Cvičení QM-I-3 14.10

zápis vzor. řešení během cvičení

Úloha 1: $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ $\mathcal{H}^{(1)} = \mathcal{H}^{(2)} = \mathbb{C}^2$

$$\hat S_z = \hat s_z^{(1)} \otimes \hat I + \hat I \otimes \hat s_z^{(2)}$$

$$\hat P |s_1 s_2\rangle = |s_2 s_1\rangle$$

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\mathcal{H}^{(1)} = \mathcal{L}\{|+\rangle, |-\rangle\}$$

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} = \{|s\rangle\}_{s=+,-}$$

$$\mathcal{H} = \mathcal{L}\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\} = |s_1 s_2\rangle$$

$s = \pm 1$

Řešení působením na bázi + abstraktní algebra

a) DK že $[\hat S_z, P] = 0$ tj. $\boxed{S_z P = P S_z}$

$\forall s_1 s_2$: ① $S_z P|s_1 s_2\rangle = P S_z |s_1 s_2\rangle$ ②

$S_z|s\rangle = s\frac{\hbar}{2}|s\rangle$

pro 1 částici

$S_z|+\rangle = \frac{\hbar}{2}|+\rangle$

$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$

půs. $\hat S_z$ na $|s_1 s_2\rangle$?

① $S_z P|s_1 s_2\rangle = S_z|s_2 s_1\rangle = (s_z \otimes I + I \otimes s_z)|s_2 s_1\rangle$

$|s_1\rangle \otimes |s_2\rangle$

$$= s_2\frac{\hbar}{2}|s_2 s_1\rangle + s_1\frac{\hbar}{2}|s_2 s_1\rangle = \frac{\hbar}{2}(s_1 + s_2)|s_2 s_1\rangle$$

② $\hat P \hat S_z|s_1 s_2\rangle = \hat P(s_z \otimes I + I \otimes s_z)|s_1 s_2\rangle = P\left(s_1\frac{\hbar}{2}|s_1 s_2\rangle + s_2\frac{\hbar}{2}|s_1 s_2\rangle\right)$

$$= \frac{\hbar}{2}(s_1 + s_2)|s_2 s_1\rangle$$

totéž ✓

Řešení v reprezentaci (maticově)

a) $[S_z, P] = 0 = S_z P - P S_z$ $\boxed{S_z P = \mathcal{S}_z P}$

$$P = \begin{pmatrix} \langle ++|P|++\rangle & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{array}{l} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array}$$

$P|++\rangle = |++\rangle$

$P|+-\rangle = |-+\rangle$ $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$P|-+\rangle = |+-\rangle$

$P|--\rangle = |--\rangle$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

$$PS_z = \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$S_zP = \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

tj $[P, S_z] = 0$

(b) najděte spol. bázi vl. v. maticově (tj v reprezentaci)

vl. č. P

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ ... vl. v. vl. č. $\lambda = 1$

$\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right)$

vl. č. $\lambda = -1$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: $\lambda = 1$ .. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = -1$ .. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

vl. č. S_z -- $S = \frac{\hbar}{2}(s_1 + s_2)$ .. vl. v. $|s_1 s_2\rangle$

$S = \frac{\hbar}{2}\cdot 2 = \hbar$ .. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |++\rangle$ $\qquad S = \frac{\hbar}{2}\cdot(-2) = -\hbar$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |--\rangle$

$S = 0$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda \; S$

společná sada:

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv |++\rangle$ .. $|\lambda S\rangle \equiv |1, \hbar\rangle$ — $(1, \hbar)$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv |--\rangle$ .. $|\lambda S\rangle \equiv |+1, -\hbar\rangle$ — $(1, -\hbar)$

$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ .. $|\lambda S\rangle = |1, 0\rangle$ — $(1, 0)$

$\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$ $\lambda = -1$ .. $|\lambda s\rangle = |-1,0\rangle$ $\dashv (-1,0)$

$\boxed{\text{VSKO} \quad \hat{P}, \hat{S}}$ $|s_1 s_2\rangle$ .. jiná báze $|\lambda, s\rangle$

$\boxed{\text{ÚLOHA 2}}$ $|\psi\rangle = \underline{|z:+\rangle} \otimes |x:+\rangle$ $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$

$\begin{pmatrix}1\\0\end{pmatrix}$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

++ +− −+ −− $S_z \pm \frac{\hbar}{2}$

v reprez S_z

$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}$ $\Leftarrow$ $\begin{pmatrix}1\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\0\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\end{pmatrix}$

$|+\rangle \otimes \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|+-\rangle$

a) měření $S_z^{(2)}$ pravd. a stav po měření?

