptotically predictable**, which means i) asymptotically flat and ii) with the closure of the causal past of future null infinity being **globally hyperbolic** (more precisely, there has to exist an open region of "conformal" space-time – obtained from the original one by a conformal transformation – which is globally hyperbolic and which contains the whole closure of the causal past of future null infinity of the original space-time). Practically, this means the absence of naked singularities.

It turns out that the event horizon is necessarily a 3D achronal (space-like or light-like) hypersurface. Worth to have in mind that the event horizon is a *global* concept in the sense that one has to know *all* the space-time up to future null infinity in order to decide whether and where there is a black hole. There exist *quasi-local* definitions of horizons, mostly based on Penrose's **(marginally) trapped surfaces** (compact 2D submanifolds such that both outgoing and ingoing orthogonal geodesic null congruences have non-positive expansions). In a dynamical situation, the various horizons may differ (and they *typically* differ from the instantaneous event horizon), but for stationary space-times they all coincide and reduce to the Killing horizon of the respective time Killing field. The strong rigidity theorem is the crucial result in this direction.

The strong rigidity theorem [Hawking 1972]

- Have a stationary event horizon in a space-time which is analytic, where the fundamental matter fields obey well-behaved hyperbolic equations and where the weak energy condition holds. The null generator of such a horizon coincides there with a certain Killing vector field of the whole space-time, hence it is a Killing horizon.

- There are only two possibilities: either the (Killing) field which generates the horizon is hypersurface-orthogonal (i.e., the region outside the black hole is *static*), or there exists another, axial Killing vector field (an asymptotically space-like field with closed orbits and non-empty set of fixed points [the **axis**]).

So let us consider **stationary and** – necessarily – **axisymmetric black holes**. Denote by $t^\mu$ and $\phi^\mu$ the time and the axial Killing fields. We assume the two symmetries commute, as it is always the case in an asymptotically flat space-time. Denote by $t$ and $\phi$ the parameters of the time and the axial symmetries. Physically, $t$ represents proper time of a rest observer at infinity, while $\phi$ represents the azimuthal angle (supposed to range from $0$ to $2\pi$ in the sense of the source rotation). The Killing fields then write $t^\mu = \frac{\partial x^\mu}{\partial t}$, $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$. We (thus) assume there exists a regular symmetry axis where the azimuthal circumference of the orbits of $\phi^\mu$ (which amounts to $2\pi g_{\phi\phi}$) shrinks to zero, i.e. where $g_{\phi\phi} \equiv g_{\mu\nu}\phi^\mu\phi^\nu$ vanishes.

Corollary [explaining why "rigidity" theorem]:

Angular velocity of the stationary axisymmetric horizon is constant all over the horizon.

Proof: The Killing field ($h^\mu$) claimed to coincide, on the horizon, with the latter's null generator has to be a linear combination of $t^\mu$ and $\phi^\mu$, since otherwise it would indicate the existence of a third independent symmetry (which however is assumed *not* to be present). With a suitable normalization, one can thus write $h^\mu = t^\mu + \omega_{\rm H}\phi^\mu$. The quantity $\omega_{\rm H}$ has to be constant (even off the horizon), because otherwise $h^\mu$ would not satisfy the Killing equation. Now, since the horizon has to itself stay invariant under the action of $t^\mu$ and/or $\phi^\mu$, these vectors have to be tangent to it (being space-like in general), and since $h^\mu$ should be null on the horizon, they should also be normal to $h^\mu$ there. Hence, on the horizon, multiplication of the above relation by $h_\mu$ yields $0=0$, while multiplications by $\phi_\mu$ and by $t_\mu$ yield

$$\omega_{\rm H} = -\frac{\phi_\mu t^\mu}{\phi_\alpha\phi^\alpha} \equiv -\frac{g_{\mu\nu}t^\mu\phi^\nu}{g_{\alpha\beta}\phi^\alpha\phi^\beta} \equiv -\frac{g_{t\phi}}{g_{\phi\phi}}\,, \qquad \omega_{\rm H} = -\frac{t_\mu t^\mu}{t_\alpha\phi^\alpha} \equiv -\frac{g_{\mu\nu}t^\mu t^\nu}{g_{\alpha\beta}t^\alpha\phi^\beta} \equiv -\frac{g_{tt}}{g_{t\phi}}\,,$$

where the metric components are evaluated *on the horizon*. The first expression indicates that $\omega_{\rm H}$ can be interpreted as the angular velocity of the horizon with respect to infinity. Equating the two expressions, one obtains

$$0 = -g_{tt}g_{\phi\phi} + (g_{t\phi})^2 = g_{\phi\phi}(-g_{tt} - g_{t\phi}\omega_{\rm H}) \equiv g_{\phi\phi}N^2\,.$$

Indeed, it is easy to check that this $N^2$ coincides with the one introduced before in Sections 1.2 and 1.3, since

$$-N^2 \equiv h_\mu h^\mu = (t_\mu + \omega_{\rm H}\phi_\mu)(t^\mu + \omega_{\rm H}\phi^\mu) = g_{tt} + 2g_{t\phi}\omega_{\rm H} + g_{\phi\phi}\omega_{\rm H}^2 = g_{tt} + g_{t\phi}\omega_{\rm H}\,.$$

## 1.5 Warnings

Consider such a thought: let $h^\mu$ be a Killing-horizon null geodesic generator. Then the l.h. side of $h^{\mu;\alpha}h_\alpha = \kappa_{\rm H}h^\mu$ can be written as $-h^{\alpha;\mu}h_\alpha$, because $h^\mu$ is Killing. And $h^\mu$ is also null, hence $h^{\alpha;\mu}h_\alpha = 0$. So what is it all about?

- One may make such mistakes at many places of the Killing-horizon calculations. The point is that $h^\mu$ is only null at the very horizon – this property in fact *defines* the horizon, it is (and has to be) *specific* for it. So $(-N^2)^{;\mu} \equiv (h^\alpha h_\alpha)^{;\mu} = 2h^{\alpha;\mu}h_\alpha$ is *non-zero*. Indeed, remember this was especially claimed/assumed in the Vishveshwara theorem.

- The Killing property of $h^\mu$ is a subtle point, too. It is factually only required at the horizon. Actually, recall the careful wording of the strong rigidity theorem: "The null generator of such a horizon coincides there with a certain Killing vector field of the whole space-time..." So one may either base the study on a vector field which is Killing *everywhere*, albeit it may be unphysical (e.g. space-like) at some regions; or, one may know of some "reasonable" vector field (possibly with certain favourable properties) which however is *only* Killing at the very horizon. While the first (everywhere Killing) option is the $h^\mu = t^\mu + \omega_{\rm H}\phi^\mu$ field (which is space-like above a certain radius), we will see the second option (Killing only at the horizon) is the field $\mathcal{n}^\mu = t^\mu + \omega\phi^\mu$, with $\omega := -\frac{t_\mu\phi^\mu}{\phi_\alpha\phi^\alpha}$ (which is time-like everywhere outside the horizon). We will see such a field, for example, has the nice property that it is *everywhere* vorticity-free.

- Indeed, the vanishing vorticity (i.e., hypersurface orthogonality) is another "issue": vorticity is *also* only required to vanish for the *horizon* generator. Of the above two fields, the everywhere-Killing one, $h^\mu = t^\mu + \omega_{\rm H}\phi^\mu$, is only vorticity-free at the horizon, whereas $\mathcal{n}^\mu = t^\mu + \omega\phi^\mu$ will be shown to have zero vorticity *everywhere*.

Clearly, one has to always take care of *where*, how generally the given property is being employed. In particular, the most precarious is to use any of the properties (potentially only applying at the very horizon) *under a derivative*. In such a case, one should always check carefully how the given quantity behaves under a given differentiation.

## 1.6 Orthogonal transitivity and circularity

We have mentioned how one has to be careful about what holds "in a volume" (perhaps even everywhere) and what only holds on the horizon. Another source of confusion may be how general is the host space-time. The basic statements concerning Killing horizons really just require that a certain Killing vector field becomes null on a null hypersurface. However, if one considers – most naturally – the Killing field representing stationarity, the strong rigidity theorem claims that stationarity also implies axisymmetry (if not staticity), though under the very strong assumption of the metric *analyticity*. In an asymptotically flat case, the two symmetries commute, so one naturally arrives at *stationary and axisymmetric space-times*.

Within that setting, the crucial results (such as the rigidity theorem or the zeroth law of black-hole thermodynamics we will derive later) follow by Einstein equations and some of the energy conditions (plus possibly the analyticity). Another possibility is to assume that the space-time moreover is *orthogonally transitive*. Then those conclusions follow by purely geometric means, irrespectively of the field equations. When speaking of stationary and axisymmetric space-times, people often suppose this additional property automatically, yet it is by no means *granted*, so we somewhat go into it below.

The planes locally spanned by two independent smooth vector fields are called the **surfaces of transitivity** of the continuous group generated by these fields. The group is said to be **orthogonally transitive**, if there exists a family of complementary-dimension integral surfaces everywhere orthogonal to the surfaces of transitivity. Now, consider the continuous group generated by stationarity and axisymmetry. Its surfaces of transitivity are locally spanned by $t^\mu$ and $\phi^\mu$, and the question is *whether there exist global "meridional" surfaces*, everywhere orthogonal to both of them.

