CIRCULAR HORIZONS

notes for BH-thermodynamics course

Some people think that everything important lives on a boundary. In physics, the idea that behaviour of the "bulk" can be described by theories formulated on its boundary is called the **holographic principle**. After a lunch, I tend to doubt about it, yet it might at least work for black holes. Actually, the holographic viewpoint was inspired by the proposal (in 1972) by Jacob Bekenstein to connect the entropy of a black hole with the proper area of the horizon. In terms of such a quantity, he formulated the second law of black-hole thermodynamics. Were Bekenstein right, the usual first-law term $T\mathrm{d}S$ would require that the black hole have **temperature**, proportional to the quantity called **surface gravity**. Bardeen, Carter and Hawking subsequently (1973) wrote the paper "The four laws of black hole mechanics" where they added, for a stationary and axisymmetric black hole, the other three laws.

As it is e.g. clear from Carter's thorough lecture on stationary black holes in the famous Les Houches '72 Summer School proceedings, the authors of the paper themselves did *not* think their laws were more than an *analogy* of thermodynamics. In particular, Hawking decided to show that black holes do *not* emit thermal radiation, in order to demonstrate that despite formal analogy, the "temperature" of the horizon does not have its usual sense, and so neither the area *really* represents the black-hole entropy. Employing the quantum-field theory on a (classical) curved background, he managed to show (1974) just the opposite: black holes do emit thermal radiation, in accord with the black-body formula.[1] The road to black-hole thermodynamics was open.

Today, the thermodynamics of black holes – of something "built from pure geometry" – is considered a key to deep connections between gravitational, quantum and statistical physics. Since the gravitational degrees of freedom should also contribute to the relevant quantities, a complete picture also requires to quantize these (not only the other fields "on a background"). The area is thus being studied heavily as one of the promising targets – and tests – of any attempt to quantize gravitation. Bekenstein's proposals continue to inspire it.

Below, as the starting point, we mention several notions and results from the geometry and physics of black-hole horizons. We naturally restrict to non-dynamical, stationary horizons. In the electro-vacuum and asymptotically flat case, this necessarily means the Kerr-Newman family of solutions according to the uniqueness theorems, but we will at least be slightly more general and will consider stationary and axially symmetric (in fact so-called circular) setting.

Although many definitions of **black-hole horizons** exist, in a generic situation it is not entirely clear what "a black hole" actually means. Anyway, the **laws of black-hole thermodynamics** start from a *stationary and axially symmetric* situation: the zeroth law states that on a stationary horizon the **surface gravity** is everywhere the same, and the first law fixes the "first thermodynamic law" for changes of basic quantities (mass, horizon area, charge and spin) characterizing the transition of the black hole between two close stationary states. The word "axisymmetric" has disappeared somehow, yet it is there automatically: if a stationary black hole is not static, i.e. if it is *rotating*, then it *has to be* axisymmetric, since otherwise – with some bump on it – it would emit gravitational waves, so it could not be stationary.

[1] Black hole is quite an ideal black body, isn't it...

1.1 Horizons in circular space-times (physical approach)

In stationary and axisymmetric space-times (at least when these two symmetries commute, which is always ensured in an asymptotically flat case), all those various horizon definitions coincide and reduce to the important notion of the **Killing horizon** – a connected null hypersurface generated by a certain Killing vector field. The Killing horizons are the simplest type of horizons for which thermodynamic quantities are being introduced and for which the laws of thermodynamics are being voiced. Below, however, we first approach the horizon from physics side.

1.1.1 Stationary circular orbits and their light cones

Consider an asymptotically flat space-time with two commuting Killing vector fields, of which – at least "far away" – one is time-like with open orbits (t^μ) and one is space-like with closed orbits (ϕ^μ). Omitting important assumptions about the existence of the symmetry axis and its regularity ("elementary flatness"), denote by t the parameter of the time symmetry and by ϕ the parameter of the axial symmetry. Physically, t represents proper time of a rest observer at infinity, while ϕ represents the azimuthal angle standardly ranging from 0 to 2π (in the sense of the source rotation). The Killing fields then write $t^\mu = \frac{\partial x^\mu}{\partial t}$, $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$.

If the space-time is also **orthogonally transitive**, that is, if integral meridional planes exist (as e.g. in the vacuum case, see later), the metric can be written as

$$\boxed{\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + g_{\phi\phi}\,(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2}\,, \tag{1.1}$$

where

$$N^2 := -g_{tt} - g_{t\phi}\omega \qquad \text{and} \qquad \omega := -\frac{g_{t\phi}}{g_{\phi\phi}}$$

are called the **lapse function** and the **dragging angular velocity**, and r and θ cover the meridional planes (alternatively, one might use cylindrical-type ρ and z coordinates, for example). Worth to note that the whole "Killing" part of the metric has invariant meaning since it is given by scalar products of the Killing fields,

$$g_{tt} = g_{\mu\nu}t^\mu t^\nu\,, \quad g_{t\phi} = g_{\mu\nu}t^\mu \phi^\nu\,, \quad g_{\phi\phi} = g_{\mu\nu}\phi^\mu \phi^\nu\,.$$

The same is of course inherited by N^2, ω and all other quantities possibly defined from that part of the metric.