$S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\rightarrow$ $\hat{I} \otimes \hat{S}_z = \underline{\begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&-1\end{pmatrix}}\frac{\hbar}{2}$

$S_z^{(2)}|s_1 s_2\rangle = \frac{\hbar}{2} s_2 |s_1 s_2\rangle$

$\frac{\hbar}{2}\begin{pmatrix}1&&&0\\&-1&&\\&&1&\\0&&&-1\end{pmatrix}$

$\begin{pmatrix}1\begin{pmatrix}1&0\\0&-1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&-1\end{pmatrix}\\0\begin{pmatrix}1&0\\0&-1\end{pmatrix} & 1\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{pmatrix}\frac{\hbar}{2}$

	$|++\rangle$	$\frac{\hbar}{2}$
−	$|+-\rangle$	$-\frac{\hbar}{2}$
	$|-+\rangle$	$+\frac{\hbar}{2}$
−	$|--\rangle$	$-\frac{\hbar}{2}$

$\frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$

$\frac{\hbar}{2}$ ... $P_+ = |++\rangle\langle++| + |-+\rangle\langle-+|$

$\boxed{P_+\begin{pmatrix}1&&&\\&0&&\\&&1&\\&&&0\end{pmatrix}}$ $-\frac{\hbar}{2}$ $\boxed{\begin{pmatrix}0&&&\\&1&&\\&&0&\\&&&1\end{pmatrix} = P_-}$

MĚŘENÍ:

můžeme měřit

$$S_z^{(2)} = \begin{cases} +\frac{\hbar}{2} & p_+ = \langle\psi|P_+|\psi\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2} \\ -\frac{\hbar}{2} & p_- = \langle\psi|P_-|\psi\rangle = \frac{1}{2} \end{cases}$$

$$\frac{1}{\sqrt{2}}(1\,1\,0\,0)\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

stav po měření

$$+\frac{\hbar}{2} \quad |\psi'\rangle = \hat{P}_+|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \xrightarrow{n} \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$-\frac{\hbar}{2} \quad |\psi'\rangle = \hat{P}_-|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \xrightarrow{n} \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

---

a) abstraktní zápisy

$$|\psi\rangle = |+\rangle\otimes|x+\rangle = |+\rangle\otimes\frac{1}{\sqrt{2}}\left(|+\rangle+|-\rangle\right)$$

$$S_z^{(2)} = \hat{I}\otimes\hat{S}_z$$

$$\hookrightarrow \frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|$$

$$= \left(\frac{\hbar}{2}\right)\underbrace{\hat{I}\otimes|+\rangle\langle+|}_{P_+}\left(-\frac{\hbar}{2}\right)\underbrace{\hat{I}\otimes|-\rangle\langle-|}_{P_-}$$

$\langle+|-\rangle = 0$

měření:

$$\frac{\hbar}{2} \quad |\psi'\rangle = P_+|\psi\rangle = \hat{I}\otimes|+\rangle\langle+|\left(|+\rangle\otimes\frac{1}{\sqrt{2}}\left(|+\rangle+|-\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}|+\rangle|+\rangle = \frac{1}{\sqrt{2}}|++\rangle$$

$$-\frac{\hbar}{2} \quad |\psi'\rangle = P_-|\psi\rangle$$

$|-\rangle\langle-|$ --- $|\psi'\rangle = \frac{1}{\sqrt{2}}|+-\rangle$

$$p_+ = \langle\psi|\psi'\rangle = \frac{1}{\sqrt{2}}\langle+|\otimes\left(\langle+|+\langle-|\right)\left(\frac{1}{\sqrt{2}}|++\rangle\right)$$

$= \frac{1}{2} \underset{1}{\langle + | + \rangle} \left((\langle + | + \langle - |) | + \rangle \right) = \frac{1}{2}$

$\underset{1}{\langle + | + \rangle} + \underset{0}{\langle - | + \rangle}$

b) nejdříve měření $S_x^{(1)}$ pak a)

$S_x^{(1)} = S_x \otimes I = \frac{\hbar}{2} \left[|x+\rangle\langle x+| \otimes I - |x-\rangle\langle x-| \otimes I \right]$

$\hookrightarrow \frac{\hbar}{2} |x+\rangle\langle x+| - \frac{\hbar}{2} |x-\rangle\langle x-|$

$|\psi\rangle = |+\rangle \otimes |x+\rangle$

1. měření $S_x^{(1)}$

měř. 2

$\frac{\hbar}{2}$. $|\psi'\rangle = P_+^{1x} |\psi\rangle = \left(|x+\rangle\langle x+| \otimes I \right) \left(|+\rangle \otimes |x+\rangle \right) \langle \psi'' \rangle$ 0

$|\psi'\rangle = \left(|x+\rangle \underset{\alpha}{\langle x+|+\rangle} \right) \otimes |x+\rangle$ $\quad |\psi'\rangle$ 0

$-\frac{\hbar}{2}$.