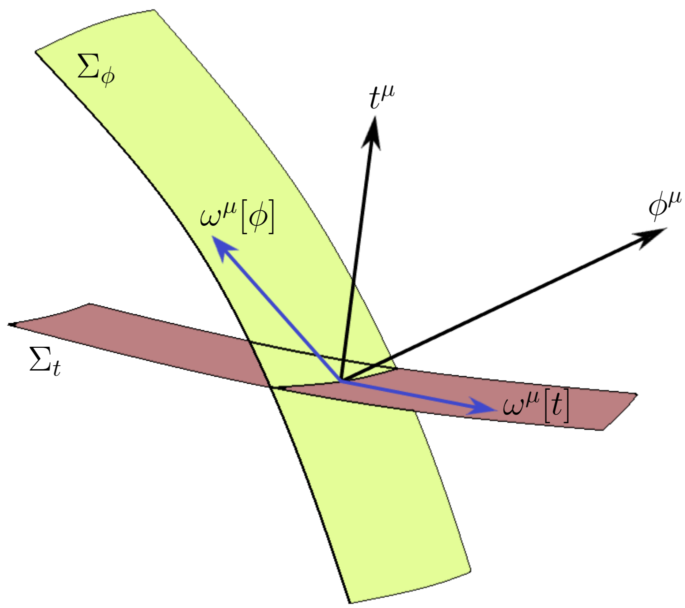


**Figure 1.1** *Two independent vector fields, $t^\mu$ and $\phi^\mu$, their orthogonal hypersurfaces $\Sigma_t$ and $\Sigma_\phi$, and their vorticity vectors $\omega^\mu[t]$ and $\omega^\mu[\phi]$ (which are orthogonal to them, so tangent to $\Sigma(t)$ and $\Sigma(\phi)$, respectively). If the intersection $\Sigma(t) \cap \Sigma(\phi)$ is integrable, one speaks of **orthogonal transitivity** of the symmetry group generated by $t^\mu$ and $\phi^\mu$.*

In the case of just one vector field $V^\mu$, Frobenius theorem says that its local orthogonal hyperplanes $\Sigma(V)$ are integrable if and only if $V^\mu$ has zero vorticity – when the latter does not "entwine" about itself. Here, in the case of two fields, $t^\mu$ and $\phi^\mu$, wanted is the integrability of the local planes given by intersections $\Sigma(t)\cap\Sigma(\phi)$. Such planes would certainly be integrable if both the normal fields $t^\mu$ and $\phi^\mu$ had zero vorticities. However, even weaker condition is clearly sufficient: the vorticity of $t^\mu$ must not have a component in the direction of $\phi^\mu$, and the vorticity of $\phi^\mu$ must not have a component in the direction of $t^\mu$. Therefore, one demands

$$\omega^\mu[t]\phi_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_\mu t_{\nu;\rho}t_\sigma = 0 \qquad \text{and} \qquad \omega[\phi]^\mu t_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}t_\mu\phi_{\nu;\rho}\phi_\sigma = 0\ ;$$

more often it is being written as

$$\phi_{[\mu}t_{\nu;\rho}t_{\sigma]} = 0 \qquad \text{and} \qquad t_{[\mu}\phi_{\nu;\rho}\phi_{\sigma]} = 0\ .$$

If these conditions are satisfied, the thus existing global meridional planes can be covered by coordinates $x^1$, $x^2$, e.g. by $r$ and $\theta$ (or $\rho$ and $z$), such that the metric does not contain the terms

$g_{t1}$, $g_{t2}$, $g_{1\phi}$ and $g_{2\phi}$. The latter can also be voiced in such a way that the metric is invariant under the inversion $t \to -t$, $\phi \to -\phi$; one says that the symmetry group is *invertible* (strictly speaking, this property is slightly stronger than the orthogonal transitivity, see Carter 1969).

Proposition

In a general stationary and axisymmetric case, the vorticity vector of $n^\mu = t^\mu + \omega\phi^\mu$ satisfies

$$\omega^\mu[n] := \frac{1}{2}\,\epsilon^{\mu\nu\kappa\lambda} n_{\nu;\kappa} n_\lambda = \frac{\phi^\mu}{g_{\phi\phi}}\,(\phi_\iota\omega^\iota[t] - \omega\, t_\iota\omega^\iota[\phi])\,.$$

Proof:

$$\begin{aligned} n_{\nu;\kappa} n_\lambda &= (t_{\nu;\kappa} + \omega\phi_{\nu;\kappa} + \phi_\nu\omega_{;\kappa})(t_\lambda + \omega\phi_\lambda) = \\ &= t_{\nu;\kappa}t_\lambda + \omega\, t_{\nu;\kappa}\phi_\lambda + \omega\phi_{\nu;\kappa}t_\lambda + \omega^2\phi_{\nu;\kappa}\phi_\lambda + \omega_{;\kappa}\phi_\nu t_\lambda + \omega\omega_{;\kappa}\phi_\nu\phi_\lambda\,. \end{aligned}$$

The last term can be omitted since it is symmetric in $(\nu, \lambda)$ and thus does not contribute to $n_{[\nu;\kappa} n_{\lambda]}$. To the above, it is suitable to substitute the gradient of $\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = -\frac{t_\iota\phi^\iota}{\phi_\sigma\phi^\sigma}$ ,

$$\begin{aligned} \omega_{;\kappa} &= -\frac{(g_{t\phi})_{;\kappa}}{g_{\phi\phi}} + \frac{g_{t\phi}}{g_{\phi\phi}}\frac{(g_{\phi\phi})_{;\kappa}}{g_{\phi\phi}} = -\frac{(g_{t\phi})_{;\kappa} + \omega(g_{\phi\phi})_{;\kappa}}{g_{\phi\phi}} = -\frac{(t_\iota\phi^\iota)_{;\kappa} + \omega(\phi_\iota\phi^\iota)_{;\kappa}}{g_{\phi\phi}} = \\ &= -\frac{-2t_{\kappa;\iota}\phi^\iota - 2\omega\phi_{\kappa;\iota}\phi^\iota}{g_{\phi\phi}} = \frac{2\phi^\iota}{g_{\phi\phi}}\,(t_{\kappa;\iota} + \omega\phi_{\kappa;\iota})\,, \end{aligned}$$

where we have used that $t^\mu$ and $\phi^\mu$ are Killings and that they commute ($t^\iota\phi_{\kappa;\iota} = t_{\kappa;\iota}\phi^\iota$),

$$\begin{aligned} (t_\iota\phi^\iota)_{;\kappa} &= t_{\iota;\kappa}\phi^\iota + t^\iota\phi_{\iota;\kappa} = -t_{\kappa;\iota}\phi^\iota - t^\iota\phi_{\kappa;\iota} = -2t_{\kappa;\iota}\phi^\iota\,, \\ (\phi_\iota\phi^\iota)_{;\kappa} &= 2\phi_{\iota;\kappa}\phi^\iota = -2\phi_{\kappa;\iota}\phi^\iota\,. \end{aligned}$$

Substituting to $n_{\nu;\kappa} n_\lambda$ above (without the last term already), we have

$$\begin{aligned} n_{\nu;\kappa} n_\lambda &= t_{\nu;\kappa}t_\lambda + \omega\, t_{\nu;\kappa}\phi_\lambda + \omega\phi_{\nu;\kappa}t_\lambda + \omega^2\phi_{\nu;\kappa}\phi_\lambda + \frac{2\phi^\iota}{g_{\phi\phi}}\,(t_{\kappa;\iota} + \omega\phi_{\kappa;\iota})\phi_\nu t_\lambda = \\ &= \frac{\phi^\iota}{g_{\phi\phi}}\,(t_{\nu;\kappa}t_\lambda\phi_\iota - t_{\nu;\kappa}\phi_\lambda t_\iota + \omega\phi_{\nu;\kappa}t_\lambda\phi_\iota - \omega\phi_{\nu;\kappa}\phi_\lambda t_\iota + 2t_{\kappa;\iota}\phi_\nu t_\lambda + 2\omega\phi_{\kappa;\iota}\phi_\nu t_\lambda)\,. \end{aligned}$$

Imposing the anti-symmetrization $[\nu\kappa\lambda]$, we write separately the terms with $t_{\mu;\alpha}$ and $\phi_{\mu;\alpha}$ ,

$$\begin{aligned} t_{[\nu;\kappa}t_{\lambda]}\phi_\iota - t_{[\nu;\kappa}\phi_{\lambda]}t_\iota + 2\phi_{[\nu}t_\lambda t_{\kappa];\iota} &= \phi_\iota t_{[\nu;\kappa}t_{\lambda]} - t_{\iota;[\nu}t_\kappa\phi_{\lambda]} + t_{[\nu}\phi_\kappa t_{\lambda];\iota} - t_\iota\phi_{[\nu}t_{\kappa;\lambda]} = \\ &= 4\,\phi_{[\iota}t_{\nu;\kappa}t_{\lambda]}\,, \\ \phi_{[\nu;\kappa}t_{\lambda]}\phi_\iota - \phi_{[\nu;\kappa}\phi_{\lambda]}t_\iota + 2\phi_{[\nu}t_\lambda\phi_{\kappa];\iota} &= \phi_\iota\phi_{[\nu;\kappa}t_{\lambda]} - \phi_{\iota;[\nu}t_\kappa\phi_{\lambda]} + t_{[\nu}\phi_\kappa\phi_{\lambda];\iota} - t_\iota\phi_{[\nu}\phi_{\kappa;\lambda]} = \\ &= 4\,\phi_{[\iota}\phi_{\nu;\kappa}t_{\lambda]} = -4\,t_{[\iota}\phi_{\nu;\kappa}\phi_{\lambda]}\,, \end{aligned}$$

thus finding that

$$n_{[\nu;\kappa} n_{\lambda]} = \frac{4\phi^\iota}{g_{\phi\phi}}\left(\phi_{[\iota}t_{\nu;\kappa}t_{\lambda]} - \omega\, t_{[\iota}\phi_{\nu;\kappa}\phi_{\lambda]}\right)$$

$$\Rightarrow \quad \omega^\mu[n] = \frac{1}{2}\,\epsilon^{\mu\nu\kappa\lambda} n_{[\nu;\kappa} n_{\lambda]} = \frac{2\phi^\iota}{g_{\phi\phi}} \left( \epsilon^{\mu\nu\kappa\lambda} \phi_{[\iota} t_{\nu;\kappa} t_{\lambda]} - \omega \epsilon^{\mu\nu\kappa\lambda} t_{[\iota} \phi_{\nu;\kappa} \phi_{\lambda]} \right).$$