Consider a family of stationary motions along the circles $r = \mathrm{const}$, $\theta = \mathrm{const}$, i.e. with four-velocity given by combination of the Killing fields,

$$u^\mu = \frac{t^\mu + \Omega\phi^\mu}{|t^\mu + \Omega\phi^\mu|} = \frac{t^\mu + \Omega\phi^\mu}{\sqrt{-g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa)}}\,, \tag{1.2}$$

where the azimuthal angular velocity $\Omega := \frac{\mathrm{d}\phi}{\mathrm{d}t}$ is constant in time (it may however depend on r and θ). In the above coordinates (t, r, θ, ϕ) adapted to the symmetries,

$$t^\mu = \delta^\mu_t\,, \quad \phi^\mu = \delta^\mu_\phi\,, \quad g_{\iota\kappa}(t^\iota + \Omega\phi^\iota)(t^\kappa + \Omega\phi^\kappa) = g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = -N^2 + g_{\phi\phi}(\Omega - \omega)^2\,,$$

so the four-velocity has components

$$u^\mu = u^t(1,0,0,\Omega)\,, \quad u^t = \frac{1}{\sqrt{N^2 - g_{\phi\phi}(\Omega-\omega)^2}}\,. \tag{1.3}$$

The angular velocity with respect to an asymptotic inertial system, Ω, cannot be arbitrary – too large values would correspond to super-luminal (space-like) motion. The interval of time-like motion has boundaries where u^μ can no longer be normalized by any real u^t, i.e. at the roots of $g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = 0$:

$$\Omega_{\rm max,min} = \frac{-g_{t\phi} \pm \sqrt{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \frac{N}{\sqrt{g_{\phi\phi}}}\,. \tag{1.4}$$

1.1.2 Horizon as the limit for circular motions

Hence, the light cone of stationary circular motions, expressed in terms of Ω, is centred by the dragging angular velocity ω, extending from it equally to both sides. The time-like option for Ω closes up at $N=0$, which (thus) clearly represents a horizon. The observers orbiting there with $\Omega=\omega$ are null already, representing the null generators of the horizon. Below, N^2 becomes *negative* and N thus imaginary.

Within the stationary circular orbits, the value $\Omega = \omega$ identifies the congruence of so-called Zero Angular Momentum Observers. Indeed, the (specific) azimuthal angular momentum (for *any* motion, not just the stationary circular one) reads

$$u_\phi = g_{\phi t}u^t + g_{\phi\phi}u^\phi = g_{\phi\phi}u^t(\Omega-\omega)\,,$$

so it vanishes if and only if $\Omega = \omega$. The ZAMO congruence has four-velocity and four-acceleration

$$u^\mu = \frac{1}{N}(1,0,0,\omega)\,, \qquad a_\alpha = \frac{N_{,\alpha}}{N}\,.$$

The field $\xi^\mu = t^\mu + \omega\phi^\mu$ also has other simple properties:[2]

- Scalar products:

$$g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega = -N^2$$

  (hence, ξ^μ is null where $N^2=0$, i.e. on the horizon),

$$\phi_\mu\xi^\mu = \phi_\mu(t^\mu+\omega\phi^\mu) = g_{t\phi} + g_{\phi\phi}\omega = 0\,, \quad t_\mu\xi^\mu = t_\mu(t^\mu+\omega\phi^\mu) = g_{tt} + g_{t\phi}\omega = -N^2\,.$$

- Covariant version:

$$\xi_\alpha = g_{\alpha\mu}\xi^\mu = g_{\alpha\mu}(t^\mu+\omega\phi^\mu) = g_{\alpha t} + g_{\alpha\phi}\omega = -N^2\delta^t_\alpha = -N^2 t_{,\alpha}$$

[2] Note that it is *not* a Killing field, because ω is *not* constant – it depends on both r and θ.

$\Rightarrow \xi^\mu$ is orthogonal to the hypersurfaces $t = \mathrm{const} \;\Rightarrow\;$ it has zero twist, $\omega^\mu[\xi] = 0$. The latter follows from Frobenius theorem; actually, it is clear immediately,

$$\omega^\mu[\xi] := \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\xi_{\nu;\kappa}\xi_\lambda = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}(\xi_{\nu,\kappa} - \Gamma^\iota{}_{\underline{\kappa\nu}}\xi_\iota)\xi_\lambda = \frac{1}{2}N^2\epsilon^{\mu\nu\kappa\lambda}(N^2 t_{,\nu})_{,\kappa}t_{,\lambda} = 0\,.$$

In passing, the Frobenius theorem says that hypersurface orthogonality and zero twist are *equivalent properties* – indeed, this property is well known: ω^μ represents rotation (curl) of ξ_ν within the (hyper)surface orthogonal to ξ_λ; if that rotation is zero, the field ξ_ν has to be proportional to a gradient of some scalar field. Writing it out in components, $\omega^\mu = 0$ means $\xi_{\nu,\kappa} = \xi_{\kappa,\nu}$, which are exactly the integrability conditions for the equation $-f\mathrm{d}\Phi = \xi_\nu \mathrm{d}x^\nu$, i.e. $-f\Phi_{,\nu} = \xi_\nu$, with f and Φ some scalars ($f \equiv N^2$ and $\Phi \equiv t$ in our case).

- Expansion tensor:

$$\Theta_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \xi_{(\alpha;\beta)} = \left(\delta^\alpha_\mu + \frac{\xi^\alpha\xi_\mu}{N^2}\right)\left(\delta^\beta_\nu + \frac{\xi^\beta\xi_\nu}{N^2}\right)\xi_{(\alpha;\beta)} =$$

$$= \xi_{(\mu;\nu)} + \xi_{(\mu;\beta)}\frac{\xi^\beta\xi_\nu}{N^2} + \xi_{(\alpha;\nu)}\frac{\xi^\alpha\xi_\mu}{N^2} + \xi_{(\alpha;\beta)}\frac{\xi^\alpha\xi^\beta\xi_\mu\xi_\nu}{N^4}\,.$$

But t^μ and ϕ^μ are Killings, so

$$\xi_{(\alpha;\beta)} = \cancel{t_{(\alpha;\beta)}} + \omega\cancel{\phi_{(\alpha;\beta)}} + \phi_{(\alpha}\omega_{,\beta)}\,,$$

and $\phi_\alpha\xi^\alpha = 0$, $\omega_{,\beta}\xi^\beta = 0$, hence $\xi_{(\alpha;\beta)}\xi^\beta = 0$ and thus we are left with

$$\Theta_{\mu\nu} = \xi_{(\mu;\nu)} = \phi_{(\mu}\omega_{,\nu)}\,.$$

This also implies that the expansion scalar vanishes, $\Theta := \Theta^\nu_\nu = 0$, so the expansion tensor is "pure shear", $\Theta_{\mu\nu} = \sigma_{\mu\nu}$. (Note again that this is *not* automatic, because ξ^μ is *not* Killing.)

