CVIČENÍ-I-1 : pozn & vzor. řešení

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$\boxed{U1}$ $\langle A^+\phi|\psi\rangle = \langle\phi|A\psi\rangle \quad \forall\phi,\psi \qquad$ (Def)

a) $c\cdot$(Def): $\quad c\langle A^+\phi|\psi\rangle = c\langle\phi|A\psi\rangle$

$\downarrow$ Axiom: antilin $\quad \downarrow$ Axiom: lin

$\langle c^*A^+\phi|\psi\rangle = \langle\phi|cA\psi\rangle$ tj. $B=cA$; $B^+=c^*A^+$ splňuje

b) (Def) + (Def $A\to B$): $\langle A^+\phi + B^+\phi|\psi\rangle = \langle\phi|A\psi+B\psi\rangle$

+ užití definice součtu oper.

c) (Def) platí $\forall|\psi\rangle$ tj. i pro $|\psi\rangle \to B|\psi\rangle$:

$$\langle B^+A^+\phi|\psi\rangle \xleftarrow[|\phi\rangle\to A^+\phi]{(\text{Def } B)} \langle A^+\phi|B\psi\rangle = \langle\phi|AB\psi\rangle$$

d) (Def)* : $\langle\psi|A^+\phi\rangle = \langle A\psi|\phi\rangle$ ... def pro $B=A^+$; $B^+=A$

s prohozeným ozn. $\psi\leftrightarrow\phi$

e) Ověření definice dosazením a použití def. vnějšího součinu:

$$\langle\phi|A\psi\rangle = \langle\phi|\,|u\rangle\langle v|\psi\rangle = \langle v|\psi\rangle\langle\phi|u\rangle$$

$$\langle A^+\phi|\psi\rangle = \langle |v\rangle\langle u|\phi|\psi\rangle = \langle u|\phi\rangle^*\langle v|\psi\rangle = \langle v|\psi\rangle\langle\phi|u\rangle = \quad ✓$$

$\boxed{U2}$ a,b) pravidlo U1c) : $(L^+L)^+ = L^+L^{++} = L^+L$ ✓ + pravidlo U1d)

$(LL^+) = L^{++}L^+ = LL^+$ ✓

c) pravidlo U1b) : $(L+L^+)^+ = L^+ + L$

d) -"- + U1a) : $[i(L-L^+)]^+ = -i(L^+-L) = i(L-L^+)$

e) $(LAL^+)^+ = L^{++}A^+L^+ = LAL^+$

f) $(i[A,B])^+ = -i(BA-AB) = i[A,B]$

g) $(AB+BA)^+ = BA+AB \qquad$ f) .. pravidlo U1c)

$\boxed{U3}$ $$L = \frac{1}{2}\{L+L^+ + L-L^+\} = \underbrace{\frac{1}{2}(L+L^+)}_{A} + i\,\underbrace{\frac{i}{2}(L^+-L)}_{B}$$

$\boxed{U4}$ 1) $\langle\phi_1|A|\phi_1\rangle + \langle\phi_2|A|\phi_2\rangle + \langle\phi_1|A|\phi_2\rangle + \langle\phi_2|A|\phi_1\rangle = $ %. $A\to B$

2) $\langle\phi_1|A|\phi_2\rangle - \langle\phi_2|A|\phi_2\rangle + i\langle\phi_1|A|\phi_2\rangle - i\langle\phi_2|A|\phi_1\rangle = $ %. $A\to B$

první dva členy můžeme vyškrtnout díky $\langle\phi_1|A|\phi_1\rangle = \langle\phi_1|B|\phi_1\rangle$

pak druhou rovnici násobíme i a sečteme

U5

A1: $(\psi|\phi) = (\phi|\psi)^*$ ... $(A,B) = \sum_n \langle n|A^+Bn\rangle = \sum_n \langle A^+Bn|n\rangle^* = \sum_n \langle n|B^+A|n\rangle^*$

$= (B,A)^*$ ✓

A2: $(\phi, c_1\psi_1 + c_2\psi_2) = c_1(\phi|\psi_1) + c_2(\phi,\psi_2)$

$(A, c_1B + c_2B) = c_1(A,B_1) + c_2(A,B_2)$ ... ($\forall$ člen $\sum_n$ splňuje)