For each value of the free index $\mu$, the summation over $\iota$ only yields non-trivial contribution for $\iota = \mu$, so the formula can be written as

$$\begin{aligned} \omega^\mu[n] &= \frac{2\phi^\mu}{g_{\phi\phi}} \left( \epsilon^{\iota\nu\kappa\lambda} \phi_{[\iota} t_{\nu;\kappa} t_{\lambda]} - \omega \epsilon^{\iota\nu\kappa\lambda} t_{[\iota} \phi_{\nu;\kappa} \phi_{\lambda]} \right) = \\ &= \frac{\phi^\mu}{2g_{\phi\phi}} \left( \epsilon^{\iota\nu\kappa\lambda} \phi_\iota t_{[\nu;\kappa} t_{\lambda]} - \omega \epsilon^{\iota\nu\kappa\lambda} t_\iota \phi_{[\nu;\kappa} \phi_{\lambda]} \right) = \frac{\phi^\mu}{g_{\phi\phi}} \left( \phi_\iota \omega^\iota[t] - \omega\, t_\iota \omega^\iota[\phi] \right). \qquad \square \end{aligned}$$

Corollary: Space-time is circular if and only if $n^\mu = t^\mu + \omega\phi^\mu$ is hypersurface-orthogonal.
Proof: First, the circularity conditions $\phi_\iota \omega^\iota[t] = 0$, $t_\iota \omega^\iota[\phi] = 0$ immediately make $\omega^\mu[n]$ vanish, so $n^\mu$ is then hypersurface-orthogonal. Equally clearly, if $n^\mu$ is hypersurface-orthogonal, i.e. $n_{[\nu;\kappa} n_{\lambda]} = 0$, then

$$0 = \phi_{[\mu} n_{\nu;\kappa} n_{\lambda]} = \phi_{[\mu} n_{\nu;\kappa} t_{\lambda]} = \phi_{[\mu} t_{\nu;\kappa} t_{\lambda]} + \omega \phi_{[\mu} \phi_{\nu;\kappa} t_{\lambda]},$$

because $n_{\nu;\kappa} = t_{\nu;\kappa} + \omega\phi_{\nu;\kappa} + \phi_\nu \omega_{,\kappa}$ (so the last term cancels in anti-symmetrization with $\phi_\mu$).

Theorem [Papapetrou, 1966]

The conditions $\phi_{[\mu} t_\nu t_{\kappa;\lambda]} = 0$, $t_{[\mu} \phi_\nu \phi_{\kappa;\lambda]} = 0$ are equivalent to the conditions

$$\boxed{\phi_{[\mu} t_\lambda R_{\kappa]\nu} t^\nu = 0, \qquad t_{[\mu} \phi_\lambda R_{\kappa]\nu} \phi^\nu = 0}.$$

Proof: The derivation is the same for both the conditions, and since it starts from properties which hold for *any* Killing vector field, we will at this stage denote the latter generically as $\xi^\mu$. Multiplying the definition $\omega_\mu[\xi] = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} \xi^{\nu;\kappa} \xi^\lambda$ by $\epsilon^{\mu\beta\gamma\delta}$ and using the formula (1.4), i.e., explicitly,

$$\epsilon_{\mu\nu\kappa\lambda} \epsilon^{\mu\beta\gamma\delta} = -\delta^\beta_\nu \delta^\gamma_\kappa \delta^\delta_\lambda - \delta^\delta_\nu \delta^\beta_\kappa \delta^\gamma_\lambda - \delta^\gamma_\nu \delta^\delta_\kappa \delta^\beta_\lambda + \delta^\beta_\nu \delta^\delta_\kappa \delta^\gamma_\lambda + \delta^\gamma_\nu \delta^\beta_\kappa \delta^\delta_\lambda + \delta^\delta_\nu \delta^\gamma_\kappa \delta^\beta_\lambda,$$

we easily obtain

$$\omega_\mu \epsilon^{\mu\beta\gamma\delta} = -\xi^\beta \xi^{[\gamma;\delta]} - \xi^\delta \xi^{[\beta;\gamma]} - \xi^\gamma \xi^{[\delta;\beta]} = -\xi^\beta \xi^{\gamma;\delta} - \xi^\delta \xi^{\beta;\gamma} - \xi^\gamma \xi^{\delta;\beta} = -\xi^{\{\beta} \xi^{\gamma;\delta\}}.$$

Differentiation of the latter by $x^\beta$ yields

$$\begin{aligned} \omega_{\mu;\beta} \epsilon^{\mu\beta\gamma\delta} &= -(\xi^\beta \xi^{\gamma;\delta})_{;\beta} - (\xi^\delta \xi^{\beta;\gamma})_{;\beta} - (\xi^\gamma \xi^{\delta;\beta})_{;\beta} = \\ &= -\cancel{\xi^\beta{}_{;\beta}} \xi^{\gamma;\delta} - \xi^\beta \xi^{\gamma;\delta}{}_\beta - \cancel{\xi^\delta{}_{;\beta} \xi^{\beta;\gamma}} - \xi^\delta \xi^{\beta;\gamma}{}_\beta - \cancel{\xi^\gamma{}_{;\beta} \xi^{\delta;\beta}} - \xi^\gamma \xi^{\delta;\beta}{}_\beta = \\ &= -\xi_\beta \xi^{\gamma;\delta\beta} + \xi^\delta \square \xi^\gamma - \xi^\gamma \square \xi^\delta = -\cancel{\xi_\beta R^{\gamma\delta\beta\sigma} \xi_\sigma} + \xi^\gamma R^\delta_\beta \xi^\beta - \xi^\delta R^\gamma_\beta \xi^\beta, \end{aligned} \tag{1.9}$$

where the Killing property and the formulas (1.1), (1.2) have been employed. Multiplication of this relation by $\epsilon_{\alpha\nu\gamma\delta}$ leads to twice the same term on the right-hand side, while on the left one has

$$\omega_{\mu;\beta} \epsilon^{\mu\beta\gamma\delta} \epsilon_{\alpha\nu\gamma\delta} = 2\,\omega_{\mu;\beta} (\delta^\mu_\nu \delta^\beta_\alpha - \delta^\mu_\alpha \delta^\beta_\nu) = 2(\omega_{\nu;\alpha} - \omega_{\alpha;\nu}) \equiv 4\,\omega_{[\nu;\alpha]},$$

so we arrive at the formula for gradient of (any) Killing-vector twist,

$$\boxed{\omega_{[\nu;\alpha]} = \frac{1}{2}\,\epsilon_{\alpha\nu\gamma\delta}\,\xi^{\gamma} R^{\delta}_{\beta}\,\xi^{\beta}}\,. \tag{1.10}$$

Now, let us specify to our $\xi^{\mu} \equiv t^{\mu}$, $\omega_{\mu} \equiv \omega_{\mu}[t]$ case (with $\phi^{\mu}$ the second existing Killing field) and consider the derivative

$$\begin{aligned}\left(\phi^{\nu}\omega_{\nu}\right)_{,\alpha} &= \phi^{\nu}{}_{;\alpha}\omega_{\nu} + \phi^{\nu}\omega_{\nu;\alpha} = \phi^{\nu}{}_{;\alpha}\omega_{\nu} + \phi^{\nu}\omega_{\alpha;\nu} + 2\phi^{\nu}\omega_{[\nu;\alpha]} = \\ &= \cancel{(\pounds_{\phi}\omega_{\alpha})} + \epsilon_{\alpha\nu\gamma\delta}\,\phi^{\nu}t^{\gamma}R^{\delta}_{\beta}\,t^{\beta}\,.\end{aligned} \tag{1.11}$$

This result confirms that

$$\phi_{[\mu}t_{\nu}t_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad \left(\phi^{\nu}\omega_{\nu}[t]\right)_{,\alpha} = 0 \quad \Longleftrightarrow \quad \phi_{[\mu}t_{\lambda}R_{\kappa]\nu}t^{\nu} = 0\,.$$

Similarly one would verify that

$$t_{[\mu}\phi_{\nu}\phi_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad \left(t^{\nu}\omega_{\nu}[\phi]\right)_{,\alpha} = 0 \quad \Longleftrightarrow \quad t_{[\mu}\phi_{\lambda}R_{\kappa]\nu}\phi^{\nu} = 0\,.$$

The opposite implications are also based on the relation (1.11). Since $\phi_{\mu} = g_{\mu\alpha}\phi^{\alpha} = g_{\mu\phi}$ vanishes on the symmetry axis,[3] also trivial there is $\omega^{\mu}[\phi] = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\phi_{\nu}\phi_{\kappa;\lambda}$. Consequently, both the invariants $\phi_{\nu}\omega^{\nu}[t]$ and $t_{\nu}\omega^{\nu}[\phi]$ vanish on the axis as well. Now, if the space-time satisfies $\phi_{[\mu}t_{\lambda}R_{\kappa]\nu}t^{\nu} = 0$ and $t_{[\mu}\phi_{\lambda}R_{\kappa]\nu}\phi^{\nu} = 0$, implying that the gradients of both the invariants are everywhere zero, $\left(\phi_{\nu}\omega^{\nu}[t]\right)_{,\alpha} = 0$ and $\left(t_{\nu}\omega^{\nu}[\phi]\right)_{,\alpha} = 0$, then the invariants are themselves zero everywhere, which is the orthogonal-transitivity condition. □

$\boxed{\text{Corollaries}}$: Since the orthogonal-transitivity properties trivially hold for the metric tensor,

$$\phi_{[\mu}t_{\lambda}g_{\kappa]\nu}t^{\nu} = \phi_{[\mu}t_{\lambda}t_{\kappa]} = 0\,, \qquad t_{[\mu}\phi_{\lambda}g_{\kappa]\nu}\phi^{\nu} = t_{[\mu}\phi_{\lambda}\phi_{\kappa]} = 0\,,$$

one may use Einstein equations and translate the Ricci-based conditions to

$$\phi_{[\mu}t_{\lambda}T_{\kappa]\nu}t^{\nu} = 0\,, \qquad t_{[\mu}\phi_{\lambda}T_{\kappa]\nu}\phi^{\nu} = 0\,. \tag{1.12}$$

They can be summarized (added) in $\phi_{[\mu}t_{\lambda}T_{\kappa]\nu}n^{\nu} = 0$ or $\phi_{[\mu}n_{\lambda}T_{\kappa]\nu}n^{\nu} = 0$, with $n^{\mu} = t^{\mu} + \omega\phi^{\mu}$ (remember that $\phi_{\mu}n^{\mu} = 0$, so the latter is even "nicer").