Let us now confirm the importance of $N = 0$ from a more geometrical perspective. We will show how it naturally arises as the main property of a Killing horizon. Before embarking on that point, let us first recall several properties of Killing vector fields.

1.2 Selected properties of the Killing vector fields

The Killing vector fields ξ^μ fix directions along which the space-time metric does not change. This is naturally expressed in terms of the Lie derivative. The Lie derivative is a very "low level" operation, it needs neither the connection and nor the metric, yet if we speak of the Lie derivative *of the metric*, the latter has to be there, right? Let us also assume the space-time is equipped with the Levi-Civita connection as it is standard in GR. Then the (Killing) equation for the Killing fields writes

$$0 = \pounds_\xi g_{\mu\nu} = g_{\mu\nu,\iota}\xi^\iota + \xi^\iota{}_{,\mu}g_{\iota\nu} + \xi^\iota{}_{,\nu}g_{\mu\iota} = \cancel{g_{\mu\nu;\iota}}\xi^\iota + \xi^\iota{}_{;\mu}g_{\iota\nu} + \xi^\iota{}_{;\nu}g_{\mu\iota} = \xi_{\nu;\mu} + \xi_{\mu;\nu} =: 2\xi_{(\mu;\nu)}\,.$$

The Killing vector fields have many simple properties, of which we only mention the clearly vanishing expansion scalar, i.e. $\xi^{\mu}{}_{;\mu} = 0$. Next, let us mention an important relation between Killing vector fields and curvature. In the Ricci identity for ξ_ν,

$$\xi_{\nu;\kappa\lambda} - \xi_{\nu;\lambda\kappa} = R^{\sigma}{}_{\nu\kappa\lambda}\xi_\sigma ,$$

we anti-commute, by the Killing equation, ν and λ in the second term, and then we write the relation together with its cyclic permutations:

$$\begin{aligned} \xi_{\nu;\kappa\lambda} + \xi_{\lambda;\nu\kappa} &= R^{\sigma}{}_{\nu\kappa\lambda}\xi_\sigma \, , \\ \xi_{\lambda;\nu\kappa} + \xi_{\kappa;\lambda\nu} &= R^{\sigma}{}_{\lambda\nu\kappa}\xi_\sigma \, , \\ \xi_{\kappa;\lambda\nu} + \xi_{\nu;\kappa\lambda} &= R^{\sigma}{}_{\kappa\lambda\nu}\xi_\sigma \, . \end{aligned}$$

Now add the first and the last equation, while subtracting the middle one (for example),

$$\begin{aligned} 2\xi_{\nu;\kappa\lambda} &= (R^{\sigma}{}_{\nu\kappa\lambda} + R^{\sigma}{}_{\kappa\lambda\nu} - R^{\sigma}{}_{\lambda\nu\kappa})\xi_\sigma = \left(\cancel{R^{\sigma}{}_{\{\nu\kappa\lambda\}}} - 2R^{\sigma}{}_{\lambda\nu\kappa}\right)\xi_\sigma = -2R^{\sigma}{}_{\lambda\nu\kappa}\xi_\sigma \\ \Longrightarrow \quad \xi_{\nu;\kappa\lambda} &= -R^{\sigma}{}_{\lambda\nu\kappa}\xi_\sigma = R_{\nu\kappa\lambda\sigma}\xi^{\sigma} \, . \end{aligned} \tag{1.5}$$

- Corollary: by contraction of this equation, one has

  $$\boxed{\xi_{\nu;}{}^{\kappa}{}_{\kappa} \equiv \Box\xi_\nu = -R^{\sigma}{}_{\nu}\xi_\sigma} \, . \tag{1.6}$$

  If the Ricci tensor vanishes, this corresponds to the (de Rham) wave equation for the EM four-potential (otherwise the signs at the curvature terms are opposite). Regarding also that the Killing fields automatically satisfy the "Lorenz condition" $\xi^{\mu}{}_{;\mu} = 0$, one infers the following: in space-times with $R^{\mu\nu} = 0$, the knowledge of a Killing vector implies the knowledge of a possible EM four-potential. (Note that the corresponding EM field must be a *test* field, because otherwise the Ricci tensor would be $R^{\mu\nu} = 8\pi T^{\mu\nu}_{\rm EM}$ rather than zero.)

- Another corollary: projecting (1.5) twice on a tangent vector $u^\mu = \frac{{\rm d}x^\mu}{{\rm d}\tau}$ of any geodesic ($\frac{{\rm D}u^\mu}{{\rm d}\tau} = 0$), we find

  $$\xi_{\nu;\kappa\lambda}u^\kappa u^\lambda = \frac{{\rm D}\xi_{\nu;\kappa}}{{\rm d}\tau}\,u^\kappa = \frac{{\rm D}(\xi_{\nu;\kappa}u^\kappa)}{{\rm d}\tau} = \frac{{\rm D}^2\xi_\nu}{{\rm d}\tau^2} = R_{\nu\kappa\lambda\sigma}u^\kappa u^\lambda \xi^\sigma \, . \tag{1.7}$$

  That means, the Killing vectors satisfy the geodesic-deviation equation.