$\frac{1}{\sqrt{2}}$ 1

$(\alpha) |x+\rangle \otimes |x+\rangle \xrightarrow{n} |x+\rangle \otimes |x+\rangle$

$\alpha = \langle x+|+\rangle = \langle + | x+ \rangle^* = \frac{1}{\sqrt{2}}$ $\quad \| \psi' \|^2 = \langle \psi' | \psi' \rangle = \underset{1}{\langle x+|x+\rangle} \cdot \underset{1}{\langle x+|x+\rangle}$

$\frac{\hbar}{2}$ ... $|\psi'\rangle = P_+^{1x} |\psi\rangle \xrightarrow{\text{norm}} |x+\rangle |x+\rangle$

$-\frac{\hbar}{2}$ ... $|\psi'\rangle = P_-^{1x} |\psi\rangle \xrightarrow{\text{norm}} |x-\rangle |x+\rangle$

---

měření a v horní větvi ... stav před měř.

$|x+\rangle |x+\rangle$

$$S_z^2 = \frac{\hbar}{2}\left(\hat{I}\otimes|+\rangle\langle+| - \hat{I}\otimes|-\rangle\langle-|\right)$$

$\frac{\hbar}{2}$ stav .. $P_+|\psi'\rangle = |\psi''\rangle = \left(\hat{I}\otimes|+\rangle\langle+|\right)|x+\rangle|x+\rangle$

$= |x+\rangle|+\rangle\,\mathcal{B}$ $\qquad \mathcal{B} = \langle+|x+\rangle = \frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}|x+\rangle|+\rangle.$

$-\frac{\hbar}{2}$ $\qquad P_-|\psi'\rangle = |\psi''\rangle = \frac{1}{\sqrt{2}}|x+\rangle|-\rangle$

pravděpodobnosti: $p_+ = \||\psi''\rangle\|^2 = \frac{1}{2}$

$$\||\psi''\rangle\|^2 \equiv \langle\psi''|\psi''\rangle = \langle\psi'|P_+^\dagger P_+|\psi'\rangle = \underbrace{\langle\psi'|P_+|\psi'\rangle}_{p_+}$$

$P_+^\dagger = P_+ \qquad P_+^2 = P_+$

$$\frac{1}{\sqrt{2}}\langle x+|\langle+|\,\frac{1}{\sqrt{2}}|x+\rangle|+\rangle = \frac{1}{2}\underbrace{\langle x+|x+\rangle}_{=1}\underbrace{\langle+|+\rangle}_{=1}$$

měř. 1 S_x^1 $\qquad p$ $\qquad$ měř. 2 S_z^2

$|\psi\rangle$ $\qquad |+\rangle|x+\rangle$ (1, 2)

- $\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|\psi'\rangle = |x+\rangle|x+\rangle$
  - $\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|x+\rangle|+\rangle$
  - $-\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|x+\rangle|-\rangle$
- $-\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|\psi'\rangle = |x-\rangle|x+\rangle$
  - $\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|x-\rangle|+\rangle$
  - $-\frac{\hbar}{2}$ .. $\frac{1}{2}$ .. $|x-\rangle|-\rangle$

$\boxed{\text{U3}}$

$$|\psi\rangle = |+\rangle \overset{\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)}{|x+\rangle} + |x{:}+\rangle|+\rangle$$

$$\frac{1}{\sqrt{2}}\Big(|+\rangle\big(|+\rangle+|-\rangle\big) + \big(|+\rangle+|-\rangle\big)|+\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{|++\rangle + |+-\rangle + |++\rangle}\, + |-+\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(2|++\rangle + |+-\rangle + |-+\rangle\Big)$$

$$\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\\0\end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \leftarrow$$

$\begin{pmatrix}1\\1\end{pmatrix}$ ... $|x+\rangle$ ... vl. v. S_x ... $+\frac{\hbar}{2}$

$\begin{pmatrix}1\\-1\end{pmatrix}$ $|x-\rangle$ .. vl. v. S_x .. $-\frac{\hbar}{2}$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = 1+1 = 2$ .. $\frac{1}{\sqrt{\|\ \|}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix} = |+\rangle$ $\quad \begin{pmatrix}0\\1\end{pmatrix} = |-\rangle$

$$\begin{pmatrix}1\\1\end{pmatrix} = |+\rangle + |-\rangle$$

... $|x+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big)$