Immediately clear is that *vacuum* stationary and axisymmetric space-times are necessarily orthogonally transitive. Actually, every space-time is such in which sources move purely along stationary circular trajectories (along the Killing directions, i.e. with four-velocity satisfying $u_{[\nu}t_{\kappa}\phi_{\lambda]} = 0$). This is illustrated on an ideal fluid, $T_{\kappa\nu} = (\rho + P)u_{\kappa}u_{\lambda} + Pg_{\kappa\lambda}$: the second part is circular automatically and the first one has to satisfy $\phi_{[\mu}t_{\lambda}u_{\kappa]} = 0$, resp. $t_{[\mu}\phi_{\lambda}u_{\kappa]} = 0$ (which is the same). The stationary axisymmetric space-times which are orthogonally transitive are thus called **circular space-times**.

[3] On a regular axis, $g_{\phi\phi} \equiv \phi_{\mu}\phi^{\mu}$ has to vanish since it determines proper circumference about the axis (along a circular orbit of $\phi^{\mu}$ at some given radius). This is *not* due to $\phi^{\mu}$ becoming null (light-like) there, but because $\phi_{\mu} = g_{\mu\phi}$ shrinks there to *zero* (while $\phi^{\mu} = \partial x^{\mu}/\partial\phi$ everywhere).

### 1.6.1 Circular (i.e. "Kerr-like") metric

If the stationary and axisymmetric space-time is also circular, the metric can be written as

$$\boxed{\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + g_{\phi\phi}\,(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2}\,, \tag{1.13}$$

with $N^2 := -g_{tt} - g_{t\phi}\omega$ and $\omega := -\frac{g_{t\phi}}{g_{\phi\phi}}$ the **lapse function** and the **dragging angular velocity**, and $r$ and $\theta$ covering the meridional planes (alternatively, one might e.g. use the $\rho$ and $z$ coordinates of cylindrical type). Worth to repeat once more that the whole "Killing" part of the metric has invariant meaning since it is given by scalar products of the Killing fields,

$$g_{tt} = g_{\mu\nu}t^\mu t^\nu\,, \quad g_{t\phi} = g_{\mu\nu}t^\mu \phi^\nu\,, \quad g_{\phi\phi} = g_{\mu\nu}\phi^\mu \phi^\nu\,.$$

The same is of course inherited by $N^2$, $\omega$ and all other quantities possibly defined from that part of the metric.

The speciality of circular metrics, within generic stationary and axisymmetric ones, is that the meridional surfaces are *globally* orthogonal to $t^\mu$ and $\phi^\mu$, so the coordinates can be tied to them so that $g_{t1}$, $g_{t2}$, $g_{1\phi}$ and $g_{2\phi}$ vanish *everywhere* $(1, 2 \neq t, \phi)$.

### 1.6.2 Stationary circular orbits and their light cones

Consider a family of stationary motions along the circles $r = \mathrm{const}$, $\theta = \mathrm{const}$, i.e. with four-velocity given by combination of the Killing fields,

$$u^\mu = \frac{t^\mu + \Omega\phi^\mu}{|t^\mu + \Omega\phi^\mu|} = \frac{t^\mu + \Omega\phi^\mu}{\sqrt{-g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa)}}\,, \tag{1.14}$$

where the azimuthal angular velocity $\Omega := \frac{\mathrm{d}\phi}{\mathrm{d}t}$ is constant in time (it may however depend on $r$ and $\theta$). In the above coordinates $(t, r, \theta, \phi)$ adapted to the symmetries,

$$t^\mu = \delta^\mu_t, \;\; \phi^\mu = \delta^\mu_\phi, \quad g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa) = g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = -N^2 + g_{\phi\phi}(\Omega - \omega)^2\,,$$

so the four-velocity has components

$$u^\mu = u^t(1, 0, 0, \Omega)\,, \quad u^t = \frac{1}{\sqrt{N^2 - g_{\phi\phi}(\Omega - \omega)^2}}\,. \tag{1.15}$$

The angular velocity with respect to an asymptotic inertial system, $\Omega$, cannot be arbitrary – too large values would correspond to super-luminal (space-like) motion. The interval of time-like motion has boundaries where $u^\mu$ can no longer be normalized by any real $u^t$, i.e. at the roots of $g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = 0$:

$$\Omega_{\mathrm{max,min}} = \frac{-g_{t\phi} \pm \sqrt{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \frac{N}{\sqrt{g_{\phi\phi}}}\,. \tag{1.16}$$

### 1.6.3 Horizon as a limit for circular motions

Hence, the light cone of stationary circular motions, expressed in terms of $\Omega$, is centred by the dragging angular velocity $\omega$, extending from it equally to both sides. The time-like option for $\Omega$ closes up at $N = 0$, which (thus) clearly represents a horizon. The observers orbiting there with $\Omega = \omega$ are null already, being proportional to the null generators of the horizon. Below, $N^2$ becomes *negative* and $N$ thus imaginary.

### 1.6.4 Stationary circular congruence with zero angular momentum

Within the stationary circular orbits, the value $\Omega = \omega$ identifies the congruence of so-called Zero Angular Momentum Observers. Indeed, the (specific) azimuthal angular momentum (for *any* motion, not just the stationary circular one) reads

$$u_\phi = g_{\phi t}u^t + g_{\phi\phi}u^\phi = g_{\phi\phi}u^t(\Omega - \omega)\,,$$

so it vanishes if and only if $\Omega = \omega$. The ZAMO congruence has four-velocity and four-acceleration

$$u^\mu = \frac{t^\mu + \omega\phi^\mu}{N} = \frac{1}{N}(1,0,0,\omega) \equiv n^\mu\,, \qquad a_\alpha := u_{\alpha;\beta}u^\beta = \frac{N_{,\alpha}}{N}\,.$$

#### Notation etc

In the GR course, we have denoted $\xi^\mu := t^\mu + \omega\phi^\mu$, but that is somewhat misleading, because by $\xi^\mu$ we generally denote Killing vector fields. And, it *is* crucial in the present course to distinguish between Killing and non-Killing fields. So let us switch to the notation $\mathcal{n}^\mu := t^\mu + \omega\phi^\mu$. Consistence with the notation used in 3+1 splitting would suggest to use $Nn^\mu$, because in that way the quantities $N$, $n^\mu$ and $t^\mu$ would exactly correspond to what in 3+1 is denoted so ... $N$ is lapse, $n^\mu$ is the normal to the hypersurfaces $t = \mathrm{const}$ (see below) and $t^\mu$ is the “time vector"; and, the shift vector $N^\mu$, also important in 3+1, corresponds to $N^\mu \equiv -\omega\phi^\mu$. However, $Nn^\mu$ is graphically not ideal at times, so let us adhere to $\mathcal{n}^\mu$, with the remark that we will nowhere use the shift vector $N^\mu$, so there should be no confusion. The ZAMO congruence represents the most privileged stationary circular congruence, time-like everywhere from radial infinity down to the horizon. [This is not to claim that ZAMO is *the only* such privileged subfamily. For example, there exist **Carter (canonical) observers**, also time-like down to the horizon, and exclusive in that their angular velocity $\Omega$ is the same as that of principal null directions of the Weyl tensor (so the latter are purely radial relative to them). However, the Carter observers only exist in algebraically special space-times, not in *any* stationary and axisymmetric space-time as ZAMOs.]

<u>The field $\mathcal{n}^\mu = t^\mu + \omega\phi^\mu$ also has other nice properties:</u>[4]

- Scalar products:

$$g_{\mu\nu}\mathcal{n}^\mu\mathcal{n}^\nu = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega = -N^2$$

[4] Note that it is *not* a Killing field, because $\omega$ is *not* constant – it depends on both $r$ and $\theta$.

(hence, $\mathfrak{n}^\mu$ is null where $N^2=0$, i.e. on the horizon),

$$\phi_\mu \mathfrak{n}^\mu = \phi_\mu(t^\mu + \omega\phi^\mu) = g_{t\phi} + g_{\phi\phi}\omega = 0\,, \quad t_\mu \mathfrak{n}^\mu = t_\mu(t^\mu + \omega\phi^\mu) = g_{tt} + g_{t\phi}\omega = -N^2\,.$$

- Covariant version:

$$\mathfrak{n}_\alpha = g_{\alpha\mu}\mathfrak{n}^\mu = g_{\alpha\mu}(t^\mu + \omega\phi^\mu) = g_{\alpha t} + g_{\alpha\phi}\omega = -N^2\delta^t_\alpha = -N^2 t_{,\alpha}\,, \quad n_\alpha = -Nt_{,\alpha}$$

$\Rightarrow n^\mu$ is orthogonal to the hypersurfaces $t=\mathrm{const}$ $\Rightarrow$ it has zero twist, $\omega^\mu[n]=0$. The latter follows from Frobenius theorem; actually, it is clear immediately,

$$\omega^\mu[n] := \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} n_{\nu;\kappa} n_\lambda = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}(n_{\nu,\kappa} - \Gamma^\iota{}_{\underline{\kappa\nu}} n_\iota) n_\lambda = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}(Nt_{,\nu})_{,\kappa} Nt_{,\lambda} = 0\,.$$

Memory from GR: the Frobenius theorem says that hypersurface orthogonality and zero twist are *equivalent properties* – indeed, this property is well known: $\omega^\mu[n]$ represents rotation (curl) of $n_\nu$ within the (hyper)surface orthogonal to $n_\lambda$; if that rotation is zero, the field $n_\nu$ has to be proportional to a gradient of some scalar field. In components, $\omega^\mu[n]=0$ means $n_{\nu,\kappa}=n_{\kappa,\nu}$, which exactly are the integrability conditions for the equation $-f\mathrm{d}\Phi = n_\nu \mathrm{d}x^\nu$, i.e. $-f\Phi_{,\nu}=n_\nu$, with $f$ and $\Phi$ some scalars ($f\equiv N$ and $\Phi\equiv t$ in our case).