- Yet another corollary: differentiating equation (1.5), one obtains an equation symbolically looking as $\nabla\nabla\nabla\xi = -\xi\nabla R - R\nabla\xi$; differentiating once more, one has $\nabla\nabla\nabla\nabla\xi = -\xi\nabla\nabla R - 2\nabla R\nabla\xi - R\nabla\nabla\xi$, where $\nabla\nabla\xi$ can be expressed from (1.5); etc etc...: whenever the 2nd derivative arises on the r.h. side, one substitutes from (1.5), thus gradually expressing all the derivatives ($\geqslant$ 2nd) in terms of ξ^μ and its gradient. In other words, thanks to equation (1.5), the entire Taylor expansion of ξ^μ is fully determined by ξ^μ and its gradient. Hence, the Killing exercise can in principle have as many independent solutions as the number of the "initial conditions" ξ^μ and $\xi_{\mu;\nu}$; and these are 4+6, since $\xi_{\mu;\nu}$ is antisymmetric. So, in a 4D space-time, there may at most exist 10 independent Killing fields. In a general dimension d, it is $d(d+1)/2$.

1.3 A useful formula

Have a vector field ξ^μ. Denote

$$g_{\mu\nu}\xi^\mu\xi^\nu =: -N^2 \quad \Rightarrow \quad (-N^2)_{;\alpha} = 2\xi_{\mu;\alpha}\xi^\mu\,, \quad \text{and} \;\; \omega^\mu := \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\xi_{\nu;\kappa}\xi_\lambda \;\; (\text{vorticity, twist})\,.$$

Using the famous relation

$$\epsilon_{\mu\nu\kappa\lambda}\epsilon^{\mu\alpha\beta\gamma} = -3!\,\delta_\nu^{[\alpha}\delta_\kappa^\beta\delta_\lambda^{\gamma]}\,, \tag{1.8}$$

we obtain a useful formula

$$\begin{aligned}
4\omega_\mu\omega^\mu &= \epsilon_{\mu\nu\kappa\lambda}\xi^{\nu;\kappa}\xi^\lambda\,\epsilon^{\mu\alpha\beta\gamma}\xi_{\alpha;\beta}\xi_\gamma = \\
&= -\left(\delta_\nu^\alpha\delta_\kappa^\beta\delta_\lambda^\gamma + \delta_\nu^\gamma\delta_\kappa^\alpha\delta_\lambda^\beta + \delta_\nu^\beta\delta_\kappa^\gamma\delta_\lambda^\alpha - \delta_\nu^\beta\delta_\kappa^\alpha\delta_\lambda^\gamma - \delta_\nu^\gamma\delta_\kappa^\beta\delta_\lambda^\alpha - \delta_\nu^\alpha\delta_\kappa^\gamma\delta_\lambda^\beta\right)\xi^{\nu;\kappa}\xi^\lambda\xi_{\alpha;\beta}\xi_\gamma = \\
&= \xi^{\alpha;\beta}\xi_{\alpha;\beta}N^2 - \xi^{\gamma;\alpha}\xi^\beta\xi_{\alpha;\beta}\xi_\gamma - \xi^{\beta;\gamma}\xi^\alpha\xi_{\alpha;\beta}\xi_\gamma \\
&\quad - \xi^{\beta;\alpha}\xi_{\alpha;\beta}N^2 + \xi^{\gamma;\beta}\xi^\alpha\xi_{\alpha;\beta}\xi_\gamma + \xi^{\alpha;\gamma}\xi^\beta\xi_{\alpha;\beta}\xi_\gamma = \\
&= 2N^2\xi^{[\alpha;\beta]}\xi_{\alpha;\beta} - 2\xi^{[\gamma;\alpha]}\xi_\gamma\xi_{\alpha;\beta}\xi^\beta + 2\xi^{[\gamma;\beta]}\xi_\gamma\xi_{\alpha;\beta}\xi^\alpha\,.
\end{aligned}$$

Relabelling $\alpha \leftrightarrow \beta$ in the last term, we have

$$4\omega_\mu\omega^\mu = 2N^2\xi^{[\alpha;\beta]}\xi_{\alpha;\beta} + 4\xi^{[\gamma;\alpha]}\xi_\gamma\xi_{[\beta;\alpha]}\xi^\beta\,. \tag{1.9}$$

Hence, if ξ^μ is a Killing field, i.e. if $\xi_{[\alpha;\beta]} = \xi_{\alpha;\beta}$, we obtain the relation

$$\boxed{4\omega_\mu\omega^\mu = 2N^2\xi^{\alpha;\beta}\xi_{\alpha;\beta} + (N^2)^{;\alpha}(N^2)_{;\alpha}}\,. \tag{1.10}$$

1.4 Killing horizon: theorem by Vishveshwara

Proposition: Two orthogonal null vectors are necessarily proportional to each other.

Proof: Orthogonality is a local property, and one can at every point work in a locally Minkowskian frame where $g_{\mu\nu} = \eta_{\mu\nu}$. So, have a non-trivial null vector k^μ and some other non-trivial vector V^μ orthogonal to k^μ:

$$\begin{aligned}
0 &= \eta_{\mu\nu}k^\mu k^\nu = -(k^0)^2 + k^2 \quad \Rightarrow \quad (k^0)^2 = k^2\,, \\
0 &= \eta_{\mu\nu}k^\mu V^\nu = -k^0V^0 + \vec{k}\cdot\vec{V} \quad \Rightarrow \quad V^0 = \frac{\vec{k}\cdot\vec{V}}{k^0} \quad \Rightarrow \quad (V^0)^2 = \frac{(\vec{k}\cdot\vec{V})^2}{k^2}\,,
\end{aligned}$$

where $k^2 := \vec{k}\cdot\vec{k} \equiv \eta_{ij}k^ik^j = \delta_{ij}k^ik^j$ and likewise for V^2. Then