A3: $(A,A) \geq 0$; rovnost jen pro $A=0$

$(A,A) = \sum_n \langle n|A^+A|n\rangle = \sum_n \langle An|An\rangle \geq 0$

rovnost jen pro $A|n\rangle = 0 \ \forall n$ tj; $A=0$ neboť $\{|n\rangle\}$ je báze

označme $\sigma_0 \equiv I$ .. pak platí $\sigma_i^2 = I = \sigma_i^+\sigma_i$ (hermit)

tj; $(\sigma_i,\sigma_i) = \mathrm{Tr}\, I = 2$ ... normování $\frac{1}{\sqrt2}\sigma_i$

$(I,\sigma_i) = \mathrm{Tr}\,\sigma_i = 0$; $i,j>0$: $\sigma_i\sigma_j = \varepsilon_{ijk}\, i\sigma_k$ .. $\to (\sigma_i,\sigma_j) = 0$ pro $i \neq j$

tj ON báze na 4D prostoru ... $\frac{1}{\sqrt2}\sigma_i$ .. koef = skal. součin

tj; $M = \sum \alpha_i\sigma_i$; $\alpha_0 = \frac{1}{\sqrt2}\mathrm{Tr}\, M$; $i>0$: $\alpha_i = \frac{1}{\sqrt2}\mathrm{Tr}(\sigma_i M)$ $(\sigma_i^+ = \sigma_i)$

U6

$|\sigma_x - \lambda I| = \lambda^2 - 1 = 0 \to \lambda = \pm1$; $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \pm\begin{pmatrix}a\\b\end{pmatrix} \to |x+\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$ $|x-\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

$I = |x+\rangle\langle x+| + |x-\rangle\langle x-| = \begin{pmatrix}1\\1\end{pmatrix}(1\,1)\cdot\frac12 + \begin{pmatrix}1\\-1\end{pmatrix}(1\,-1)\cdot\frac12 = \frac12\left(\begin{pmatrix}1&1\\1&1\end{pmatrix} + \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right)$

$|\sigma_y - \lambda I| = \lambda^2 - 1 = 0 \to \lambda = \pm1$ $\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \pm\begin{pmatrix}a\\b\end{pmatrix} \to |y+\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix}$ $|y-\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix}$

$I = \frac12\begin{pmatrix}1\\i\end{pmatrix}(1\,-i) + \frac12\begin{pmatrix}1\\-i\end{pmatrix}(1\,+i) = \frac12\begin{pmatrix}1&-i\\i&1\end{pmatrix} + \frac12\begin{pmatrix}1&i\\-i&1\end{pmatrix}$

$\sigma_z \to \begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}0\\1\end{pmatrix}$ $I = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}$

$|y+\rangle = |x+\rangle\langle x+|y+\rangle + |x-\rangle\langle x-|y+\rangle = \frac{1+i}{2}|x+\rangle + \frac{1-i}{2}|x-\rangle$

$|y-\rangle = |x+\rangle\langle x+|y-\rangle + |x-\rangle\langle x-|y-\rangle = \frac{1-i}{2}|\cancel{y}+\rangle + \frac{1+i}{2}|x-\rangle$

U7 $\det S \neq 0$ (LNZ)

Nejlépe odvodit 1-operátor nejdříve: $I = \sum_{ij} a_{ij}|\phi_i\rangle\langle\phi_j|$

podmínka $I|\phi_k\rangle = |\phi_k\rangle$ dá $a_{ij} = (S^{-1})_{ij}$

Pak $|\psi\rangle = I|\psi\rangle = \sum_{ij}(S^{-1})_{ij}|\phi_i\rangle\langle\phi_j|\psi\rangle$ tj; $c_i = \sum_j (S^{-1})_{ij}\langle\phi_j|\psi\rangle$:

$\langle\psi|\psi'\rangle = \langle\psi|I|\psi'\rangle = \sum_{ij}\langle\psi|\phi_i\rangle(S^{-1})_{ij}\langle\phi_j|\psi'\rangle$