Reminder: as proved before in a coordinate-independent manner, this item only holds in the circular case. Indeed, otherwise $\mathfrak{n}_\alpha = g_{\alpha t} + g_{\alpha\phi}\omega$ may also have $r$ and/or $\theta$ components, and thus not to be orthogonal to $t=\mathrm{const}$.

- Expansion tensor:

$$\begin{aligned}\Theta_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu \mathfrak{n}_{(\alpha;\beta)} = (\delta^\alpha_\mu + n^\alpha n_\mu)(\delta^\beta_\nu + n^\beta n_\nu)\mathfrak{n}_{(\alpha;\beta)} = \\ &= \mathfrak{n}_{(\mu;\nu)} + \mathfrak{n}_{(\mu;\beta)} n^\beta n_\nu + \mathfrak{n}_{(\alpha;\nu)} n^\alpha n_\mu + \mathfrak{n}_{(\alpha;\beta)} n^\alpha n^\beta n_\mu n_\nu\,.\end{aligned}$$

But $t^\mu$ and $\phi^\mu$ are Killings, so

$$\mathfrak{n}_{(\alpha;\beta)} = \cancel{t_{(\alpha;\beta)}} + \omega\cancel{\phi_{(\alpha;\beta)}} + \phi_{(\alpha}\omega_{,\beta)}\,,$$

and $\phi_\alpha n^\alpha = 0$, $\omega_{,\beta} n^\beta = 0$, hence $\mathfrak{n}_{(\alpha;\beta)} n^\beta = 0$ and thus we are left with

$$\Theta_{\mu\nu} = \mathfrak{n}_{(\mu;\nu)} = \phi_{(\mu}\omega_{,\nu)}\,.$$

This also implies that the expansion scalar vanishes, $\Theta := \Theta^\nu_\nu = 0$, so the expansion tensor is "pure shear", $\Theta_{\mu\nu}=\sigma_{\mu\nu}$. (Note again: this is *not* automatic, because $\mathfrak{n}^\mu$ is *not* Killing.)

- Comparison of $\mathfrak{n}^\mu \equiv t^\mu + \omega\phi^\mu$ with the field $h^\mu \equiv t^\mu + \omega_{\mathrm{H}}\phi^\mu$ (at generic location):

$\mathfrak{n}^\mu$ is not Killing, whereas $h^\mu$ is. Both have zero expansion. $\mathfrak{n}^\mu$ is hypersurface-orthogonal (it has zero vorticity), whereas $h^\mu$ is not. $\mathfrak{n}^\mu$ is not shear-free, whereas $h^\mu$ is (as it is clear from above, with $\omega_{\mathrm{H}}$ being constant). $\mathfrak{n}^\mu$ is orthogonal to $\phi^\mu$, whereas $h^\mu$ is not, $\phi_\mu h^\mu = \cancel{\phi_\mu \mathfrak{n}^\mu} + \phi_\mu(\omega_{\mathrm{H}} - \omega)\phi^\mu = g_{\phi\phi}(\omega_{\mathrm{H}} - \omega)$. $\mathfrak{n}^\mu$ is time-like everywhere outside the horizon, whereas $h^\mu$ is not; namely, $h^\mu$ becomes space-like above the largest real root of the equation $0 = -g_{tt} - 2g_{t\phi}\omega_{\mathrm{H}} - g_{\phi\phi}\omega_{\mathrm{H}}^2 = N^2 - g_{\phi\phi}(\omega_{\mathrm{H}} - \omega)^2$ (in the equatorial plane of a Kerr space-time, for example, it means above $r=38.9M$ for $a=0.1M$, while already above $r=2.08M$ for $a=0.9M$; for $a\to M^+$, i.e. in the extreme limit, the border shrinks to $r\to M^+$, so $h^\mu$ is *nowhere* time-like in the equatorial plane).

### 1.6.5 Circularity condition for EM field

$\boxed{\text{Lemma}}$ *Any* stationary and axisymmetric EM field $F_{\mu\nu}$ (in a space-time of the same commuting symmetries) satisfies $F_{\mu\nu}t^\mu\phi^\nu=0$, ${}^*F_{\mu\nu}t^\mu\phi^\nu=0$ (in any region which involves at least a part of the symmetry axis where $\phi_\mu=0$).

Proof:

$$\begin{aligned}(F^{\mu\nu}t_\mu\phi_\nu)_{,\alpha} \equiv (F_{\mu\nu}t^\mu\phi^\nu)_{,\alpha} &= (F_{\mu\nu}t^\mu\phi^\nu)_{;\alpha} = F_{\mu\nu;\alpha}t^\mu\phi^\nu + F_{\mu\nu}t^\mu{}_{;\alpha}\phi^\nu + F_{\mu\nu}t^\mu\phi^\nu{}_{;\alpha} = \\ &= (-F_{\alpha\mu;\nu} - F_{\nu\alpha;\mu})\,t^\mu\phi^\nu + F_{\mu\nu}t^\mu{}_{;\alpha}\phi^\nu + F_{\mu\nu}t^\mu\phi^\nu{}_{;\alpha} = \\ &= (F_{\mu\alpha;\nu}\phi^\nu + \phi^\nu{}_{;\alpha}F_{\mu\nu})\,t^\mu + (F_{\alpha\nu;\mu}t^\mu + t^\mu{}_{;\alpha}F_{\mu\nu})\,\phi^\nu = \\ &= (\cancel{\pounds_\phi F_{\mu\alpha}} - \phi^\nu{}_{;\mu}F_{\nu\alpha})\,t^\mu + (\cancel{\pounds_t F_{\alpha\nu}} - t^\mu{}_{;\nu}F_{\alpha\mu})\,\phi^\nu = \\ &= F_{\mu\alpha}(t^\mu{}_{;\nu}\phi^\nu - \phi^\mu{}_{;\nu}t^\nu) = F_{\mu\alpha}(t^\mu{}_{,\nu}\phi^\nu - \phi^\mu{}_{,\nu}t^\nu) = 0\,,\end{aligned}$$

because the Killing fields $t^\mu$ and $\phi^\mu$ are supposed to commute. The same computation works for the dual ${}^*F_{\mu\nu}$. Now, $F^{\mu\nu}t_\mu\phi_\nu$ and ${}^*F^{\mu\nu}t_\mu\phi_\nu$ necessarily are zero on the symmetry axis where $\phi_\mu=0$, so they have to vanish in any region containing at least part of the latter.

- Remark: Vanishing of $F_{\mu\nu}t^\mu\phi^\nu$ follows *immediately*,

$$F_{\mu\nu}t^\mu\phi^\nu = (A_{\nu,\mu} - A_{\mu\nu})\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial\phi} = \cancel{A_{\nu,t}}\phi^\nu - \cancel{A_{\mu,\phi}}t^\mu = 0\,.$$

$\boxed{\text{Theorem}}$ The circularity conditions are satisfied by *any* stationary and axisymmetric source-free EM field.

Proof: Take $\phi^{[\mu}t^\lambda T^{\kappa]}{}_\nu n^\nu$ (which in the circular case should vanish). Of the key expression

$$4\pi\,T^\kappa{}_\nu n^\nu = \left(F^{\kappa\iota}F_{\nu\iota} - \frac{1}{4}\,\delta^\kappa_\nu F^{\sigma\iota}F_{\sigma\iota}\right)n^\nu = F^{\kappa\iota}F_{\nu\iota}n^\nu - \frac{1}{4}\,n^\kappa F^{\sigma\iota}F_{\sigma\iota}\,,$$

the second term clearly *is* “circular” since $\phi^{[\mu}t^\lambda n^{\kappa]}=0$, so let us focus on the first term, contributing by $\phi^{[\mu}t^\lambda F^{\kappa]\iota}F_{\nu\iota}n^\nu$. The latter is circular if and only if $F^{\kappa\iota}F_{\nu\iota}n^\nu$ is proportional to $n^\kappa$ (or, more precisely, if it is a combination of $t^\kappa$ and $\phi^\kappa$).