$$\eta_{\mu\nu}V^\mu V^\nu = -(V^0)^2 + V^2 = -\frac{(\vec{k}\cdot\vec{V})^2}{k^2} + V^2 = -\frac{(kV\cos\alpha)^2}{k^2} + V^2 = V^2\sin^2\alpha\,,$$

with α the angle between $\vec{k}$ and $\vec{V}$. Therefore, V^μ is space-like in general ($\sin\alpha \neq 0$), with the special exception of $\sin\alpha = 0$ when it is null. However, the latter case means $\vec{V} = \lambda\vec{k}$ (with λ some constant), which enforces

$$V^0 \equiv \frac{\vec{k}\cdot\vec{V}}{k^0} = \frac{\lambda k^2}{k^0} = \frac{\lambda (k^0)^2}{k^0} = \lambda k^0 \quad \Longrightarrow \quad V^\mu = \lambda k^\mu \,.$$

Definition: The set $\{g_{\mu\nu}\xi^\mu\xi^\nu \equiv -N^2 = 0, \text{ with } \xi^\mu \text{ Killing and } \xi^\mu \neq 0\}$, is called a **Killing horizon**, if it is a null and connected hypersurface (or a union of such).

Lemma: On its horizon, the Killing field is proportional to the normal of that hypersurface.
Proof: On the Killing horizon, by definition, ξ^μ is null and the normal $(N^2)^{;\mu}$ as well. However, they are also *orthogonal* to each other, $(-N^2)_{;\alpha}\xi^\alpha = 2\xi_{\mu;\alpha}\xi^\mu\xi^\alpha = 2\xi_{(\mu;\alpha)}\xi^\mu\xi^\alpha = 0$. But the only vector orthogonal to a null vector is proportional to it. (Hence, ξ^μ is a **null generator** of the Killing horizon, being both tangent and orthogonal to it.)

Theorem [Vishveshwara (1968)]

Let ξ^μ be a Killing field. Then the set $\{g_{\mu\nu}\xi^\mu\xi^\nu \equiv -N^2 = 0, \text{ with } \xi^\mu \neq 0\}$ is a null hypersurface (a Killing horizon) if and only if $\omega_\mu = 0$ there, with $(N^2)_{;\alpha} \neq 0$.

Proof $\Rightarrow$: First, if the set $\{g_{\mu\nu}\xi^\mu\xi^\nu \equiv -N^2 = 0\}$ is a null hypersurface, its normal is null, i.e. $(N^2)^{;\alpha}(N^2)_{;\alpha} = 0$. Formula (1.10) then implies $\omega_\mu\omega^\mu = 0$. Second, since ξ^μ and $(N^2)^{;\mu}$ are proportional to each other there (see Lemma), if ξ^μ is non-trivial, $(N^2)^{;\mu}$ must be as well.

Proof $\Leftarrow$: If $\omega = 0$, formula (1.10) says that the set $\{N^2 = 0\}$ has to be null. Besides that, Frobenius says it is an integral hypersurface. And, if $(N^2)^{;\mu} \neq 0$ there, it implies that ξ^μ is non-trivial there as well, because – again – these two vectors have to be proportional there.

Consequence of the Lemma:

Since, on the hypersurface $\{N^2 = 0\}$, the vectors ξ^μ and $(N^2)^{;\mu}$ are proportional to each other, $(N^2)^{;\mu} = -2\xi^{\alpha;\mu}\xi_\alpha = 2\xi^{\mu;\alpha}\xi_\alpha \sim \xi^\mu$, it means ξ^μ is *geodesic* there.

1.5 Orthogonal-transitivity conditions

Have some vector field ξ^μ. Frobenius theorem says that its local orthogonal hyperplanes $\Sigma(\xi)$ are integrable if and only if ξ^μ has zero vorticity – when the latter does not "entwine" about itself. In stationary and axisymmetric space-times, the so-called **orthogonal transitivity** is the requirement that integral meridional surfaces exist, everywhere orthogonal to both independent Killing vector fields t^μ and ϕ^μ. So, wanted is the integrability of the local planes given by intersections $\Sigma(t) \cap \Sigma(\phi)$. Such planes would certainly be integrable if both the normal fields t^μ and ϕ^μ had zero vorticities. However, even weaker condition is clearly sufficient: the vorticity of t^μ must not have a component in the direction of ϕ^μ, and the vorticity