Let us make use of the standard decomposition of the expression $F^{\kappa\iota}F_{\nu\iota}$ in terms of a certain time-like field (“observer”) and the corresponding electric and magnetic field. We will specifically consider the ZAMO observer (which always exists and is time-like all the way down to the horizon),

$$u^\mu = \frac{1}{N}(t^\mu + \omega\phi^\mu) \equiv n^\mu\,, \qquad E_\mu := F_{\mu\nu}n^\nu, \quad B_\mu := -{}^*F_{\mu\nu}n^\nu \equiv \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}n^\nu F^{\kappa\lambda}\,,$$

which is reciprocal to

$$F_{\mu\nu} = n_\mu E_\nu - E_\mu n_\nu + \epsilon_{\mu\nu\rho\sigma}n^\rho B^\sigma\,, \qquad {}^*F_{\mu\nu} = B_\mu n_\nu - n_\mu B_\nu + \epsilon_{\mu\nu\rho\sigma}n^\rho E^\sigma\,,$$

and thus giving rise to[5]

$$F^{\kappa\iota}F^\nu{}_\iota = n^\kappa n^\nu E^2 + (g^{\kappa\nu}+n^\kappa n^\nu)\,B^2 - E^\kappa E^\nu - B^\kappa B^\nu + n^\kappa\big(\vec E\times\vec B\big)^\nu + n^\nu\big(\vec E\times\vec B\big)^\kappa\,,$$

[5] See GR-course lecture notes; again multiplication of *epsilons* over one index is the key.

with $\left(\vec{E}\times\vec{B}\right)^{\mu} := \epsilon^{\mu\sigma\alpha\beta} E_\sigma n_\alpha B_\beta$ . Since $n_\nu n^\nu = -1$, $(\delta^\kappa_\nu + n^\kappa n_\nu) n^\nu = 0$, $E_\nu n^\nu = 0$, $B_\nu n^\nu = 0$ and $(\vec{E}\times\vec{B})_\nu n^\nu = 0$, the multiplication of the above $F^{\kappa\iota}F_{\nu\iota}$ by $n^\nu$ yields

$$F^{\kappa\iota}F_{\nu\iota}n^\nu = -n^\kappa E^2 - \left(\vec{E}\times\vec{B}\right)^{\kappa} .$$

The first term is clearly circular, being proportional to $n^\kappa$. So compute

$$\begin{aligned}
&4\pi\,\epsilon_{\alpha\mu\lambda\kappa} t^\lambda T^\kappa{}_\nu n^\nu = \epsilon_{\alpha\mu\lambda\kappa} t^\lambda F^{\kappa\iota} F_{\nu\iota} n^\nu = -\epsilon_{\alpha\mu\lambda\kappa} n^\lambda \left(\vec{E}\times\vec{B}\right)^{\kappa} = \\
&\qquad = -\epsilon_{\alpha\mu\lambda\kappa}\epsilon^{\kappa\beta\gamma\delta} n^\lambda E_\beta n_\gamma B_\gamma = \ldots = -E_\mu B_\nu + B_\mu E_\nu \\
\Longrightarrow \quad & 4\pi\,\epsilon_{\alpha\mu\lambda\kappa}\phi^\mu t^\lambda T^\kappa{}_\nu n^\nu \equiv 4\pi\,\epsilon_{\alpha\mu\lambda\kappa}\phi^{[\mu} t^\lambda T^{\kappa]}{}_\nu n^\nu = \\
&\qquad = -\phi^\mu E_\mu B_\nu + \phi^\mu B_\mu E_\nu \equiv -(\cancel{\phi^\mu F_{\mu\iota} n^\iota})\, B_\nu - (\cancel{\phi^\mu {}^*\!F_{\mu\iota} n^\iota})\, E_\nu = 0 \,,
\end{aligned}$$

because both the expressions in parentheses vanish due to the above Lemma.

- Remark: We have nowhere cared about the sources, so we in fact have not used the source-free assumption. However, it is clear that the possible sources would have to move in a stationary manner along circular orbits in order that they themselves satisfy the circularity conditions. Indeed, it is known that if currents also had poloidal components, the resulting EM field would *not* be circular, so if the field is "exact" (dynamical), neither the space-time would be such (see [Gourgoulhon]).

## 1.7 The weak rigidity theorem

Theorem [Carter 1969]

In a circular space-time, the dragging angular velocity $\omega$ is constant ($=: \omega_{\mathrm{H}}$) all over the surface $\{-N^2 = 0\}$, so the latter is a Killing horizon. (Namely, $h^\mu \equiv t^\mu + \omega_{\mathrm{H}}\phi^\mu$ thus *is* a Killing field which on the horizon coincides with $\mathscr{n}^\mu \equiv t^\mu + \omega\phi^\mu$.)

Proof:

- First, if the two Killing fields commute, $t_{\alpha,\mu}\phi^\mu = \phi_{\alpha,\mu}t^\mu$ (which is automatic in the asymptotically flat case), it also means $t_{\alpha;\mu}\phi^\mu = \phi_{\alpha;\mu}t^\mu$. Let us use it in

$$\begin{aligned}
g_{t\phi;\alpha} &= (t_\mu\phi^\mu)_{;\alpha} = t_{\mu;\alpha}\phi^\mu + t^\mu\phi_{\mu;\alpha} = -t_{\alpha;\mu}\phi^\mu - t^\mu\phi_{\alpha;\mu} = -2t^\mu\phi_{\alpha;\mu} \\
g_{\phi\phi;\alpha} &= (\phi_\mu\phi^\mu)_{;\alpha} = 2\phi_{\mu;\alpha}\phi^\mu = -2\phi_{\alpha;\mu}\phi^\mu \\
\Longrightarrow \quad \omega_{,\alpha} &= \left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right)_{;\alpha} = \frac{-g_{t\phi;\alpha}g_{\phi\phi} + g_{t\phi}g_{\phi\phi;\alpha}}{(g_{\phi\phi})^2} = \frac{-g_{t\phi;\alpha} - \omega g_{\phi\phi;\alpha}}{g_{\phi\phi}} = \\
&= \frac{2\phi_{\alpha;\mu}}{g_{\phi\phi}}\,(t^\mu + \omega\phi^\mu) \equiv \frac{2N}{g_{\phi\phi}}\,\phi_{\alpha;\mu}n^\mu .
\end{aligned}$$

- It is clear that $\omega_{,\alpha}t^\alpha = 0$ and $\omega_{,\alpha}\phi^\alpha = 0$ (thus $\omega_{,\alpha}n^\alpha = 0$), so we only need to check the derivative of $\omega$ in the plane perpendicular to both $t^\mu$ and $\phi^\mu$,

$$\epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma \omega_{,\delta} = \frac{2N}{g_{\phi\phi}}\,\epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma \phi_{\delta;\mu}n^\mu \equiv \frac{2N}{g_{\phi\phi}}\,\epsilon^{\alpha\beta\gamma\delta}\phi_{[\beta}t_\gamma\phi_{\delta];\mu}n^\mu .$$

- One of the circularity conditions reads $0 = 4!\,\phi_{[\beta}t_\gamma\phi_{\delta;\mu]} = 3!\,\phi_{\{[\beta}t_\gamma\phi_{\delta];\mu\}}$ (braces mean cyclic permutation in the enclosed indices, without a prefactor), from where one can express

$$
\begin{aligned}
&3!\,\phi_{[\beta}t_\gamma\phi_{\delta];\mu} = 3!\,\phi_{[\mu}t_\beta\phi_{\gamma];\delta} - 3!\,\phi_{[\delta}t_\mu\phi_{\beta];\gamma} + 3!\,\phi_{[\gamma}t_\delta\phi_{\mu];\beta} = \\
&= \phi_\mu t_\beta\phi_{\gamma;\delta} + \phi_\gamma t_\mu\phi_{\beta;\delta} + \phi_\beta t_\gamma\phi_{\mu;\delta} - \phi_\beta t_\mu\phi_{\gamma;\delta} - \phi_\gamma t_\beta\phi_{\mu;\delta} - \phi_\mu t_\gamma\phi_{\beta;\delta} \\
&\quad - \phi_\delta t_\mu\phi_{\beta;\gamma} - \phi_\beta t_\delta\phi_{\mu;\gamma} - \phi_\mu t_\beta\phi_{\delta;\gamma} + \phi_\mu t_\delta\phi_{\beta;\gamma} + \phi_\beta t_\mu\phi_{\delta;\gamma} + \phi_\delta t_\beta\phi_{\mu;\gamma} \\
&\quad + \phi_\gamma t_\delta\phi_{\mu;\beta} + \phi_\mu t_\gamma\phi_{\delta;\beta} + \phi_\delta t_\mu\phi_{\gamma;\beta} - \phi_\delta t_\gamma\phi_{\mu;\beta} - \phi_\mu t_\delta\phi_{\gamma;\beta} - \phi_\gamma t_\mu\phi_{\delta;\beta} = \\
&= 2\phi_\mu t_{\{\beta}\phi_{\gamma;\delta\}} - 2t_\mu\phi_{\{\beta}\phi_{\gamma;\delta\}} - 2\phi_{\{\beta}t_{[\gamma}\phi_{\delta]\};\mu}\,.
\end{aligned}
$$

Now multiply this by $\eta^\mu$, using $\phi_\mu\eta^\mu = 0$, $t_\mu\eta^\mu = -N^2$ and $\phi_{\delta;\mu}\eta^\mu = \frac{1}{2}\,g_{\phi\phi}\omega_{,\delta}$:

$$
\phi_{[\beta}t_\gamma\phi_{\delta];\mu}\eta^\mu = \frac{1}{3!}\left(2N^2\phi_{\{\beta}\phi_{\gamma;\delta\}} - g_{\phi\phi}\phi_{\{\beta}t_{[\gamma}\omega_{,\delta]\}}\right) = \frac{1}{2}\left(2N^2\phi_{[\beta}\phi_{\gamma;\delta]} - g_{\phi\phi}\phi_{[\beta}t_\gamma\omega_{,\delta]}\right).
$$

Finally, multiplication by $\epsilon^{\alpha\beta\gamma\delta}$ yields

$$
\begin{aligned}
&\epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma\omega_{,\delta} = \frac{2\epsilon^{\alpha\beta\gamma\delta}}{g_{\phi\phi}}\,\phi_{[\beta}t_\gamma\phi_{\delta];\mu}\eta^\mu = \frac{\epsilon^{\alpha\beta\gamma\delta}}{g_{\phi\phi}}\left(2N^2\phi_{[\beta}\phi_{\gamma;\delta]} - g_{\phi\phi}\phi_{[\beta}t_\gamma\omega_{,\delta]}\right) \\
&\Longrightarrow \quad \epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma\omega_{,\delta} = \frac{N^2}{g_{\phi\phi}}\,\epsilon^{\alpha\beta\gamma\delta}\phi_\beta\phi_{\gamma;\delta}\,. \qquad (1.17)
\end{aligned}
$$

- Hence, on the $N^2 = 0$ hypersurface, the derivative of $\omega$ in the complementary (meridional) plane vanishes as well, so $\omega$ is constant there.

Lemma The generator of a Killing horizon is a principal null vector of Weyl (there on the horizon).

Proof: Above, we have learnt that the shear of the $\eta^\mu \equiv t^\mu + \omega\phi^\mu$ field reads $\sigma_{\mu\nu} = \phi_{(\mu}\omega_{,\nu)}$. The rigidity theorems, in any of the versions, say that on the horizon $\omega_{,\nu}$ vanishes, hence also the shear of $\eta^\mu$. However, on the horizon, $\eta^\mu$ coincides with the geodesic null Killing generator $h^\mu$. Hence, according to the Goldberg-Sachs theorem (see e.g. GR-course lecture notes), the generator of a Killing horizon is tangent *there* to the repeated principal null congruence of the Weyl tensor. (Worth to add that this property only holds on the horizon. Sure, we might have started directly from $h^\mu \equiv t^\mu + \omega_{\rm H}\phi^\mu$ which is Killing and shear-free *everywhere*, yet even this field is only null at the very horizon.)

Note that this property is not so "obvious", because the horizon is a feature of the metric and its first derivatives, while the principal null directions are given by *curvature*. Indeed, curvature generally does not behave in any special way on black-hole horizons.

## 1.8 Acceleration scalar and the black-hole surface gravity

Due to the equivalence principle, "gravitational acceleration" cannot in general be represented by an invariant quantity. However, in circular space-times, there *is* such a possibility – it is provided by gradient of the "Killing" part of the metric (which in turn follows by scalar products of the Killing vector fields, and thus has an invariant character).

### 1.8.1 Physical picture

Intuitively, the strength of the field can be characterized by magnitude of four-acceleration which some suitable observers need in order to "keep themselves at a given orbit". Such a concept is of course ambiguous, but in circular space-times there do exist certain symmetry-privileged orbits – the stationary circular ones we mentioned at the beginning. In particular, a natural choice is the ZAMO congruence having $\Omega = \omega$, which is time-like everywhere down to the horizon and has zero angular momentum with respect to infinity. [Whereas, for instance, it is not possible to consider *static* ($\Omega=0$) congruence for this purpose, since that is only time-like outside the static limit given by $g_{tt}=0$.]

Let us first compute the four-acceleration of a *generic* stationary circular orbit in a circular space-time,

$$a_\mu = \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - \Gamma^\iota{}_{\mu\kappa} u_\iota u^\kappa = -\Gamma_{\iota\mu\kappa} u^\iota u^\kappa = -\frac{1}{2}(g_{\iota\mu,\kappa}+g_{\kappa\iota,\mu}-g_{\mu\kappa,\iota})u^\iota u^\kappa = -\frac{1}{2}\, g_{\kappa\iota,\mu} u^\iota u^\kappa =$$
$$= -\frac{1}{2}(u^t)^2(g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2) = \frac{1}{2}\,\frac{g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2}{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2}\,, \tag{1.18}$$

where we have used the stationarity of the motion, thus constancy of $u_\mu$ along the orbit, and symmetry of $u^\iota u^\kappa$ due to which the term $(g_{\iota\mu,\kappa} - g_{\mu\kappa,\iota})$ antisymmetric in $(\iota, \kappa)$ drops out in the multiplication. The main aspect is that the "Killing" components $a_t$ and $a_\phi$ vanish.

Now specifically for the ZAMO sub-family, i.e. for $u^\mu = n^\mu$, or $\Omega = \omega \equiv -g_{t\phi}/g_{\phi\phi}$ :

$$g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega = -N^2\,,$$
$$g_{tt,\mu} + 2g_{t\phi,\mu}\omega + g_{\phi\phi,\mu}\omega^2 = (g_{tt} + g_{t\phi}\omega)_{,\mu} = (-N^2)_{,\mu} = -2NN_{,\mu}\,,$$

so one obtains

$$a_\mu = \frac{1}{2}\,\frac{-2NN_{,\mu}}{-N^2} = \frac{N_{,\mu}}{N}\,. \tag{1.19}$$

The lapse is often being expressed in terms of the gravitational potential $\Phi$, as $N = e^\Phi$; then the ZAMO's acceleration is just $a_\mu = \Phi_{,\mu}$ .

However, in the limit $N \to 0^+$, all the time-like range of stationary circular motions (and ZAMO family in particular) go over to the null generators of the horizon – the photons which just stay on the horizon, keeping constant $r$ and $\theta$ while orbiting with $\Omega = \omega_{\mathrm{H}}$ in the azimuthal direction. No other time-like or light-like world-line can lie on the horizon. This means that in the horizon limit ($N \to 0$), the magnitude of the circular-orbit acceleration undoubtedly diverges; on the ZAMO acceleration which we plan to use it is seen at first sight. Yet there is a natural way how to regularize such a divergence: multiply the acceleration by $N$. This has a clear meaning since $N$ represents the dilation factor between the proper time of ZAMO and the Killing time $t$. Indeed, from (1.15) we have

$$u^t \equiv \frac{\mathrm{d}t}{\mathrm{d}\hat\tau} = \frac{1}{N} \quad\Longrightarrow\quad N_{,\mu} = Na_\mu = \frac{a_\mu}{u^t} \equiv \frac{\mathrm{D}u_\mu}{\mathrm{d}\hat\tau}\,\frac{\mathrm{d}\hat\tau}{\mathrm{d}t}\,.$$

One thus obtains the ZAMO acceleration taken "with respect to the asymptotic inertial time".

So let us define the acceleration scalar $\kappa$ and the black-hole surface gravity $\kappa_{\rm H}$ by

$$\kappa^2 := N^2 g^{\mu\nu} a_\mu a_\nu = g^{\mu\nu} N_{,\mu} N_{,\nu}\,, \qquad \kappa^2_{\rm H} := \lim_{N\to 0} \kappa^2\,. \tag{1.20}$$

Remark: Horizon vacuumness once more, from circularity:

Circularity conditions (1.12), i.e. $\phi_{[\mu} t_\lambda T_{\kappa]\nu} t^\nu = 0$ and $t_{[\mu} \phi_\lambda T_{\kappa]\nu} \phi^\nu = 0$, can be summed to $\phi_{[\mu} t_\lambda T_{\kappa]\nu} \mathcal{N}^\nu = 0$, since $\mathcal{N}^\mu \equiv t^\mu + \omega\phi^\mu$. Write this out (without $1/3!$) and multiply it by $\mathcal{N}^\kappa$, while remembering that $\phi_\kappa \mathcal{N}^\kappa = 0$ and $t_\kappa \mathcal{N}^\kappa = -N^2$,

$$\begin{aligned} 0 &= (\phi_\mu t_\lambda T_{\kappa\nu} + \phi_\kappa t_\mu T_{\lambda\nu} + \phi_\lambda t_\kappa T_{\mu\nu} - \phi_\lambda t_\mu T_{\kappa\nu} - \phi_\kappa t_\lambda T_{\mu\nu} - \phi_\mu t_\kappa T_{\lambda\nu})\, \mathcal{N}^\nu \mathcal{N}^\kappa = \\ &= 2\phi_{[\mu} t_{\lambda]} T_{\kappa\nu} \mathcal{N}^\nu \mathcal{N}^\kappa + 2N^2 \phi_{[\mu} T_{\lambda]\nu} \mathcal{N}^\nu\,. \end{aligned}$$

The fields $\phi^\mu$ and $t^\mu$ are independent, and on the horizon $N = 0$, so $T_{\kappa\nu}\mathcal{N}^\nu\mathcal{N}^\kappa$ has to vanish there, which, according to Einstein equations, implies the same for $R_{\kappa\nu}\mathcal{N}^\nu\mathcal{N}^\kappa$ (because $g_{\kappa\nu}\mathcal{N}^\nu\mathcal{N}^\kappa = 0$ there).

### 1.8.2 Zeroth law of black-hole thermodynamics

In a thermal equilibrium, the temperature is the same within the whole system. Similarly, in a stationary state, the surface gravity is constant all over the black-hole horizon. Let us show a different proof than those given above. Namely, most of the above results (a rigidity theorem in particular) either follow as a *purely geometric* property (independent of the theory of gravitation and of energy conditions) if one assumes that the space-time is circular, or they are obtained for a *generic* space-time from the field equations and energy conditions. (One speaks of **weak and strong** versions of the results, respectively.) Up to now, we have been assuming the circularity where necessary, only having employed the Einstein equations for the translation between $R_{\mu\nu}$ and $T_{\mu\nu}$ in the circularity conditions and to understand the horizon circumstance $R_{\mu\nu}h^\mu h^\nu = 0$. Here we base the proof on the field equations and the energy-dominance conditions. (And then we also offer an alternative one.)

Theorem [Bardeen, Carter & Hawking 1973] $\kappa_{\rm H}$ is constant all over a Killing horizon.