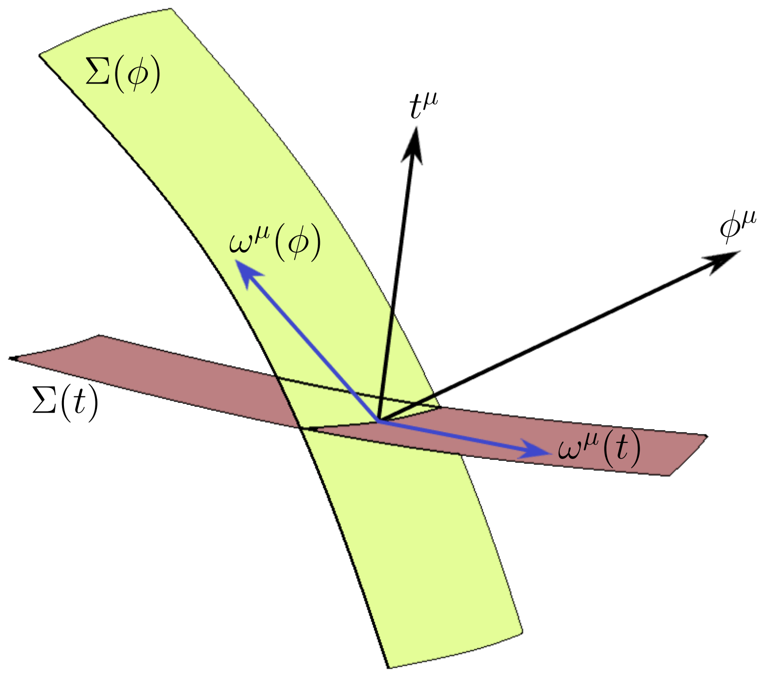

Figure 1.1 *Two independent vector fields, t^μ and ϕ^μ, their orthogonal hypersurfaces $\Sigma(t)$ and $\Sigma(\phi)$, and their vorticity vectors $\omega^\mu[t]$ and $\omega^\mu[\phi]$ (which are orthogonal to them, so tangent to $\Sigma(t)$ and $\Sigma(\phi)$, respectively). If the intersection $\Sigma(t) \cap \Sigma(\phi)$ is integrable, one speaks of* ***orthogonal transitivity*** *of the two fields.*

of ϕ^μ must not have a component in the direction of t^μ. Therefore, one demands

$$\omega^\mu[t]\phi_\mu = \epsilon^{\mu\nu\rho\sigma}\phi_\mu t_{\nu;\rho}t_\sigma = 0 \qquad \text{and} \qquad \omega[\phi]^\mu t_\mu = \epsilon^{\mu\nu\rho\sigma}t_\mu \phi_{\nu;\rho}\phi_\sigma = 0\,;$$

more often it is being written as

$$\phi_{[\mu}t_{\nu;\rho}t_{\sigma]} = 0 \qquad \text{and} \qquad t_{[\mu}\phi_{\nu;\rho}\phi_{\sigma]} = 0\,.$$

If these conditions are satisfied, the thus existing global meridional planes can be covered by coordinates x^1, x^2, for example by r and θ (or ρ and z), such that the metric does not contain the terms g_{t1}, g_{t2}, $g_{1\phi}$ and $g_{2\phi}$.

Theorem [Papapetrou, 1966]

The conditions $\phi_{[\mu}t_\nu t_{\kappa;\lambda]} = 0$, $t_{[\mu}\phi_\nu\phi_{\kappa;\lambda]} = 0$ are equivalent to the conditions

$$\boxed{\phi_{[\mu}t_\lambda R_{\kappa]\nu}t^\nu = 0\,, \qquad t_{[\mu}\phi_\lambda R_{\kappa]\nu}\phi^\nu = 0}\,.$$

Proof: The derivation is the same for both the conditions, and since it starts from properties which hold for *any* Killing vector field, we will at this stage denote the latter generically as ξ^μ. Multiplying the definition $\omega_\mu[\xi] = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\xi^{\nu;\kappa}\xi^\lambda$ by $\epsilon^{\mu\beta\gamma\delta}$ and using the formula (1.8), i.e., explicitly,

$$\epsilon_{\mu\nu\kappa\lambda}\epsilon^{\mu\beta\gamma\delta} = -\delta^\beta_\nu\delta^\gamma_\kappa\delta^\delta_\lambda - \delta^\delta_\nu\delta^\beta_\kappa\delta^\gamma_\lambda - \delta^\gamma_\nu\delta^\delta_\kappa\delta^\beta_\lambda + \delta^\beta_\nu\delta^\delta_\kappa\delta^\gamma_\lambda + \delta^\gamma_\nu\delta^\beta_\kappa\delta^\delta_\lambda + \delta^\delta_\nu\delta^\gamma_\kappa\delta^\beta_\lambda\,,$$

we easily obtain

$$\omega_\mu \epsilon^{\mu\beta\gamma\delta} = -\xi^\beta \xi^{[\gamma;\delta]} - \xi^\delta \xi^{[\beta;\gamma]} - \xi^\gamma \xi^{[\delta;\beta]} = -\xi^\beta \xi^{\gamma;\delta} - \xi^\delta \xi^{\beta;\gamma} - \xi^\gamma \xi^{\delta;\beta} = -\xi^{\{\beta} \xi^{\gamma;\delta\}} \,.$$

Differentiation of the latter by x^β yields

$$\begin{aligned}
\omega_{\mu;\beta}\epsilon^{\mu\beta\gamma\delta} &= -(\xi^\beta \xi^{\gamma;\delta})_{;\beta} - (\xi^\delta \xi^{\beta;\gamma})_{;\beta} - (\xi^\gamma \xi^{\delta;\beta})_{;\beta} = \\
&= -\cancel{\xi^\beta \xi^{\gamma;\delta}{}_{\beta}} - \cancel{\xi^\delta{}_{;\beta} \xi^{\beta;\gamma}} - \xi^\delta \xi^{\beta;\gamma}{}_{\beta} - \cancel{\xi^\gamma{}_{;\beta} \xi^{\delta;\beta}} - \xi^\gamma \xi^{\delta;\beta}{}_{\beta} = \\
&= \xi^\delta \Box \xi^\gamma - \xi^\gamma \Box \xi^\delta = \xi^\gamma R^\delta_\beta \xi^\beta - \xi^\delta R^\gamma_\beta \xi^\beta \,, \qquad (1.11)
\end{aligned}$$

where only the Killing property and the formula (1.6) have been employed. Multiplication of this relation by $\epsilon_{\alpha\nu\gamma\delta}$ leads to twice the same term on the right-hand side, while on the left one has

$$\omega_{\mu;\beta}\epsilon^{\mu\beta\gamma\delta}\epsilon_{\alpha\nu\gamma\delta} = 2\,\omega_{\mu;\beta}(\delta^\mu_\nu \delta^\beta_\alpha - \delta^\mu_\alpha \delta^\beta_\nu) = 2(\omega_{\nu;\alpha} - \omega_{\alpha;\nu}) \equiv 4\,\omega_{[\nu;\alpha]} \,,$$

so we arrive at the formula for gradient of (any) Killing-vector twist,

$$\omega_{[\nu;\alpha]} = \frac{1}{2}\,\epsilon_{\alpha\nu\gamma\delta}\,\xi^\gamma R^\delta_\beta\,\xi^\beta \,. \qquad (1.12)$$