Proof: We know that *on a Killing horizon* its generating Killing field $h^\mu$ has zero vorticity, $\omega^\mu[h] \equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} h_{\nu;\kappa} h_\lambda = 0$, or $h_{[\nu;\kappa} h_{\lambda]} = 0$. Since $h_{[\nu;\kappa]} = h_{\nu;\kappa}$, it expands as

$$h_{\nu;\kappa} h_\lambda + h_{\lambda;\nu} h_\kappa + h_{\kappa;\lambda} h_\nu = 0 \quad\Longrightarrow\quad h_{\nu;\kappa} h_\lambda - h_{\nu;\lambda} h_\kappa = h_{\lambda;\kappa} h_\nu\,, \quad \text{i.e.}\;\; \mathcal{D}_{\lambda\kappa} h_\nu = h_{\lambda;\kappa} h_\nu\,,$$

where we have introduced $\mathcal{D}_{\lambda\kappa} := h_\lambda \nabla_\kappa - h_\kappa \nabla_\lambda$. Multiplying by $\kappa_{\rm H}$ while using its definition $h_{\nu;\mu} h^\mu = \kappa_{\rm H} h_\nu$ on the r.h. side, we have

$$\kappa_{\rm H} \mathcal{D}_{\lambda\kappa} h_\nu = \kappa_{\rm H} h_{\lambda;\kappa} h_\nu = h_{\nu;\mu} h^\mu h_{\lambda;\kappa} = h_{\nu;\mu} \mathcal{D}_{\lambda\kappa} h^\mu\,, \tag{1.21}$$

where we finally re-substituted the red term from above. Now, instead of $h_\nu$ alone (as above), apply $\mathcal{D}_{\lambda\kappa}$ to $\kappa_{\rm H} h_\nu = h_{\nu;\mu} h^\mu$, that is, write out the relation $\mathcal{D}_{\lambda\kappa}(\kappa_{\rm H} h_\nu) = \mathcal{D}_{\lambda\kappa}(h_{\nu;\mu} h^\mu)$:

$$h_\nu \mathcal{D}_{\lambda\kappa} \kappa_{\rm H} + \kappa_{\rm H} \mathcal{D}_{\lambda\kappa} h_\nu = h^\mu \mathcal{D}_{\lambda\kappa} h_{\nu;\mu} + h_{\nu;\mu} \mathcal{D}_{\lambda\kappa} h^\mu\,.$$

According to equation (1.21), the second terms on both sides are equal, so we are left with

$$\begin{aligned} h_\nu \mathcal{D}_{\lambda\kappa}\kappa_{\mathrm{H}} &= h^\mu \mathcal{D}_{\lambda\kappa} h_{\nu;\mu} \equiv h^\mu (h_\lambda \nabla_\kappa - h_\kappa \nabla_\lambda) h_{\nu;\mu} \equiv h^\mu (h_\lambda h_{\nu;\mu\kappa} - h_\kappa h_{\nu;\mu\lambda}) = \\ &= h^\mu (h_\lambda R_{\nu\mu\kappa\iota} - h_\kappa R_{\nu\mu\lambda\iota})\, h^\iota = (h_\lambda R_{\kappa\iota\nu\mu} - h_\kappa R_{\lambda\iota\nu\mu})\, h^\mu h^\iota = \\ &\equiv\ 2 h_{[\lambda} R_{\kappa]\iota\nu\mu} h^\iota h^\mu \,. \end{aligned} \tag{1.22}$$

At this stage, we might express Riemann in terms of Weyl and Ricci, and to employ the results obtained for *circular* horizons, namely that $R_{\iota\mu} h^\iota h^\mu = 0$ and $h_{[\lambda} R_{\kappa]\iota\nu\mu} h^\iota h^\mu = 0$ (exactly the condition for $h^\mu$ standing for the repeated principal null vector of Weyl). However, we wish to finish the proof *without* referring to the circularity property. Next step is to express the result in terms of the Ricci (rather than Riemann) tensor. It is possible through somewhat uncomfortable application of $\mathcal{D}_{\nu\iota}$ to the first equation $h^\iota{}_{;\kappa} h_\lambda - h^\iota{}_{;\lambda} h_\kappa = h_{\lambda;\kappa} h^\iota$:

$$\begin{aligned} & h_\nu h^\iota{}_{;\kappa\iota} h_\lambda + \bcancel{h_\nu h^\iota{}_{;\kappa} h_{\lambda;\iota}} - h_\nu h^\iota{}_{;\lambda\iota} h_\kappa - \bcancel{h_\nu h^\iota{}_{;\lambda} h_{\kappa;\iota}} - \\ & \quad - h_\iota h^\iota{}_{;\kappa\nu} h_\lambda - \xcancel{h_\iota h^\iota{}_{;\kappa} h_{\lambda;\nu}} + h_\iota h^\iota{}_{;\lambda\nu} h_\kappa + \xcancel{h_\iota h^\iota{}_{;\lambda} h_{\kappa;\nu}} \\ & = h_\nu h_{\lambda;\kappa\iota} h^\iota + h_\nu h_{\lambda;\kappa} \cancel{h^\iota{}_{;\iota}} - \xcancel{h_\iota h_{\lambda;\kappa\nu} h^\iota} - \xcancel{h_\iota h_{\lambda;\kappa} h^\iota{}_{;\nu}} \,, \end{aligned}$$

where the $\bcancel{\text{terms}}$ cancelled with respect to each other, the $\cancel{\text{terms}}$ vanish due to obvious reasons, and the $\xcancel{\text{terms}}$ sum to zero because

$$\begin{aligned} & - h_\iota h^\iota{}_{;\kappa} h_{\lambda;\nu} + h_\iota h^\iota{}_{;\lambda} h_{\kappa;\nu} + h_\iota h_{\lambda;\kappa} h^\iota{}_{;\nu} = h^\iota h_{\kappa;\iota} h_{\lambda;\nu} + h^\iota h_{\lambda;\iota} h_{\nu;\kappa} + h^\iota h_{\nu;\iota} h_{\kappa;\lambda} = \\ & = \kappa_{\mathrm{H}} (h_\kappa h_{\lambda;\nu} + h_\lambda h_{\nu;\kappa} + h_\nu h_{\kappa;\lambda}) = 0 \,. \end{aligned}$$

Thanks to the Killing property, $h^\iota{}_{;\kappa\iota} = -\Box\, h_\kappa$ and $h^\iota{}_{;\lambda\iota} = -\Box\, h_\lambda$, hence the equation yields, with (1.1) and (1.2) substituted,

$$h_\nu h_\lambda R_{\kappa\mu} h^\mu - h_\nu h_\kappa R_{\lambda\mu} h^\mu = h_\iota h_\lambda R^\iota{}_{\kappa\nu\mu} h^\mu - h_\iota h_\kappa R^\iota{}_{\lambda\nu\mu} h^\mu + h_\nu \cancel{h^\iota R_{\lambda\kappa\iota\mu} h^\mu} \,,$$

which is easily arranged as

$$h_\nu h_{[\lambda} R_{\kappa]\mu} h^\mu = -h_{[\lambda} R_{\kappa]\iota\nu\mu} h^\iota h^\mu \,.$$

Using it in equation (1.22), we obtain

$$\mathcal{D}_{\lambda\kappa}\kappa_{\mathrm{H}} = -2 h_{[\lambda} R_{\kappa]\mu} h^\mu \,. \tag{1.23}$$

(Remember this only applies at the horizon, since we started from the vorticity-free property of $h^\mu$ which only holds *there*.)

Finally (but one), recall that on a Killing horizon $R_{\kappa\mu} h^\kappa h^\mu = 0$, which implies that $R_{\kappa\mu} h^\mu$ has to be proportional to $h_\kappa$ (provided that the dominant energy condition holds). Therefore, $h_{[\lambda} R_{\kappa]\mu} h^\mu \sim h_{[\lambda} h_{\kappa]} = 0$.

And, finally: what actually have we learnt by finding $\mathcal{D}_{\lambda\kappa}\kappa_{\mathrm{H}} = 0$ ? First, as the horizon has to be the surface of transitivity of the two Killing vectors, $\kappa_{\mathrm{H}}$ has to definitely be constant *along* $h^\mu$ (we showed this explicitely within the Lemma below the Vishveshwara theorem). To prove the constancy over the horizon, it is thus sufficient to show the constancy in the tangent direction *independent of* $h^\mu$. But that is exactly given by the operator

$\mathcal{D}_{\lambda\kappa} \equiv h_\lambda \nabla_\kappa - h_\kappa \nabla_\lambda$.

Alternative proof: Recall equation (1.10), valid for *any* Killing field (in any space-time),

$$\omega_{[\nu;\alpha]}[\xi] = \frac{1}{2}\,\epsilon_{\alpha\nu\gamma\delta}\,\xi^\gamma R^\delta_\beta\,\xi^\beta\,.$$

The connection with (1.23) is obvious: putting $\xi^\mu \to h^\mu$, the r.h. sides (necessarily vanishing on the horizon) are same, only that this time it is written in terms of Levi-Civita. Hence, on the horizon of $\xi^\mu$ (here $h^\mu$), not only vanishes the latter's vorticity itself, but also the anti-symmetrized gradient of vorticity.

An easy way to make $\omega_{[\nu;\alpha]}$ zero is to take the field $n^\mu = t^\mu + \omega\phi^\mu$ (for which $\omega^\mu[n]=0$ *everywhere*). To convince oneself that *on the horizon* $\omega^\mu[n]$ yields the same value as $\omega^\mu[h]$, write

$$\omega^\mu{}_{;\alpha}[h] = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}(h_{\nu;\kappa\alpha}h_\lambda + h_{\nu;\kappa}h_{\lambda;\alpha}) = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}(R_{\nu\kappa\alpha\sigma}h^\sigma h_\lambda + h_{\nu;\kappa}h_{\lambda;\alpha})\,.$$

On the horizon, this really equals $\omega^\mu{}_{;\alpha}[n]$, because $n^\mu = h^\mu$ there and their gradients are same there as well. Indeed,

$$n_{\nu;\kappa} = t_{\nu;\kappa} + \omega\phi_{\nu;\kappa} + \phi_\nu\omega_{,\kappa}\,, \quad \text{while} \quad h_{\nu;\kappa} = t_{\nu;\kappa} + \omega_{\rm H}\phi_{\nu;\kappa}\,,$$

which however are same, because, on the horizon, $\omega = \omega_{\rm H}$ and $\omega_{,\kappa} = 0$ (rigidity theorem).

Proof from circularity: Above, the zeroth law has been obtained from geometry (zero vorticity of $h^\mu$ on the horizon), from Einstein equations ($R_{\alpha\beta}h^\alpha h^\beta = 0$ on the horizon) and from the dominant energy condition ($T_{\alpha\beta}h^\beta$ causal). *In circular space-times*, the theorem follows as a *purely geometrical* fact, independent of the field equations and of energy conditions.

Sorry, I don't much like the usual proof, but have not yet been able to provide a better one :-)