Now, let us specify to our $\xi^\mu \equiv t^\mu$, $\omega_\mu \equiv \omega_\mu[t]$ case (with ϕ^μ the second existing Killing field) and consider the derivative

$$\begin{aligned}
(\phi^\nu \omega_\nu)_{,\alpha} &= \phi^\nu{}_{;\alpha}\omega_\nu + \phi^\nu \omega_{\nu;\alpha} = \phi^\nu{}_{;\alpha}\omega_\nu + \phi^\nu \omega_{\alpha;\nu} + 2\phi^\nu \omega_{[\nu;\alpha]} = \\
&= \cancel{(\pounds_\phi \omega_\alpha)} + \epsilon_{\alpha\nu\gamma\delta}\,\phi^\nu t^\gamma R^\delta_\beta\, t^\beta \,. \qquad (1.13)
\end{aligned}$$

This result confirms that

$$\phi_{[\mu} t_\nu t_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad (\phi^\nu \omega_\nu[t])_{,\alpha} = 0 \quad \Longleftrightarrow \quad \phi_{[\mu} t_\lambda R_{\kappa]\nu} t^\nu = 0 \,.$$

Similarly one would verify that

$$t_{[\mu} \phi_\nu \phi_{\kappa;\lambda]} = 0 \quad \Longrightarrow \quad (t^\nu \omega_\nu[\phi])_{,\alpha} = 0 \quad \Longleftrightarrow \quad t_{[\mu} \phi_\lambda R_{\kappa]\nu} \phi^\nu = 0 \,.$$

The opposite implications are also based on the relation (1.13). Since $\phi_\mu = g_{\mu\phi}$ vanishes on the symmetry axis,[3] also trivial there is $\omega^\mu[\phi] = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\phi_\nu \phi_{\kappa;\lambda}$. Consequently, both the invariants $\phi_\nu \omega^\nu[t]$ and $t_\nu \omega^\nu[\phi]$ vanish on the axis as well. Now, if the space-time satisfies $\phi_{[\mu} t_\lambda R_{\kappa]\nu} t^\nu = 0$ and $t_{[\mu} \phi_\lambda R_{\kappa]\nu} \phi^\nu = 0$, implying that the gradients of both the invariants are everywhere zero, $(\phi_\nu \omega^\nu[t])_{,\alpha} = 0$ and $(t_\nu \omega^\nu[\phi])_{,\alpha} = 0$, then the invariants are themselves zero everywhere, which is the orthogonal-transitivity condition. □

[3] On a regular axis, $g_{\phi\phi} \equiv \phi_\mu \phi^\mu$ has to vanish since it determines proper circumference about the axis (along a circular orbit of ϕ^μ at some given radius). This is *not* due to ϕ^μ becoming null (light-like) there, but because $\phi_\mu = g_{\mu\phi}$ shrinks there to *zero* (while $\phi^\mu = \partial x^\mu / \partial \phi$ everywhere).

Corollaries: Since the orthogonal-transitivity properties trivially hold for the metric tensor,

$$\phi_{[\mu}t_\lambda g_{\kappa]\nu}t^\nu = \phi_{[\mu}t_\lambda t_{\kappa]} = 0\,, \qquad t_{[\mu}\phi_\lambda g_{\kappa]\nu}\phi^\nu = t_{[\mu}\phi_\lambda \phi_{\kappa]} = 0\,,$$

one may use Einstein equations and translate the Ricci-based conditions to

$$\phi_{[\mu}t_\lambda T_{\kappa]\nu}t^\nu = 0\,, \qquad t_{[\mu}\phi_\lambda T_{\kappa]\nu}\phi^\nu = 0\,. \tag{1.14}$$

Immediately clear is that *vacuum* stationary and axisymmetric space-times are necessarily orthogonally transitive. Actually, every space-time is such in which sources move purely along stationary circular trajectories (along the Killing directions, i.e. with four-velocity satisfying $u^{[\nu}t^\kappa\phi^{\lambda]}=0$). This is illustrated on an ideal fluid, $T_{\kappa\nu} = (\rho + P)u_\kappa u_\lambda + P g_{\kappa\lambda}$: the second part is circular automatically and the first one has to satisfy $\phi_{[\mu}t_\lambda u_{\kappa]} = 0$, resp. $t_{[\mu}\phi_\lambda u_{\kappa]} = 0$ (which is the same). The stationary axisymmetric space-times which are orthogonally transitive are thus called **circular space-times**.

1.6 The weak rigidity theorem

Theorem [Carter 1969]

In a circular space-time, the dragging angular velocity ω is constant all over the surface $\{-N^2=0\}$, so the latter is a Killing horizon.

Proof:

- First, if the two Killing fields commute, $t_{\alpha,\mu}\phi^\mu = \phi_{\alpha,\mu}t^\mu$ (which is automatic in the asymptotically flat case), it also means $t_{\alpha;\mu}\phi^\mu = \phi_{\alpha;\mu}t^\mu$. Let us use it in

$$\begin{aligned} g_{t\phi;\alpha} &= (t_\mu\phi^\mu)_{;\alpha} = t_{\mu;\alpha}\phi^\mu + t^\mu\phi_{\mu;\alpha} = -t_{\alpha;\mu}\phi^\mu - t^\mu\phi_{\alpha;\mu} = -2t^\mu\phi_{\alpha;\mu} \\ g_{\phi\phi;\alpha} &= (\phi_\mu\phi^\mu)_{;\alpha} = 2\phi_{\mu;\alpha}\phi^\mu = -2\phi_{\alpha;\mu}\phi^\mu \\ \Longrightarrow \quad \omega_{,\alpha} &= \left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right)_{;\alpha} = \frac{-g_{t\phi;\alpha}g_{\phi\phi} + g_{t\phi}g_{\phi\phi;\alpha}}{(g_{\phi\phi})^2} = \frac{-g_{t\phi;\alpha} - \omega g_{\phi\phi;\alpha}}{g_{\phi\phi}} = \\ &= \frac{2\phi_{\alpha;\mu}}{g_{\phi\phi}}\,(t^\mu + \omega\phi^\mu) \equiv \frac{2}{g_{\phi\phi}}\,\phi_{\alpha;\mu}\xi^\mu\,. \end{aligned}$$

- It is clear that $\omega_{,\alpha}t^\alpha = 0$ and $\omega_{,\alpha}\phi^\alpha = 0$ (thus $\omega_{,\alpha}\xi^\alpha = 0$), so we only need to check the derivative of ω in the plane perpendicular to both t^μ and ϕ^μ,

$$\epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma \omega_{,\delta} = \frac{2}{g_{\phi\phi}}\,\epsilon^{\alpha\beta\gamma\delta}\phi_\beta t_\gamma \phi_{\delta;\mu}\xi^\mu \equiv \frac{2}{g_{\phi\phi}}\,\epsilon^{\alpha\beta\gamma\delta}\phi_{[\beta}t_\gamma\phi_{\delta];\mu}\xi^\mu\,.$$

- One of the circularity conditions reads $0=4!\,\phi_{[\beta}t_\gamma\phi_{\delta;\mu]}=3!\,\phi_{\{[\beta}t_\gamma\phi_{\delta];\mu\}}$ (braces mean cyclic permutation in the enclosed indices, without a prefactor), from where one can express

$$3!\,\phi_{[\beta}t_\gamma\phi_{\delta];\mu} = 3!\,\phi_{[\mu}t_\beta\phi_{\gamma];\delta} - 3!\,\phi_{[\delta}t_\mu\phi_{\beta];\gamma} + 3!\,\phi_{[\gamma}t_\delta\phi_{\mu];\beta} =$$

$$
\begin{aligned}
&= \phi_\mu t_\beta \phi_{\gamma;\delta} + \phi_\gamma t_\mu \phi_{\beta;\delta} + \phi_\beta t_\gamma \phi_{\mu;\delta} - \phi_\beta t_\mu \phi_{\gamma;\delta} - \phi_\gamma t_\beta \phi_{\mu;\delta} - \phi_\mu t_\gamma \phi_{\beta;\delta} \\
&\quad - \phi_\delta t_\mu \phi_{\beta;\gamma} - \phi_\beta t_\delta \phi_{\mu;\gamma} - \phi_\mu t_\beta \phi_{\delta;\gamma} + \phi_\mu t_\delta \phi_{\beta;\gamma} + \phi_\beta t_\mu \phi_{\delta;\gamma} + \phi_\delta t_\beta \phi_{\mu;\gamma} \\
&\quad + \phi_\gamma t_\delta \phi_{\mu;\beta} + \phi_\mu t_\gamma \phi_{\delta;\beta} + \phi_\delta t_\mu \phi_{\gamma;\beta} - \phi_\delta t_\gamma \phi_{\mu;\beta} - \phi_\mu t_\delta \phi_{\gamma;\beta} - \phi_\gamma t_\mu \phi_{\delta;\beta} = \\
&= 2\phi_\mu t_{\{\beta} \phi_{\gamma;\delta\}} - 2t_\mu \phi_{\{\beta} \phi_{\gamma;\delta\}} - \phi_{\{\beta} t_\gamma \phi_{\delta\};\mu} + \phi_{\{\beta} t_\delta \phi_{\gamma\};\mu} \ .
\end{aligned}
$$

Now multiply this by ξ^μ, using the relations $\phi_\mu \xi^\mu = 0$, $t_\mu \xi^\mu = -N^2$ and $\phi_{\delta;\mu}\xi^\mu = \frac{1}{2}\, g_{\phi\phi}\omega_{,\delta}$:

$$
\phi_{[\beta} t_\gamma \phi_{\delta];\mu} \xi^\mu = \frac{1}{3!} \left(2N^2 \phi_{\{\beta} \phi_{\gamma;\delta\}} - \frac{1}{2}\, g_{\phi\phi} \phi_{\{\beta} t_\gamma \omega_{,\delta\}} + \frac{1}{2}\, g_{\phi\phi} \phi_{\{\beta} t_\delta \omega_{,\gamma\}} \right) .
$$

Finally, multiplication by $\epsilon^{\alpha\beta\gamma\delta}$ removes the $\frac{1}{3!}$ factor (antisymmetrization is $\frac{1}{n!}$ times cyclic permutation in the antisymmetrized indices), so

$$
\epsilon^{\alpha\beta\gamma\delta} \phi_\beta t_\gamma \omega_{,\delta} = \frac{2}{g_{\phi\phi}}\, \epsilon^{\alpha\beta\gamma\delta} \left(2N^2 \phi_{[\beta} \phi_{\gamma;\delta]} - g_{\phi\phi} \phi_{[\beta} t_\gamma \omega_{,\delta]} \right) \tag{1.15}
$$

$$
\implies \quad 3\epsilon^{\alpha\beta\gamma\delta} \phi_\beta t_\gamma \omega_{,\delta} = \frac{4N^2}{g_{\phi\phi}}\, \epsilon^{\alpha\beta\gamma\delta} \phi_\beta \phi_{\gamma;\delta} \ . \tag{1.16}
$$

- Hence, on the $N^2 = 0$ hypersurface, the derivative of ω in the complementary (meridional) plane vanishes as well, so ω is constant there.